Mathematical Proofs

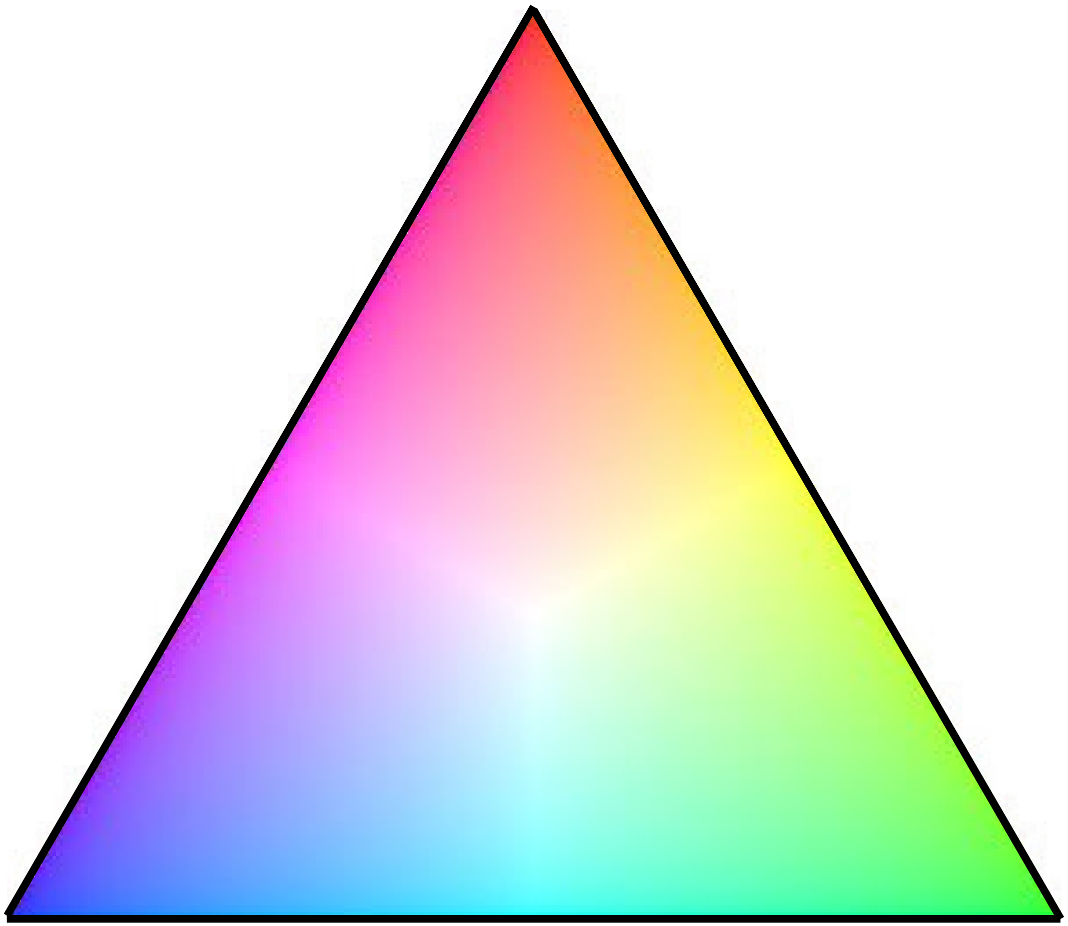
Outline for Today

- ***How to Write a Proof***
  - Synthesizing definitions, intuitions, and conventions.
- ***Proofs on Numbers***
  - Working with odd and even numbers.
- ***Universal and Existential Statements***
  - Two important classes of statements.
- ***Variable Ownership***
  - Who owns what?

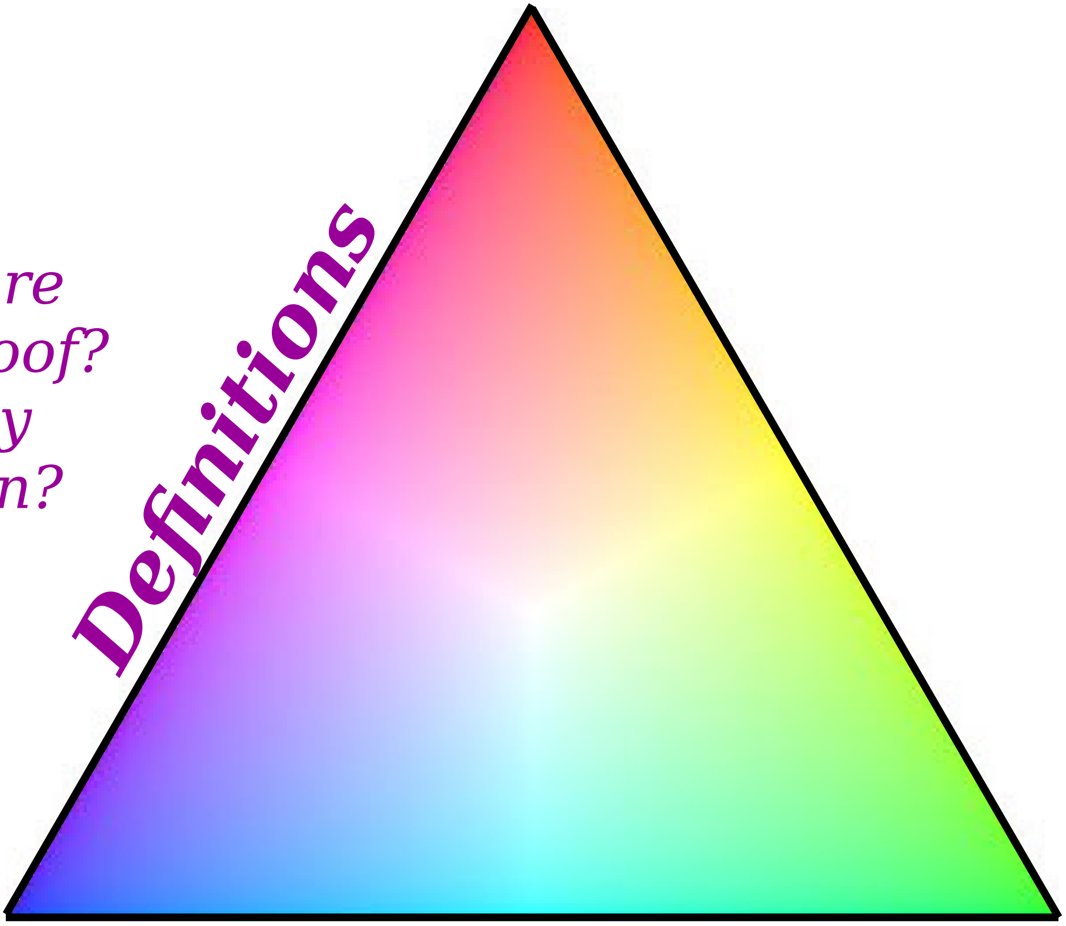
What is a Proof?

A ***proof*** is an argument that demonstrates why a conclusion is true, subject to certain standards of truth.

A ***mathematical proof*** is an argument that demonstrates why a mathematical statement is true, following the rules of mathematics.



What terms are used in this proof? What do they formally mean?



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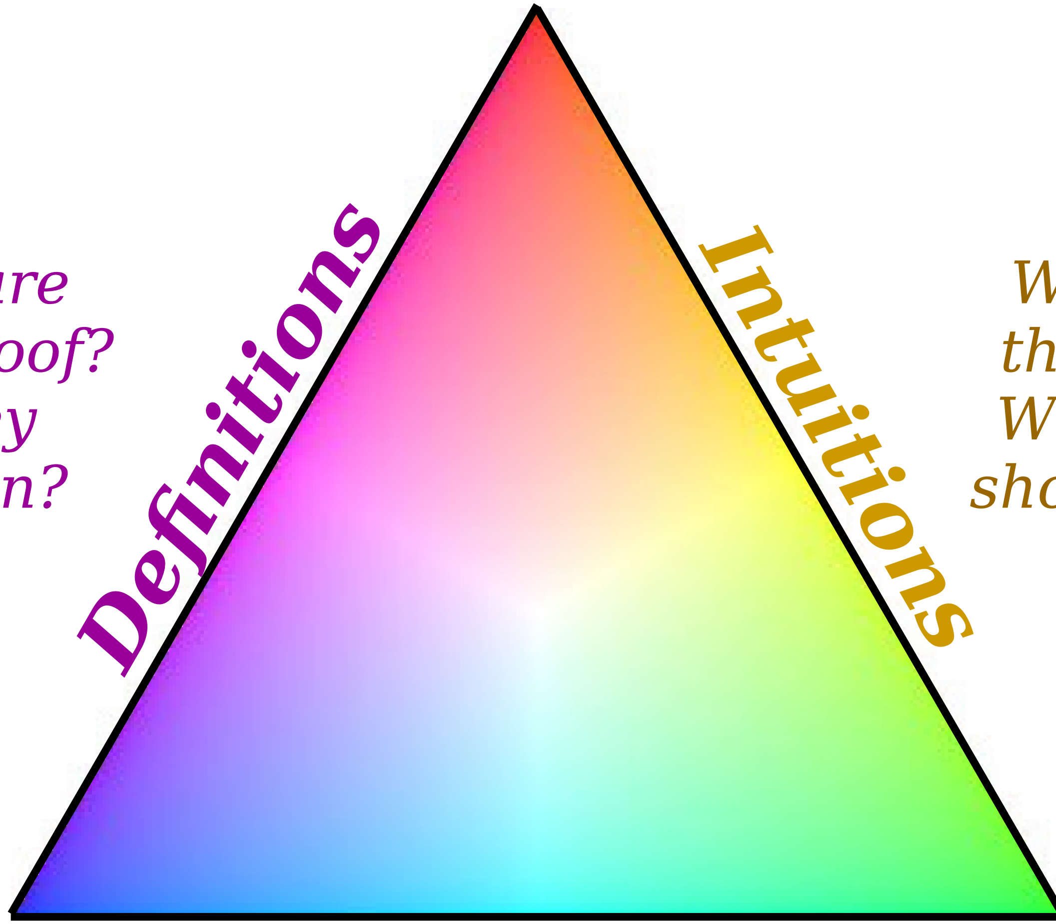

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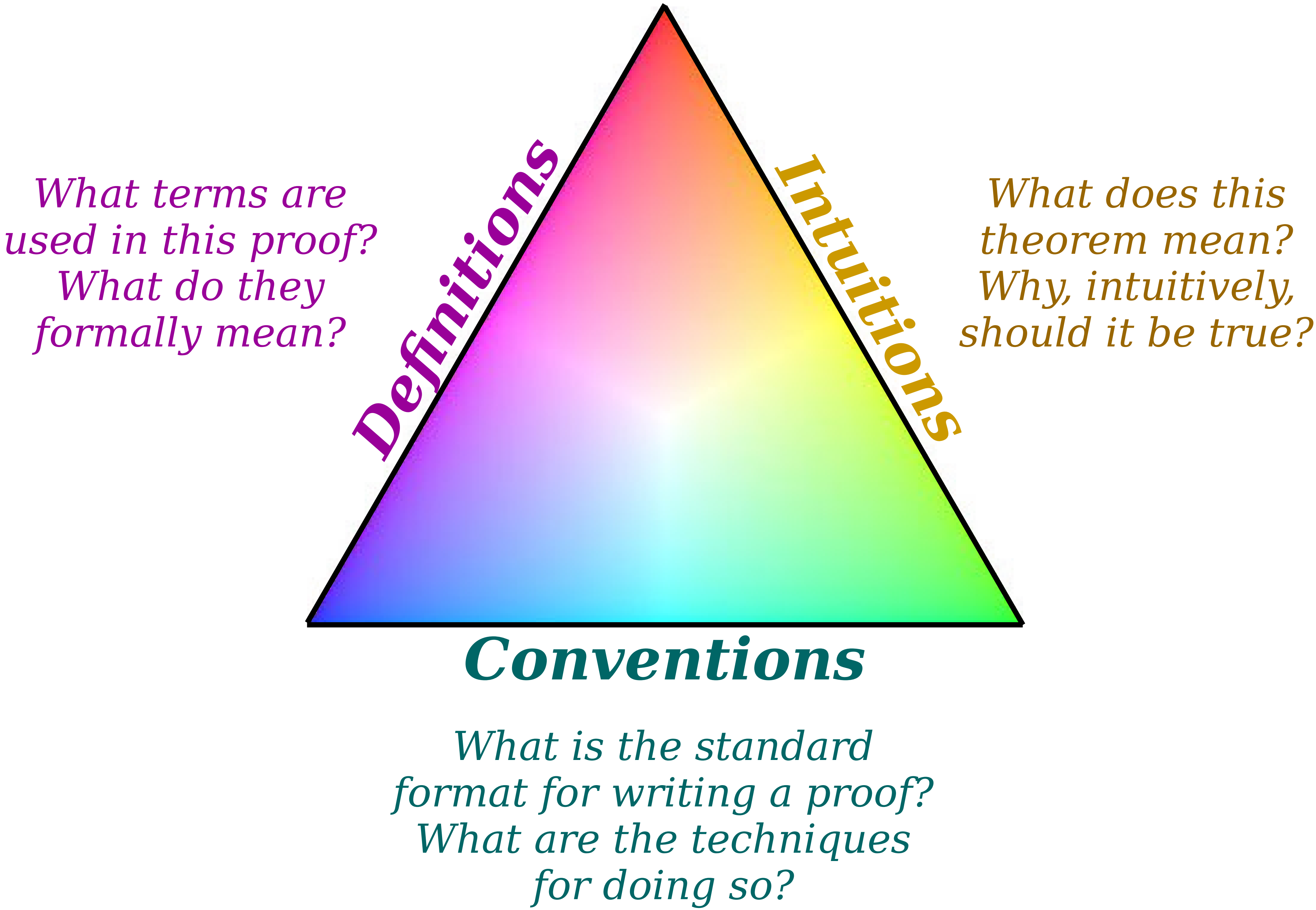
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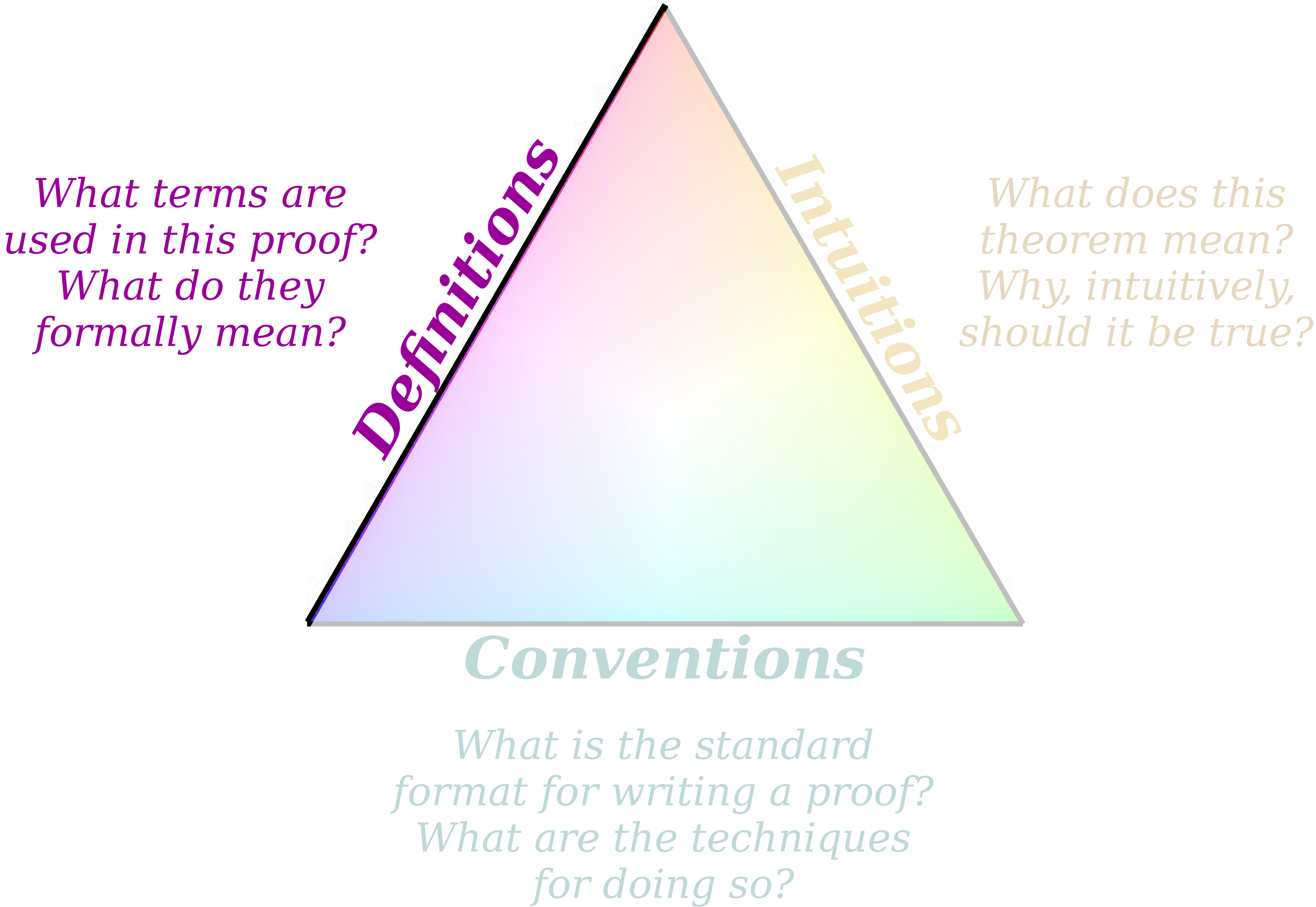
Writing our First Proof

Theorem: If n is an even integer, then n^2 is even.



