Mathematical Proofs

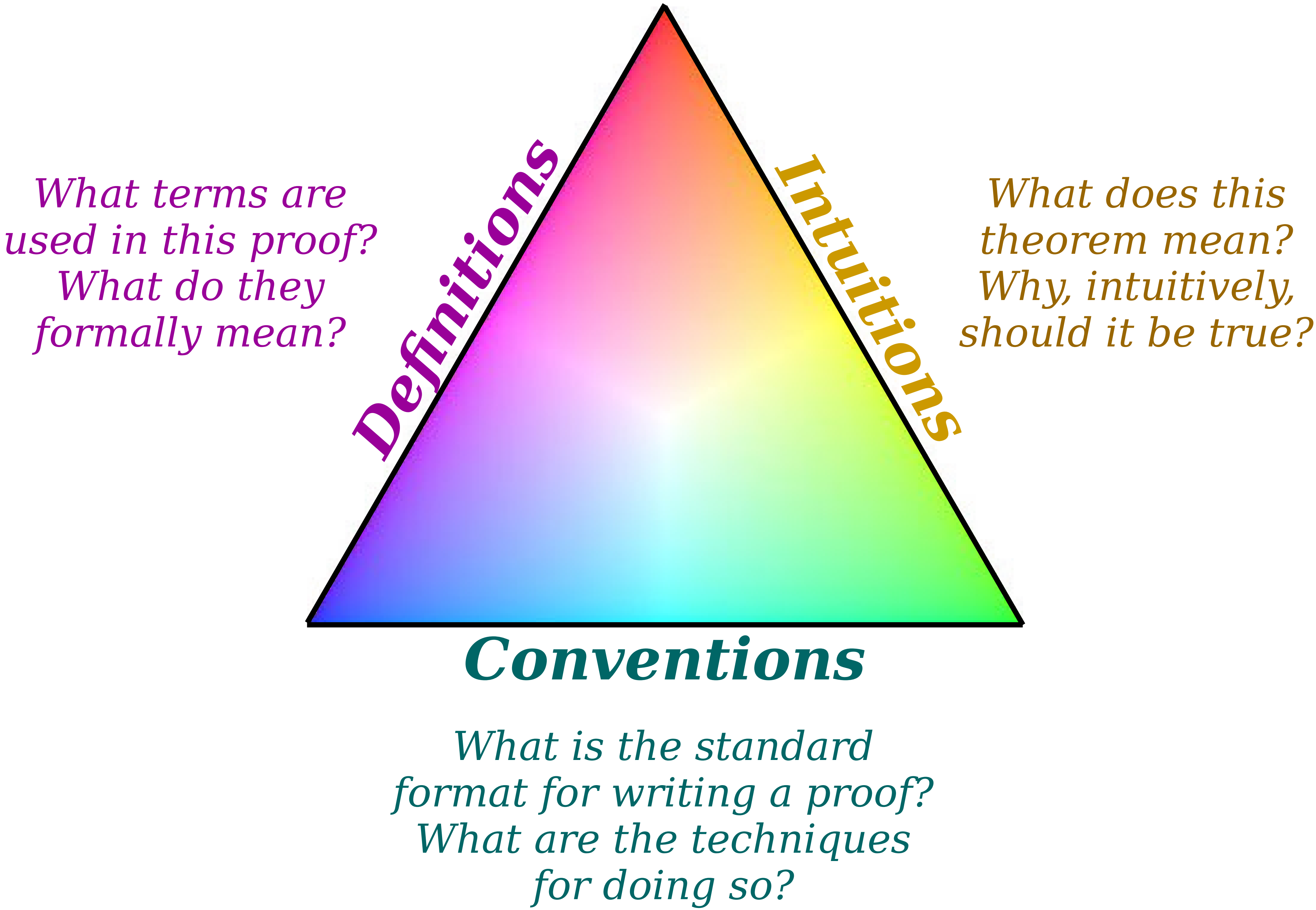
Outline for Today

- ***How to Write a Proof***
  - Synthesizing definitions, intuitions, and conventions.
- ***Proofs on Numbers***
  - Working with odd and even numbers.
- ***Universal and Existential Statements***
  - Two important classes of statements.
- ***Variable Ownership***
  - Who owns what?

What is a Proof?

A ***proof*** is an argument that demonstrates why a conclusion is true, subject to certain standards of truth.

A ***mathematical proof*** is an argument that demonstrates why a mathematical statement is true, following the rules of mathematics.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What terms are used in this proof? What do they formally mean?
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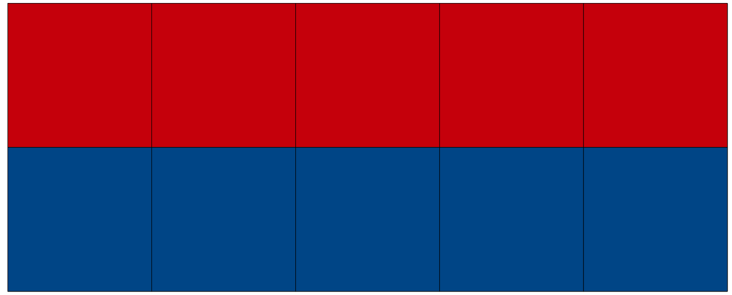
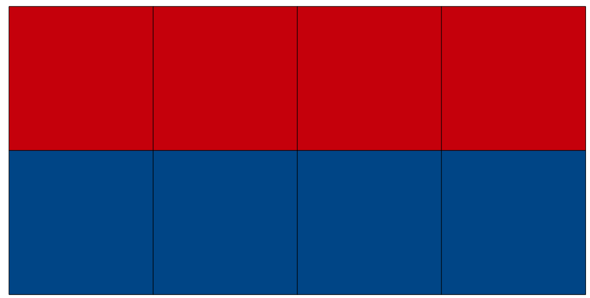
Writing our First Proof

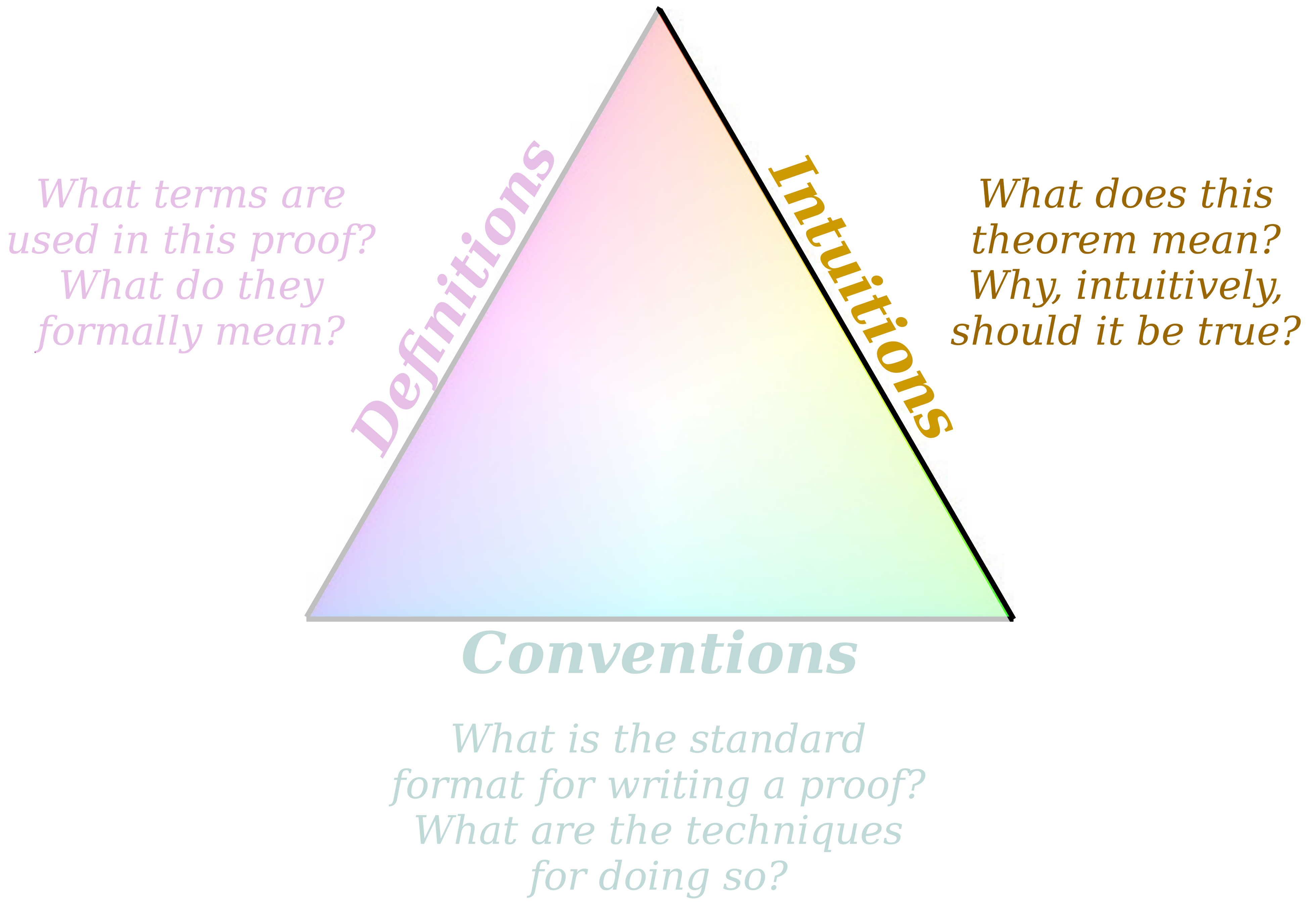
Theorem: If n is an even integer, then n^2 is even.

10  $2 \cdot \mathbf{5}$

8  $2 \cdot \mathbf{4}$

0 $2 \cdot \mathbf{0}$

An integer n is called ***even*** if
there is an integer k where $n = 2k$.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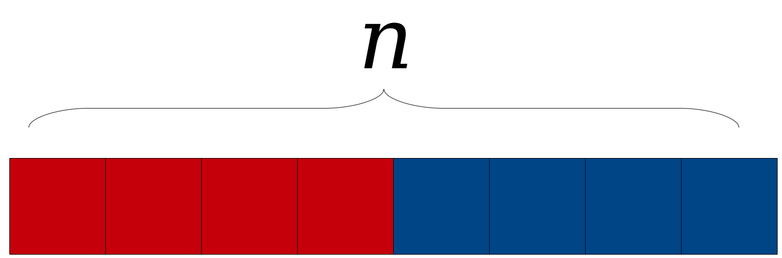
Let's Try Some Examples!

$$\begin{aligned}
2^2 &= 4 &&= 2 \cdot \mathbf{2} \\
10^2 &= 100 &&= 2 \cdot \mathbf{50} \\
0^2 &= 0 &&= 2 \cdot \mathbf{0} \\
(-8)^2 &= 64 &&= 2 \cdot \mathbf{32} \\
n^2 & &&= 2 \cdot \mathbf{?}
\end{aligned}$$

What's the pattern? How do we predict this?

Theorem: If n is an even integer, then n^2 is even.

Let's Draw Some Pictures!



