Week 6 Tutorial

Induction

Announcements

- Reminder that the first round of revisions for Midterm Exam 2 are due Sunday noon PDT.
  - We're holding extra office hours specifically to talk about the exam. Check the course calendar for details.
  - We're creating videos to offer advice on each of the problems. ***Please watch this videos before stopping by our office hours***; they cover the most common errors we saw in each problem.
- Please ensure that you're reading the feedback from TAs on your problem sets.
  - Many of the mistakes we saw on the exam were similar to errors we see on the problem sets.
  - It's hard to improve a skill if you don't get any external feedback!

Part 1: ***An Induction Game!***

Rules

- Start with a pile of n coins for some $n \geq 0$
- Players take turns removing between 1 and 5 coins from the pile.
- The player who has no more coins to remove loses the game.
- Interestingly, if the pile begins with a multiple of 6 coins in it, the second player can always win if they play correctly – give it a try!

Rules

- Start with a pile of n coins for some $n \geq 0$
- Players take turns removing between 1 and 5 coins from the pile.
- The player who has no more coins to remove loses the game.
- Interestingly, if the pile begins with a multiple of 6 coins in it, the second player can always win if they play correctly – give it a try!

1a) Play a few rounds of this game and describe the winning strategy for the second player.

Fill in answer on Gradescope!

What’s the strategy?

Some Observations

- If it’s the first player’s turn and there are no coins left, then the second player wins

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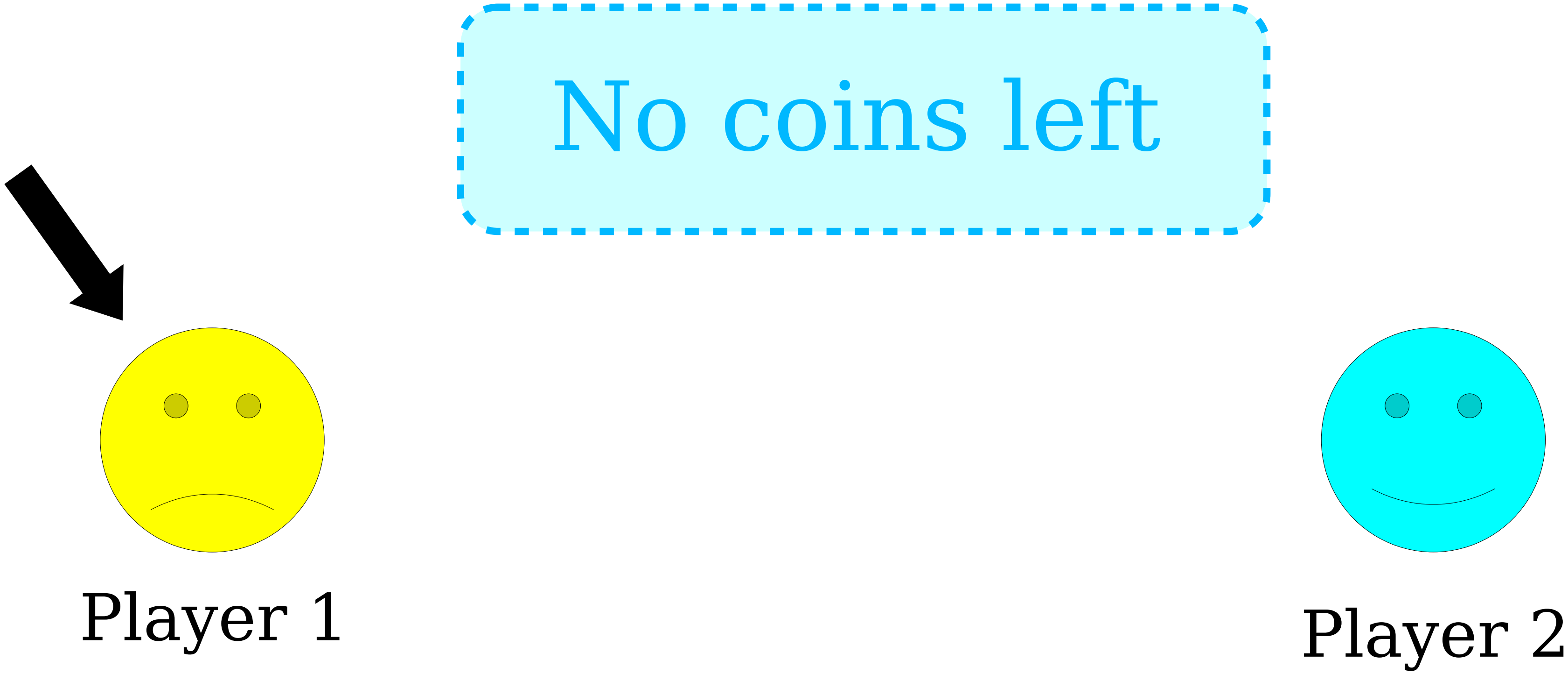
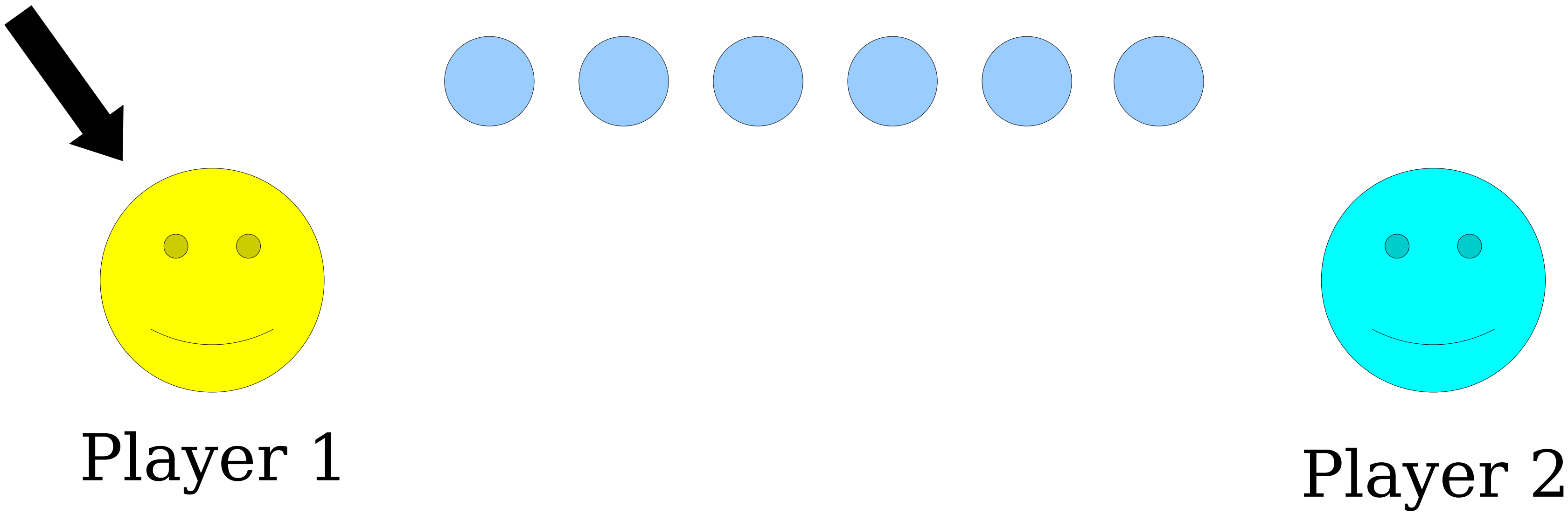