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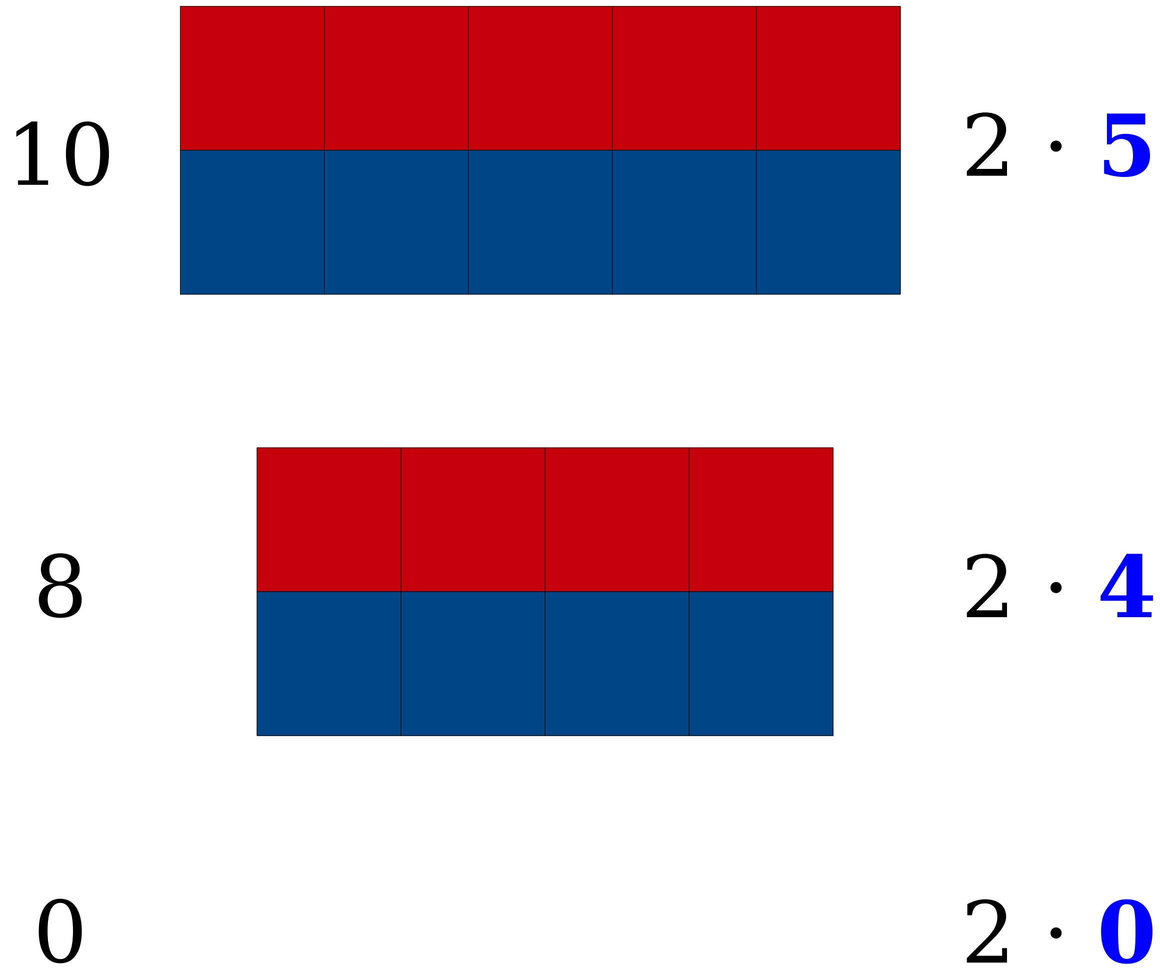
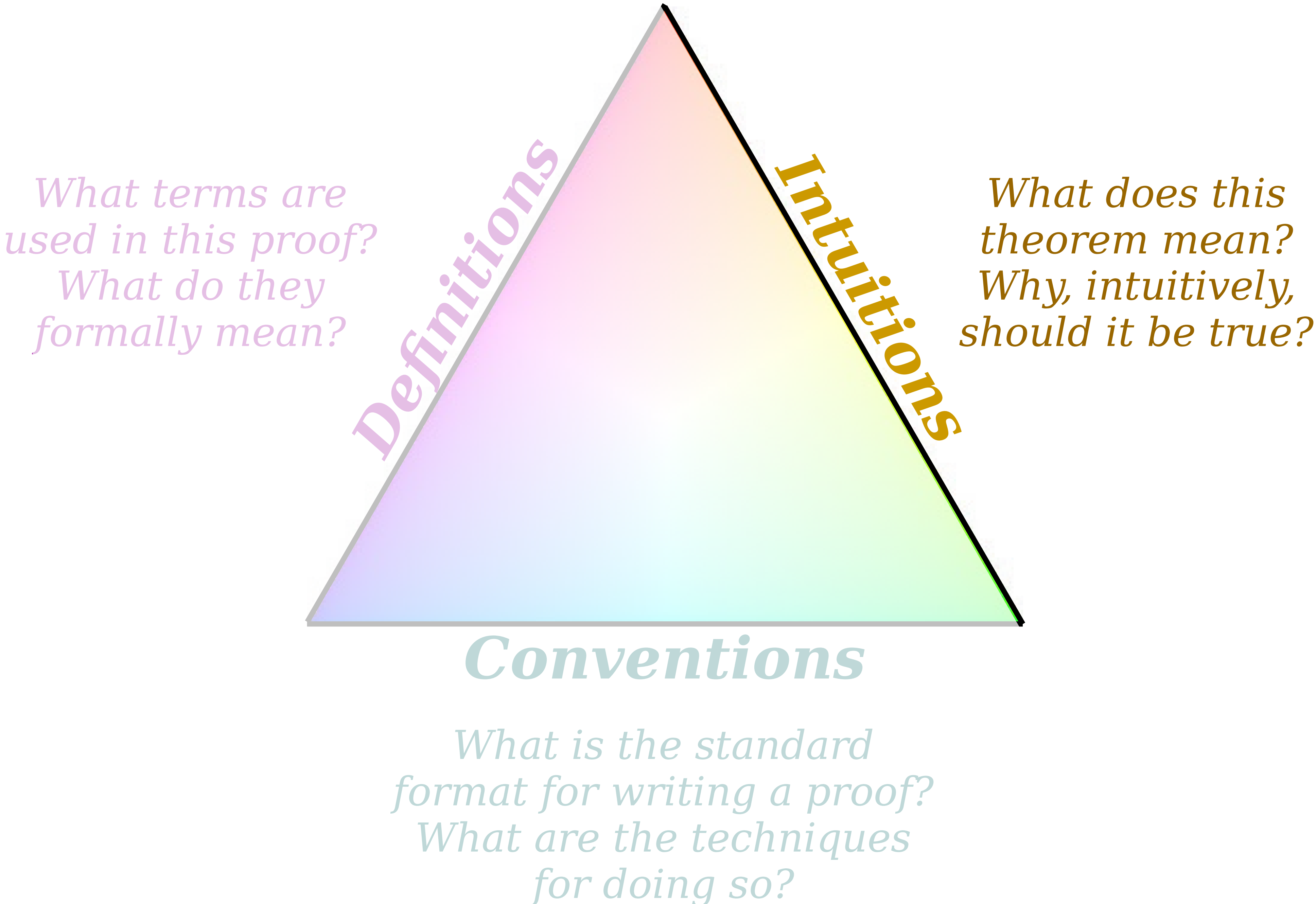


An integer n is called ***even*** if
there is an integer k where $n = 2k$.

Theorem: If n is an even integer, then n^2 is even.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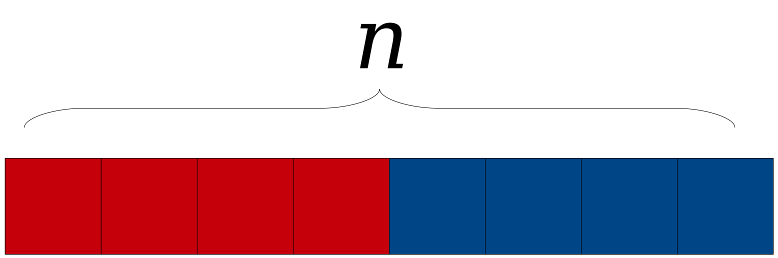
Let's Try Some Examples!

$$
\begin{aligned}
2^2 &= 4 &&= 2 \cdot \mathbf{2} \\
10^2 &= 100 &&= 2 \cdot \mathbf{50} \\
0^2 &= 0 &&= 2 \cdot \mathbf{0} \\
(-8)^2 &= 64 &&= 2 \cdot \mathbf{32} \\
n^2 & &&= 2 \cdot \mathbf{?}
\end{aligned}
$$

What's the pattern? How do we predict this?

Theorem: If n is an even integer, then n^2 is even.

Let's Draw Some Pictures!



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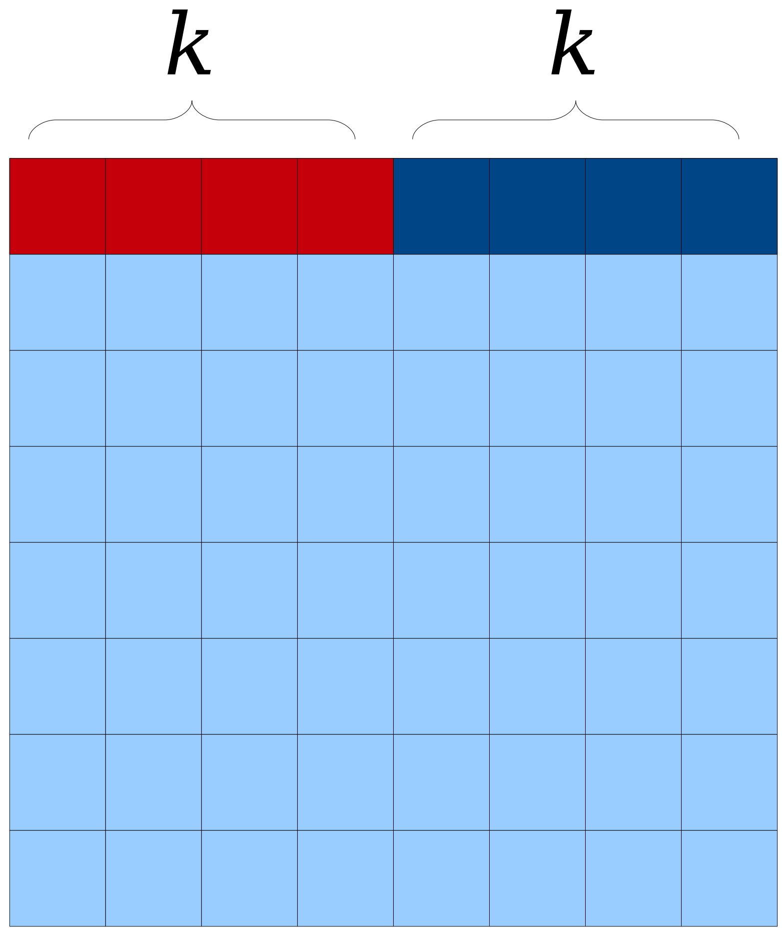

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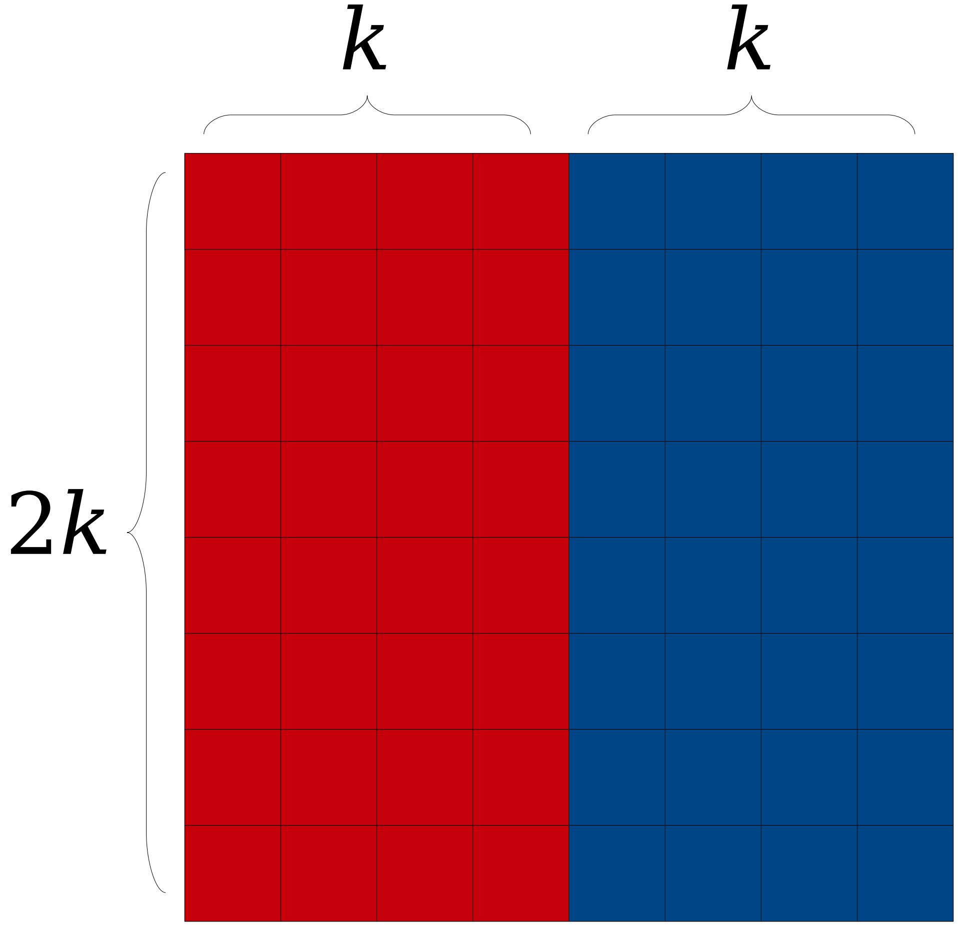

$n^2 = 2(2k^2)$

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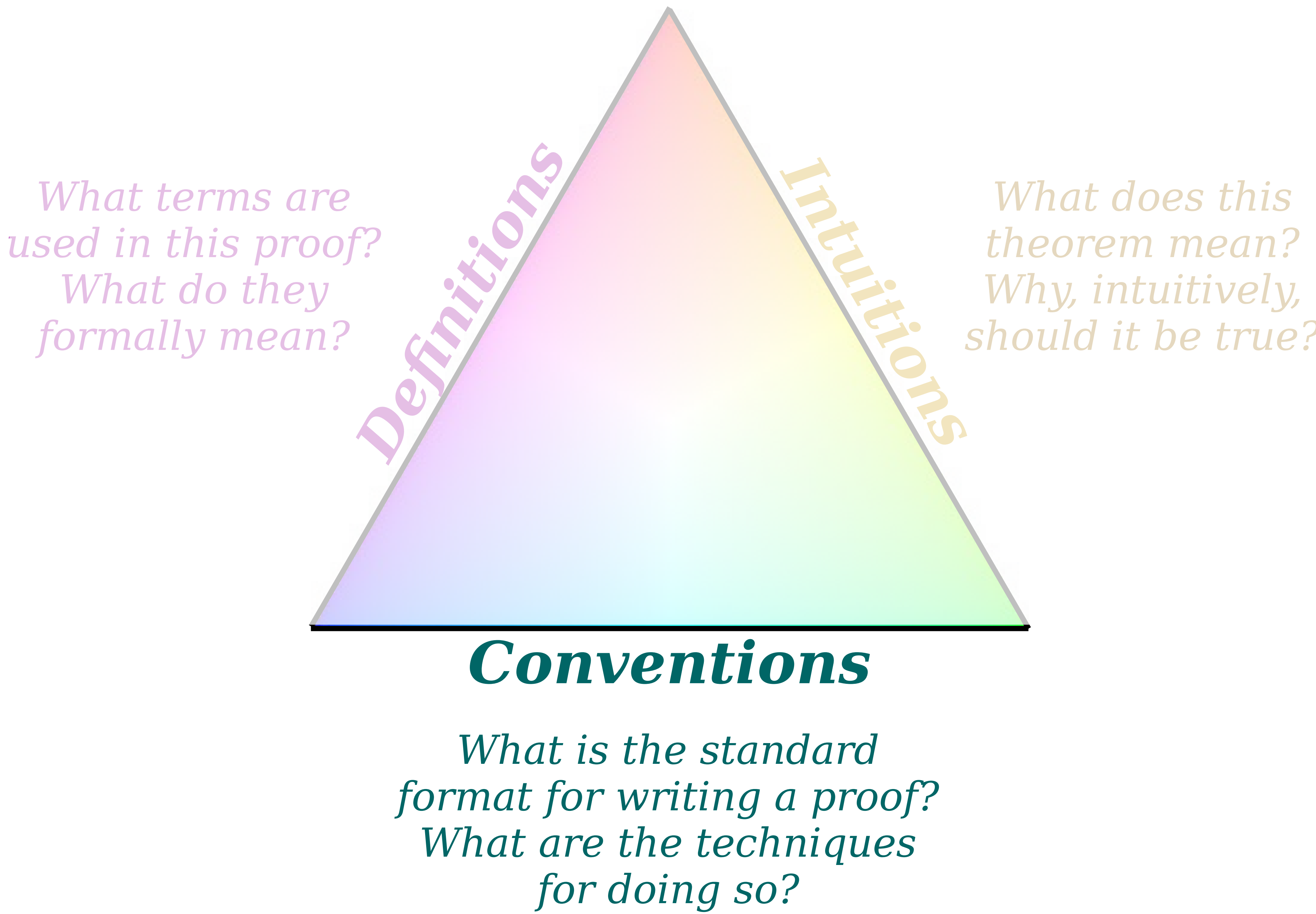
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$$n^2 = (2k)^2$$

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$$\begin{aligned} n^2 &= (2k)^2 \\ &= 4k^2 \end{aligned}$$

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This symbol means "end of proof"

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To prove a statement of the form

"If *P* is true, then *Q* is true,"

start by assuming that ***P*** is true. Here, we're inviting the reader to pick their favorite even integer.

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To prove a statement of the form

"If *P* is true, then *Q* is true,"

after assuming ***P*** is true, you need to show that ***Q*** is true. Here, we're telling the reader where we're headed.

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This is the definition of an even integer. We need to use this definition to make this proof rigorous.

From this, we se

(namely, $2k^2$) where $n^2 = 2m$. Therefore, n^2 is even, which is what we wanted to show. ■

Our First Proof!

Notice how we use the value of ***k*** that we obtained above. Giving names to quantities, allows us to manipulate them. This is similar to variables in programs.

Theorem: I

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Since n is even, there is some integer k such that $n = 2k$. This means that

$$n^2 = (2k)^2 = 4k^2 = 2(2k^2).$$

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Our ultimate goal is to prove that $\boldsymbol{n^2}$ is even. This means that we need to find some $\boldsymbol{m}$ where $\boldsymbol{n^2 = 2m}$. Here, we're explicitly showing how we can do that.

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Hey, that's what we said we were going to do! We're done now.

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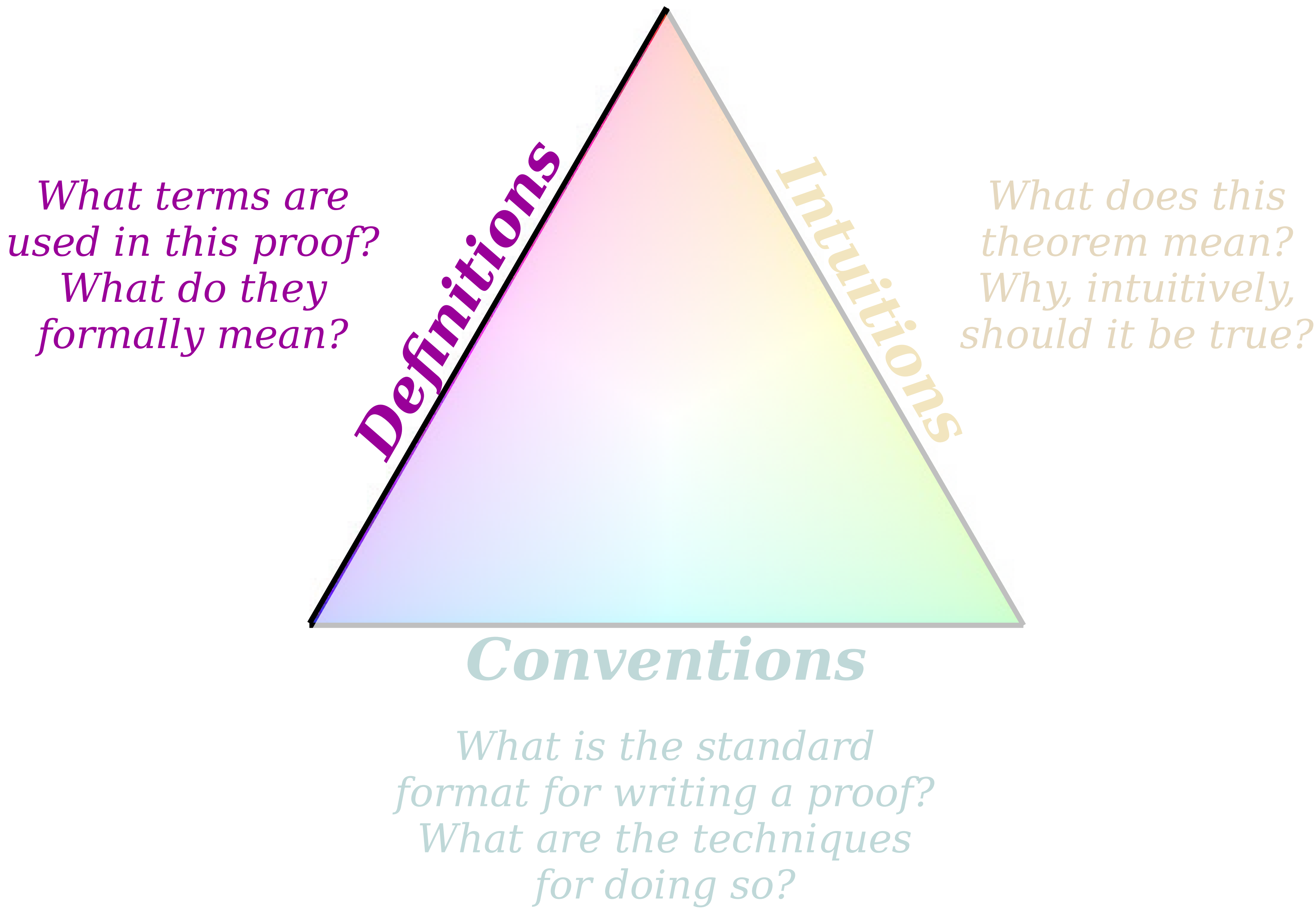
Our Next Proof