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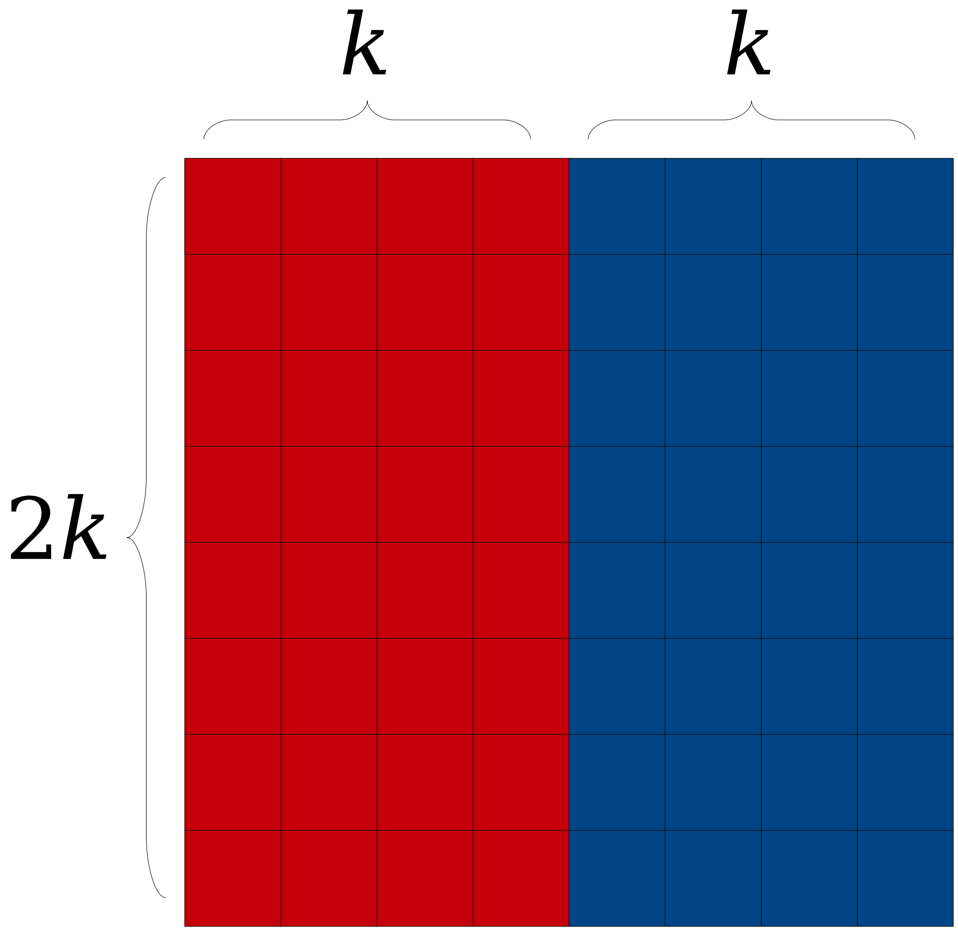
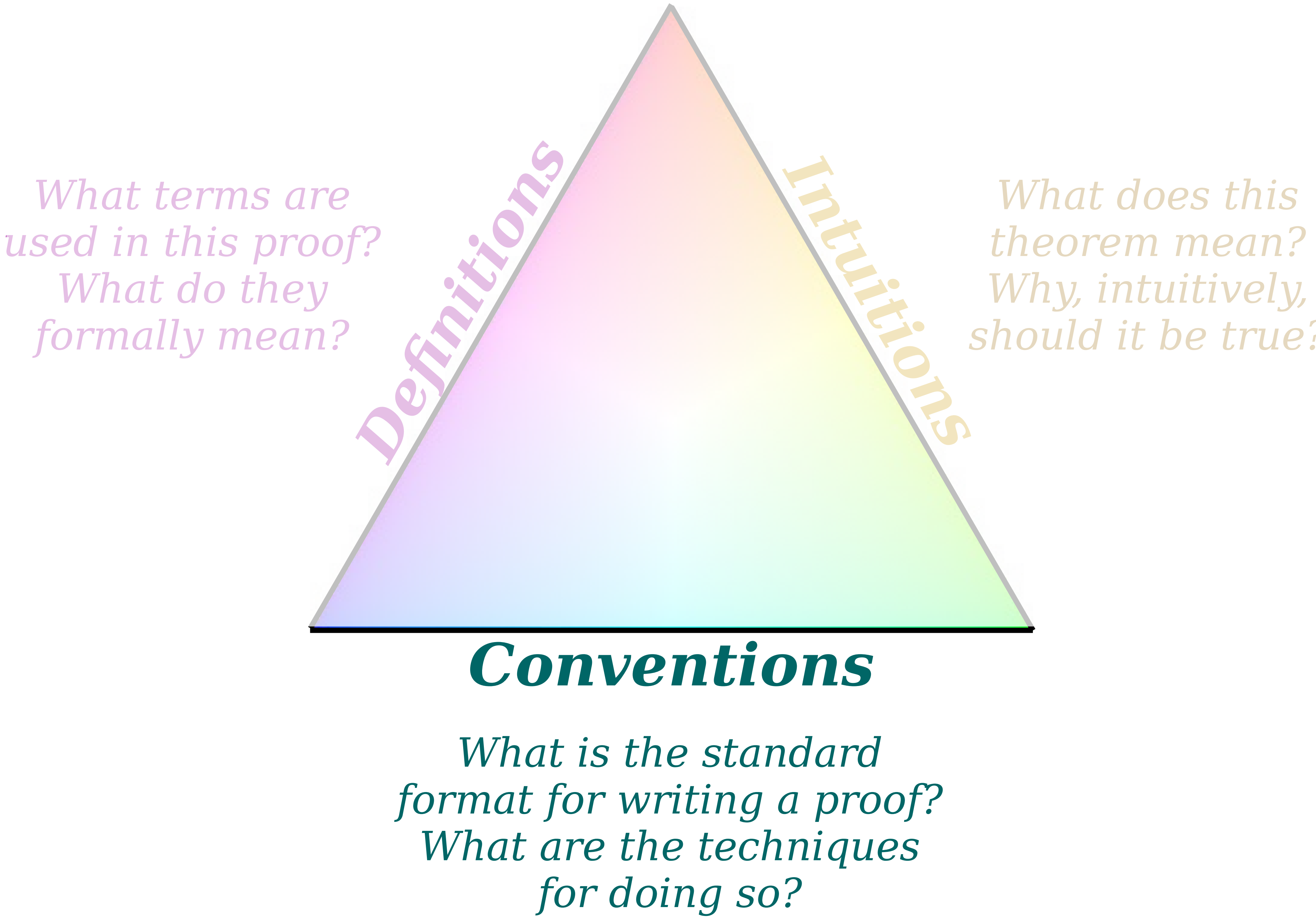


$n^2 = 2(2k^2)$

Theorem: If n is an even integer, then n^2 is even.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Our First Proof!

Theorem: If n is an even integer, then n^2 is even.

Proof: Pick an arbitrary even integer n. We need to show that n^2 is even.

Since n is even, there is some integer k such that $n = 2k$. This means that

$$\begin{aligned} n^2 &= (2k)^2 \\ &= 4k^2 \\ &= 2(2k^2). \end{aligned}$$

From this, we see that there is an integer m (namely, $2k^2$) where $n^2 = 2m$. Therefore, n^2 is even, which is what we wanted to show. ■

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This symbol means "end of proof"

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Since n is eve[...] that $n = 2k$. T[...]

From this, we [...] (namely, $2k^2$) [...] is even, which [...]

To prove a statement of the form

"If *P* is true, then *Q* is true,"

start by assuming that ***P*** is true. Here, we're inviting the reader to pick their favorite even integer.

Our First Proof!

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To prove a statement of the form

"If *P* is true, then *Q* is true,"

after assuming ***P*** is true, you need to show that ***Q*** is true. Here, we're telling the reader where we're headed.

Our First Proof!

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Proof: Pick an arbitrary even integer n. We need to show that n^2 is even.

Since n is even, there is some integer k such that $n = 2k$. This means that

n

This is the definition of an even integer. We need to use this definition to make this proof rigorous.

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(namely, $2k^2$) where $n^2 = 2m$. Therefore, n^2 is even, which is what we wanted to show. ■

Our First Proof!

Notice how we use the value of ***k*** that we obtained above. Giving names to quantities, allows us to manipulate them. This is similar to variables in programs.

Theorem: I

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> Our ultimate goal is to prove that n^2 is even. This means that we need to find some m where $n^2 = 2m$. Here, we're explicitly showing how we can do that.

$$
\begin{aligned}
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$$\begin{aligned} n^2 &= (2k)^2 \\ &= 4k^2 \\ &= 2(2k^2). \end{aligned}$$

Hey, that's what we said we were going to do! We're done now.

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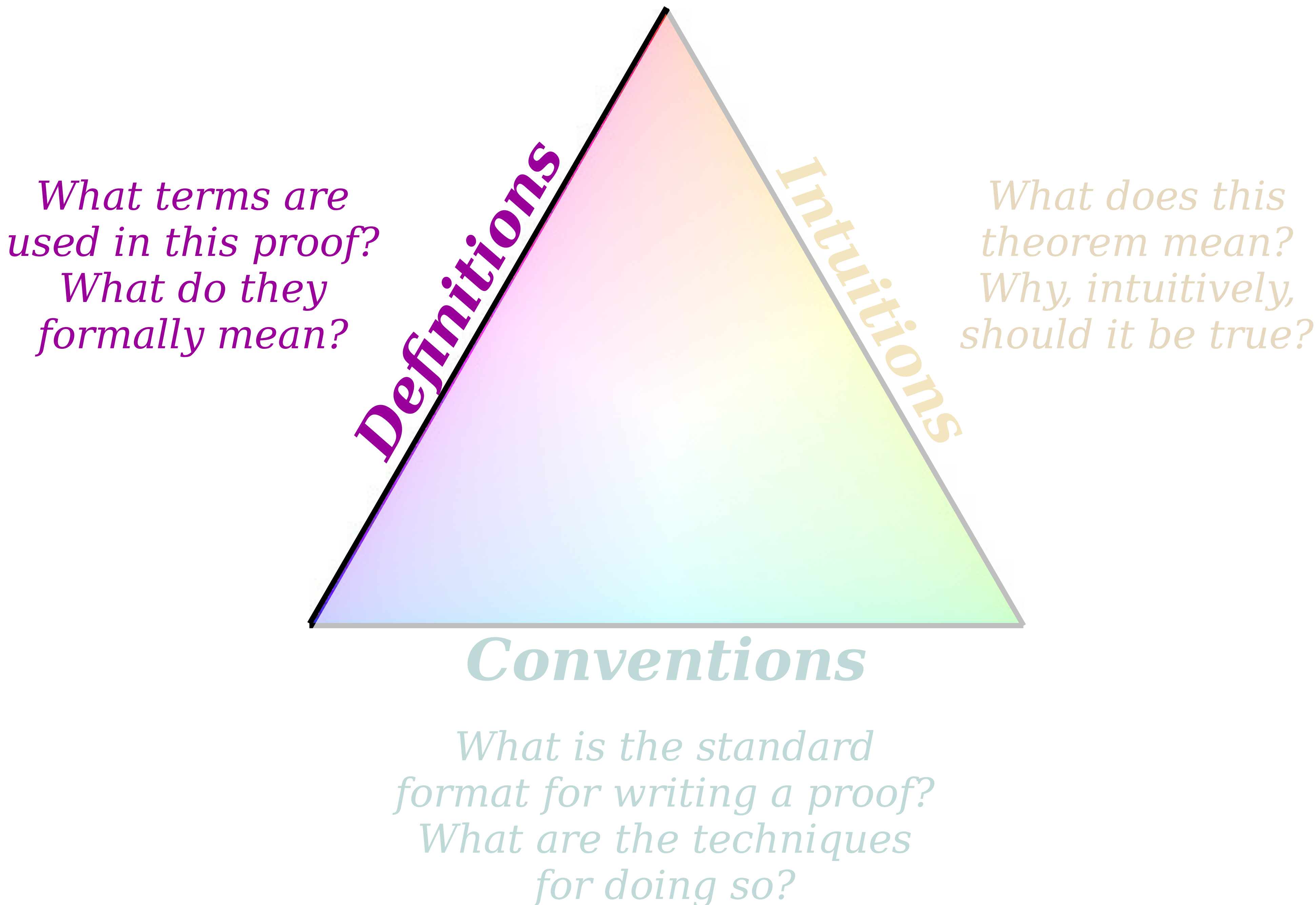
Our Next Proof

Theorem: For any integers m and n, if m and n are odd, then $m + n$ is even.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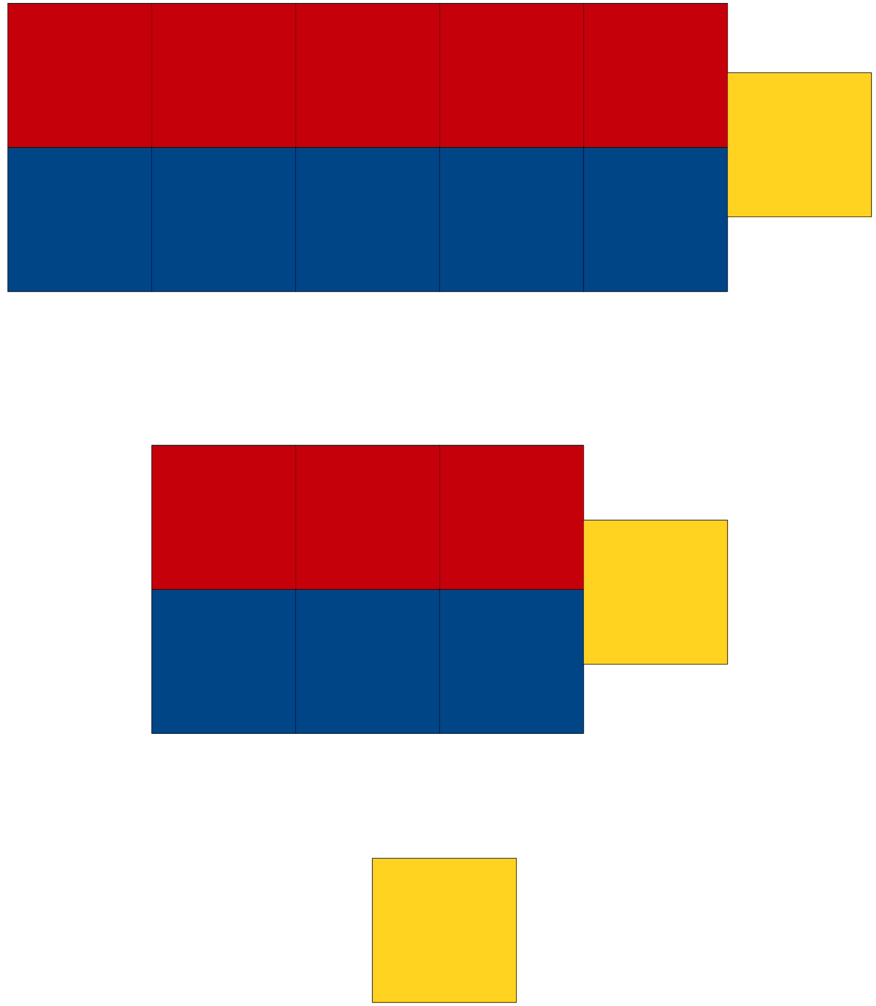