Some Observations

- If it's the first player's turn and there are no coins left, then the second player wins
- If we start with 6 coins, player 1 has to remove some but not all of the coins.



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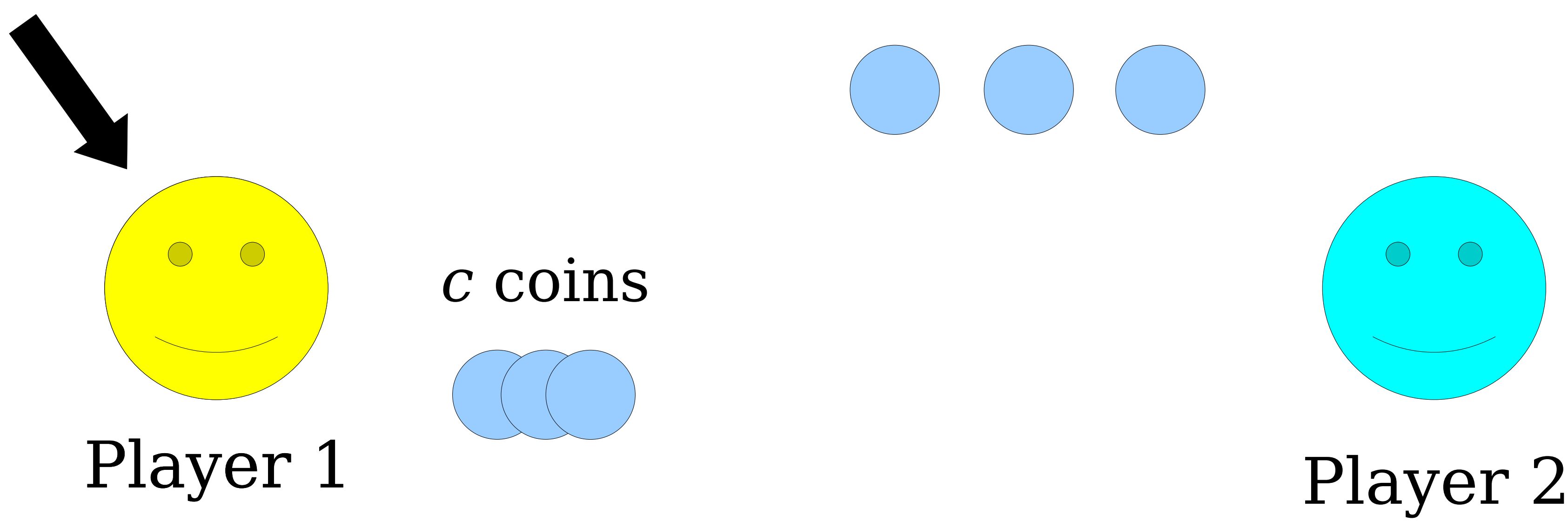
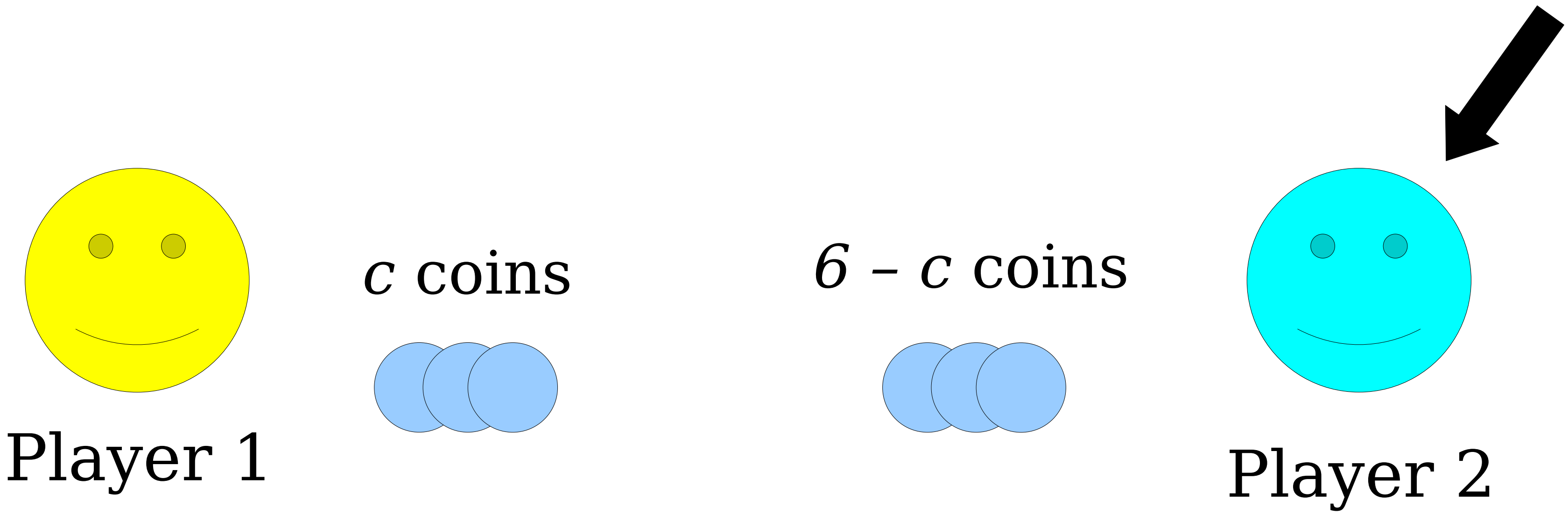