Theorem: For any integers m and n, if m and n are odd, then $m + n$ is even.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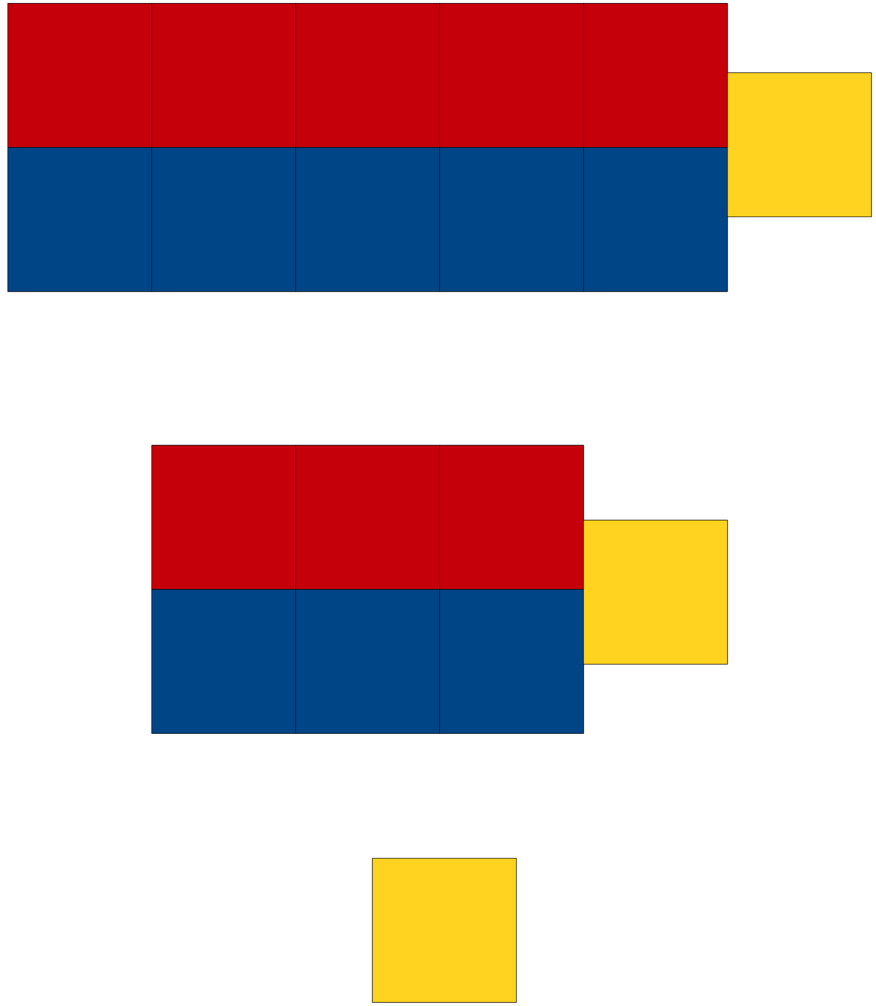


11 $\quad 2 \cdot \mathbf{5} + \mathbf{1}$

7 $\quad 2 \cdot \mathbf{3} + \mathbf{1}$

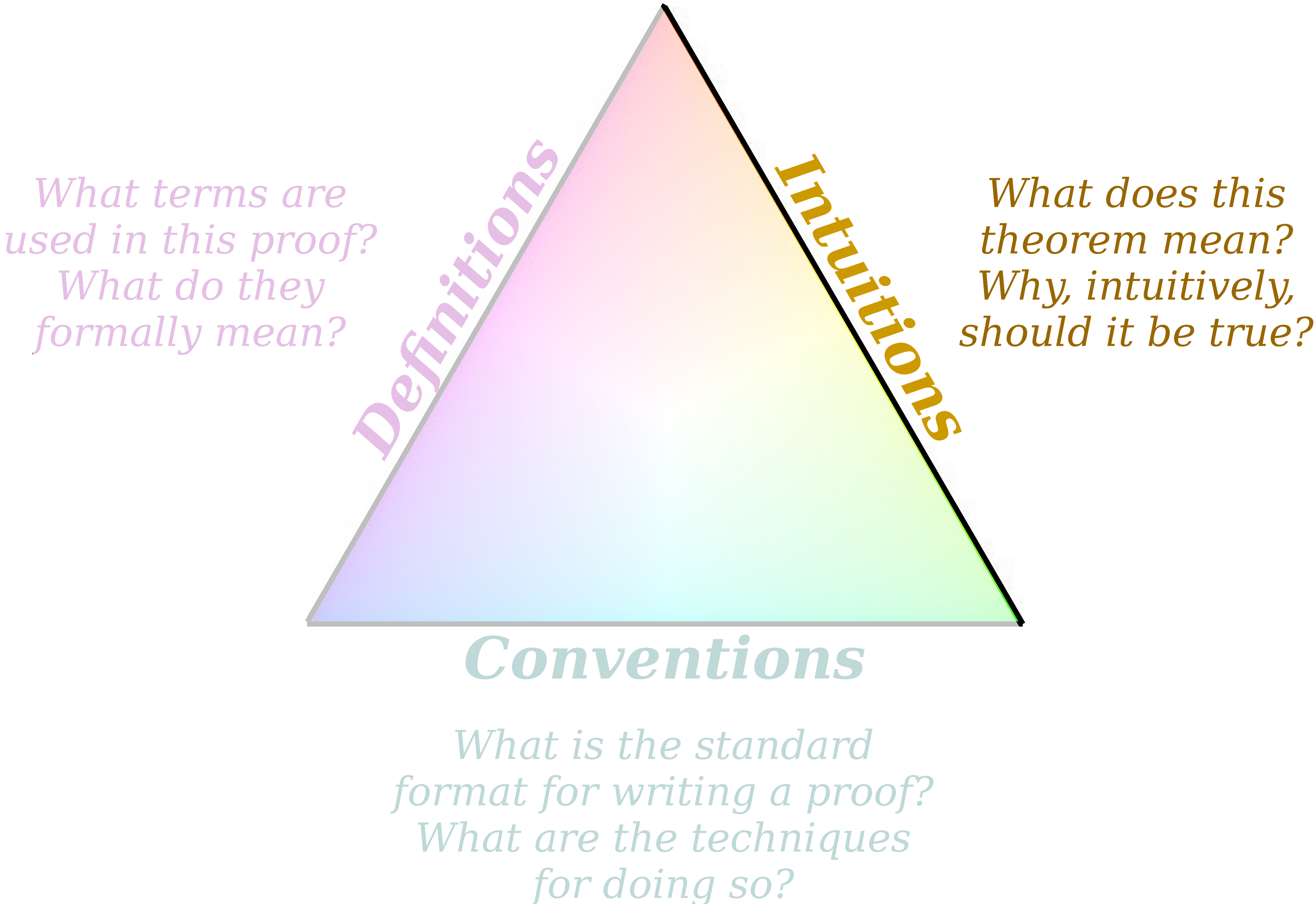
1 $\quad 2 \cdot \mathbf{0} + \mathbf{1}$

An integer n is called ***odd*** if
there is an integer k where $n = 2k+1$.

Going forward, we'll assume the following:

1. Every integer is either even or odd.
2. No integer is both even and odd.

Theorem: For any integers m and n, if m and n are odd, then $m + n$ is even.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Let’s Try Some Examples!

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\begin{aligned}
1 + 1 &= 2 &&= 2 \cdot \mathbf{1} \\
137 + 103 &= 240 &&= 2 \cdot \mathbf{120} \\
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What’s the pattern? How do we predict this?

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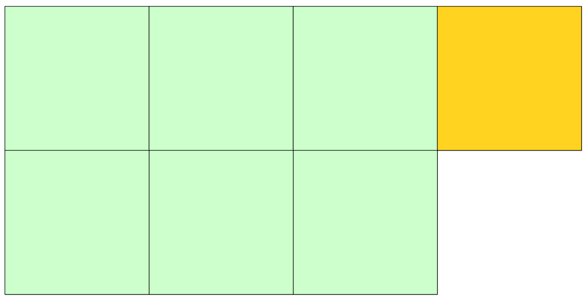

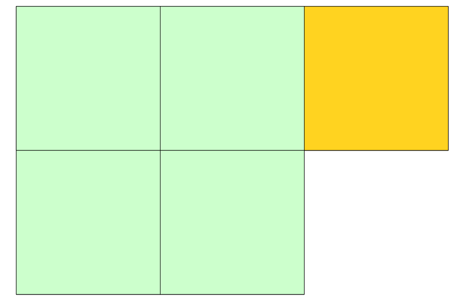

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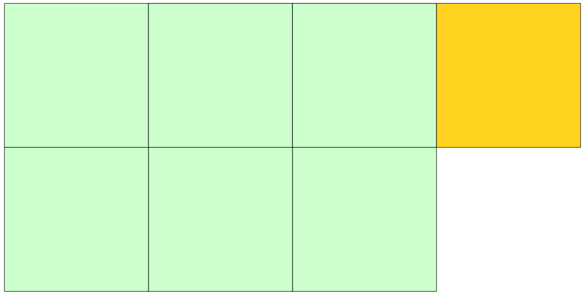

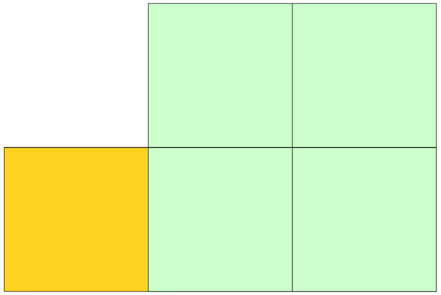

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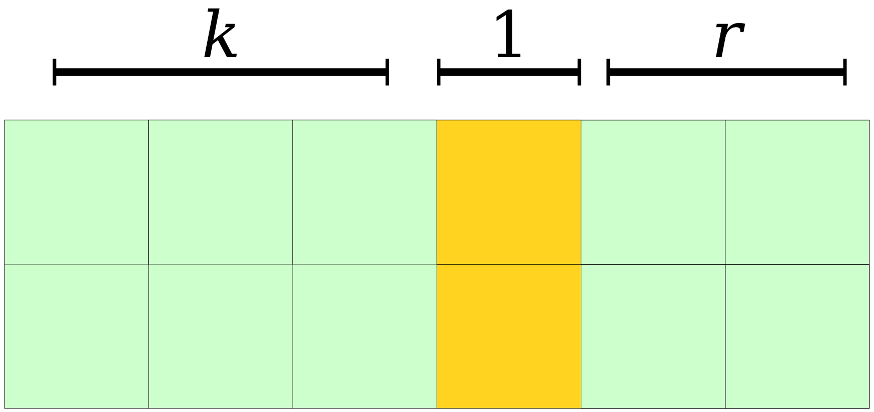
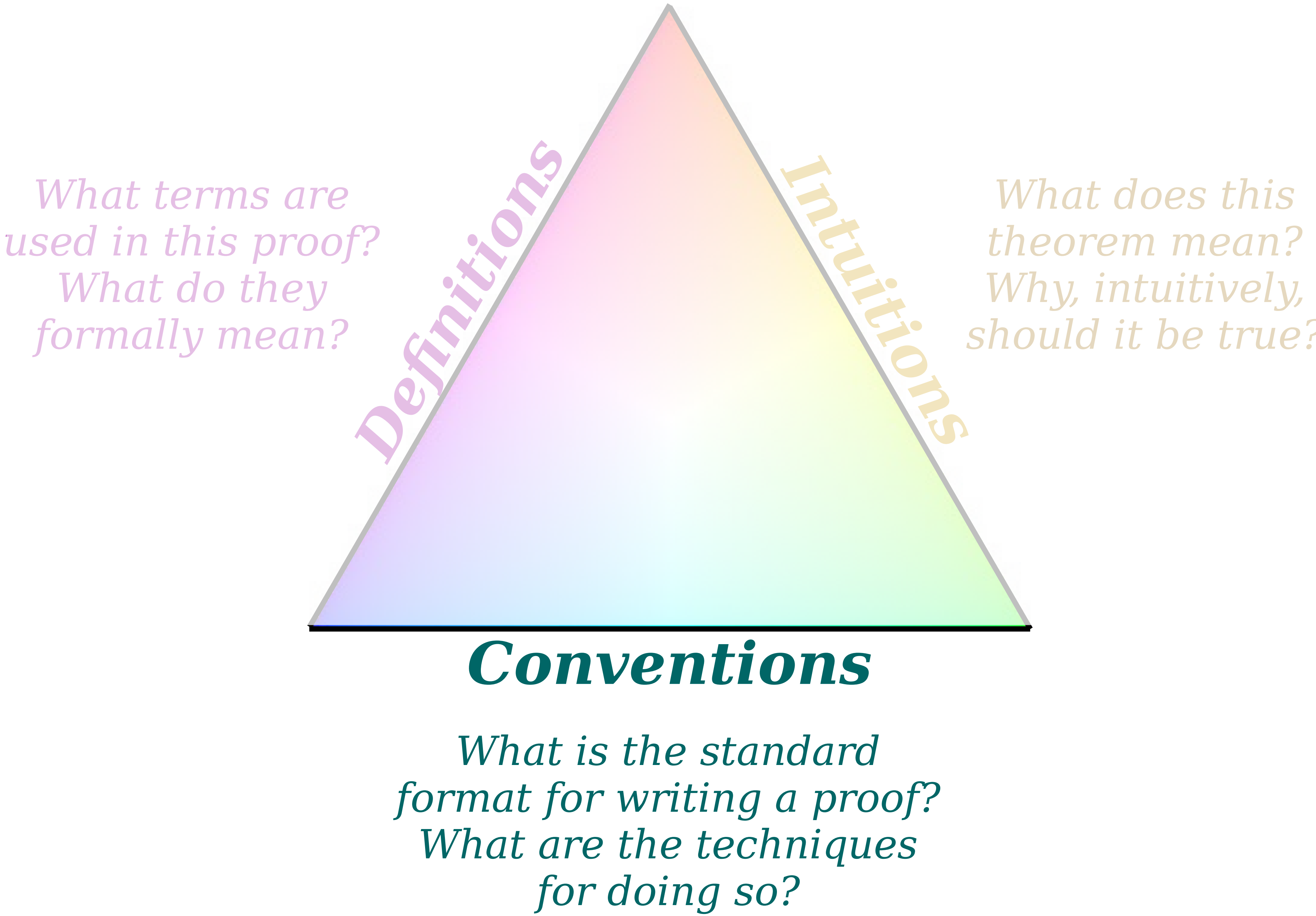


$2k+1$

$2r+1$

$$(2k+1) + (2r+1) = 2(k + r + 1)$$

Theorem: For any integers m and n, if m and n are odd, then $m+n$ is even.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$$m = 2k + 1. \qquad (1)$$

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$$m = 2k + 1. \tag{1}$$

Similarly, because n is odd there must be some integer r such that

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We ask the reader to make an *arbitrary choice*. Rather than specifying what **m** and **n** are, we're signaling to the reader that they could, in principle, supply any choices of **m** and **n** that they'd like.

By letting the reader pick **m** and **n** arbitrarily, anything we prove about **m** and **n** will generalize to all possible choices for those values.

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$$= 2(k + r + 1). \qquad (3)$$

Equation (3) tells us that there is an integer s (namely, $k + r + 1$) such that $m + n = 2s$. Therefore, we see that $m + n$ is even, as required. ■

Theorem: For any integers m and n, if m and n are odd, then $m + n$ is even.

Proof: Consider any

odd. We need to s

Since m is odd, w

Numbering these equalities lets us refer back to them later on, making the flow of the proof a bit easier to understand.

$$m = 2k + 1. \qquad (1)$$

Similarly, because n is odd there must be some integer r such that

$$n = 2r + 1. \qquad (2)$$

By adding equations (1) and (2) we learn that

$$\begin{aligned} m + n &= 2k + 1 + 2r + 1 \\ &= 2k + 2r + 2 \\ &= 2(k + r + 1). \qquad (3) \end{aligned}$$

Equation (3) tells us that there is an integer s (namely, $k + r + 1$) such that $m + n = 2s$. Therefore, we see that $m + n$ is even, as required. ■

Theorem: For any integers m and n, if m and n are odd, then $m + n$ is even.

Proof: Consider any arbitrary integers m and n where m and n are odd. We need to show that $m + n$ is even.

Since m is odd, we know that there is an integer k where

$$m = 2k + 1. \qquad (1)$$

Similarly, because n is odd there must … some integer r such that

(2)

…arn that

… $2r + 1$

… $+ 2$

… $+ 1)$. (3)

…nteger s (namely, $k + r + 1$)

…see that $m + n$ is even, as

This is a complete sentence! Proofs are expected to be written in complete sentences, so you'll often use punctuation at the end of formulas.

We recommend using the "mugga mugga" test – if you read a proof and replace all the mathematical notation with "mugga mugga," what comes back should be a valid sentence.

Theorem: For any integers m and n, if m and n are odd, then $m + n$ is even.

Proof: Consider any arbitrary integers m and n where m and n are odd. We need to show that $m + n$ is even.

Since m is odd, we know that there is an integer k where

$$m = 2k + 1. \tag{1}$$

Similarly, because n is odd there must be some integer r such that

$$n = 2r + 1. \tag{2}$$

By adding equations (1) and (2) we learn that

$$\begin{aligned} m + n &= 2k + 1 + 2r + 1 \\ &= 2k + 2r + 2 \\ &= 2(k + r + 1). \end{aligned} \tag{3}$$

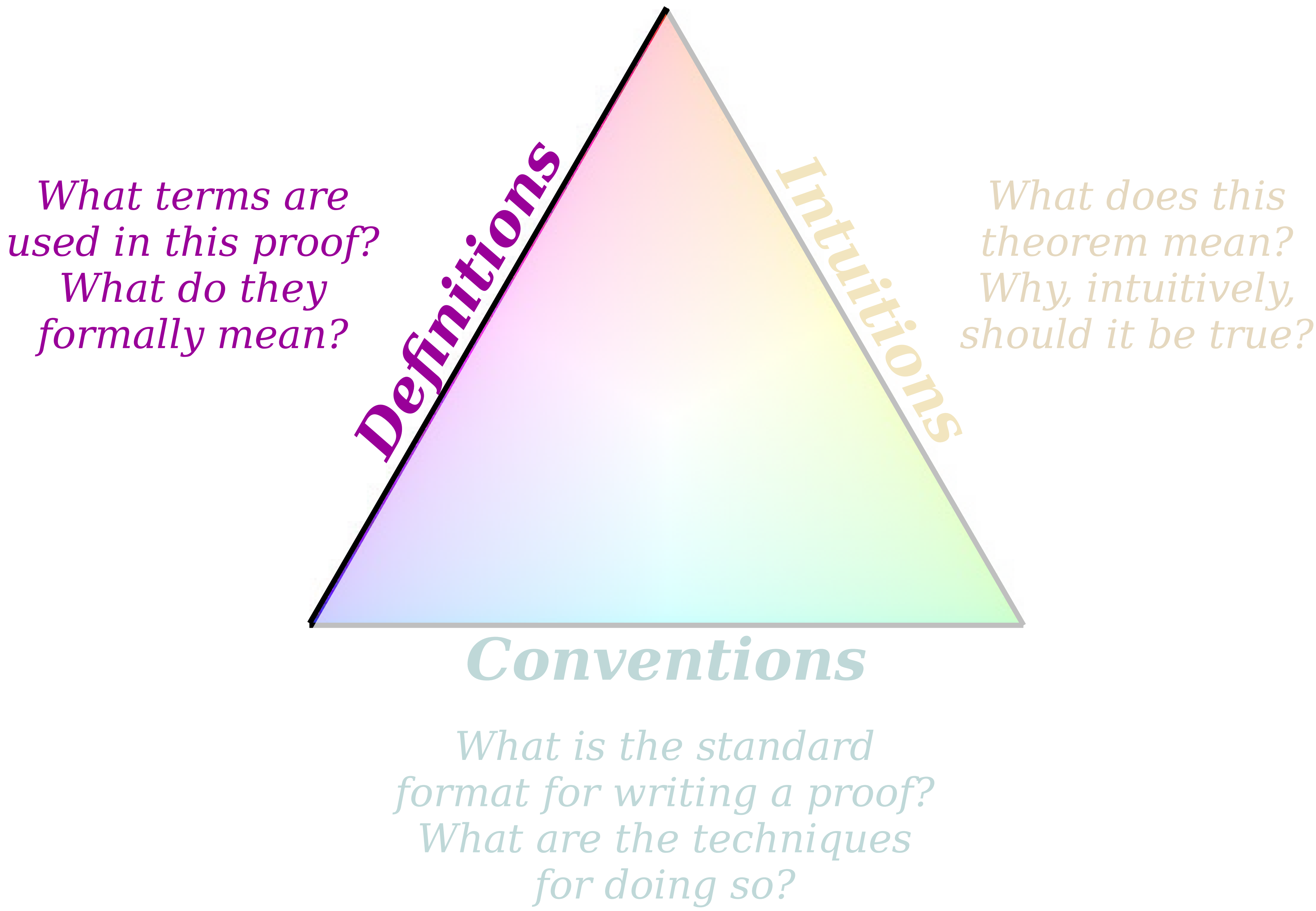
Equation (3) tells us that there is an integer s (namely, $k + r + 1$) such that $m + n = 2s$. Therefore, we see that $m + n$ is even, as required. ■

Some Little Exercises

- Here's a list of other theorems that are true about odd and even numbers:
  - ***Theorem:*** The sum and difference of any two even numbers is even.
  - ***Theorem:*** The sum and difference of an odd number and an even number is odd.
  - ***Theorem:*** The product of any integer and an even number is even.
  - ***Theorem:*** The product of any two odd numbers is odd.
- Going forward, we'll just take these results for granted. Feel free to use them in the problem sets.
- If you'd like to practice the techniques from today, try your hand at proving these results!