11 $2 \cdot \mathbf{5} + \mathbf{1}$

7 $2 \cdot \mathbf{3} + \mathbf{1}$

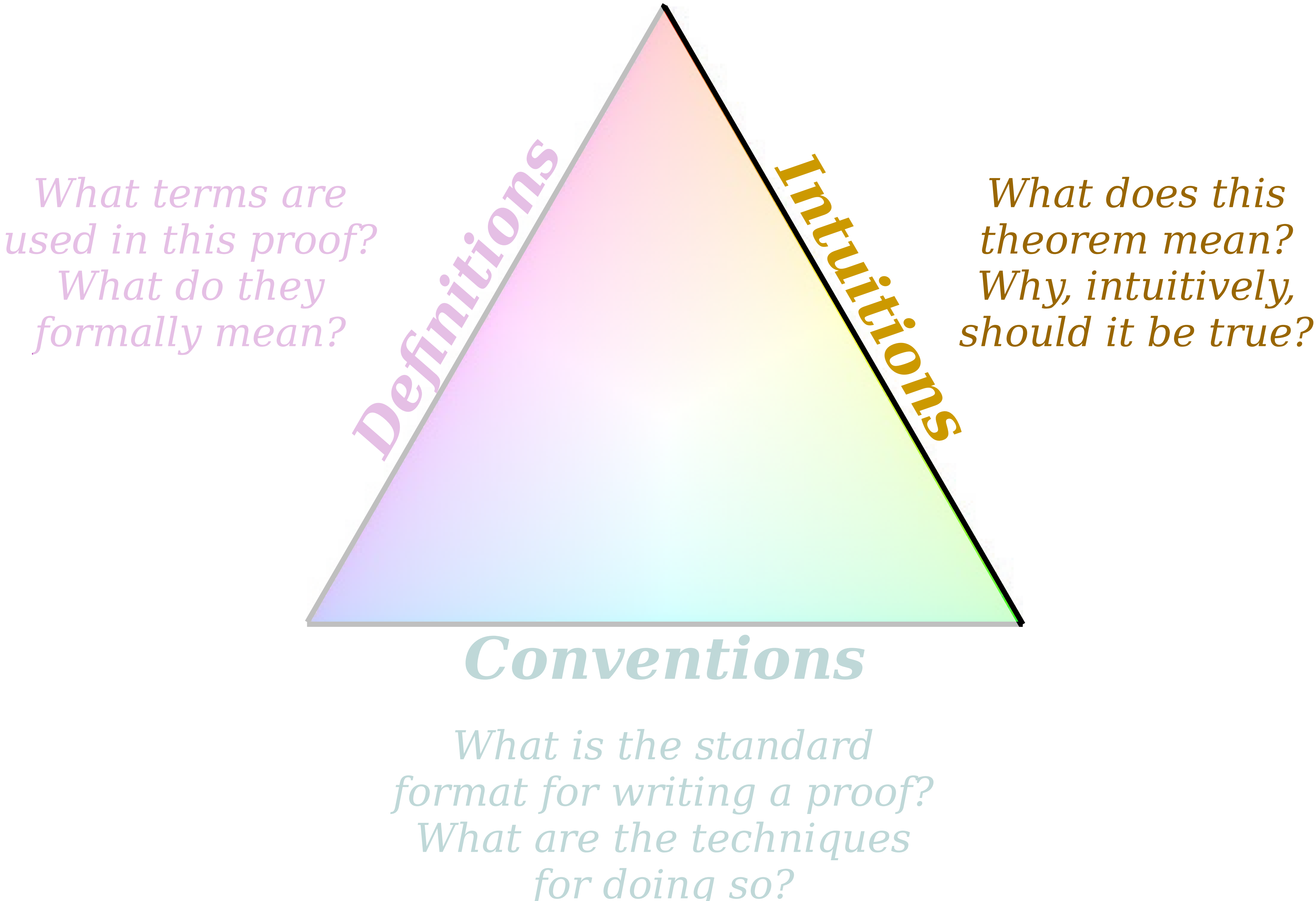
1 $2 \cdot \mathbf{0} + \mathbf{1}$

An integer n is called ***odd*** if
there is an integer k where $n = 2k+1$.

Going forward, we'll assume the following:

1. Every integer is either even or odd.
2. No integer is both even and odd.

Theorem: For any integers m and n, if m and n are odd, then $m + n$ is even.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Let's Try Some Examples!

$$\begin{aligned} 1 + 1 &= 2 &&= 2 \cdot \mathbf{1} \\ 137 + 103 &= 240 &&= 2 \cdot \mathbf{120} \\ -5 + 5 &= 0 &&= 2 \cdot \mathbf{0} \\ m + n & &&= 2 \cdot \mathbf{?} \end{aligned}$$

What's the pattern? How do we predict this?

Theorem: For any integers m and n, if m and n are odd, then $m+n$ is even.

Let's Do Some Math!

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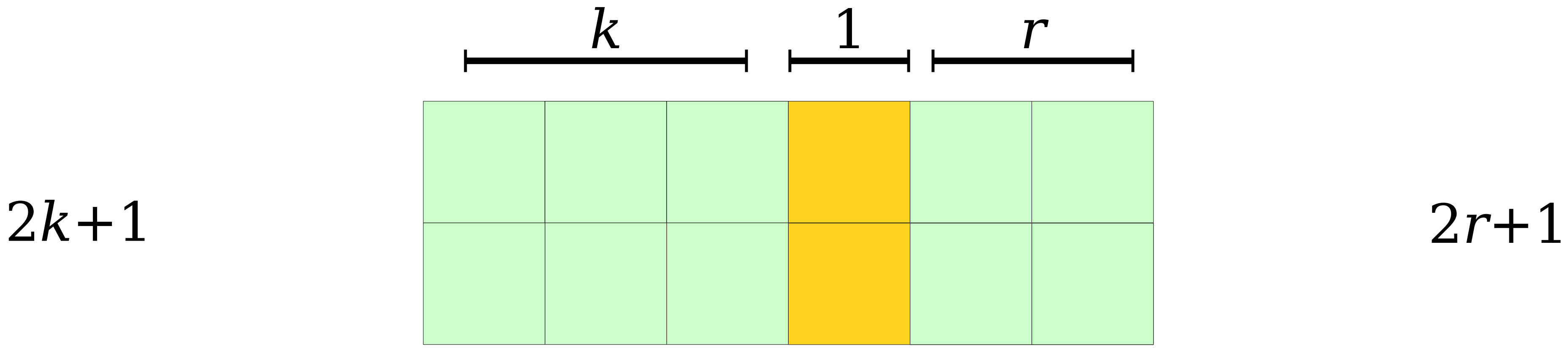

$$(2k+1) + (2r+1) = 2(k + r + 1)$$

Theorem: For any integers m and n, if m and n are odd, then $m+n$ is even.

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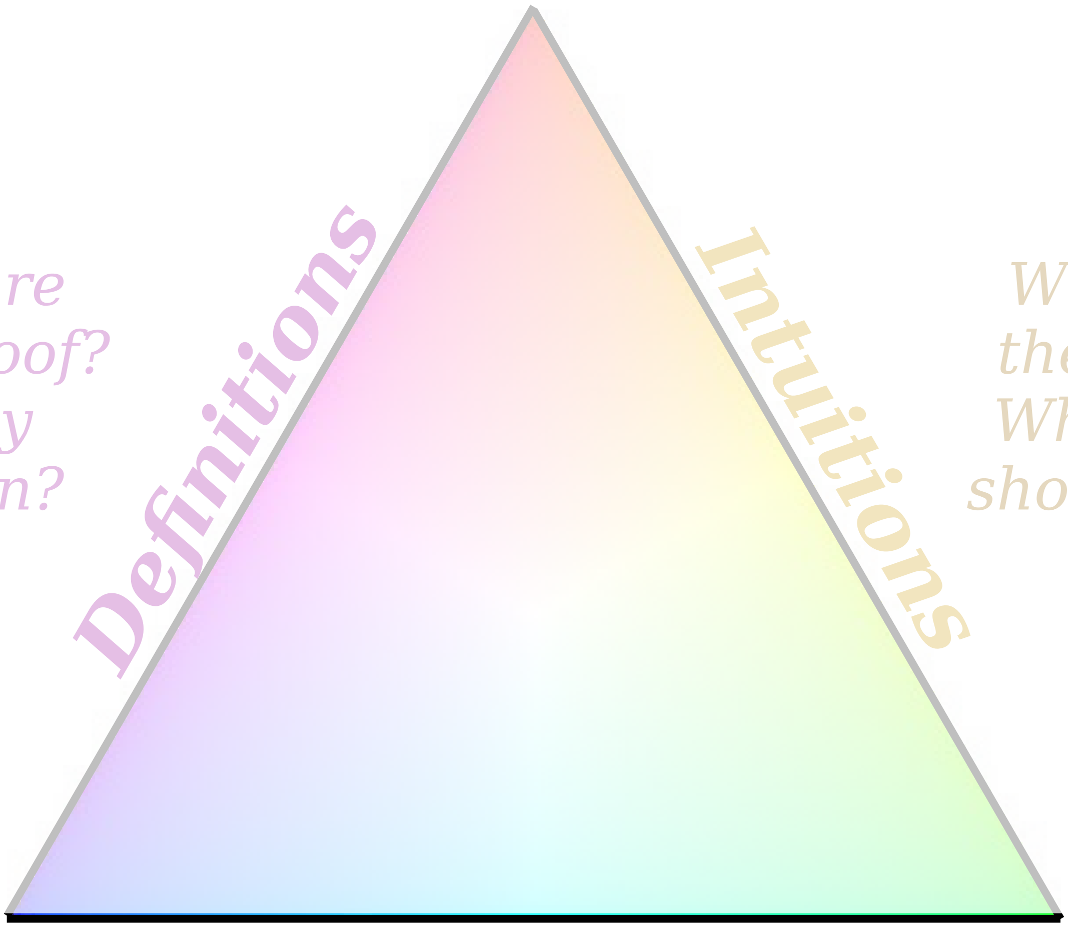

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Conventions

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Theorem: For any integers m and n, if m and n are odd, then $m + n$ is even.

Proof: Consider any arbitrary integers m and n where m and n are odd. We need to show that $m + n$ is even.

Since m is odd, we know that there is an integer k where

$$m = 2k + 1. \tag{1}$$

Similarly, because n is odd there must be some integer r such that

$$n = 2r + 1. \tag{2}$$

By adding equations (1) and (2) we learn that

$$\begin{aligned} m + n &= 2k + 1 + 2r + 1 \\ &= 2k + 2r + 2 \\ &= 2(k + r + 1). \end{aligned} \tag{3}$$

Equation (3) tells us that there is an integer s (namely, $k + r + 1$) such that $m + n = 2s$. Therefore, we see that $m + n$ is even, as required. ■

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By letting the reader pick **m** and **n** arbitrarily, anything we prove about **m** and **n** will generalize to all possible choices for those values.

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Numbering these equalities lets us refer back to them later on, making the flow of the proof a bit easier to understand.

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$$m = 2k + 1. \qquad (1)$$

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(2)

This is a complete sentence! Proofs are expected to be written in complete sentences, so you'll often use punctuation at the end of formulas.

We recommend using the "mugga mugga" test – if you read a proof and replace all the mathematical notation with "mugga mugga," what comes back should be a valid sentence.

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$2r + 1$

$+ 2$

$+ 1). \qquad (3)$

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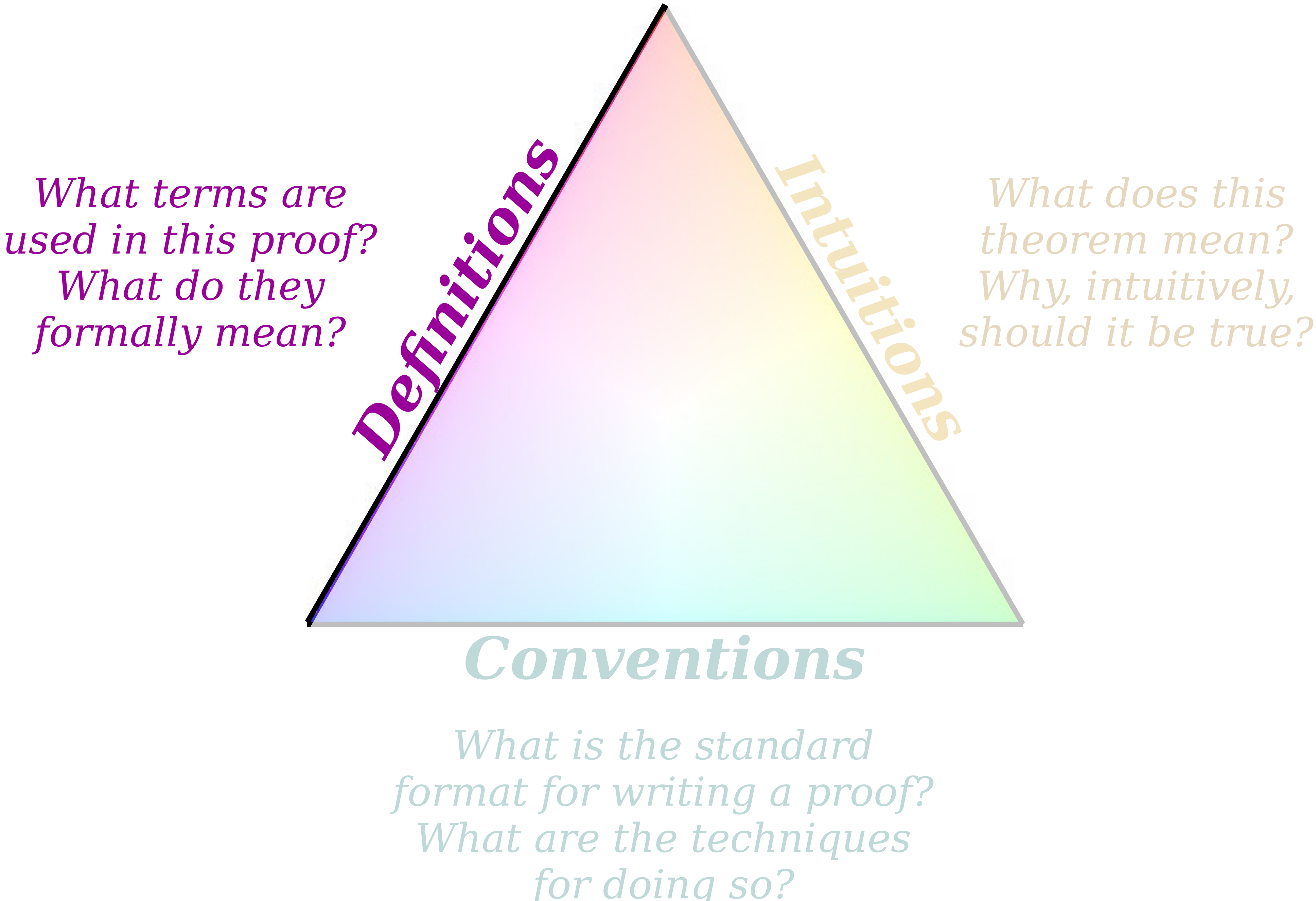
Some Little Exercises

- Here's a list of other theorems that are true about odd and even numbers:
  - ***Theorem:*** The sum and difference of any two even numbers is even.
  - ***Theorem:*** The sum and difference of an odd number and an even number is odd.
  - ***Theorem:*** The product of any integer and an even number is even.
  - ***Theorem:*** The product of any two odd numbers is odd.
- Going forward, we'll just take these results for granted. Feel free to use them in the problem sets.
- If you'd like to practice the techniques from today, try your hand at proving these results!

Universal and Existential Statements

Theorem: For any odd integer n, there exist integers r and s where $r^2 - s^2 = n$.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This result is true for every possible choice of odd integer n. It'll work for $n = 1$, $n = 137$, $n = 103$, etc.

Theorem: For any odd integer n, there exist integers r and s where $r^2 - s^2 = n$.

We aren't saying this is true for every choice of r and s. Rather, we're saying that *somewhere out there* are choices of r and s where this works.