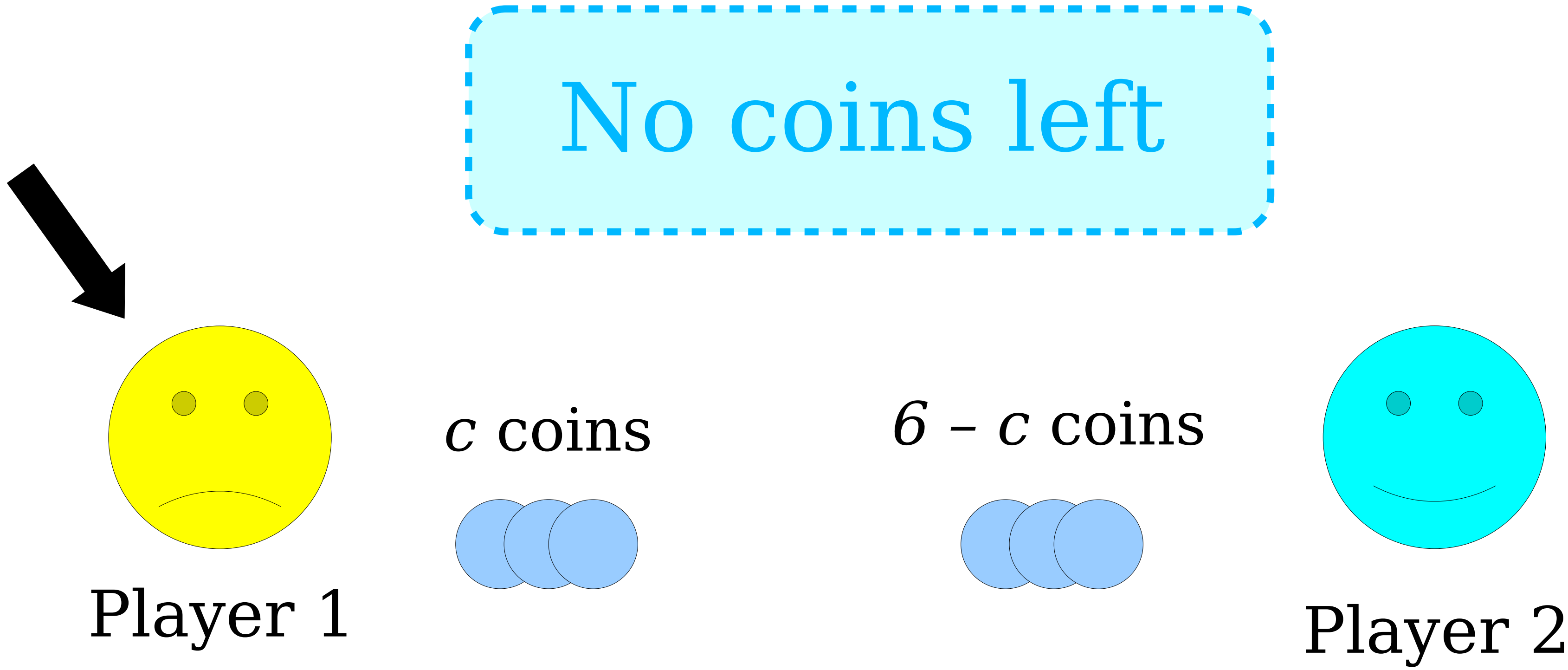
Some Observations

- If it's the first player's turn and there are no coins left, then the second player wins
- If we start with 6 coins, player 1 has to remove some but not all of the coins. Then player 2 can remove the remaining coins



Some Observations

- If it's the first player's turn and there are no coins left, then the second player wins
- If we start with 6 coins, player 1 has to remove some but not all of the coins. Then player 2 can remove the remaining coins, leaving us in a known winning state.