Universal and Existential Statements

Theorem: For any odd integer n,
there exist integers r and s where $r^2 - s^2 = n$.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This result is true for every possible choice of odd integer n. It'll work for $n = 1$, $n = 137$, $n = 103$, etc.

Theorem: For any odd integer n, there exist integers r and s where $r^2 - s^2 = n$.

We aren't saying this is true for every choice of r and s. Rather, we're saying that *somewhere out there* are choices of r and s where this works.

Universal vs. Existential Statements

- A ***universally-quantified statement*** is a statement of the form

For all *x*, `[some-property]` holds for *x*.

- We've seen how to prove these statements.
- An ***existentially-quantified statement*** is a statement of the form

There is some *x* where `[some-property]` holds for *x*.

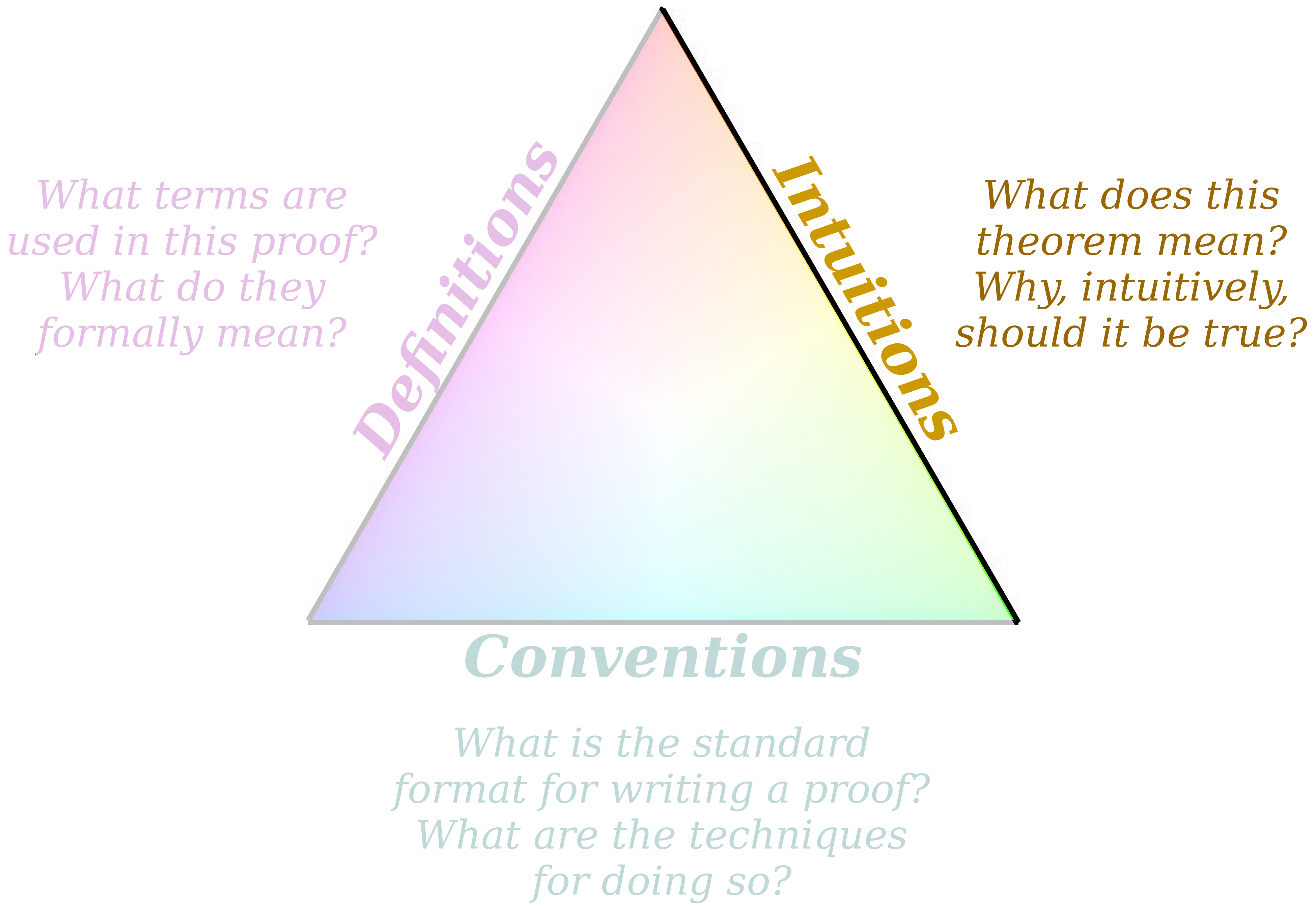
- How do you prove an existentially-quantified statement?

Proving an Existential Statement

- Over the course of the quarter, we will see several different ways to prove an existentially-quantified statement of the form

  There is an *x* where `[some-property]` holds for *x*.

- ***Simplest approach:*** Search far and wide, find an *x* that has the right property, then show why your choice is correct.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Let's Try Some Examples!

$1 = \underline{\quad}^2 - \underline{\quad}^2$

$3 = \underline{\quad}^2 - \underline{\quad}^2$

$5 = \underline{\quad}^2 - \underline{\quad}^2$

$7 = \underline{\quad}^2 - \underline{\quad}^2$

$9 = \underline{\quad}^2 - \underline{\quad}^2$

Theorem: For any odd integer *n*,
there exist integers *r* and *s* where $r^2 - s^2 = n$.

Let's Try Some Examples!

$1 = \mathbf{1}^2 - \mathbf{0}^2$

$3 = \mathbf{2}^2 - \mathbf{1}^2$

$5 = \mathbf{3}^2 - \mathbf{2}^2$

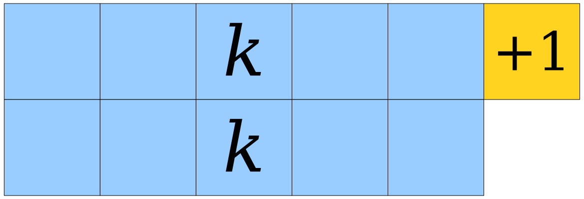
$7 = \mathbf{4}^2 - \mathbf{3}^2$

$9 = \mathbf{5}^2 - \mathbf{4}^2$

We've got a pattern – but why does this work?

Theorem: For any odd integer n, there exist integers r and s where $r^2 - s^2 = n$.

Let's Draw Some Pictures!



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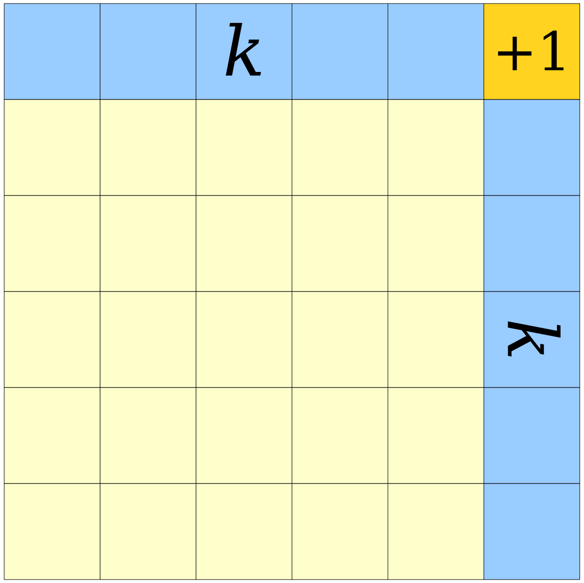
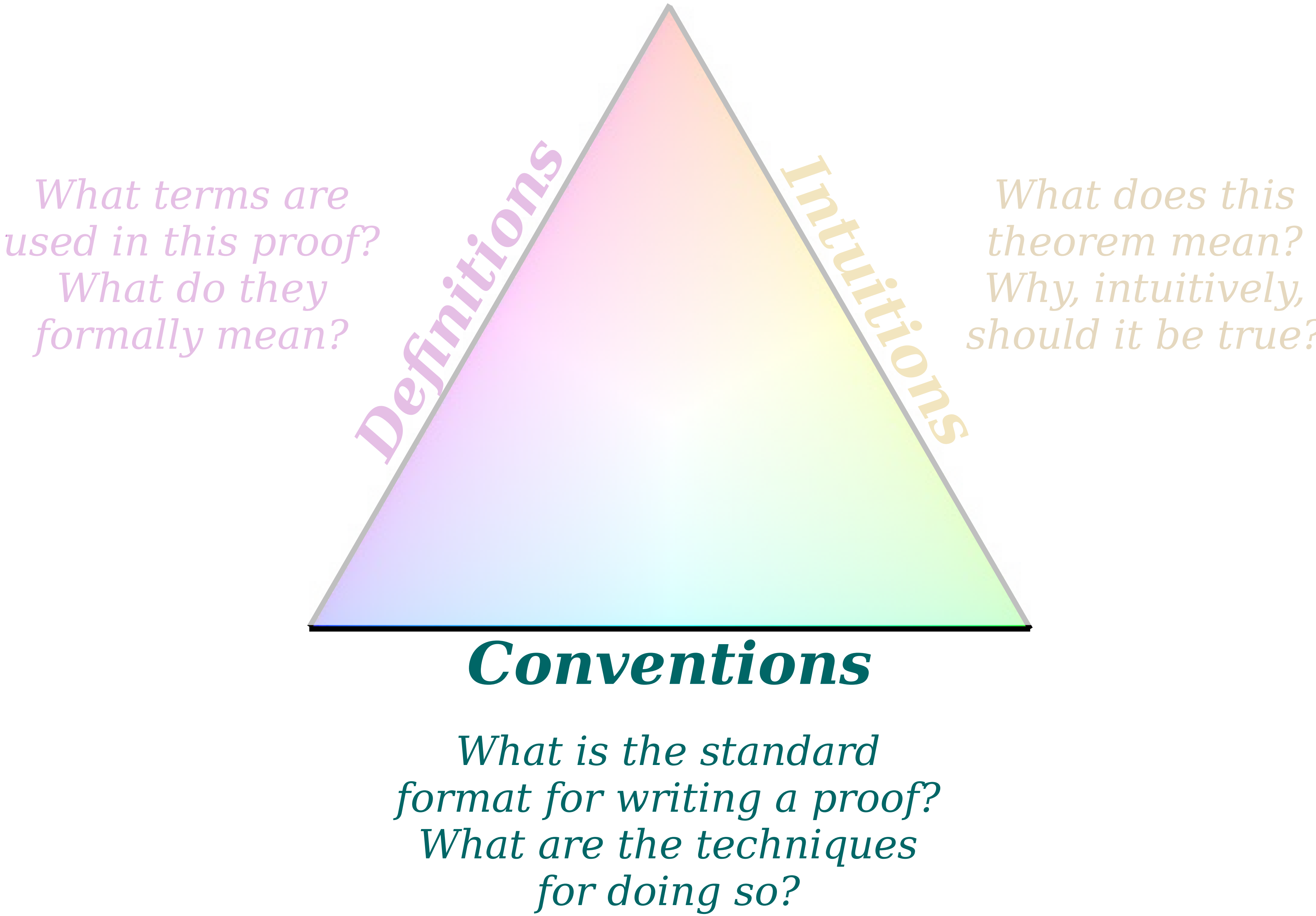


$$(k+1)^2 - k^2 = 2k+1$$

Theorem: For any odd integer n, there exist integers r and s where $r^2 - s^2 = n$.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Since n is odd, we know there is an integer k where $n = 2k + 1$. Now, let $r = k+1$ and $s = k$.

Theorem: For any odd integer n, there exist integers r and s where $r^2 - s^2 = n$.

Proof: Let n be an arbitrary odd integer. We will show that there exist integers r and s where $r^2 - s^2 = n$.

Since n is odd, we know there is an integer k where $n = 2k + 1$. Now, let $r = k+1$ and $s = k$. Then we see that

$$r^2 - s^2 \quad = \quad (k+1)^2 - k^2$$

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$$

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&= 2k + 1 \\
&= n.
\end{aligned}
$$

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$$\begin{aligned} r^2 - s^2 &= (k+1)^2 - k^2 \\ &= k^2 + 2k + 1 - k^2 \\ &= 2k + 1 \\ &= n. \end{aligned}$$

This means that $r^2 - s^2 = n$, which is what we needed to show.

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Theorem: For any odd integer n, there exist integers r and s where $r^2 - s^2 = n$.

Proof: Let n be an arbitrary odd integer. We will show that there exist integers r and s where $r^2 - s^2 = n$.

Since r [...] here
$n = 2k$ [...] e see that

> We ask the reader to make an *arbitrary choice*. Rather than specifying what ***n*** is, we're signaling to the reader that they could, in principle, supply any choice ***n*** that they'd like.

$$= \quad 2k + 1$$

$$= \quad n.$$

This means that $r^2 - s^2 = n$, which is what we needed to show. ■

Theorem: For any odd integer n, there exist integers r and s where $r^2 - s^2 = n$.

Proof: Let n be an arbitrary odd integer. We will show that there exist integers r and s where $r^2 - s^2 = n$.

Since n is odd, we know th… re $n = 2k + 1$. Now, let $r = k$… ee that

$$r^2 - s^2 = (k + \dots$$

$$= k^2 + 2k + 1 - k^2$$

$$= 2k + 1$$

$$= n.$$

This means that $r^2 - s^2 = n$, which is what we needed to show. ■

> As always, it's helpful to write out what we need to demonstrate with the rest of the proof.

Theorem: For any odd integer n, there exist integers r and s where $r^2 - s^2 = n$.

Proof: Let n be an arbitrary odd integer. We will show that there exist integers r and s where $r^2 - s^2 = n$.

Since n is odd, we know there is an integer k where $n = 2k + 1$. Now, let $r = k+1$ and $s = k$. Then we see that

$$r^2 - s^2 = (k+1)^2 \ldots$$
$$= k^2 + 2\ldots$$
$$= 2k + 1\ldots$$
$$= n.$$

This means that $r^2 - s^2 = n$, w… to show. ■

We're trying to prove an existential statement. The easiest way to do that is to just give concrete choices of the objects being sought out.

Theorem: For any odd integer n, there exist integers r and s where $r^2 - s^2 = n$.

Proof: Let n be an arbitrary odd integer. We will show that there exist integers r and s where $r^2 - s^2 = n$.

Since n is odd, we know there is an integer k where $n = 2k + 1$. Now, let $r = k+1$ and $s = k$. Then we see that

$$\begin{aligned} r^2 - s^2 &= (k+1)^2 - k^2 \\ &= k^2 + 2k + 1 - k^2 \\ &= 2k + 1 \\ &= n. \end{aligned}$$

This means that $r^2 - s^2 = n$, which is what we needed to show. ■

Time-Out for Announcements!

Working in Pairs

- Problem Set Zero is due this Friday at 2:30PM. It must be completed individually.
- After that, the remaining problem sets can be done individually or in pairs.
- We have advice about how to work effectively in pairs up on the course website – check the "Guide to Partners."
- Want to work in a pair, but don't know who to work with? Fill out ***this Google form*** and we'll connect you with a partner on Friday.

CURIS Poster Session

- CURIS is the CS department's undergraduate research program. It's a great way to get involved in research!
- There's a CURIS poster session showcasing work from the summer going on from 3PM – 5PM Friday in the Engineering Quad. Feel free to stop on by!
- Interested in seeing what research projects are open right now? Visit https://curis.stanford.edu.
- Have questions about research or how CURIS works? Email the CURIS mentors, PhD students who answer questions about research:

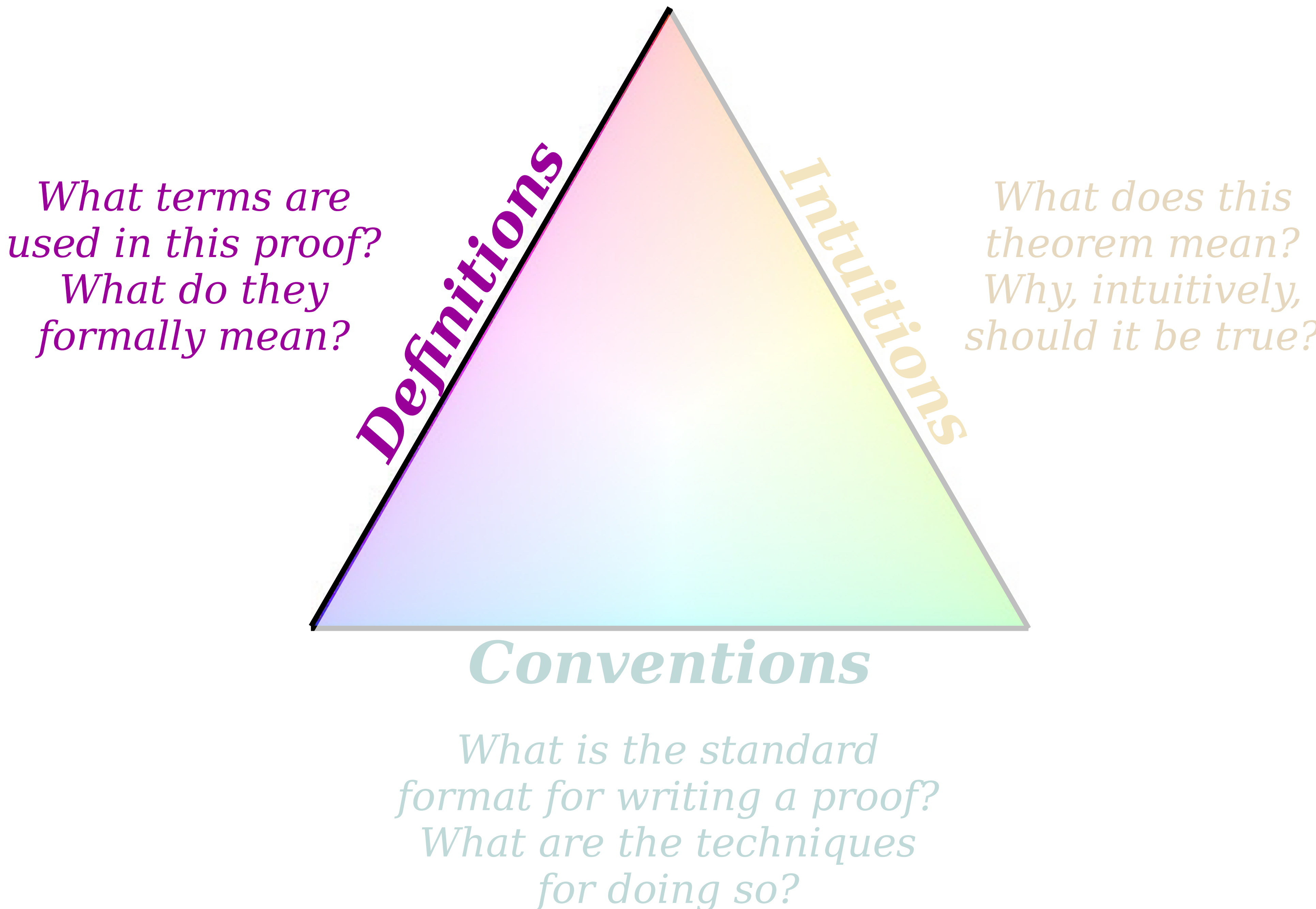
curis-mentors@cs.stanford.edu

Qt Creator Help Session

- The lovely CS106B staff have invited all y'all to join them for a Qt Creator Help Session this evening if you're having trouble getting Qt Creator up and running on your system.
- Runs ***7:00PM – 9:00PM*** in the basement of the Huang building (just around the corner from us!)
- SCPD students – please reach out to us if you need help setting things up. We'll do our best to help out.

Back to CS103!

Theorem: If n is an integer, then $\lceil n/2 \rceil + \lfloor n/2 \rfloor = n$.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Floors and Ceilings

- The notation $\lceil x \rceil$ represents the ***ceiling*** of x, the smallest integer greater than or equal to x.