Universal vs. Existential Statements

- A ***universally-quantified statement*** is a statement of the form

For all *x*, `[some-property]` holds for *x*.

- We've seen how to prove these statements.
- An ***existentially-quantified statement*** is a statement of the form

There is some *x* where `[some-property]` holds for *x*.

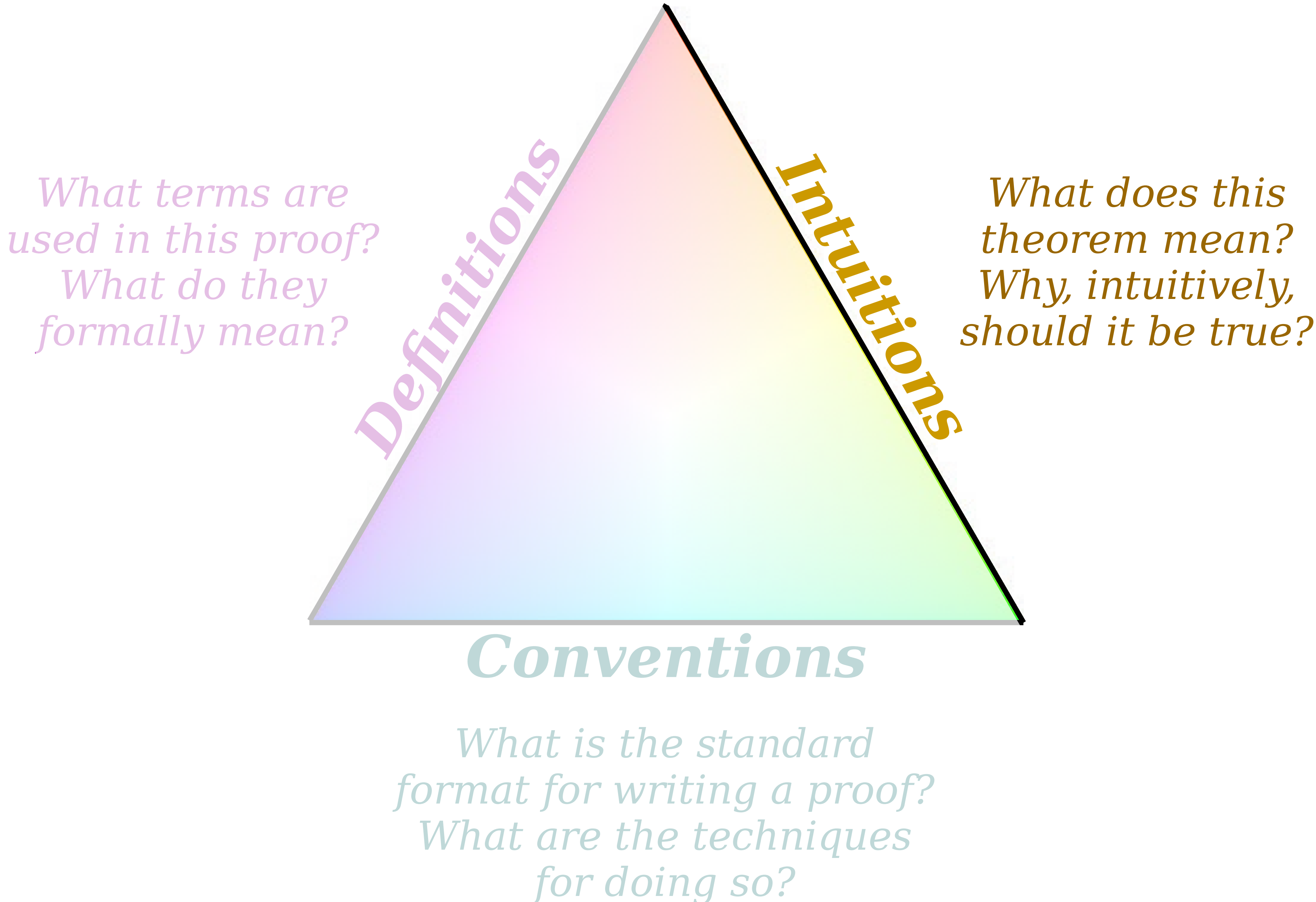
- How do you prove an existentially-quantified statement?

Proving an Existential Statement

- Over the course of the quarter, we will see several different ways to prove an existentially-quantified statement of the form

  There is an *x* where `[some-property]` holds for *x*.

- ***Simplest approach:*** Search far and wide, find an *x* that has the right property, then show why your choice is correct.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Let's Try Some Examples!

$1 = \mathbf{1}^2 - \mathbf{0}^2$

$3 = \mathbf{2}^2 - \mathbf{1}^2$

$5 = \mathbf{3}^2 - \mathbf{2}^2$

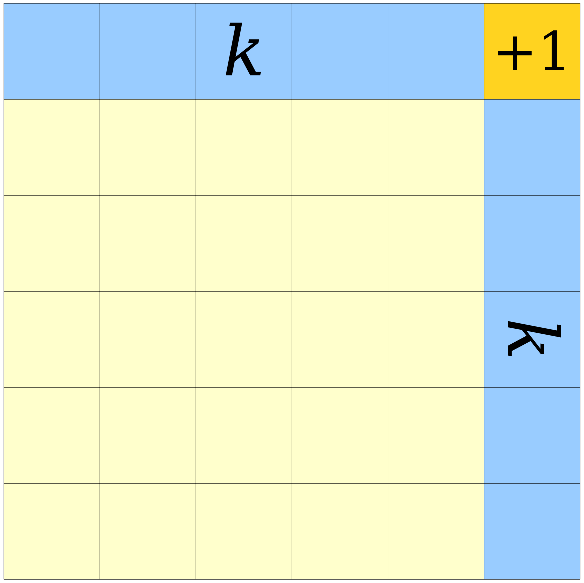
$7 = \mathbf{4}^2 - \mathbf{3}^2$

$9 = \mathbf{5}^2 - \mathbf{4}^2$

We've got a pattern – but why does this work?

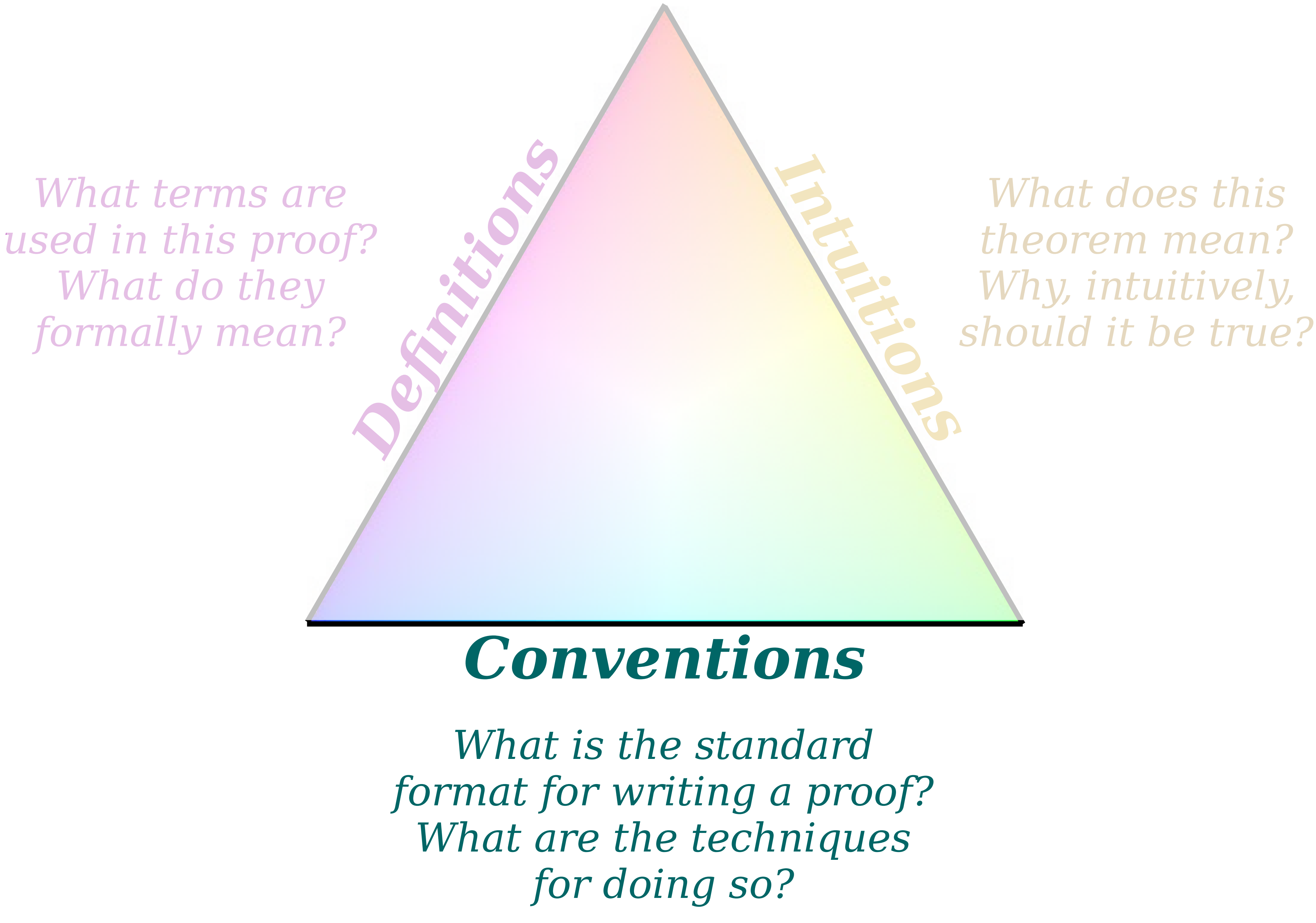
Theorem: For any odd integer n, there exist integers r and s where $r^2 - s^2 = n$.

Let’s Draw Some Pictures!



$$(k+1)^2 - k^2 = 2k+1$$

Theorem: For any odd integer n, there exist integers r and s where $r^2 - s^2 = n$.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Proof: Let n be an arbitrary odd integer. We will show that there exist integers r and s where $r^2 - s^2 = n$.

Since n is odd, we know there is an integer k where $n = 2k + 1$. Now, let $r = k+1$ and $s = k$. Then we see that

$$\begin{aligned} r^2 - s^2 &= (k+1)^2 - k^2 \\ &= k^2 + 2k + 1 - k^2 \\ &= 2k + 1 \\ &= n. \end{aligned}$$

This means that $r^2 - s^2 = n$, which is what we needed to show. ■

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We ask the reader to make an *arbitrary choice*. Rather than specifying what n is, we're signaling to the reader that they could, in principle, supply any choice n that they'd like.

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$$r^2 - s^2 = (k+ \ldots$$
$$= k^2 + 2k + 1 - k^2$$
$$= 2k + 1$$
$$= n.$$

This means that $r^2 - s^2 = n$, which is what we needed to show. ■

As always, it's helpful to write out what we need to demonstrate with the rest of the proof.

Theorem: For any odd integer n, there exist integers r and s where $r^2 - s^2 = n$.

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$$
\begin{aligned}
r^2 - s^2 &= (k+1)^2 \ldots \\
&= k^2 + 2 \ldots \\
&= 2k + 1 \\
&= n.
\end{aligned}
$$

This means that $r^2 - s^2 = n$, w… to show. ■

We're trying to prove an existential statement. The easiest way to do that is to just give concrete choices of the objects being sought out.

Theorem: For any odd integer n, there exist integers r and s where $r^2 – s^2 = n$.

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\begin{aligned}
r^2 - s^2 &= (k+1)^2 - k^2 \\
&= k^2 + 2k + 1 - k^2 \\
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&= n.
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$$

This means that $r^2 – s^2 = n$, which is what we needed to show. ■

Time-Out for Announcements!

Working in Pairs

- Problem Set Zero is due this Friday at 2:30PM. It must be completed individually.
- After that, the remaining problem sets can be done individually or in pairs.
- We have advice about how to work effectively in pairs up on the course website – check the “Guide to Partners.”
- Want to work in a pair, but don’t know who to work with? Fill out ***this Google form*** and we’ll connect you with a partner on Friday.

CURIS Poster Session

- CURIS is the CS department's undergraduate research program. It's a great way to get involved in research!
- There's a CURIS poster session showcasing work from the summer going on from 3PM – 5PM Friday in the Engineering Quad. Feel free to stop on by!
- Interested in seeing what research projects are open right now? Visit https://curis.stanford.edu.
- Have questions about research or how CURIS works? Email the CURIS mentors, PhD students who answer questions about research:

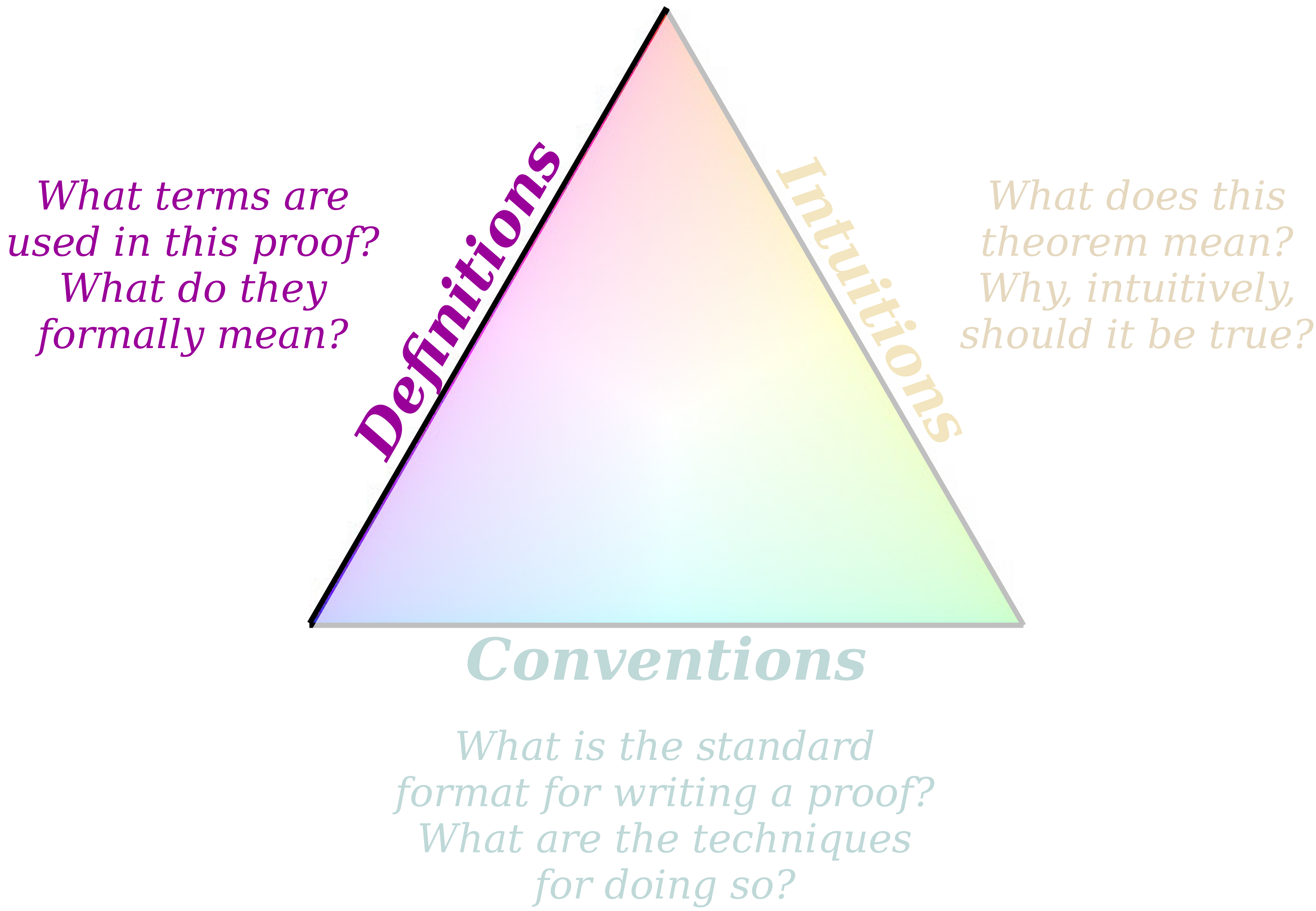
curis-mentors@cs.stanford.edu

Qt Creator Help Session

- The lovely CS106B staff have invited all y'all to join them for a Qt Creator Help Session this evening if you're having trouble getting Qt Creator up and running on your system.
- Runs ***7:00PM – 9:00PM*** in the basement of the Huang building (just around the corner from us!)
- SCPD students – please reach out to us if you need help setting things up. We'll do our best to help out.

Back to CS103!

Theorem: If n is an integer, then $\lceil n/2 \rceil + \lfloor n/2 \rfloor = n$.


What terms are used in this proof? What do they formally mean?
Definitions
Intuitions
What does this theorem mean? Why, intuitively, should it be true?
Conventions
What is the standard format for writing a proof? What are the techniques for doing so?

Floors and Ceilings

- The notation $\lceil x \rceil$ represents the ***ceiling*** of x, the smallest integer greater than or equal to x.

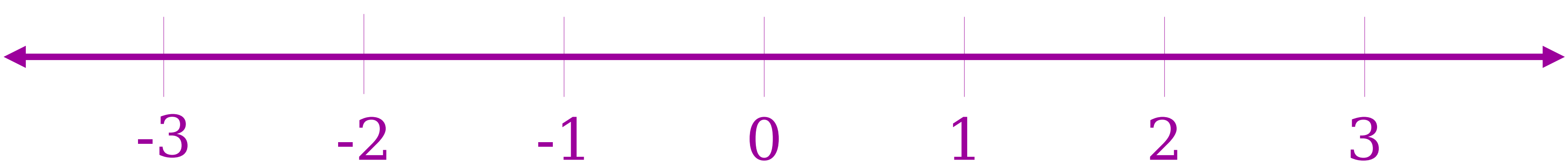
$$\lceil 1 \rceil = 1 \qquad \lceil 1.5 \rceil = 2$$

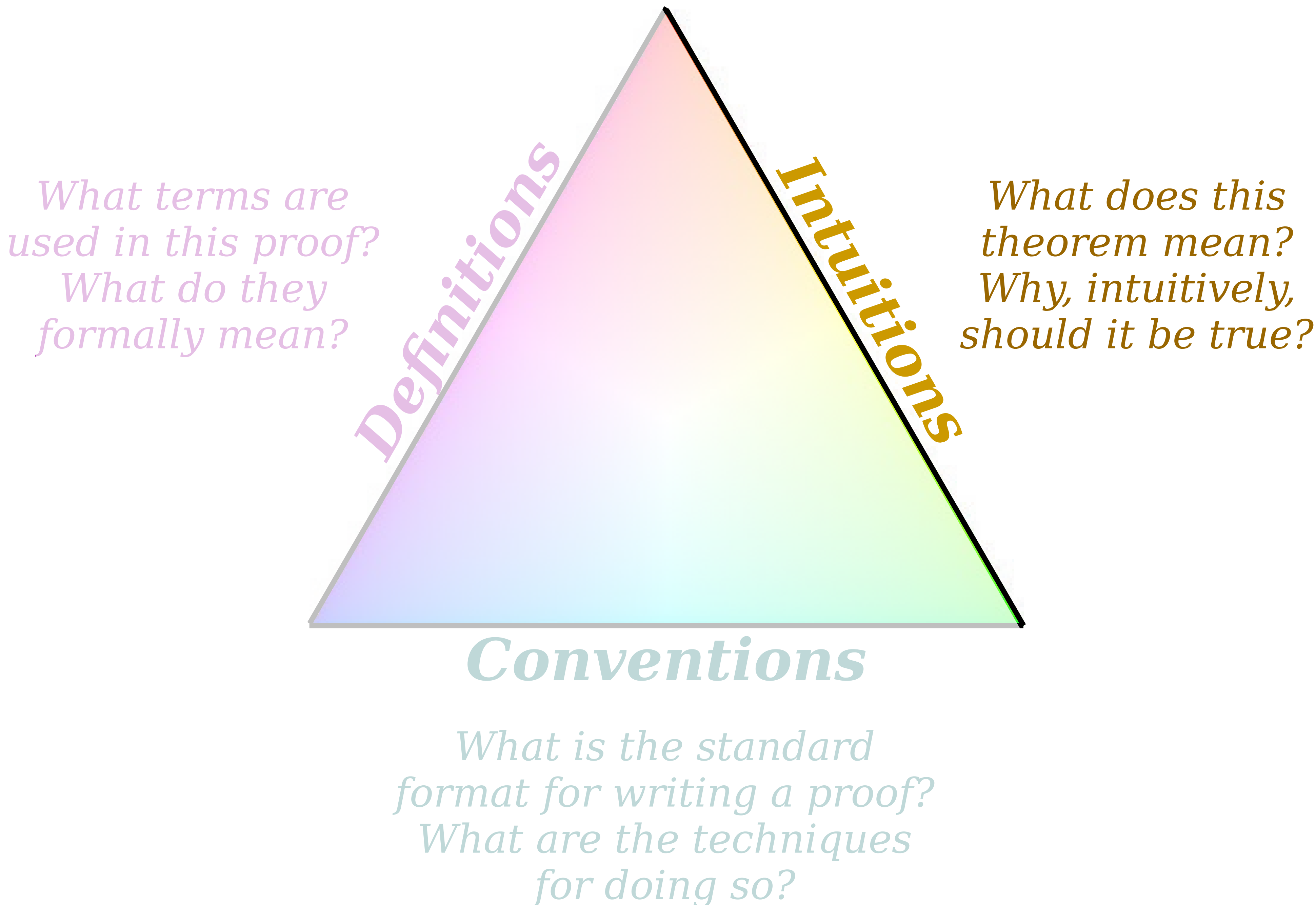
$$\lceil -1 \rceil = -1 \qquad \lceil -1.5 \rceil = -1$$

- The notation $\lfloor x \rfloor$ represents is the ***floor*** of x, the largest integer less than or equal to x.

$$\lfloor 1 \rfloor = 1 \qquad \lfloor 1.5 \rfloor = 1$$

$$\lfloor -1 \rfloor = -1 \qquad \lfloor -1.5 \rfloor = -2$$

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Let's Try Some Examples!

$$\lceil 0/2 \rceil + \lfloor 0/2 \rfloor = 0 + 0 = 0$$

$$\lceil 1/2 \rceil + \lfloor 1/2 \rfloor = 1 + 0 = 1$$

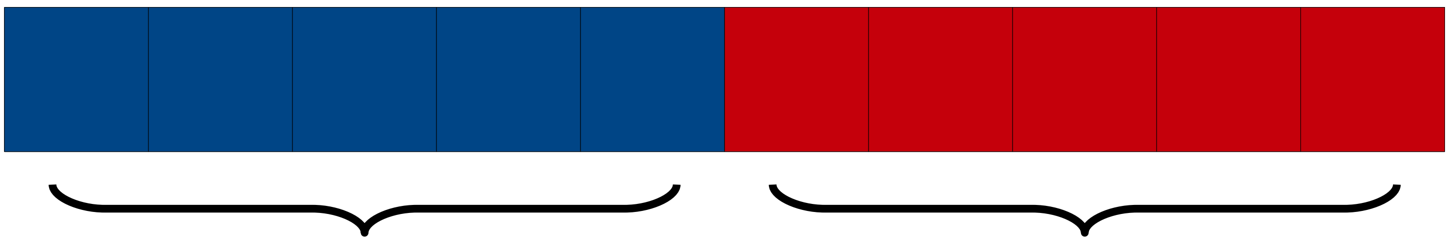
$$\lceil 2/2 \rceil + \lfloor 2/2 \rfloor = 1 + 1 = 2$$

$$\lceil 3/2 \rceil + \lfloor 3/2 \rfloor = 2 + 1 = 3$$

$$\lceil 4/2 \rceil + \lfloor 4/2 \rfloor = 2 + 2 = 4$$

Theorem: If n is an integer, then $\lceil n/2 \rceil + \lfloor n/2 \rfloor = n$.

Let’s Draw Some Pictures!

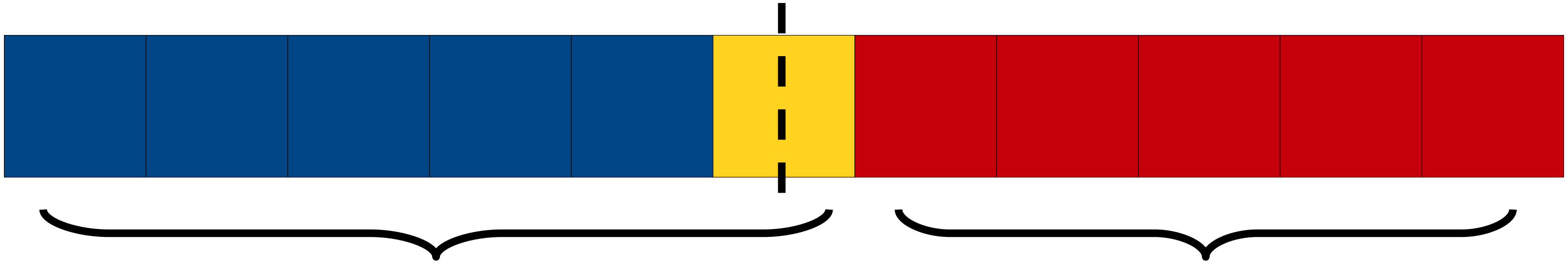


$\lceil n/2 \rceil$ $\lfloor n/2 \rfloor$

$$n = 2k$$

Theorem: If n is an integer, then $\lceil n/2 \rceil + \lfloor n/2 \rfloor = n$.

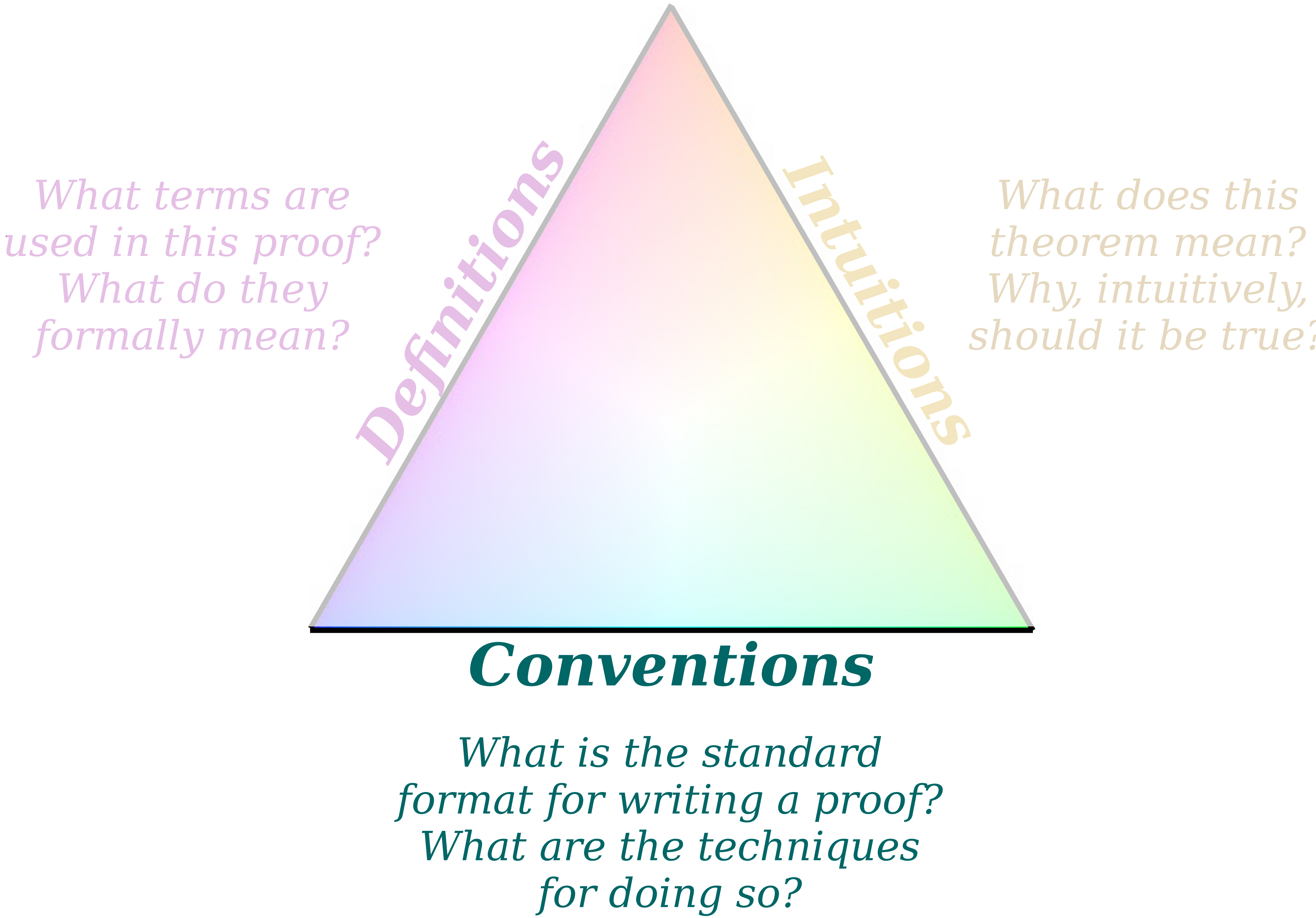
Let's Draw Some Pictures!