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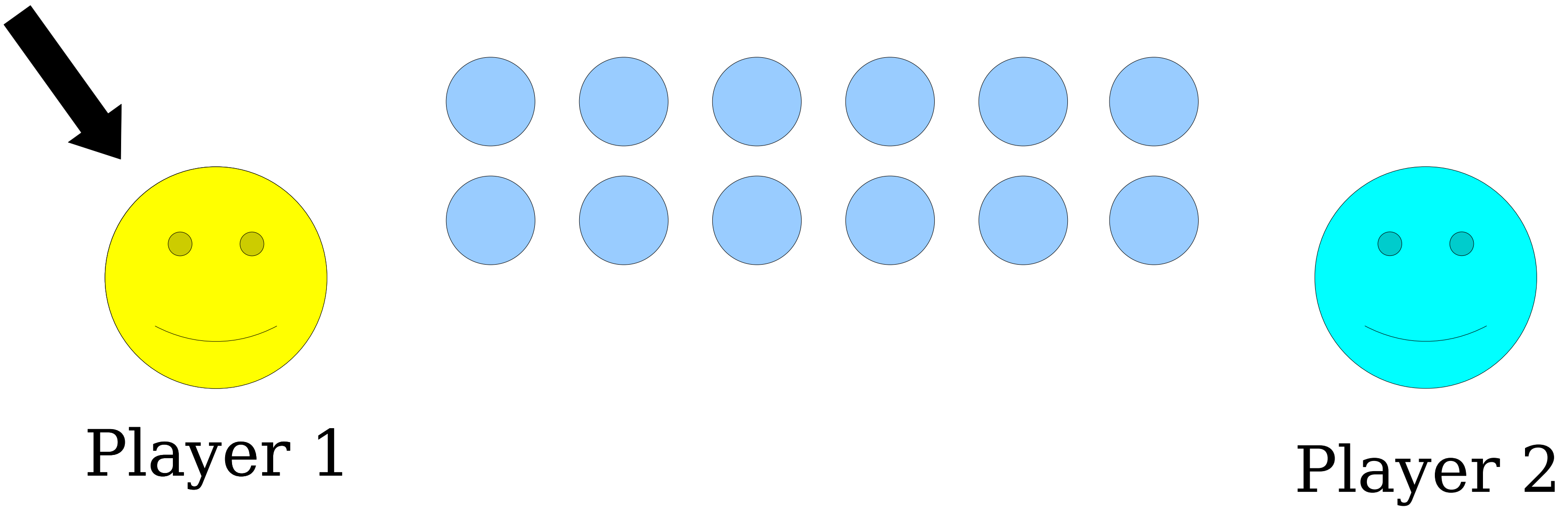
- If it's the first player's turn and there are no coins left, then the second player wins
- If we start with 6 coins, player 1 has to remove some but not all of the coins. Then player 2 can remove the remaining coins, leaving us in a known winning state.
- What happens when there are 12 coins?



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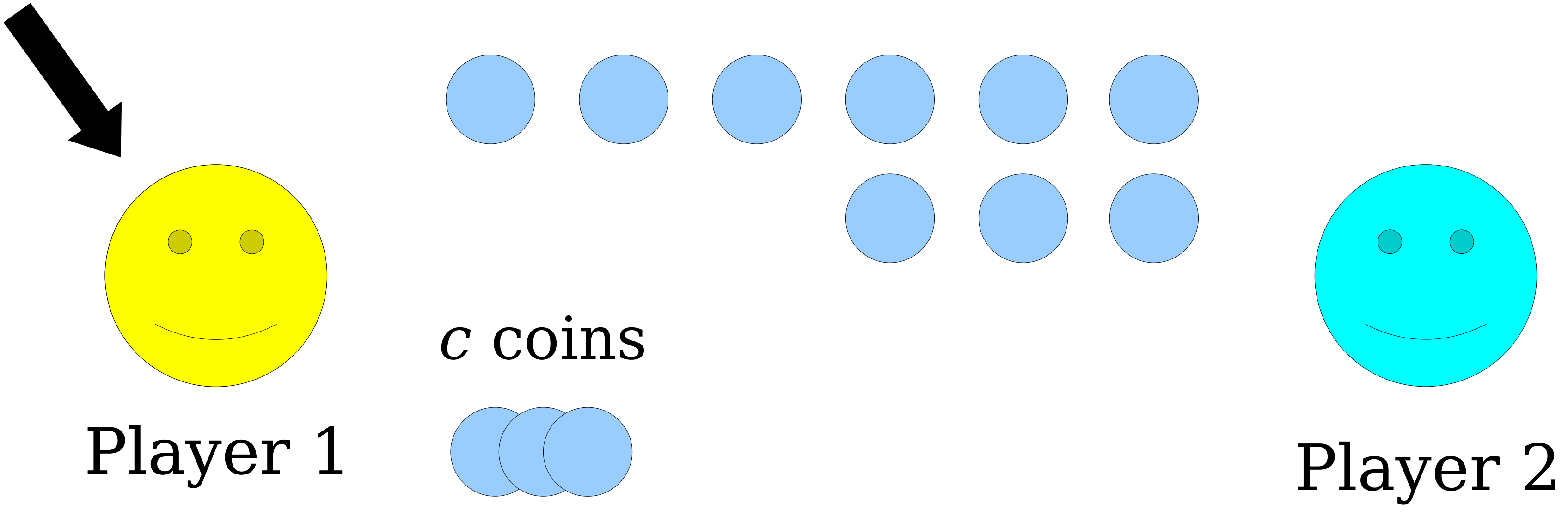
- What happens when there are 12 coins? Player 1 removes some coins