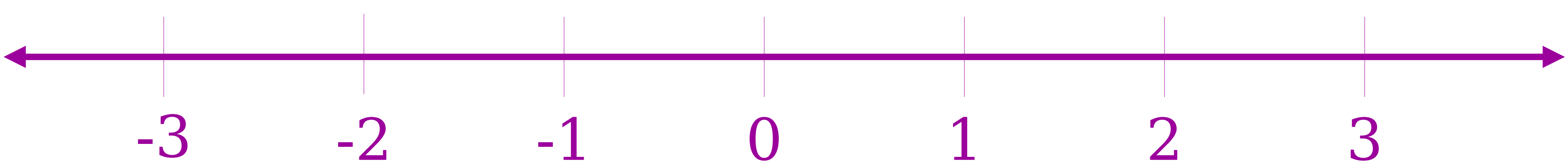
$$\lceil 1 \rceil = 1 \qquad \lceil 1.5 \rceil = 2$$

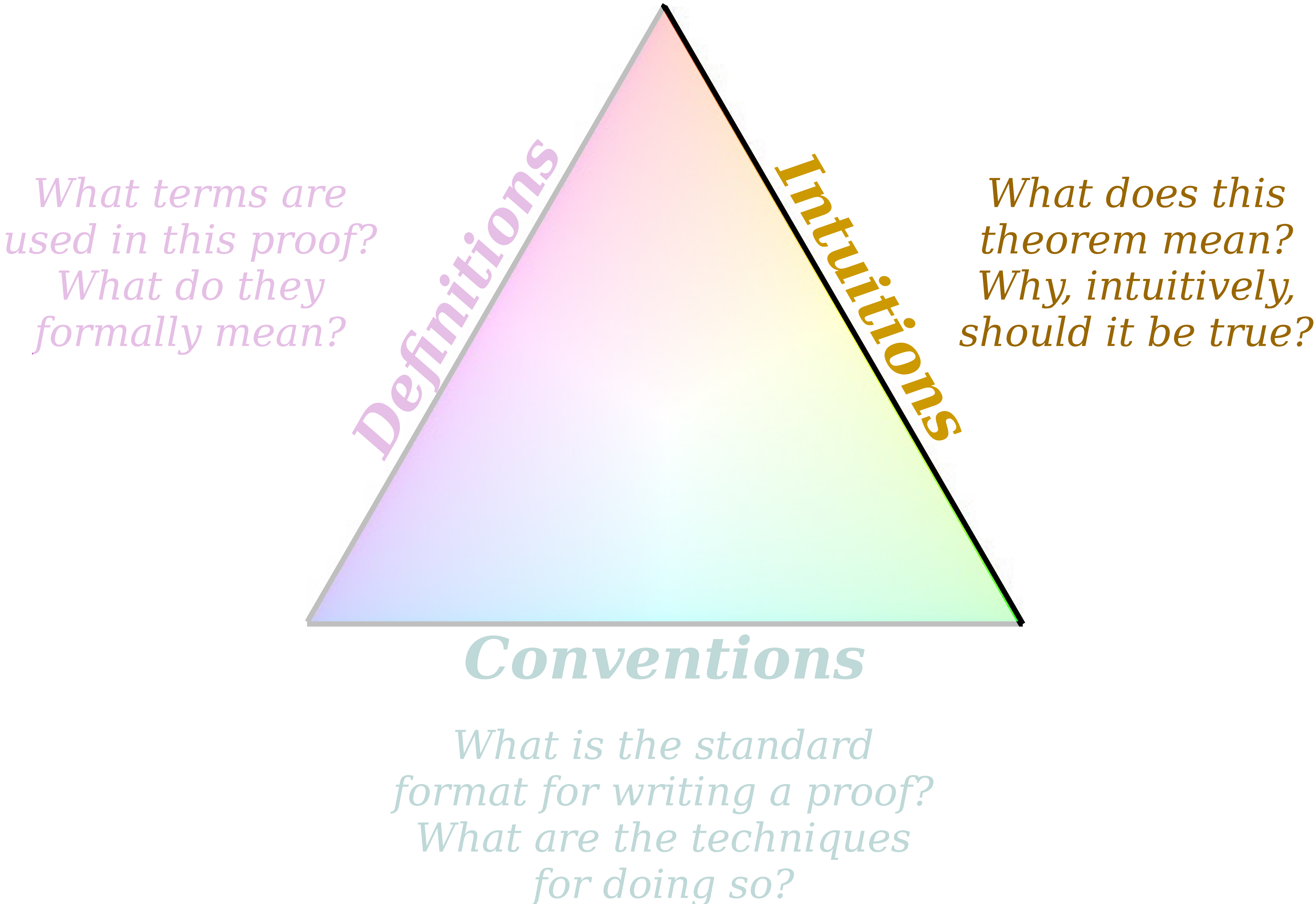
$$\lceil -1 \rceil = -1 \qquad \lceil -1.5 \rceil = -1$$

- The notation $\lfloor x \rfloor$ represents is the ***floor*** of x, the largest integer less than or equal to x.

$$\lfloor 1 \rfloor = 1 \qquad \lfloor 1.5 \rfloor = 1$$

$$\lfloor -1 \rfloor = -1 \qquad \lfloor -1.5 \rfloor = -2$$

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Let's Try Some Examples!

$$\lceil 0/2 \rceil + \lfloor 0/2 \rfloor = 0 + 0 = 0$$

$$\lceil 1/2 \rceil + \lfloor 1/2 \rfloor = 1 + 0 = 1$$

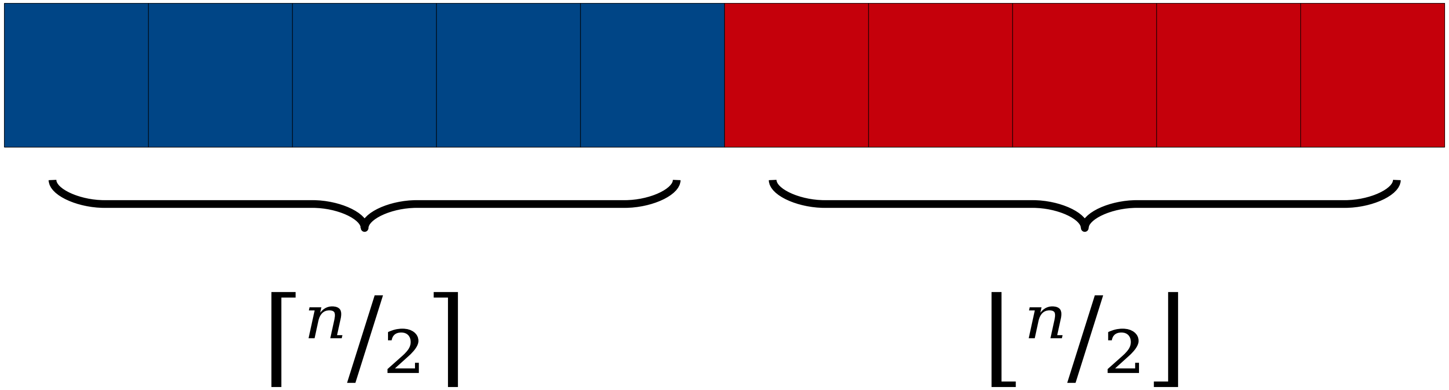
$$\lceil 2/2 \rceil + \lfloor 2/2 \rfloor = 1 + 1 = 2$$

$$\lceil 3/2 \rceil + \lfloor 3/2 \rfloor = 2 + 1 = 3$$

$$\lceil 4/2 \rceil + \lfloor 4/2 \rfloor = 2 + 2 = 4$$

Theorem: If n is an integer, then $\lceil n/2 \rceil + \lfloor n/2 \rfloor = n$.

Let's Draw Some Pictures!



$$n = 2k$$

Theorem: If n is an integer, then $\lceil n/2 \rceil + \lfloor n/2 \rfloor = n$.

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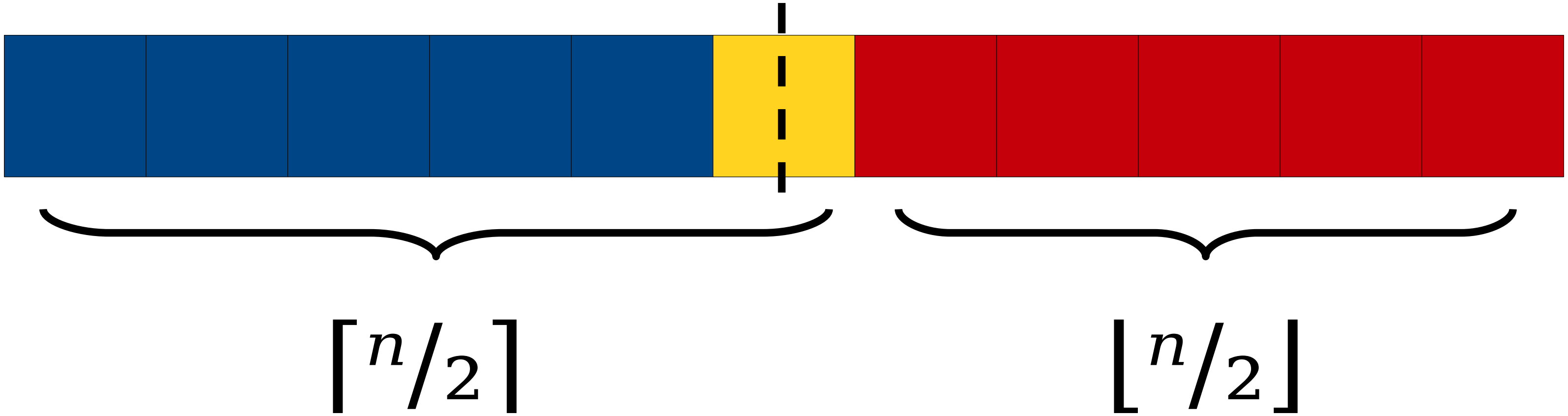
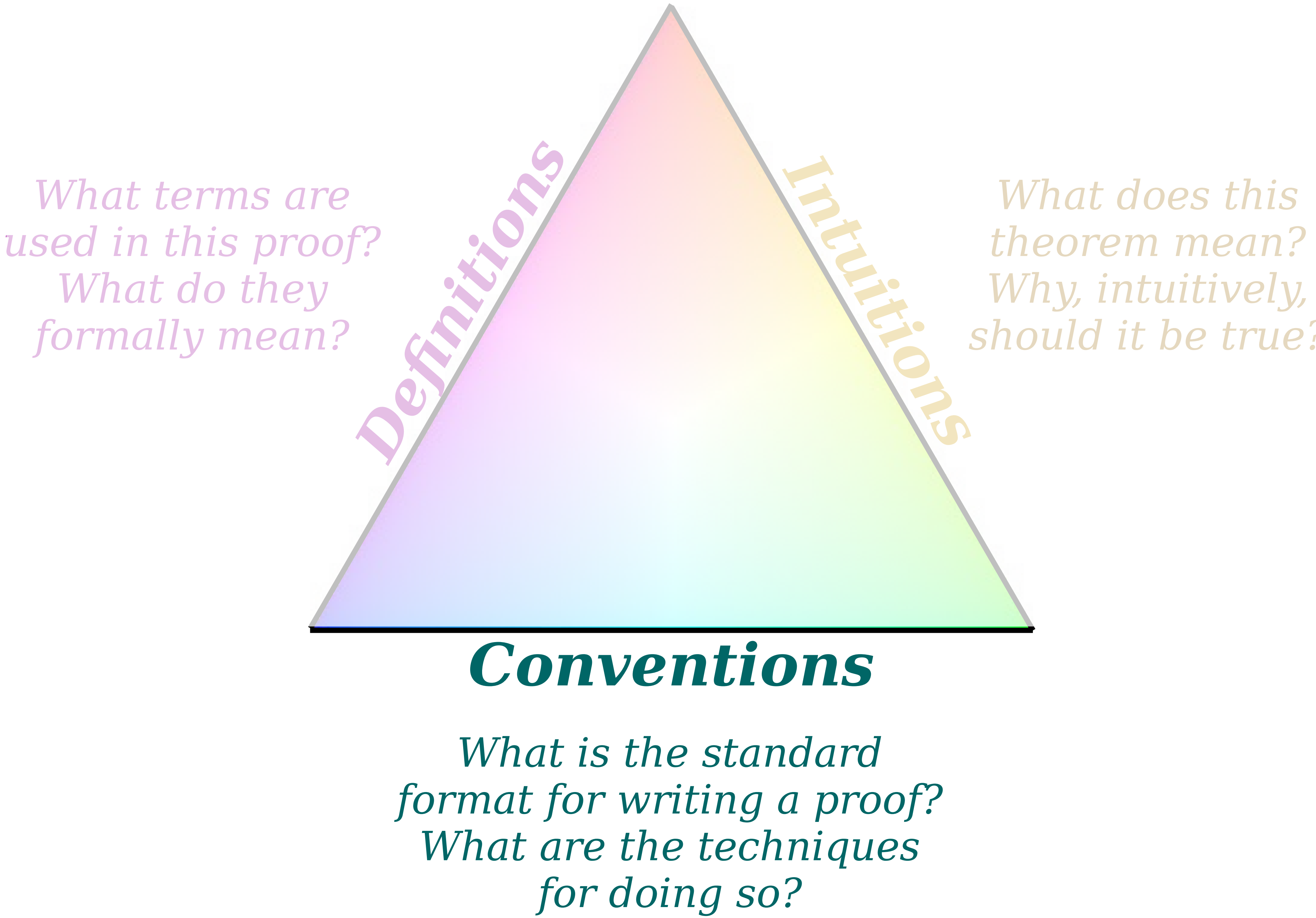


$$n = 2k + 1$$

Theorem: If n is an integer, then $\lceil n/2 \rceil + \lfloor n/2 \rfloor = n$.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Proof: Let n be an integer.

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Theorem: If n is an integer, then $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$.

Proof: Let n be an integer. We will show that $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$. To do so, we consider two cases:

Case 1: n is even.

Case 2: n is odd.

Theorem: If n is an integer, then $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$.

Proof: Let n be an integer. We will show that $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$. To do so, we consider two cases:

Case 1: n is even.

Case 2: n is odd.

This is called a *proof by cases* (or *proof by exhaustion*). We split apart into one or more cases and confirm that the result is indeed true in each of them.

(Think of it like an if/else or switch statement.)

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Case 1: n is even. This means there is an integer k such that $n = 2k$. Some algebra then tells us that

$$\left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{n}{2} \right\rfloor = \left\lceil \frac{2k}{2} \right\rceil + \left\lfloor \frac{2k}{2} \right\rfloor$$

Case 2: n is odd.

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$$= \lceil k \rceil + \lfloor k \rfloor$$

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$$= 2k$$

$$= n.$$

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$$= \lceil k \rceil + \lfloor k \rfloor$$

$$= 2k$$

$$= n.$$

Case 2: n is odd. Then there's an integer k where $n = 2k + 1$, and

$$\left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{n}{2} \right\rfloor = \left\lceil \frac{2k+1}{2} \right\rceil + \left\lfloor \frac{2k+1}{2} \right\rfloor$$

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Case 2: n is odd. Then there's an integer k where $n = 2k + 1$, and

$$\begin{aligned} \left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{n}{2} \right\rfloor &= \left\lceil \frac{2k+1}{2} \right\rceil + \left\lfloor \frac{2k+1}{2} \right\rfloor \\ &= \left\lceil k + \frac{1}{2} \right\rceil + \left\lfloor k + \frac{1}{2} \right\rfloor \end{aligned}$$

Theorem: If n is an integer, then $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$.

Proof: Let n be an integer. We will show that $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$. To do so, we consider two cases:

Case 1: n is even. This means there is an integer k such that $n = 2k$. Some algebra then tells us that

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$$= 2k$$
$$= n.$$

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In either case, we see that $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$, as required.

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At the end of a split into cases, it's a nice courtesy to explain to the reader what it was that you established in each case.

$$= n.$$

In either case, we see that $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$, as required.

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Proofs as a Dialog

Proofs as a Dialog

Let n be an arbitrary odd integer.

Since n is an odd integer, there is an integer k such that $n = 2k + 1$.

Now, let $z = k - 34$.

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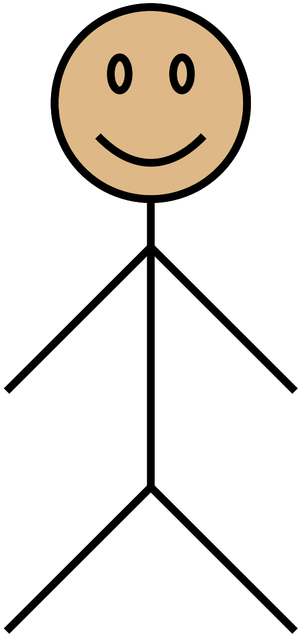

Proof Writer (You)

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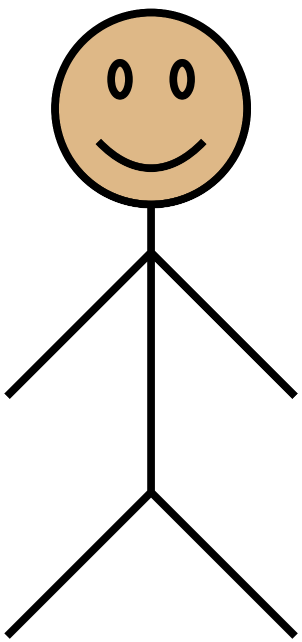
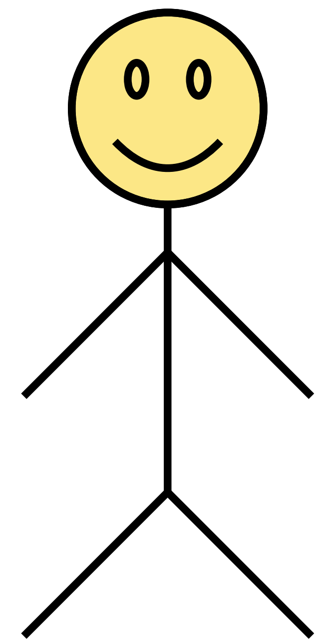


Proof Writer (You)



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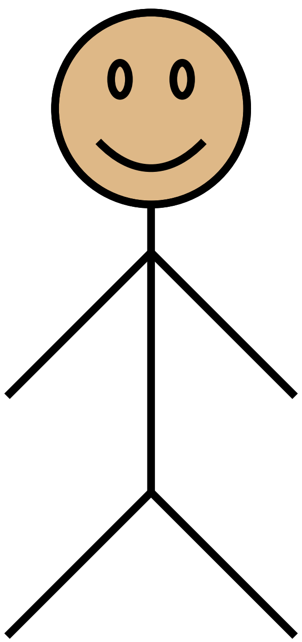

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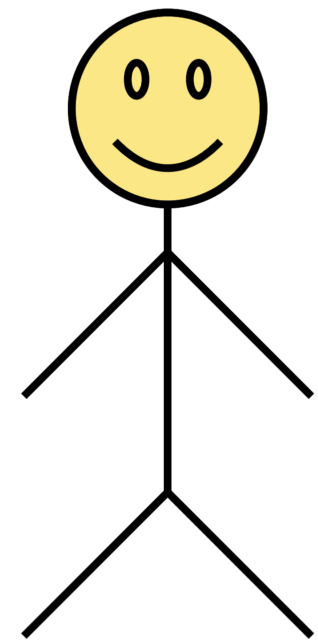

Proof Reader

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Let n be an arbitrary odd integer.

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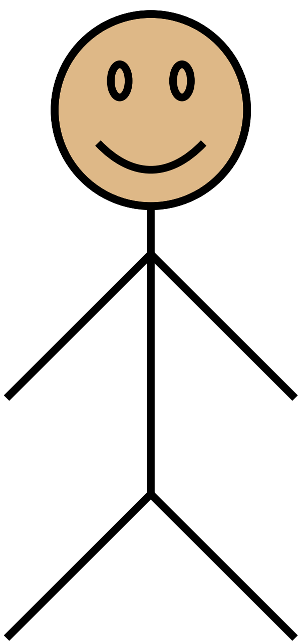
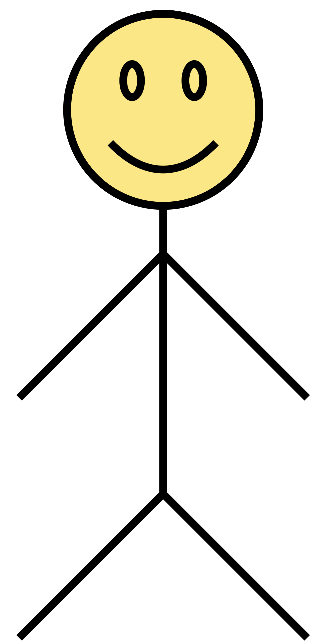
Now, let $z = k - 34$.