$\lceil n/2 \rceil$ $\lfloor n/2 \rfloor$

$$n = 2k + 1$$

Theorem: If n is an integer, then $\lceil n/2 \rceil + \lfloor n/2 \rfloor = n$.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Theorem: If n is an integer, then $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$.

Proof: Let n be an integer. We will show that $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$. To do so, we consider two cases:

Case 1: n is even.

This is called a *proof by cases* (or *proof by exhaustion*). We split apart into one or more cases and confirm that the result is indeed true in each of them.

(Think of it like an if/else or switch statement.)

Case 2: n is odd.

Theorem: If n is an integer, then $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$.

Proof: Let n be an integer. We will show that $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$. To do so, we consider two cases:

Case 1: n is even. This means there is an integer k such that $n = 2k$. Some algebra then tells us that

$$\left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{n}{2} \right\rfloor = \left\lceil \frac{2k}{2} \right\rceil + \left\lfloor \frac{2k}{2} \right\rfloor$$
$$= \lceil k \rceil + \lfloor k \rfloor$$
$$= 2k$$
$$= n.$$

Case 2: n is odd. Then there's an integer k where $n = 2k + 1$, and

$$\left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{n}{2} \right\rfloor = \left\lceil \frac{2k+1}{2} \right\rceil + \left\lfloor \frac{2k+1}{2} \right\rfloor$$

At the end of a split into cases, it's a nice courtesy to explain to the reader what it was that you established in each case.

$$= n.$$

In either case, we see that $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$, as required.

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Proof: Let n be an integer. We will show that $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$. To do so, we consider two cases:

Case 1: n is even. This means there is an integer k such that $n = 2k$. Some algebra then tells us that

$$\begin{aligned} \left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{n}{2} \right\rfloor &= \left\lceil \frac{2k}{2} \right\rceil + \left\lfloor \frac{2k}{2} \right\rfloor \\ &= \lceil k \rceil + \lfloor k \rfloor \\ &= 2k \\ &= n. \end{aligned}$$

Case 2: n is odd. Then there's an integer k where $n = 2k + 1$, and

$$\begin{aligned} \left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{n}{2} \right\rfloor &= \left\lceil \frac{2k+1}{2} \right\rceil + \left\lfloor \frac{2k+1}{2} \right\rfloor \\ &= \left\lceil k + \frac{1}{2} \right\rceil + \left\lfloor k + \frac{1}{2} \right\rfloor \\ &= (k+1) + k \\ &= 2k+1 \\ &= n. \end{aligned}$$

In either case, we see that $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$, as required. ■

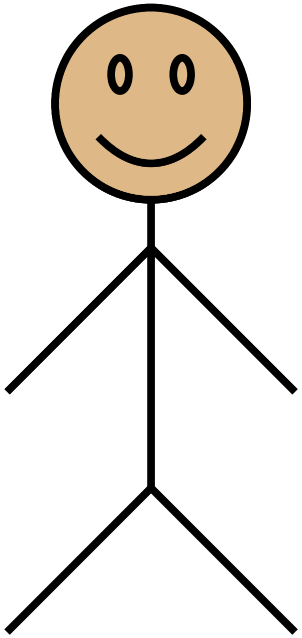
Proofs as a Dialog

Proofs as a Dialog

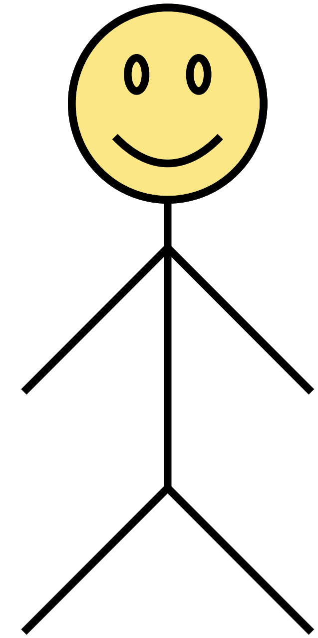
Let n be an arbitrary odd integer.

Since n is an odd integer, there is an integer k such that $n = 2k + 1$.

Now, let $z = k - 34$.



Proof Writer (You)



Proof Reader

Proofs as a Dialog

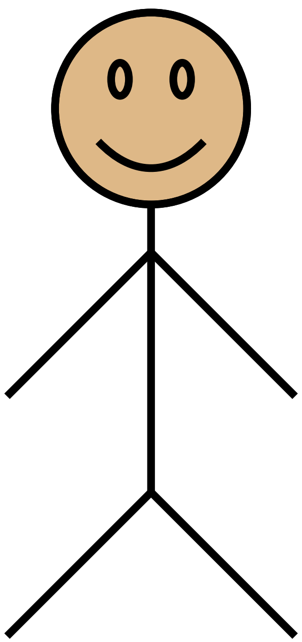
Let n be an arbitrary odd integer.

Since n is an odd integer, there is an integer k such that $n = 2k + 1$.

Now, let $z = k - 34$.

$n = 137$

Reader *Picks*



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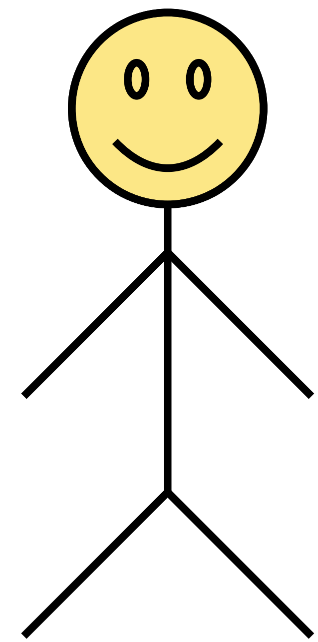

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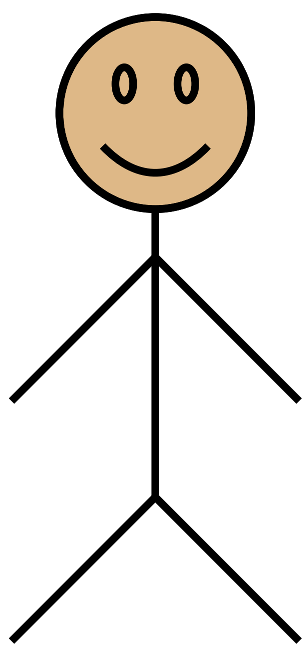


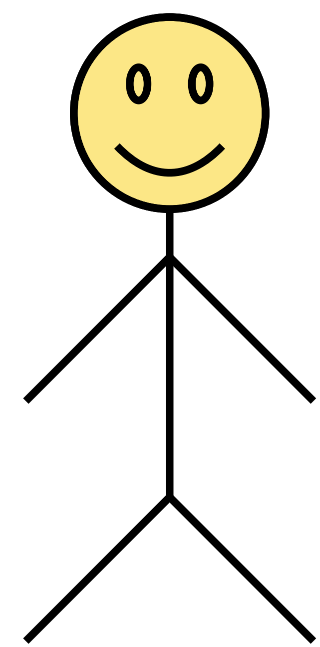
Proof Writer (You)

$k = 68$

Neither *Picks*

$n = 137$

Reader *Picks*



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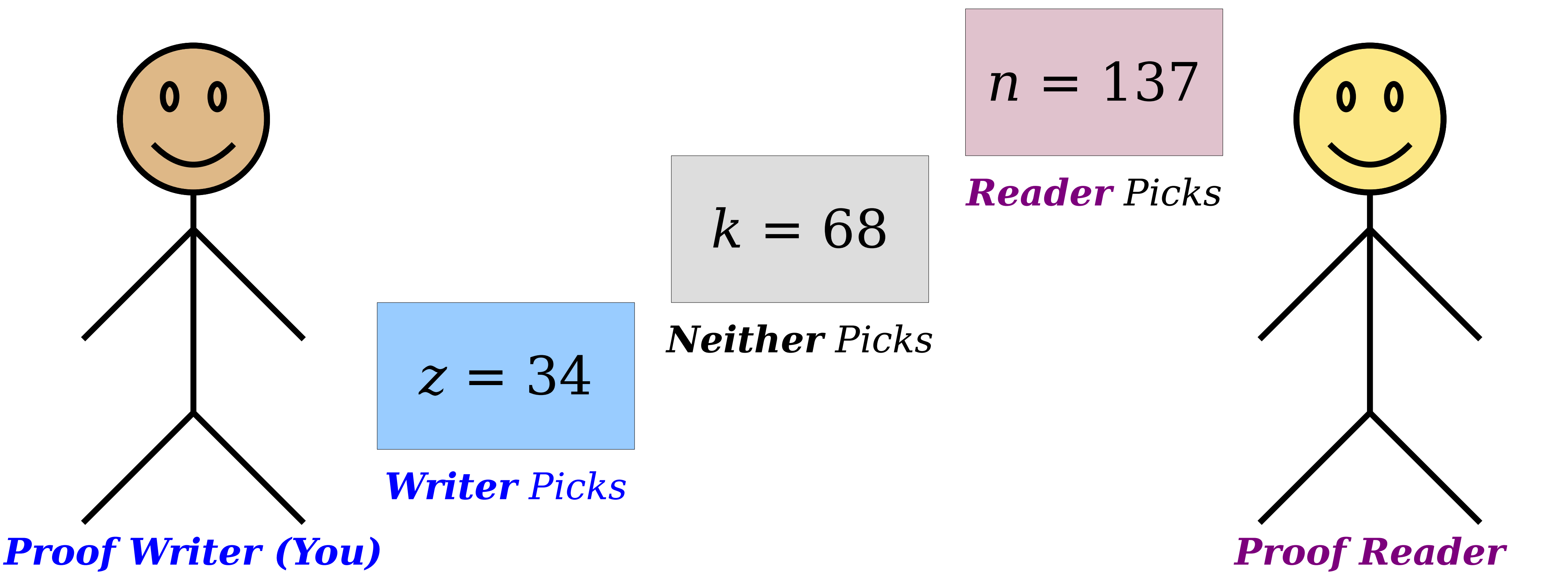

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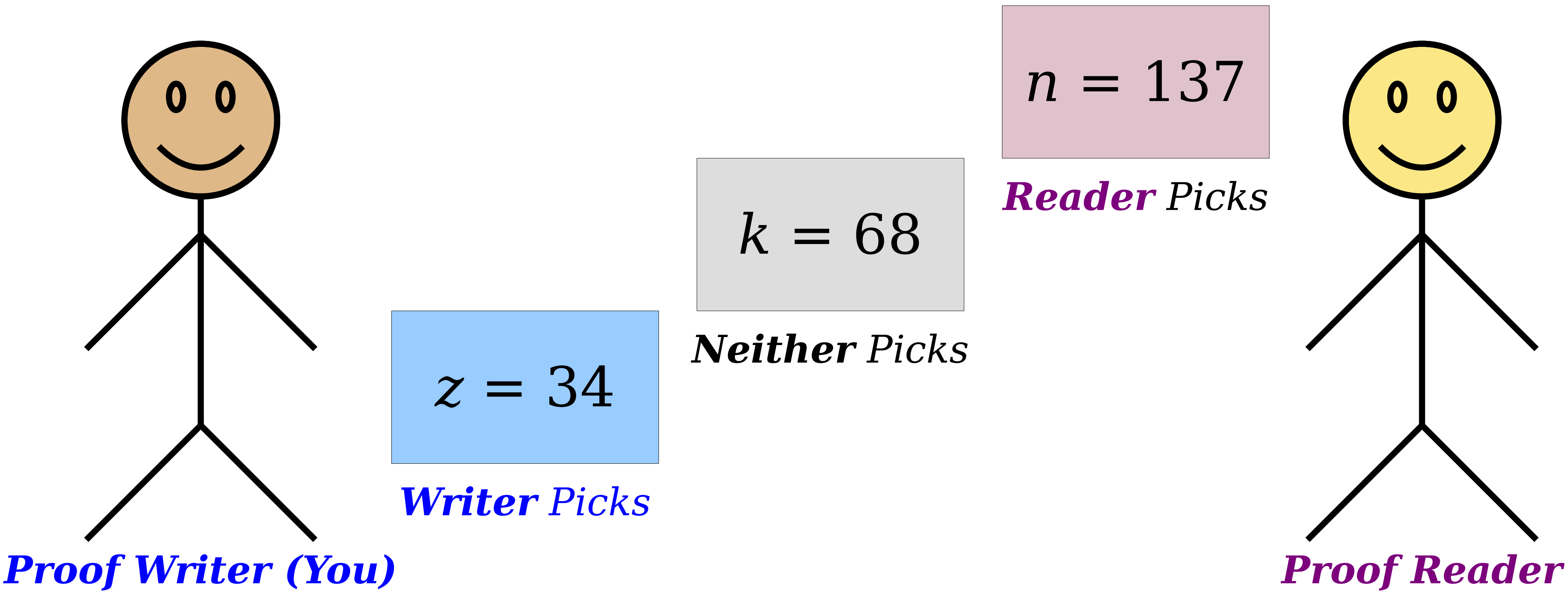
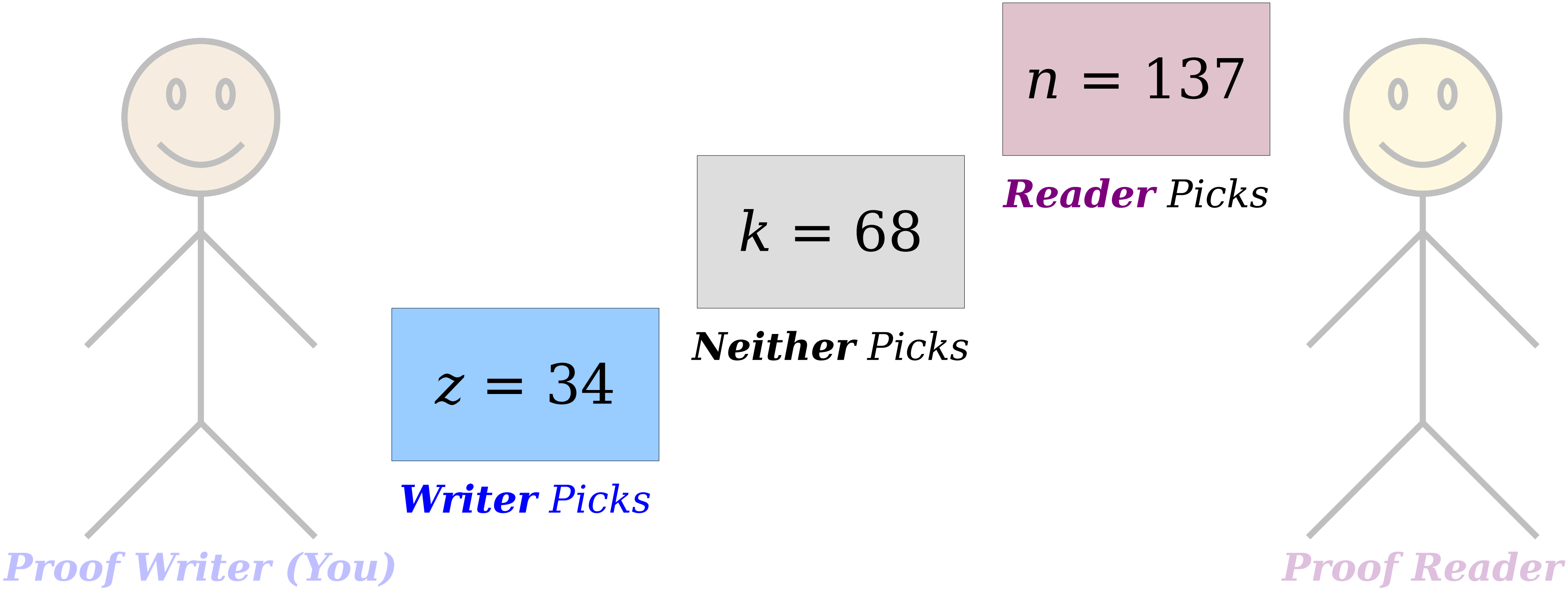