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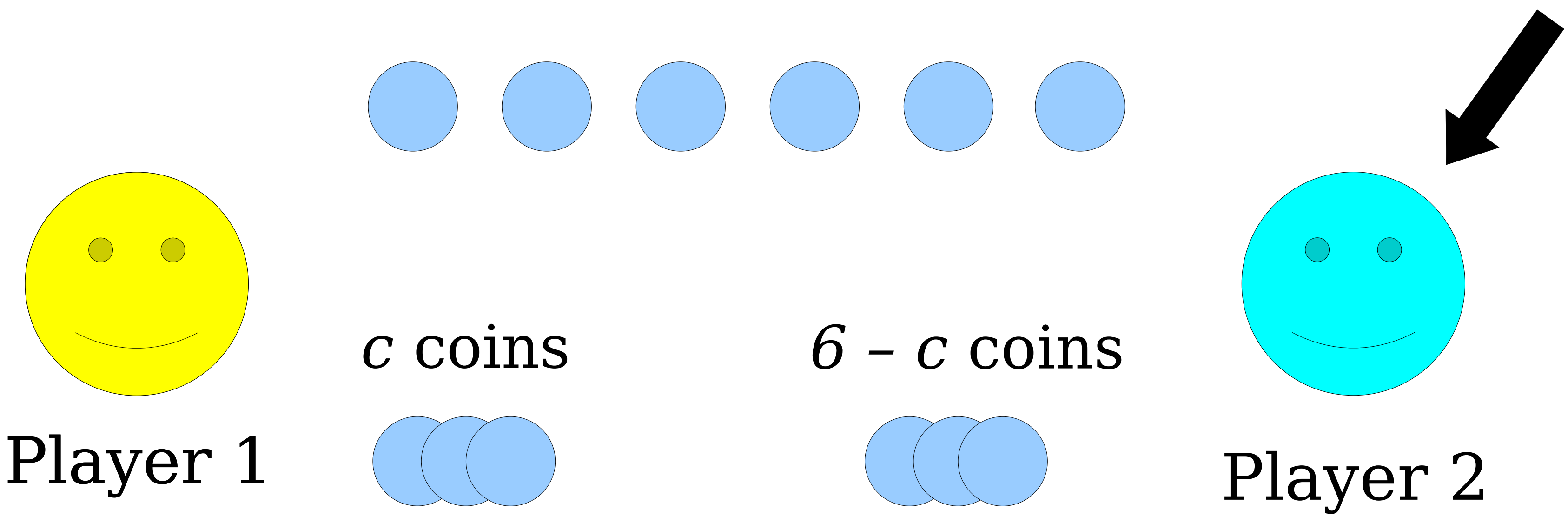
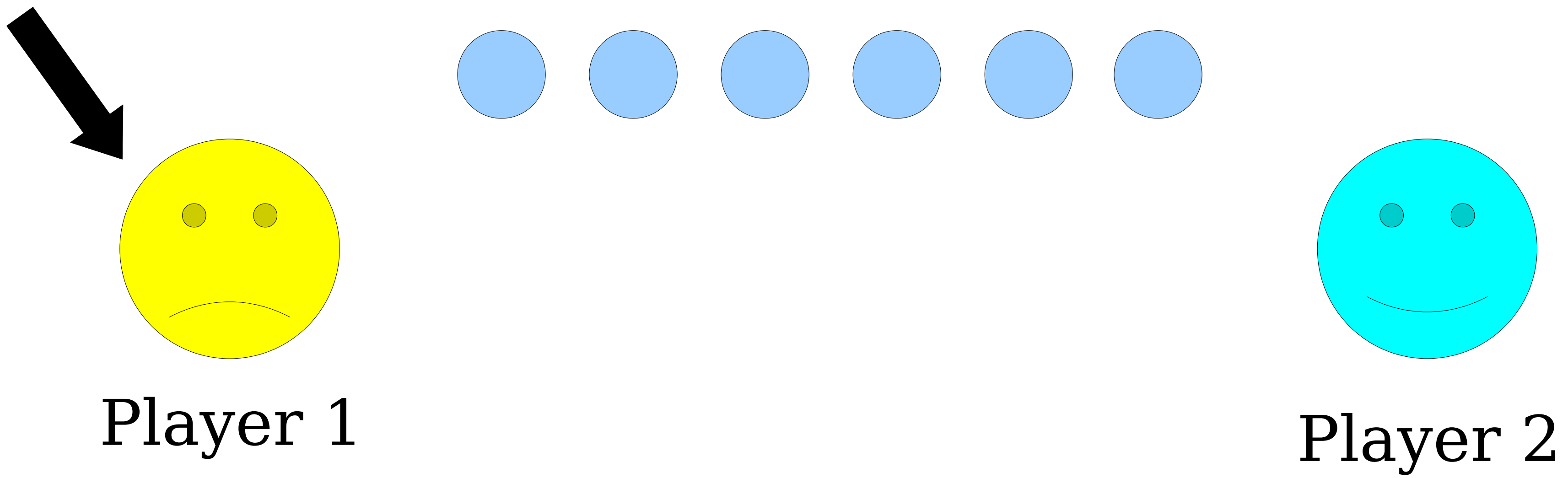
- If we start with 6 coins, player 1 has to remove some but not all of the coins. Then player 2 can remove the remaining coins, leaving us in a known winning state.
- What happens when there are 12 coins? Player 1 removes some coins, but then player 2 can remove the right number of coins to leave 6 remaining.