$n = 137$

Reader *Picks*



Proof Writer (You)



Proof Reader

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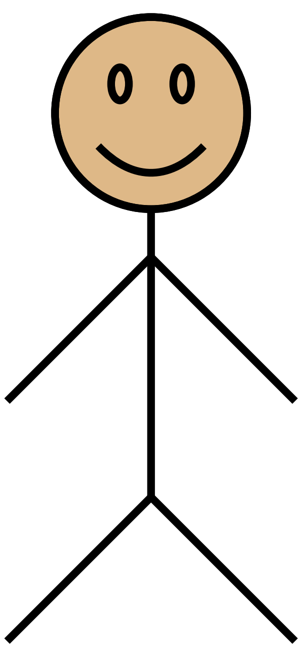
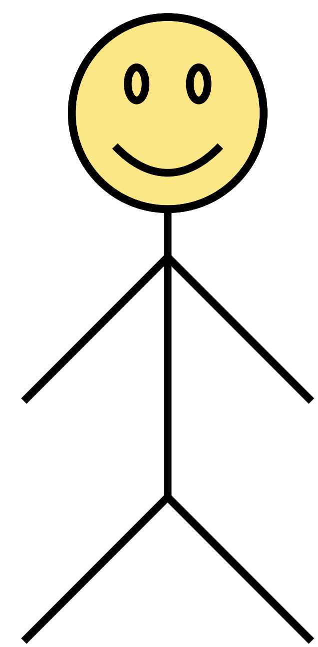


$n = 137$

Reader *Picks*



Proof Writer (You)

Proof Reader

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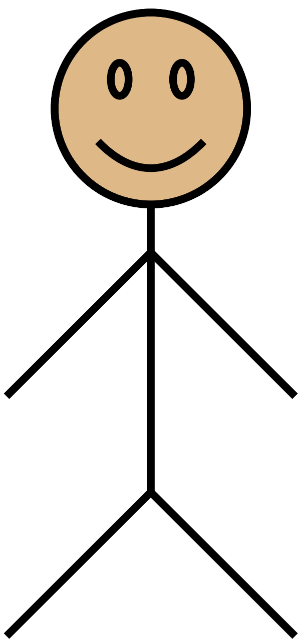


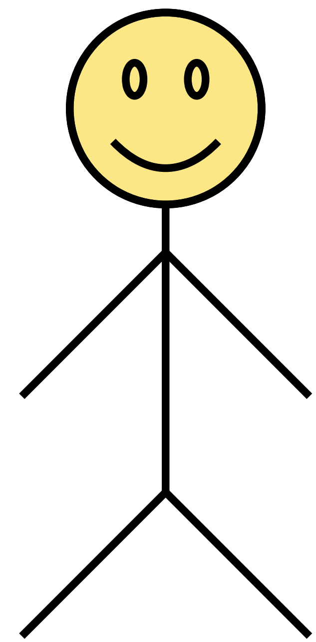
Proof Writer (You)

$k = 68$

Neither *Picks*

$n = 137$

Reader *Picks*



Proof Reader

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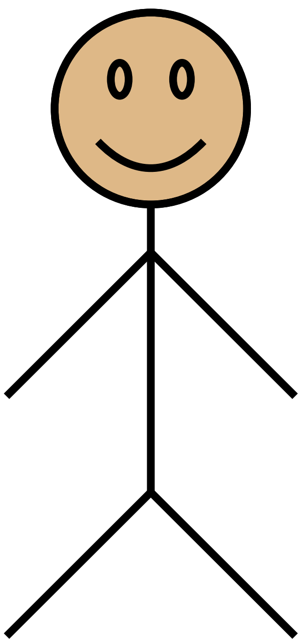
$n = 137$

Reader *Picks*

$k = 68$

Neither *Picks*



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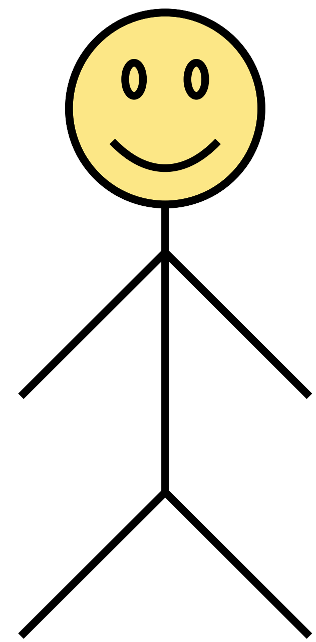

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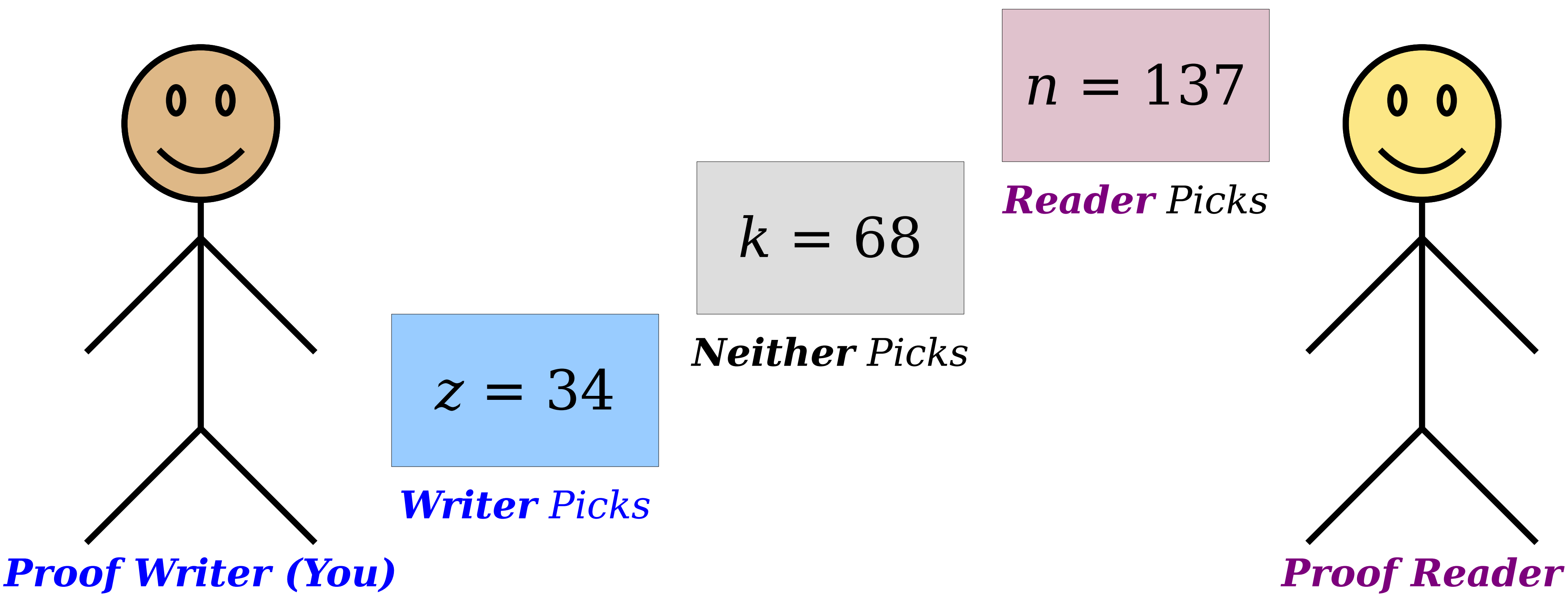

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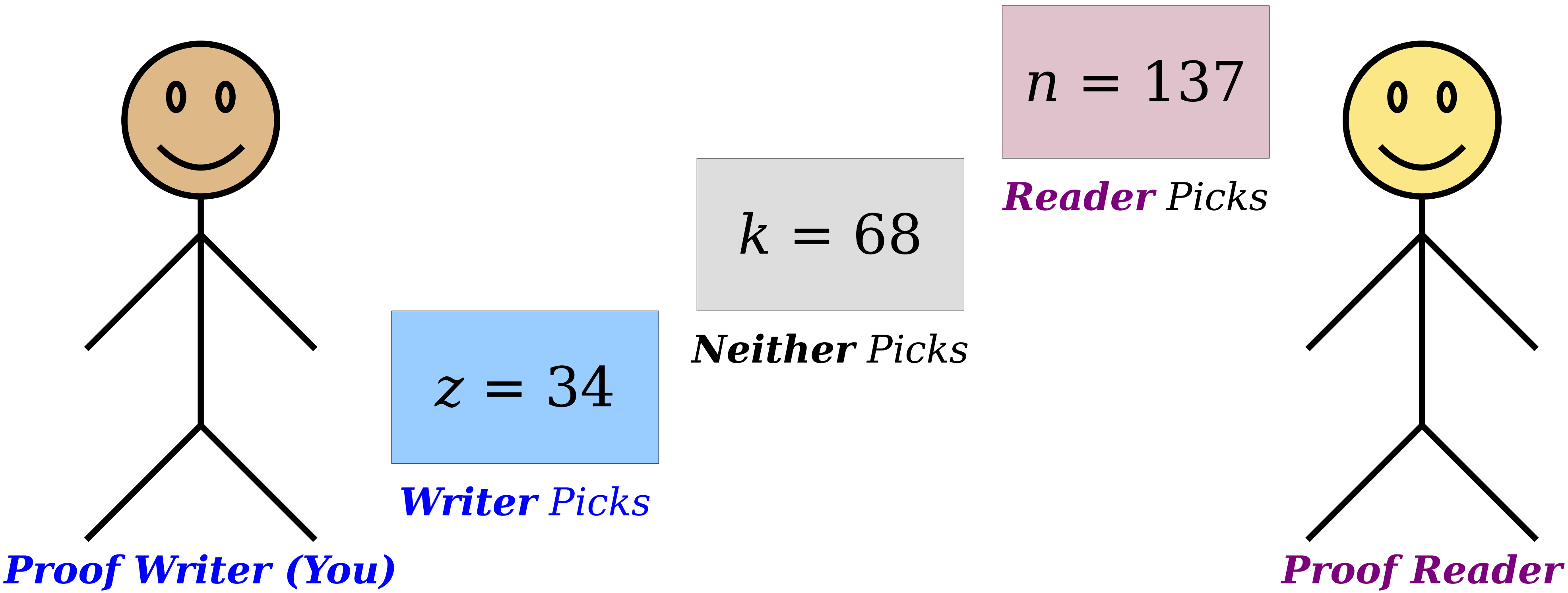

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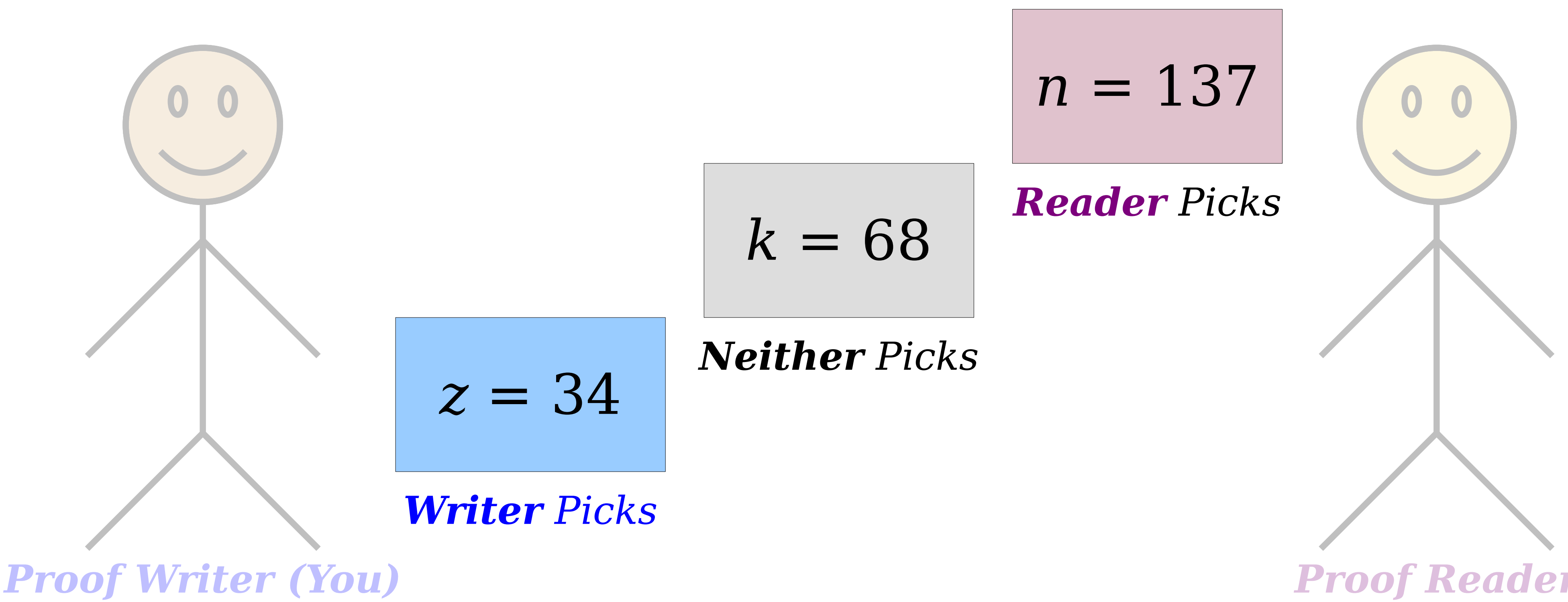
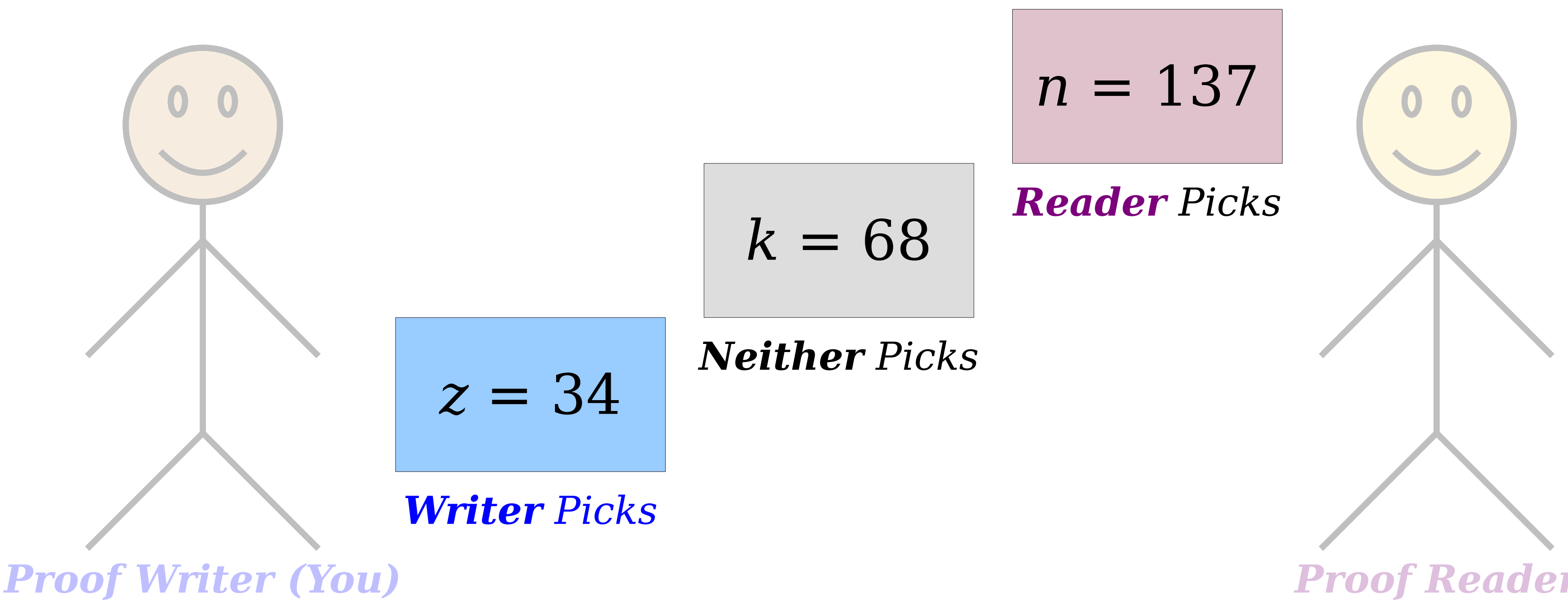


Each of these variables has a distinct, assigned value.

Each variable was either picked by the reader, picked by the writer, or has a value that can be determined from other variables.

Now, let $z = k - 34$.



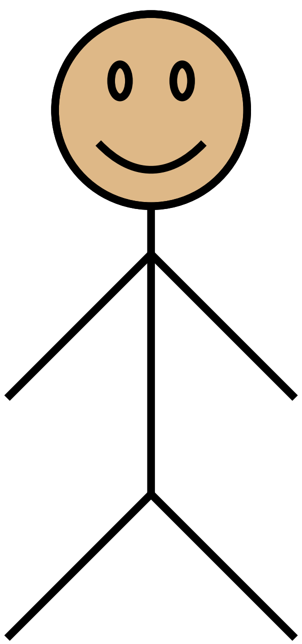
Who Owns What?

- The ***reader*** chooses and owns a value if you use wording like this:
  - Pick a natural number n.
  - Consider some $n \in \mathbb{N}$.
  - Fix a natural number n.
  - Let n be a natural number.
- The ***writer*** (you) chooses and owns a value if you use wording like this:
  - Let $r = n + 1$.
  - Pick $s = n$.
- ***Neither*** of you chooses a value if you use wording like this:
  - Since n is even, we know there is some $k \in \mathbb{Z}$ where $n = 2k$.
  - Because n is odd, there must be some integer k where $n = 2k + 1$.

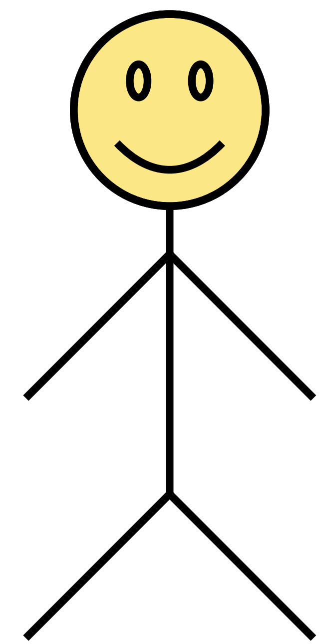
Proofs as a Dialog

Let x be an arbitrary even integer.

Then for any even x, we know that $x+1$ is odd.