Each of these variables has a distinct, assigned value.

Each variable was either picked by the reader, picked by the writer, or has a value that can be determined from other variables.

Since

Now, let $z = k - 34$.



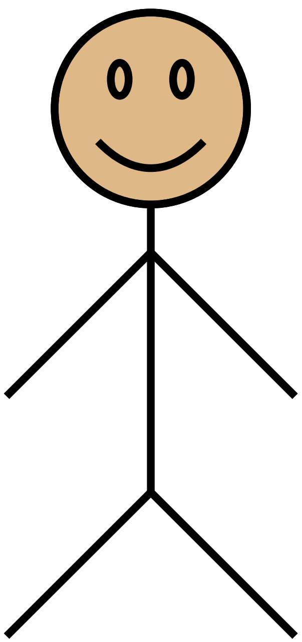
Who Owns What?

- The ***reader*** chooses and owns a value if you use wording like this:
  - Pick a natural number n.
  - Consider some $n \in \mathbb{N}$.
  - Fix a natural number n.
  - Let n be a natural number.
- The ***writer*** (you) chooses and owns a value if you use wording like this:
  - Let $r = n + 1$.
  - Pick $s = n$.
- ***Neither*** of you chooses a value if you use wording like this:
  - Since n is even, we know there is some $k \in \mathbb{Z}$ where $n = 2k$.
  - Because n is odd, there must be some integer k where $n = 2k + 1$.

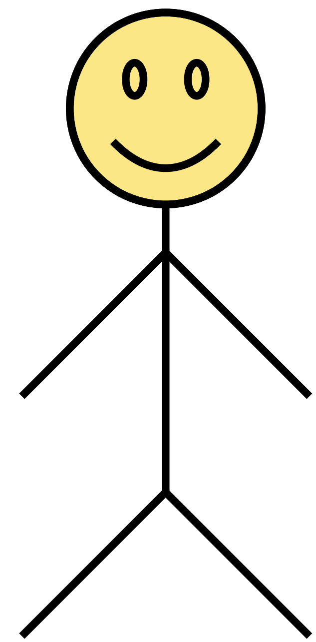
Proofs as a Dialog

Let x be an arbitrary even integer.

Then for any even x, we know that $x+1$ is odd.



Proof Writer (You)



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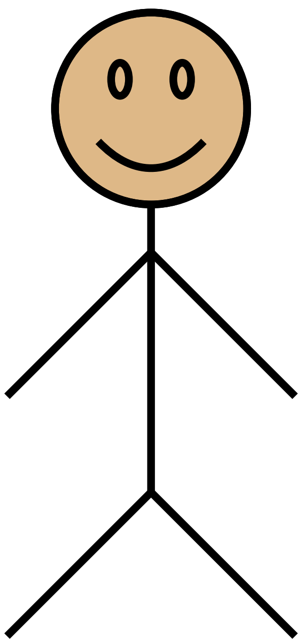
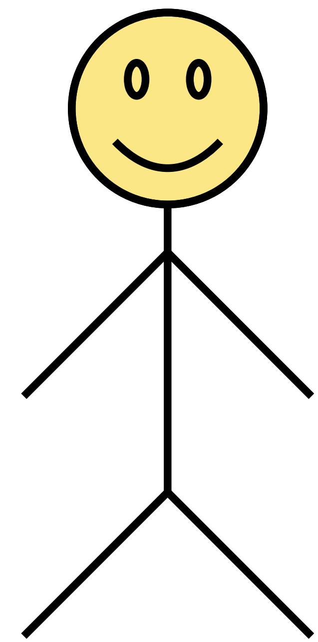


$x = 242$

Reader *Picks*



Proof Writer (You)

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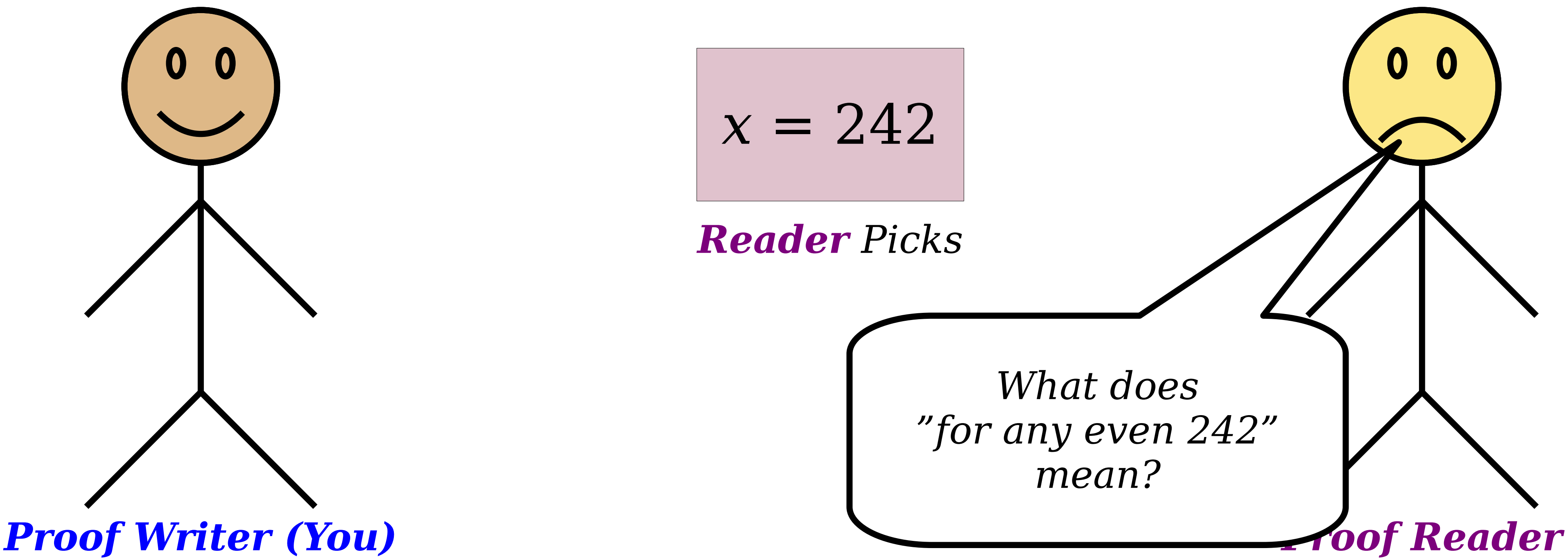

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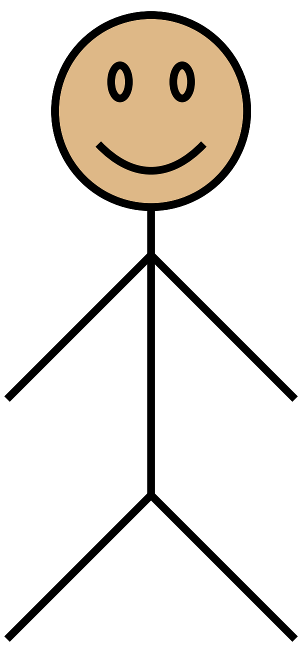

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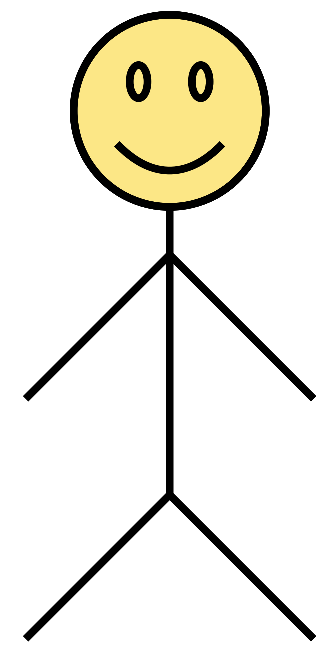

Proof Reader

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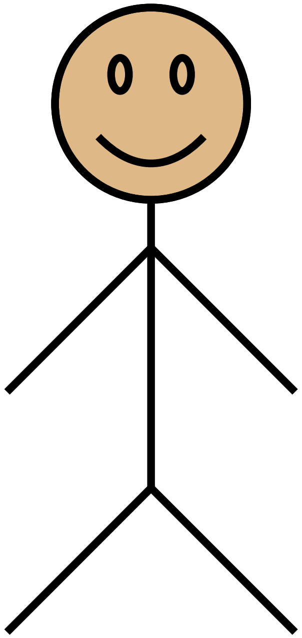
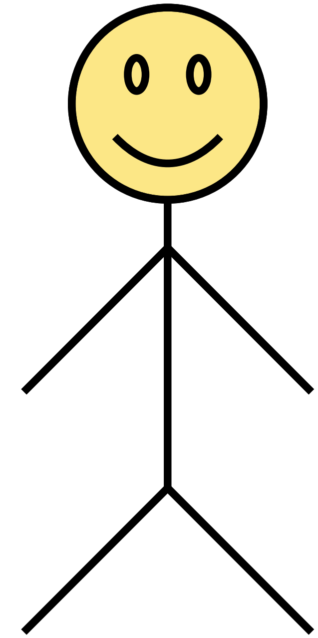


Proof Writer (You)

$x = 242$

Reader *Picks*



Proof Reader

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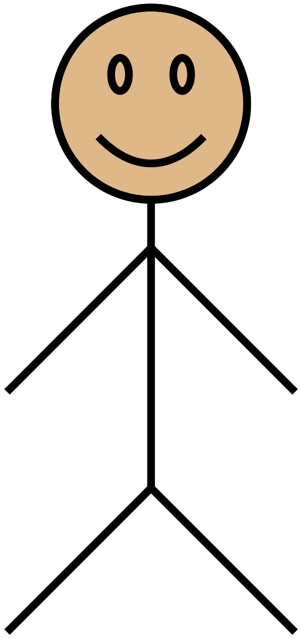
$x = 242$

Reader *Picks*



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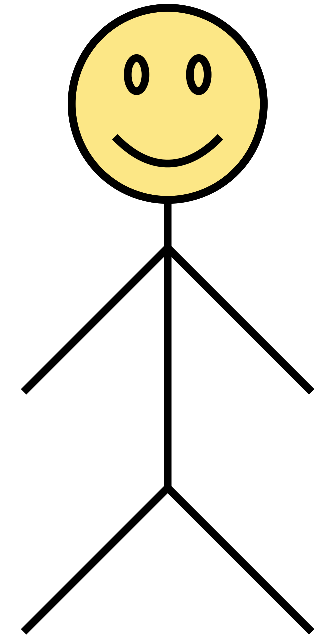


Proof Reader

Every variable needs a value.

Avoid talking about "all x" or "every x" when manipulating something concrete.

To prove something is true for any choice of a value for x, let the reader pick x.

Once you've said something like

Let x be an integer.
Consider an arbitrary $x \in \mathbb{Z}$.
Pick any x.