Some Observations

- If it’s the first player’s turn and there are no coins left, then the second player wins
- If we start with 6 coins, player 1 has to remove some but not all of the coins. Then player 2 can remove the remaining coins, leaving us in a known winning state.
- What happens when there are 12 coins? Player 1 removes some coins, but then player 2 can remove the right number of coins to leave 6 remaining. It’s player 1’s turn again and there are 6 coins, again a known winning state.



Strategy: The second player can win by making the total number of coins removed by their move and the first player's move come out to 6.

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It is a *great* idea to ***try small cases*** before jumping into a formal proof. It will be much easier to formalize the logic here now that you have a feel for how to play the game.

For all games where the number of coins is a multiple of 6, the second player can always win if they play correctly.

1b) Answer the following questions:

- What is $P(n)$?
- What is the base case?
- What is the step size?
- Is $P(n)$ universally or existentially quantified? Based on that, should we build up or build down?

Fill in answer on Gradescope!

For all games where the number of coins is a multiple of 6, the second player can always win if they play correctly.

What is $P(n)$?

What is the base case?

What is the step size?

Is $P(n)$ universally or existentially quantified? Based on that, should we build up or build down?

For all games where the number of coins is a multiple of 6, the second player can always win if they play correctly.

What is $P(n)$?

Let P(n) be the statement “if the game is played with the pile containing n coins, the second player can always win if she plays correctly.”

What is the base case?

What is the step size?

Is $P(n)$ universally or existentially quantified? Based on that, should we build up or build down?

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What is $P(n)$?

Let P(n) be the statement "if the game is played with the pile containing n coins, the second player can always win if she plays correctly."

What is the base case?

The base case is n=0, the simplest possible case of the game is when you start with no coins.

What is the step size?

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The base case is n=0, the simplest possible case of the game is when you start with no coins.

What is the step size?

We want to show the result is true for multiples of 6, so we'll take steps of size 6.

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Is $P(n)$ universally or existentially quantified? Based on that, should we build up or build down?

P(n) is universally quantified, so we should build down (start with a game of size k+6 and figure out how to reduce it to a game of size k)

What’s wrong with this proof?

Incorrect! ***Proof:*** Let $P(n)$ be the statement "if the game is played with the pile containing n coins, the second player can always win if she plays correctly." We will prove by induction that $P(n)$ holds for all natural numbers n that are multiples of 6, from which the theorem follows.

As a base case, we will prove $P(0)$, that if the game is played with a pile containing 0 coins, the second player always can win. This is true because there are no coins in the pile, so no matter what the second player does, she'll win because the first player loses.

For the inductive step, we will prove that if $P(k)$, then $P(k + 6)$: that is, the second player can always win in a game with $k+6$ coins if she plays correctly.

Suppose the game starts with k coins. By the inductive hypothesis, this means that the second player can force a win in this situation. Now we can turn this into a game of size $k+6$ by adding 6 coins and a turn where the first player removes some number c coins from the pile (where $1 \leq c \leq 5$) and a turn where the second player removes $6–c$ coins. Consequently, $P(k) \rightarrow P(k+6)$, completing the induction. ■

1c) What's wrong with this proof? Try to identify three errors the proof makes.

Fill in answer on Gradescope!

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Suppose the game start his
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can turn this into a gam here
the first player removes
$1 \leq c \leq 5$) and a turn w
Consequently, $P(k) \rightarrow P($

We need to explicitly assume $P(k)$ here. The variable k is also not properly instantiated. When you are writing an assumption or introducing variables, you need to do so using a declarative verb ("assume", "pick", "choose", etc.)

Incorrect! ***Proof:*** Let $P(n)$ be the statement "if the game is played with the pile … win if she plays correc… all natural numbers n t… ows.

As a base ca… ith a pile containing 0… rue because the… econd player does,…

For the indu… : that is, the second p… she plays correctly.

This is "building up" instead of "building down". Since the statement we're trying to prove is a universal statement (all games of size k+6 have this property), we need to start with an arbitrary game of size size k+6 instead of a game of size k.

Suppose the game starts with k coins. By the inductive hypothesis, this means that the second player can force a win in this situation. Now we can turn this into a game of size $k+6$ by adding 6 coins and a turn where the first player removes some number c coins from the pile (where $1 \leq c \leq 5$) and a turn where the second player removes $6-c$ coins. Consequently, $P(k) \to P(k+6)$, completing the induction. ■

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Fo…), then $P(k + 6)$: that is,
th… th k+6 coins if she plays
co…

Lastly, proofs should not contain first-order logic even if the definitions you're working with are given in FOL!

St… nductive hypothesis, this
m… n this situation. Now we
ca… g 6 coins and a turn where
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For the inductive step, assume for some arbitrary $k \in \mathbb{N}$ where k is a multiple of 6 that $P(k)$ is true and if the game is played with k coins, the second player can always win if she plays correctly. We will prove that $P(k + 6)$ holds: that is, the second player can always win in a game with $k + 6$ coins if she plays correctly.