Proof Writer (You)



Proof Reader

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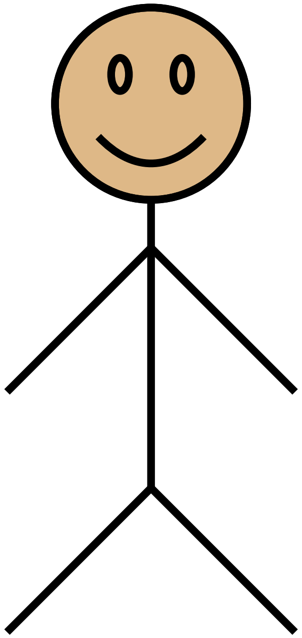

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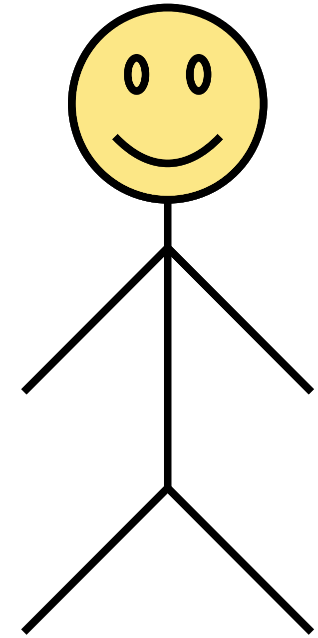

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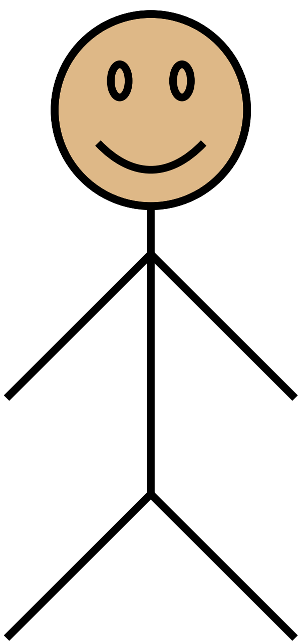
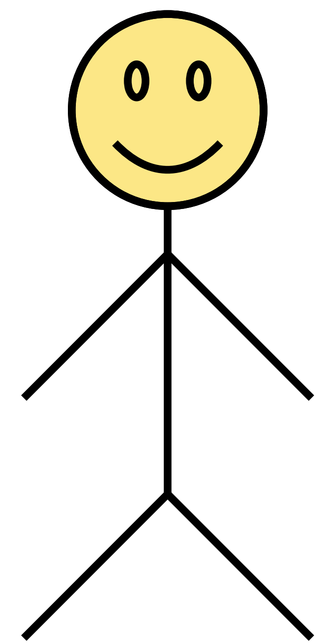


Proof Writer (You)

$x = 242$

Reader *Picks*



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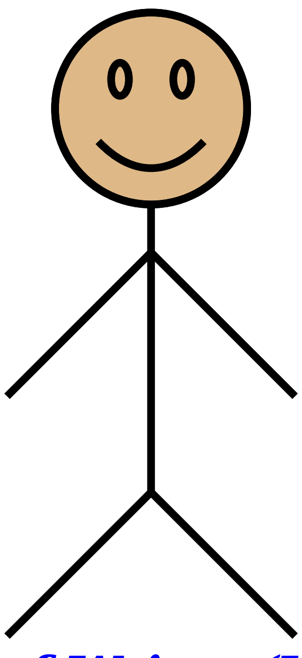

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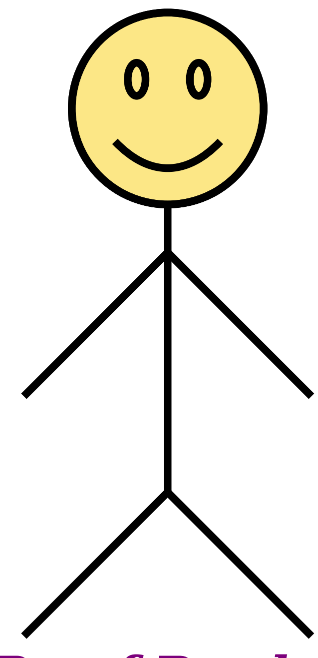

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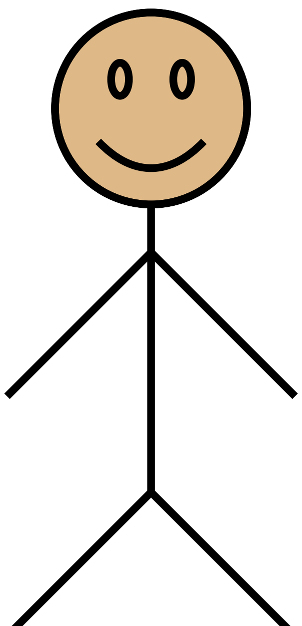

Proof Writer (You)

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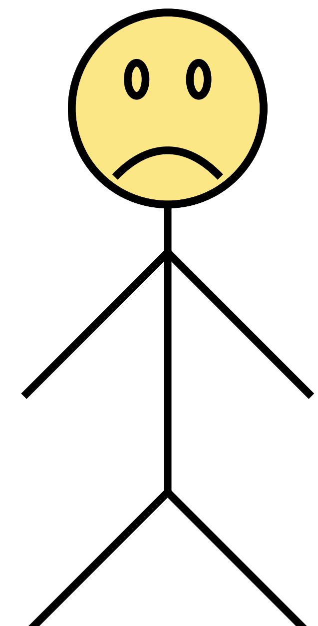

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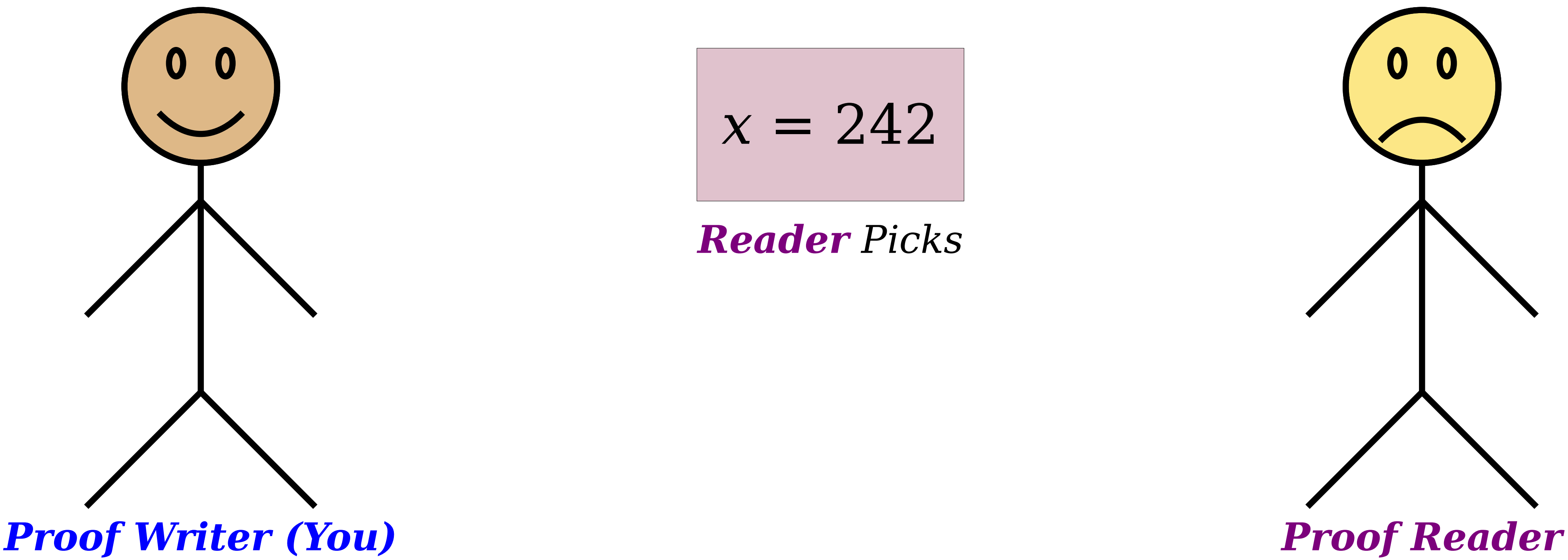

Proofs as a Dialog

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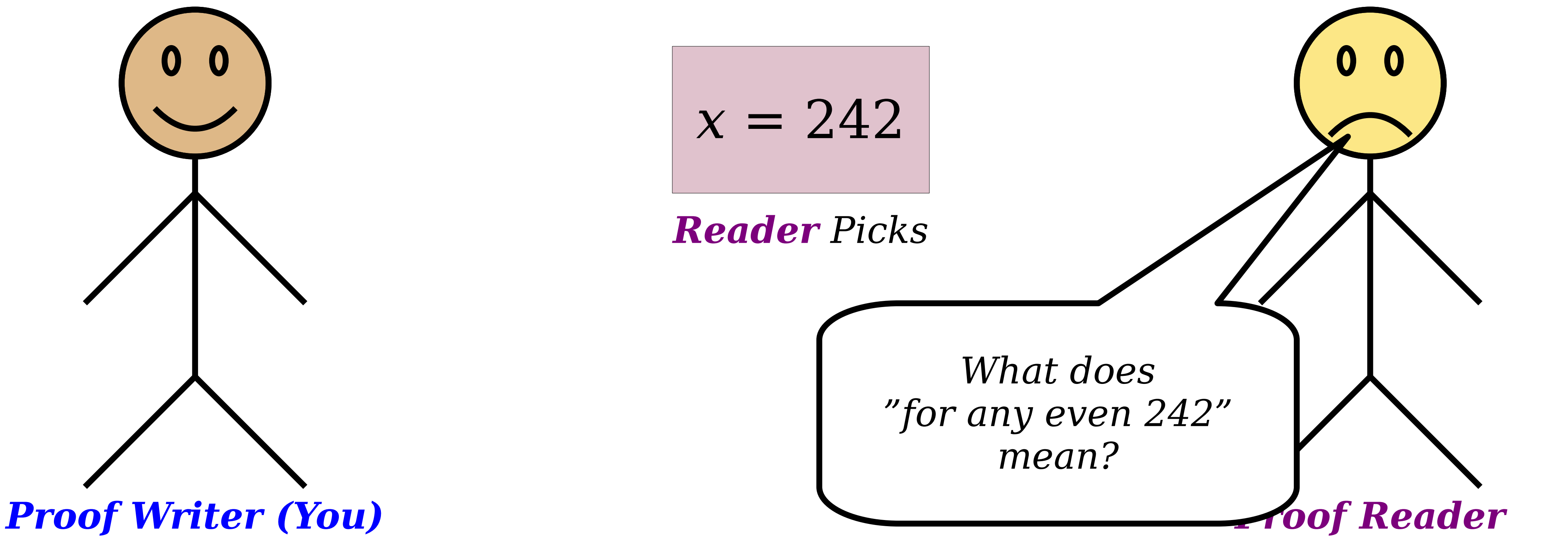

Proofs as a Dialog

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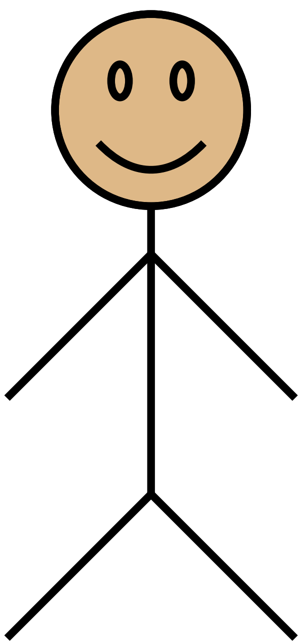

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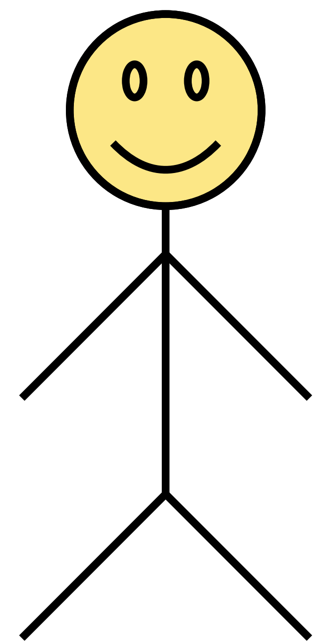

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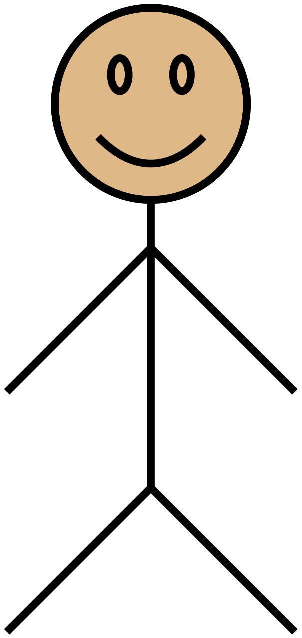

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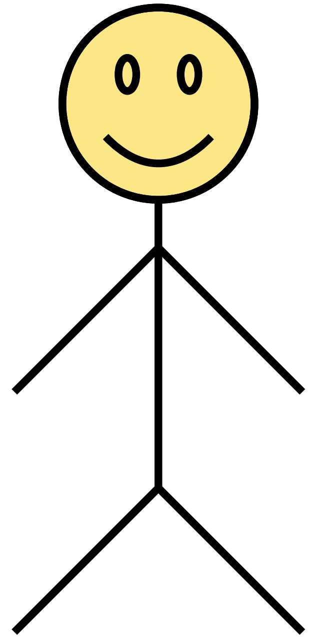

Proof Reader

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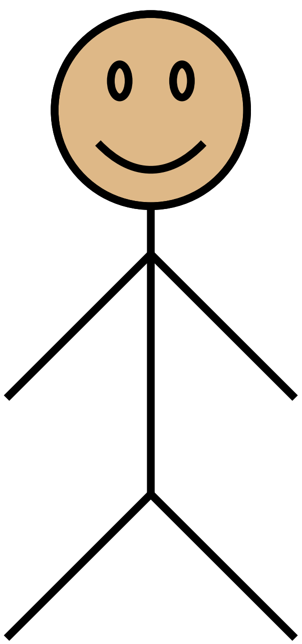
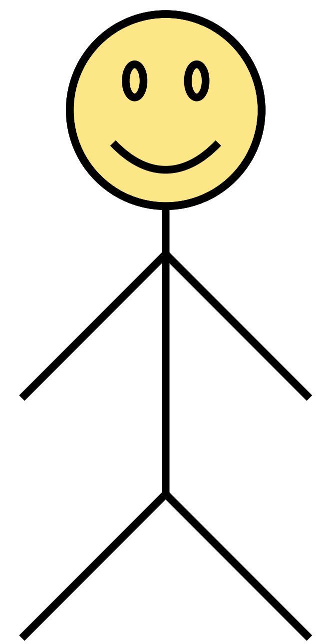


Proof Writer (You)

$x = 242$

Reader *Picks*



Proof Reader

Proofs as a Dialog

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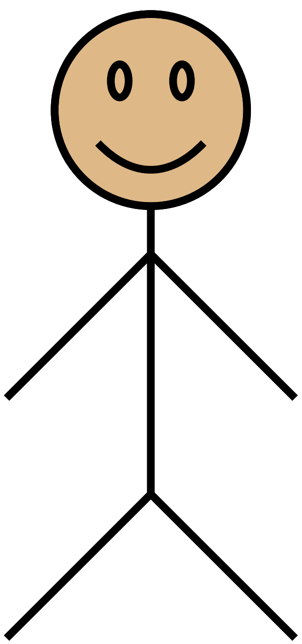

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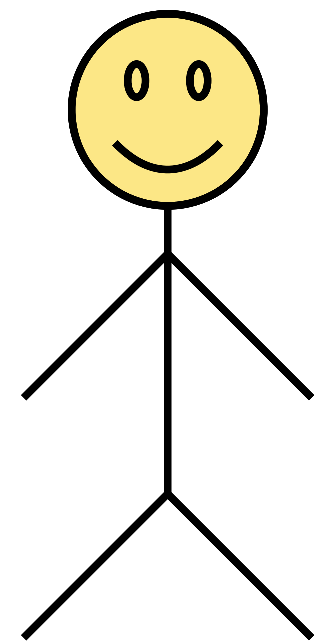


Proof Writer (You)

Proof Reader

Every variable needs a value.

Avoid talking about "all x" or "every x" when manipulating something concrete.

To prove something is true for any choice of a value for x, let the reader pick x.

Once you've said something like

Let x be an integer.
Consider an arbitrary $x \in \mathbb{Z}$.
Pick any x.

Do not say things like the following:

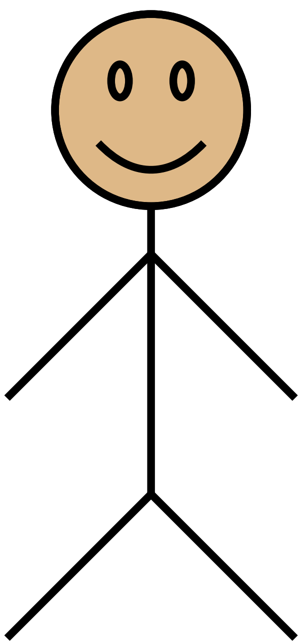
This means that ***for any*** $x \in \mathbb{Z}$ …
So ***for all*** $x \in \mathbb{Z}$ …

Proofs as a Dialog

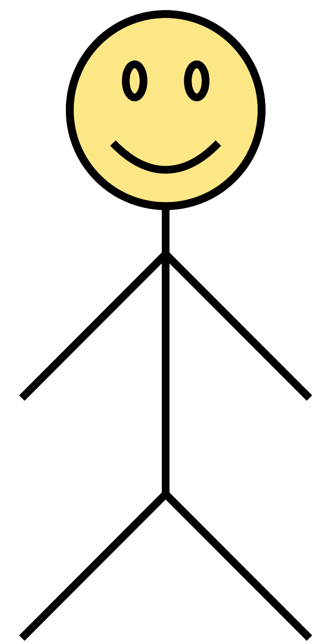
Proofs as a Dialog

Pick two integers m and n where $m+n$ is odd.

Let $n = 1$, which means that $m+1$ is odd.



Proof Writer (You)

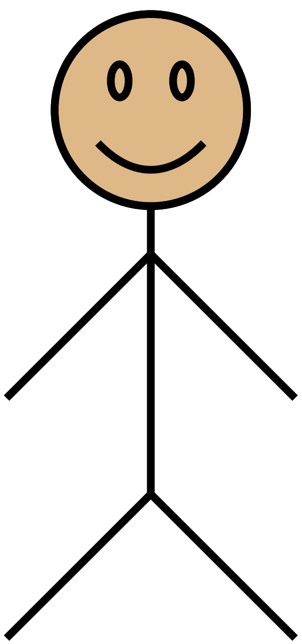


Proof Reader

Proofs as a Dialog

Pick two integers m and n where $m+n$ is odd.

Let $n = 1$, which means that $m+1$ is odd.



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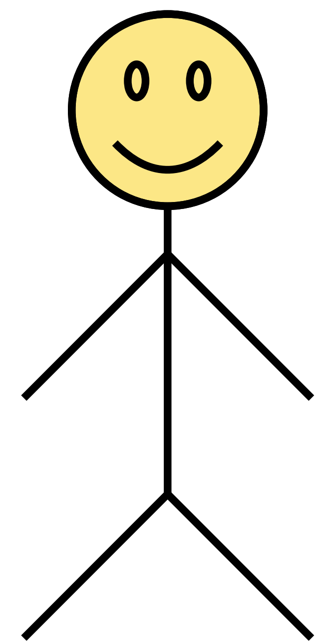

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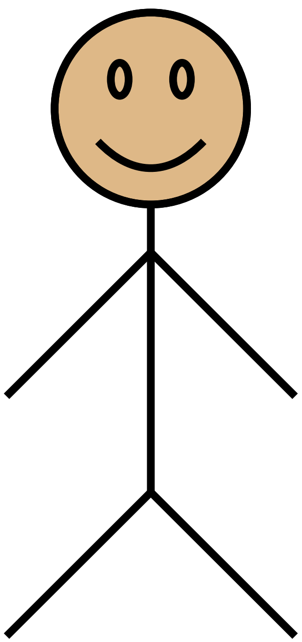


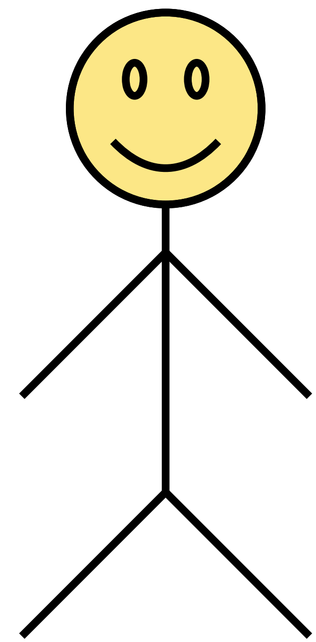
Proof Writer (You)

$m = 103$

Reader *Picks*

$n = 166$

Reader *Picks*



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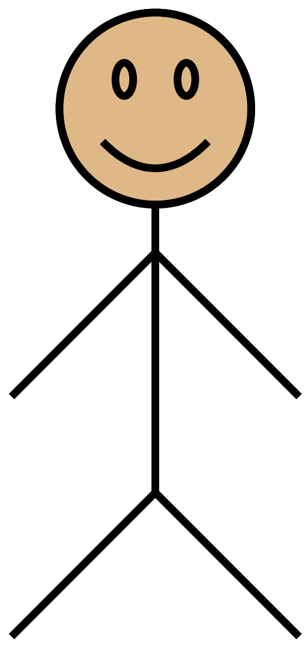

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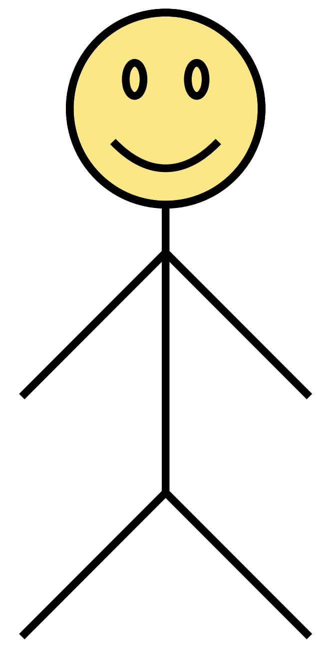

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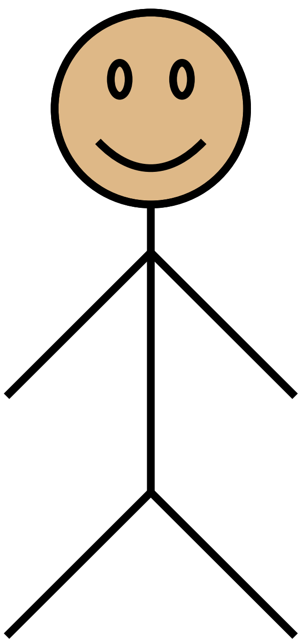