Suppose the game starts with $k+6$ coins. The first player's removes some number c coins from the pile, where $1 \leq c \leq 5$. This leaves $k+6–c$ coins remaining. Now, the second player removes $6–c$ coins. This leaves a total of $k+6–c–(6–c) = k$ coins, and it's now the first player's turn again. By the inductive hypothesis, this means that the second player can force a win in this situation, so the second player will eventually win the game. Consequently, starting with $k+6$ coins, the second player can win, so $P(k+6)$ holds, completing the induction. ■

Part 2: ***How Not to Induct***

All Horses are the Same Color

$P(n)$ = "All groups of n horses always have the same color"

All Horses are the Same Color

$P(0)$ = "All groups of 0 horses always have the same color"

Vacuously true!

Base case: $n = 0$

All Horses are the Same Color

Assume $P(k)$ = "All groups of k horses always have the same color"

Inductive hypothesis: $n = k$

All Horses are the Same Color

Prove $P(k+1)$ = “All groups of $k+1$ horses always have the same color”

Inductive hypothesis: $n = k+1$

All Horses are the Same Color

Prove $P(k+1)$ = "All groups of $k+1$ horses always have the same color"

By $P(k)$, these k horses have the same color

Inductive hypothesis: $n = k+1$

All Horses are the Same Color

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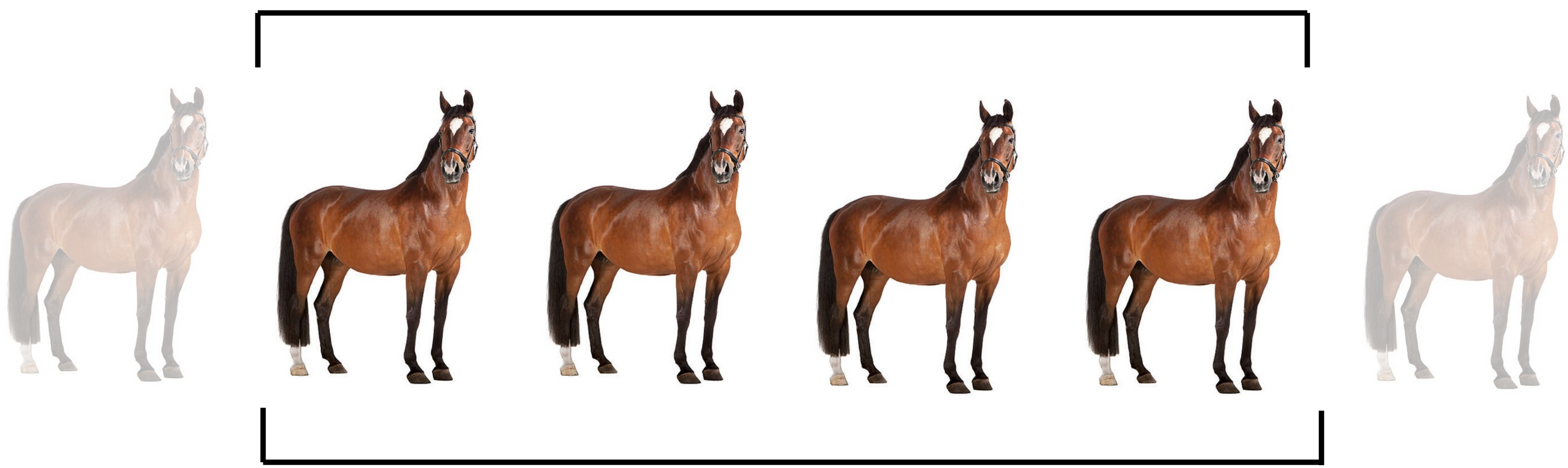
Inductive hypothesis: $n = k+1$

All Horses are the Same Color

Prove $P(k+1)$ = "All groups of $k+1$ horses always have the same color"

These horses in the middle were in both sets



Inductive hypothesis: $n = k+1$

All Horses are the Same Color

Prove $P(k+1)$ = “All groups of $k+1$ horses always have the same color”

These horses in the middle were in both sets

And we said that both horses on the ends are the same color as these overlapping horses

Inductive hypothesis: $n = k+1$

All Horses are the Same Color

Prove $P(k+1)$ = "All groups of $k+1$ horses always have the same color"

So all $k+1$ horses have the same color!

Inductive hypothesis: $n = k+1$

Incorrect! Proof: Let $P(n)$ be the statement "all groups of n horses are the same color." We will prove by induction that $P(n)$ holds for all natural numbers n, from which the theorem follows.

As our base case, we prove $P(0)$, that all groups of 0 horses are the same color. This statement is vacuously true because there are no horses.

For the inductive step, assume that for an arbitrary natural number k that $P(k)$ is true and that all groups of k horses are the same color. Now consider a group of $k+1$ horses. Exclude the last horse and look only at the first k horses. By the inductive hypothesis, these horses are the same color. Next, exclude the first horse and look only at the last k horses. Again we see by the inductive hypothesis that these horses are the same color.

Therefore, the first horse is the same color as the non-excluded horses, who in turn are the same color as the last horse. Hence the first horse excluded, the non-excluded horses, and last horse excluded are all of the same color. Thus $P(k+1)$ holds, completing the induction. ■

2) What's wrong with this proof?

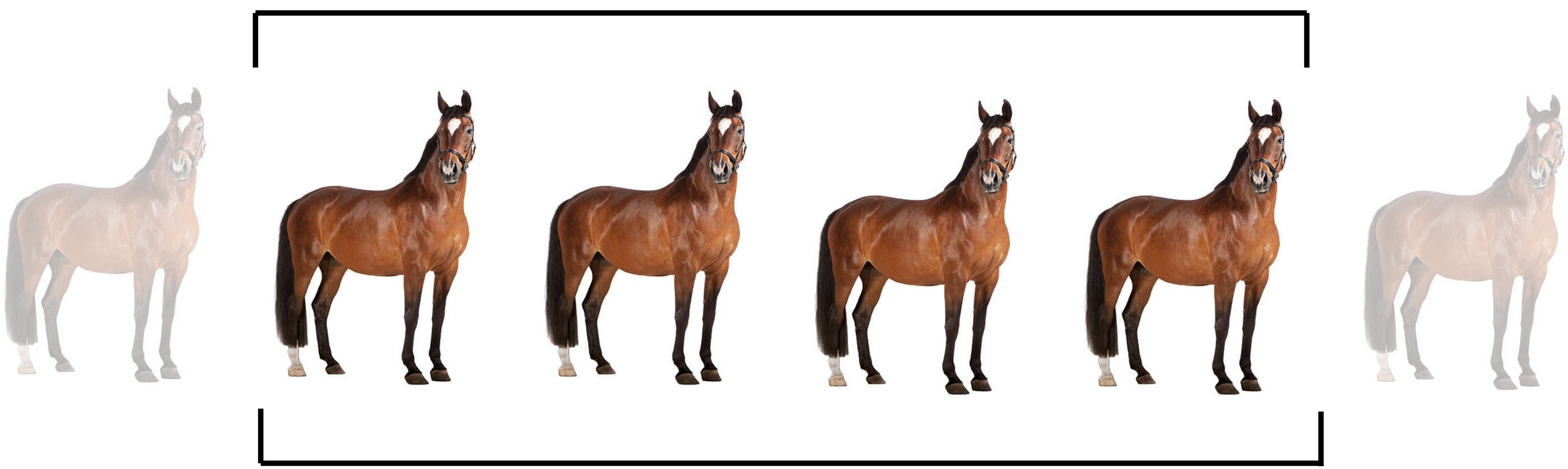
Fill in answer on Gradescope!

What’s going on here?

All Horses are the Same Color

Prove $P(k+1)$ = “All groups of $k+1$ horses always have the same color”

These horses in the middle were in both sets



Inductive hypothesis: $n = k+1$

All Horses are the Same Color

Prove $P(k+1)$ = "All groups of $k+1$ horses always have the same color"

These horses in the middle were in both sets

But what if there are no such horses?

Inductive hypothesis: $n = k+1$

All Horses are the Same Color

$P(n)$ = "All groups of n horses always have the same color"

$$P(1) \rightarrow P(2)$$

All Horses are the Same Color

$P(n)$ = "All groups of n horses always have the same color"

By $P(1)$, this 1 horse has the same color

$$P(1) \to P(2)$$

All Horses are the Same Color

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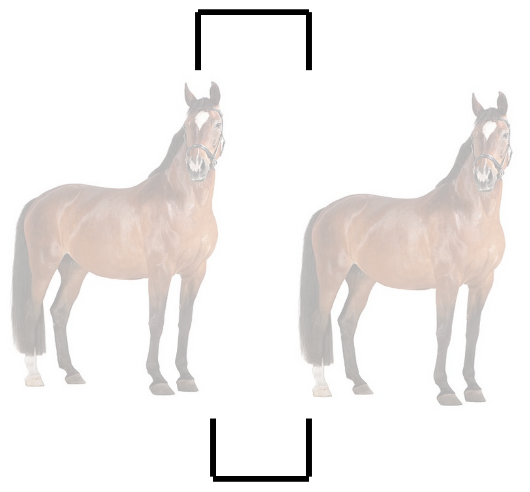
$$P(1) \rightarrow P(2)$$

All Horses are the Same Color

$P(n)$ = "All groups of n horses always have the same color"

These horses in the middle (??) were in both sets



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For the inducti | al number k
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consider a grou | nd look only at
the first k horse | es are the
same color. Ne | the last k
horses. Again v | se horses are
the same color.

The logic in our inductive step does not allow us to get from P(1) to P(2). Specifically, there are no non-excluded horses that were in both sets.

Therefore, the first horse is the same color as the non-excluded horses, who in turn are the same color as the last horse. <u>Hence the first horse excluded, the non-excluded horses, and last horse excluded are all of the same color.</u> Thus $P(k+1)$ holds, completing the induction. ■

Non-Issues with this Proof

- *"We should have proven additional base cases"*
  - A proof by induction only needs a single base case, so the fact that we only have one here is not in itself an issue.
- *"We should have used complete induction"*
  - Complete induction wouldn't have helped us here either, since our inductive step would still need to use $P(0)$ and $P(1)$ to prove $P(2)$.

Induction Debugging Tips

- Remember that induction requires two parts: the base case and the inductive step
- If you see an induction proof of a false statement, one of these pieces must be broken
- Recommendation: try playing the induction out one step at a time (Is the base case true? From the base case, does the reasoning in your inductive step allow you to conclude the next statement? What about the following statement? The one after that? etc.)

Thanks for Calling In!

Stay safe, stay healthy,
and have a good week!

See you next time.