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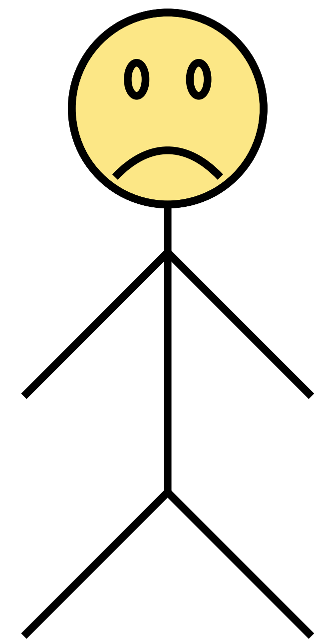

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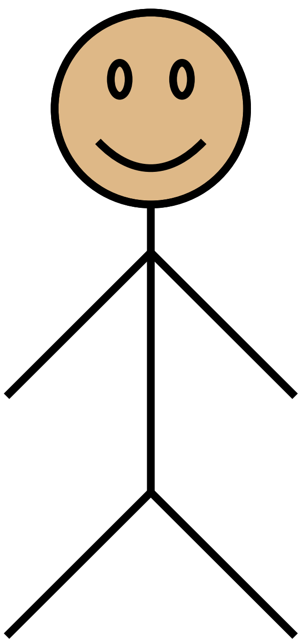

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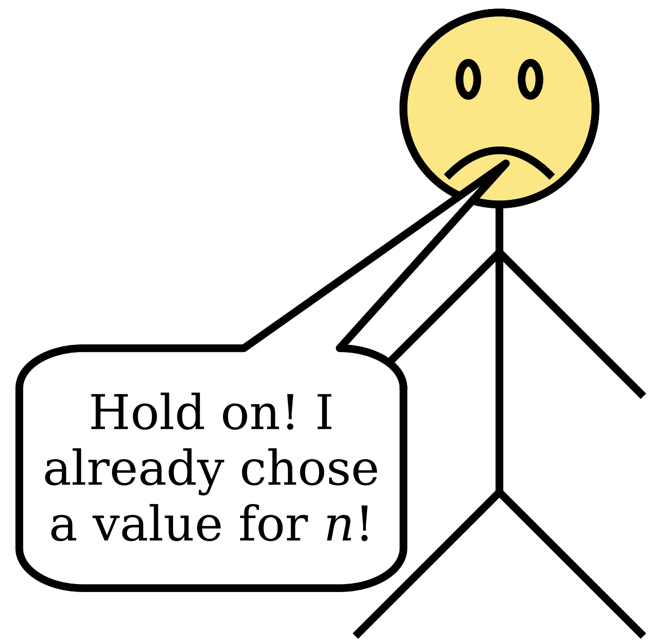
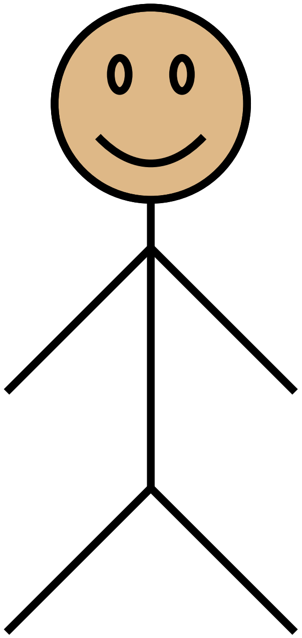


Proof Reader

Proofs as a Dialog

Let $n = 1$.

Pick any integer m where $m+1$ is odd.



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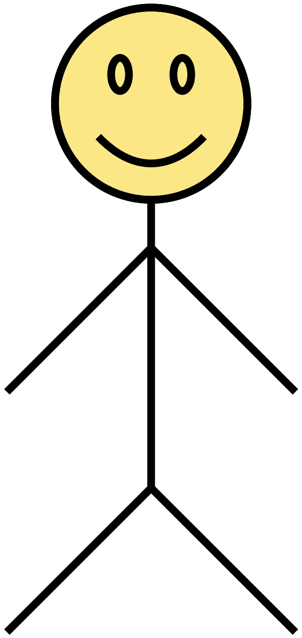

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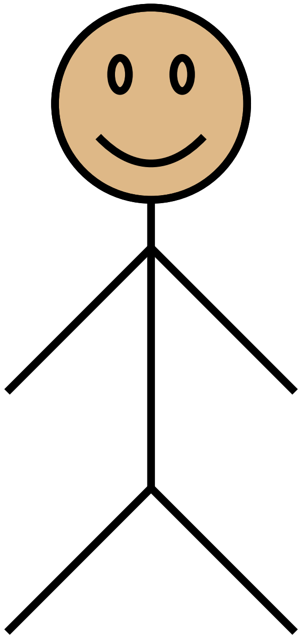

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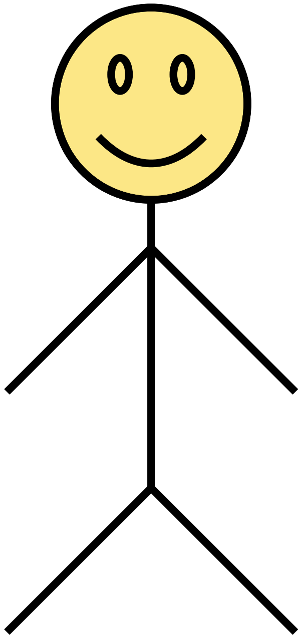

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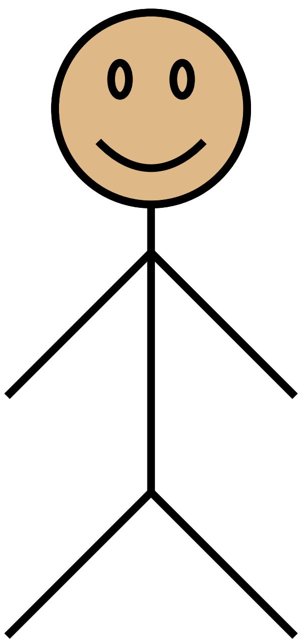
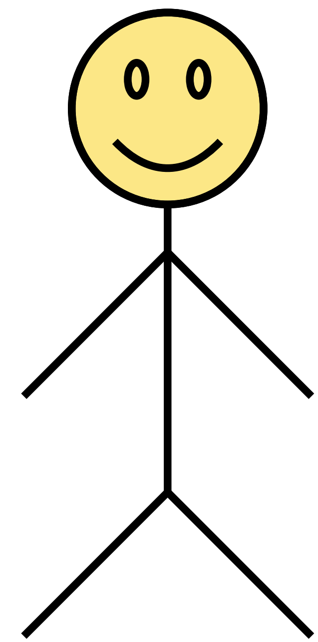


Proof Writer (You)

$n = 1$

Writer *Picks*



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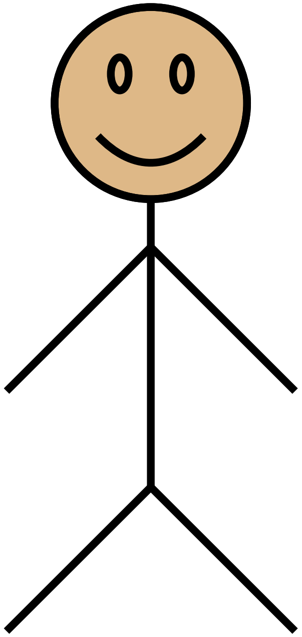
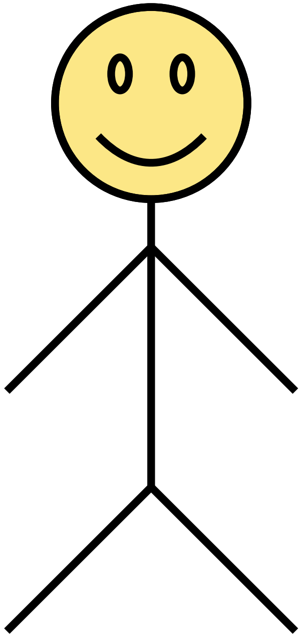


Proof Writer (You)

$n = 1$

Writer *Picks*



Proof Reader

Proofs as a Dialog

Let $n = 1$.

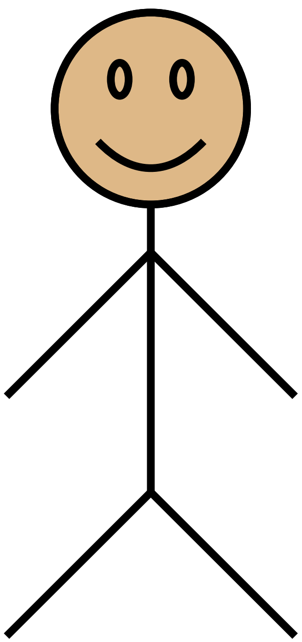
Pick any integer m where $m+1$ is odd.

$m = 166$

Reader *Picks*

$n = 1$

Writer *Picks*



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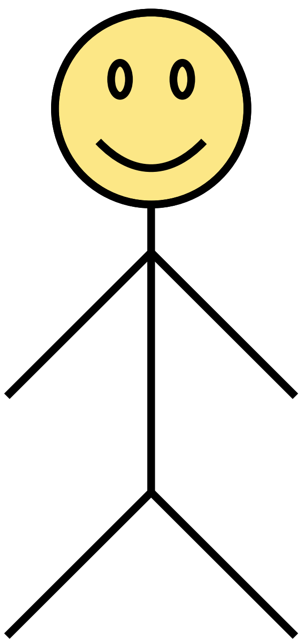

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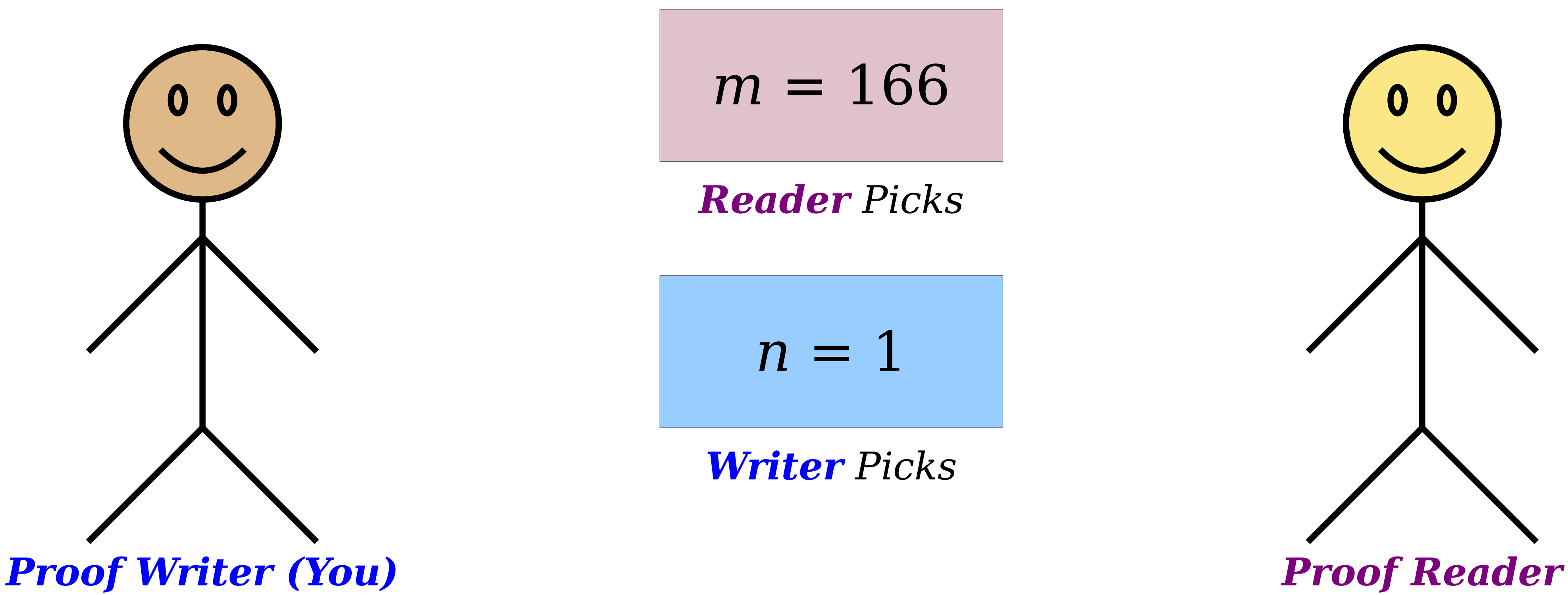
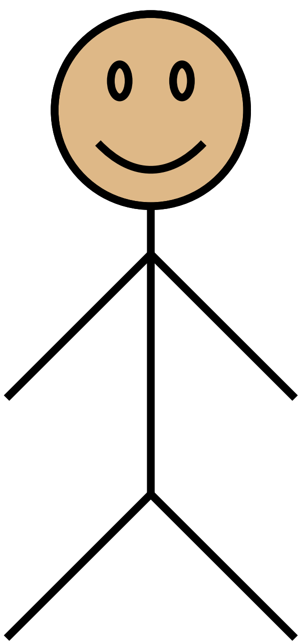


Proofs as a Dialog

Let $n = 1$.

Pick any integer m where $m+1$ is odd.

Do we even need n here?



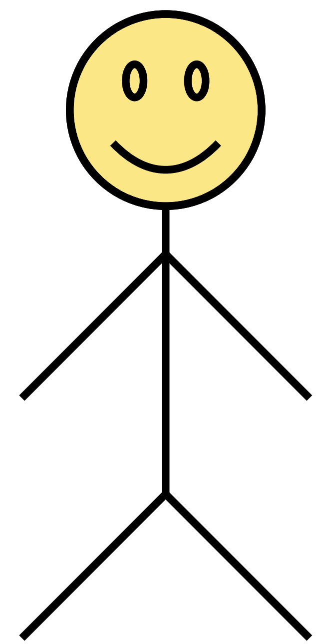
Proof Writer (You)

$m = 166$

Reader *Picks*

$n = 1$

Writer *Picks*



Proof Reader

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Proofs as a Dialog

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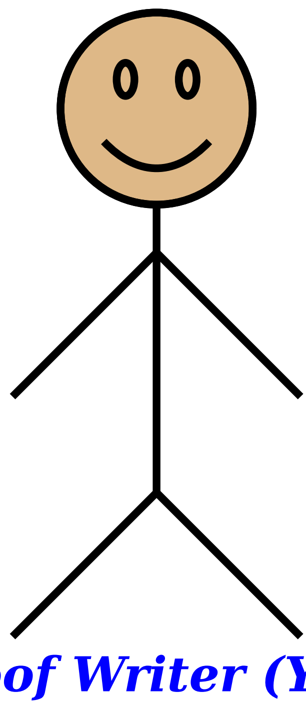
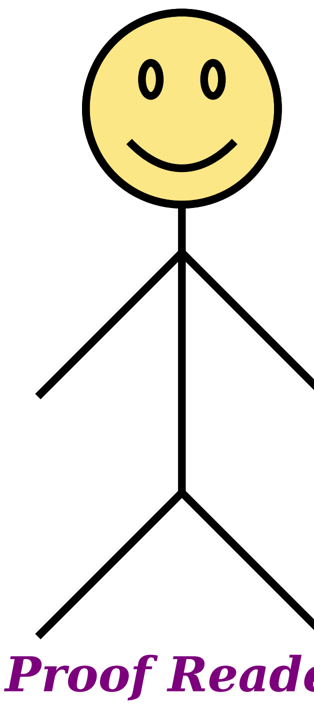
$m = 166$

Reader *Picks*



Proof Writer (You)



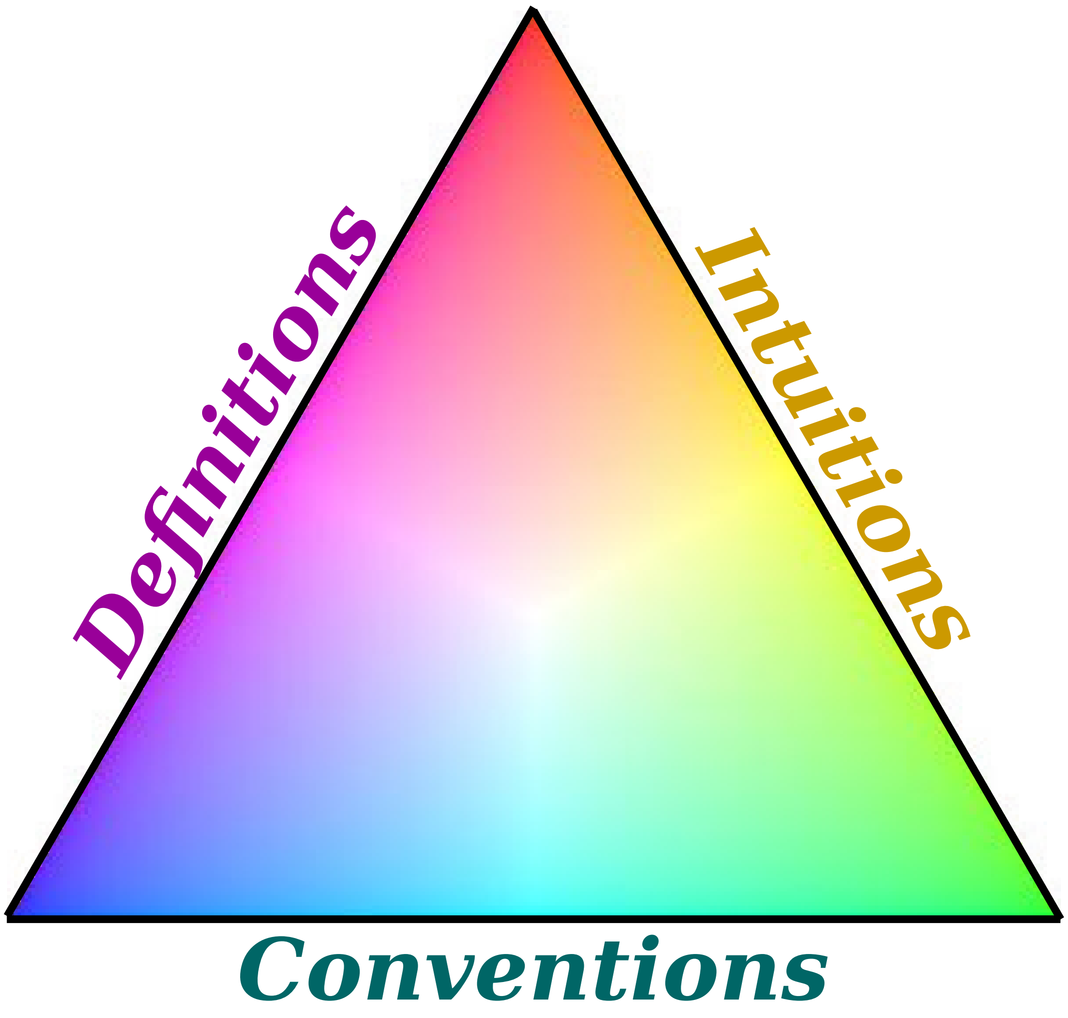
Proof Reader

Be mindful of who owns what variable.

Don’t change something you don’t own.

You don’t always need to name things, especially if they already have a name.

To Recap



Writing a good proof requires a blend of definitions, intuitions, and conventions.

An integer n is ***even*** if there is an integer k where $n = 2k$.

An integer n is ***odd*** if there is an integer k where $n = 2k+1$.

Definitions tell us what we need to do in a proof.
Many proofs directly reference these definitions.

Let's Draw Some Pictures!

Let's Do Some Math!

Let's Try Some Examples!

Building intuition for results requires creativity, trial, and error.

- Prove universal statements by making arbitrary choices.
- Prove existential statements by making concrete choices.
- Prove "If P, then Q" by assuming P and proving Q.
- Write in complete sentences.
- Number sub-formulas when referring to them.
- Summarize what was shown in proofs by cases.
- Articulate your start and end points.

Mathematical proofs have established conventions that increase rigor and readability.

Your Action Items

- ***Read "Guide to $\in$ and $\subseteq$."***
  - You'll want to have a handle on how these concepts are related, and on how they differ.
- ***Read "Guide to Proofs."***
  - This resource covers proofwriting strategies and conventions and is an essential complement to this lecture.
- ***Read "Guide to Partners."***
  - It's all about how to work effectively in pairs. Mull this over so you're ready to go for Problem Set 1.
- ***Finish and submit Problem Set 0.***
  - Don't put this off until the last minute!

Next Time

- ***Indirect Proofs***
  - How do you prove something without actually proving it?
- ***Mathematical Implications***
  - What exactly does "if P, then Q" mean?
- ***Proof by Contrapositive***
  - A helpful technique for proving implications.
- ***Proof by Contradiction***
  - Proving something is true by showing it can't be false.