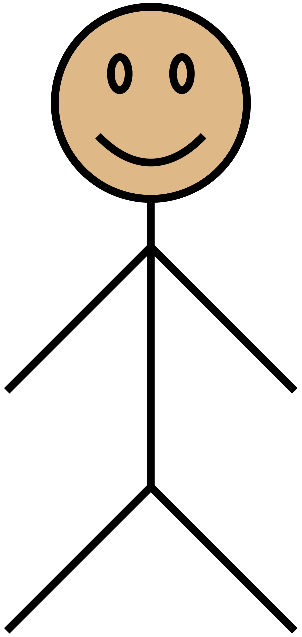
Do not say things like the following:

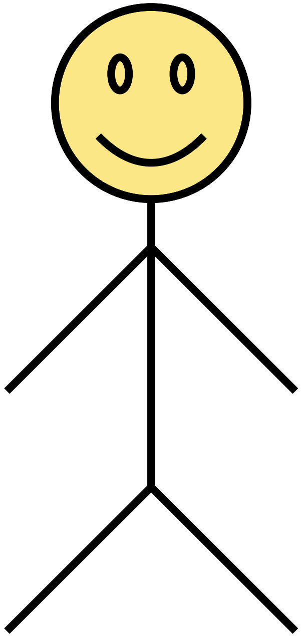
This means that ***for any*** $x \in \mathbb{Z}$ …
So ***for all*** $x \in \mathbb{Z}$ …

Proofs as a Dialog

Pick two integers m and n where $m+n$ is odd.

Let $n = 1$, which means that $m+1$ is odd.





Proofs as a Dialog

Pick two integers m and n where $m+n$ is odd.

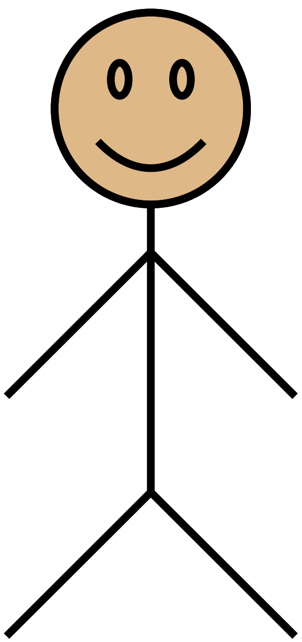
Let $n = 1$, which means that $m+1$ is odd.

$m = 103$

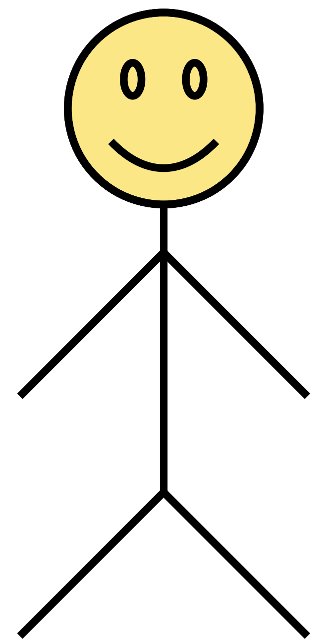
Reader *Picks*

$n = 166$

Reader *Picks*



Proof Writer (You)



Proof Reader

Proofs as a Dialog

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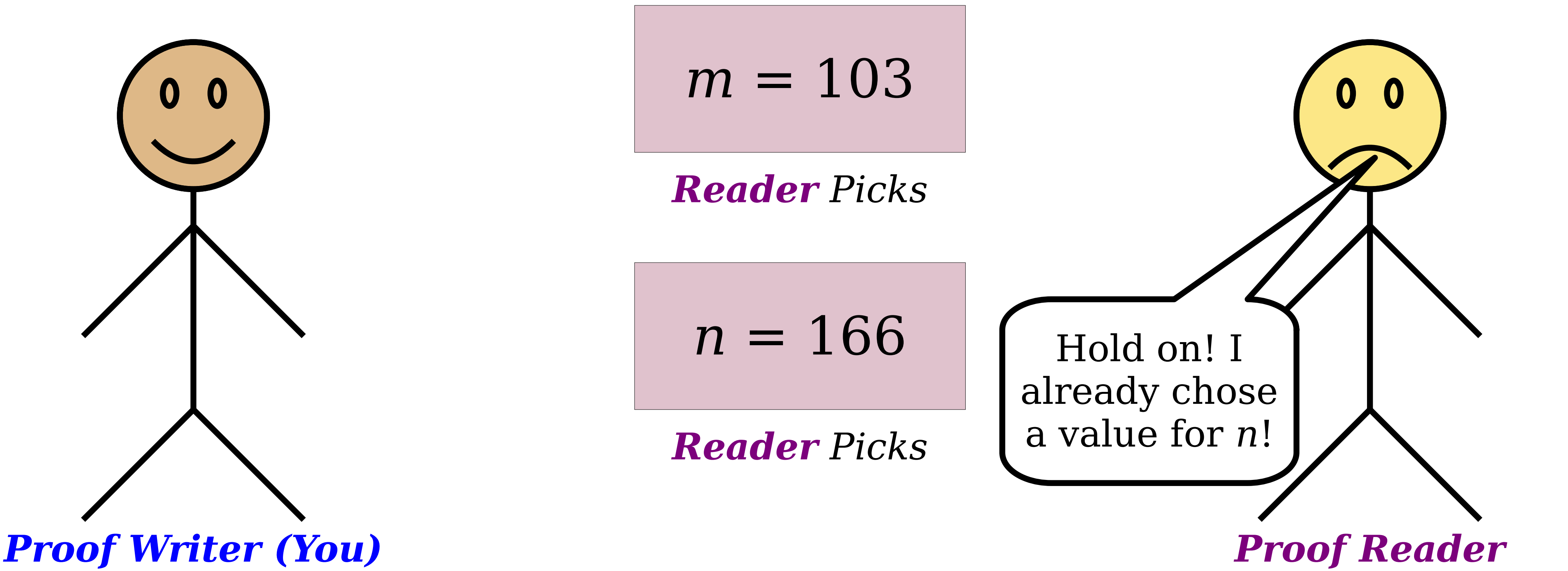
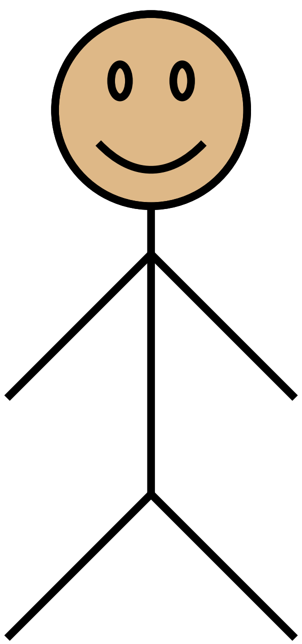


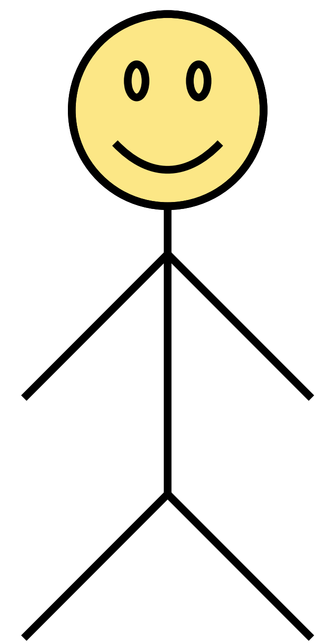
Proofs as a Dialog

Let $n = 1$.

Pick any integer m where $m+1$ is odd.



Proof Writer (You)

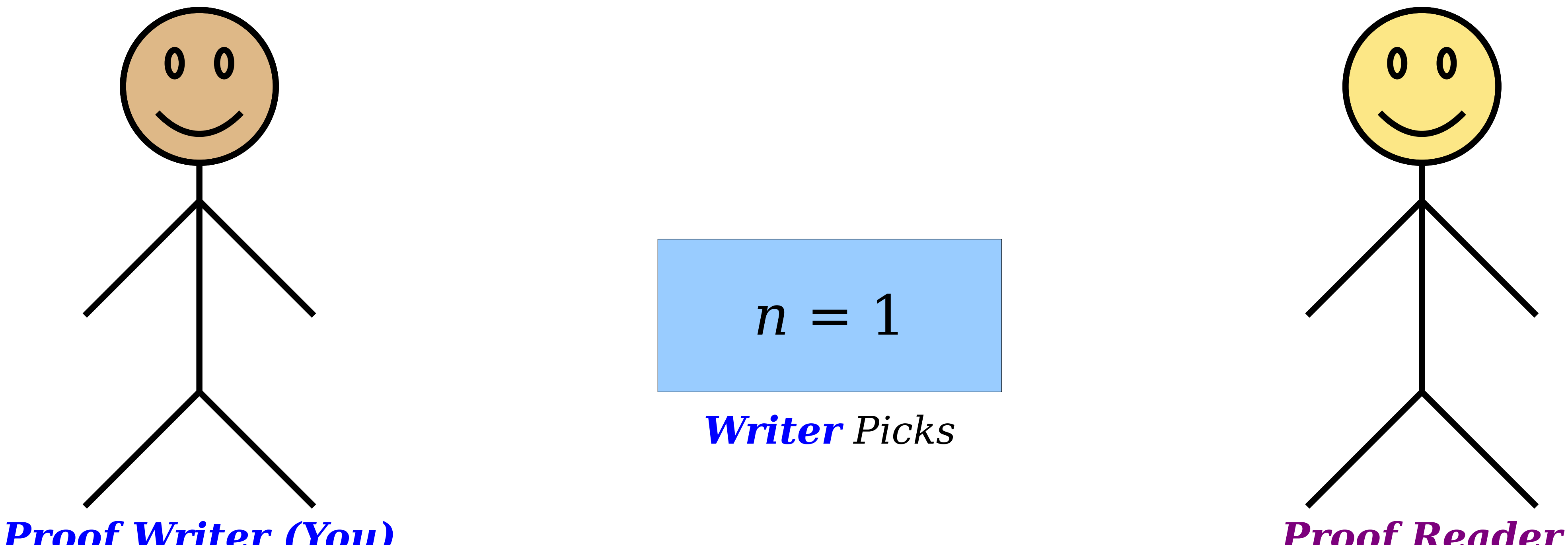


Proof Reader

Proofs as a Dialog

Let $n = 1$.

Pick any integer m where $m+1$ is odd.



Proofs as a Dialog

Let $n = 1$.

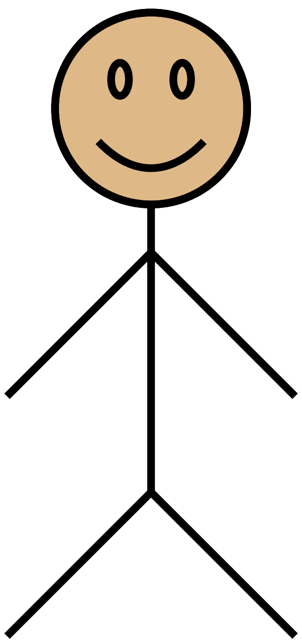
Pick any integer m where $m+1$ is odd.

$m = 166$

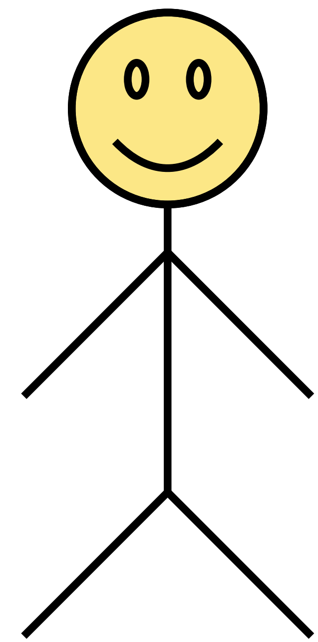
Reader *Picks*

$n = 1$

Writer *Picks*



Proof Writer (You)



Proof Reader

Proofs as a Dialog

Let $n = 1$.

Pick any integer m where $m+1$ is odd.

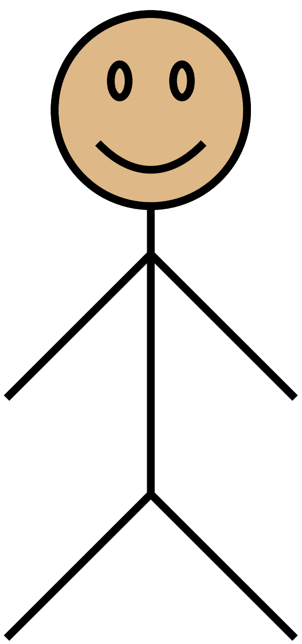
Do we even need n here?

$m = 166$

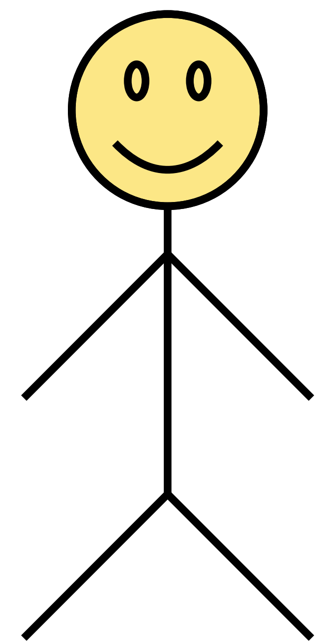
Reader *Picks*

$n = 1$

Writer *Picks*



Proof Writer (You)



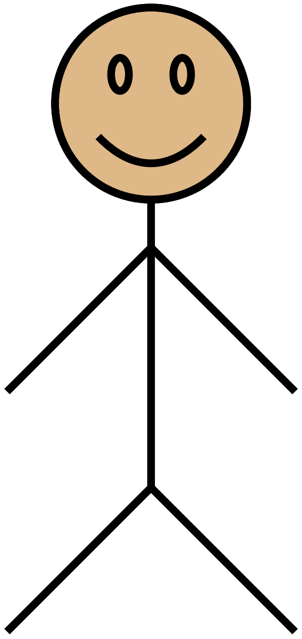
Proof Reader

Proofs as a Dialog

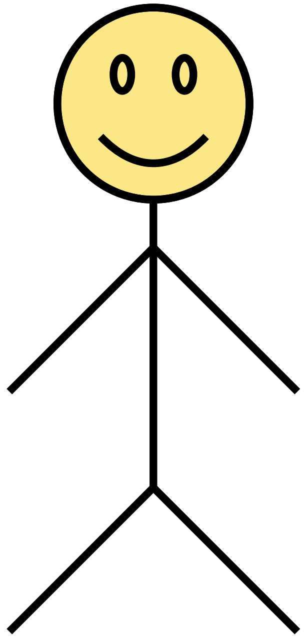
Pick any integer m where $m+1$ is odd.

$m = 166$

Reader *Picks*



Proof Writer (You)



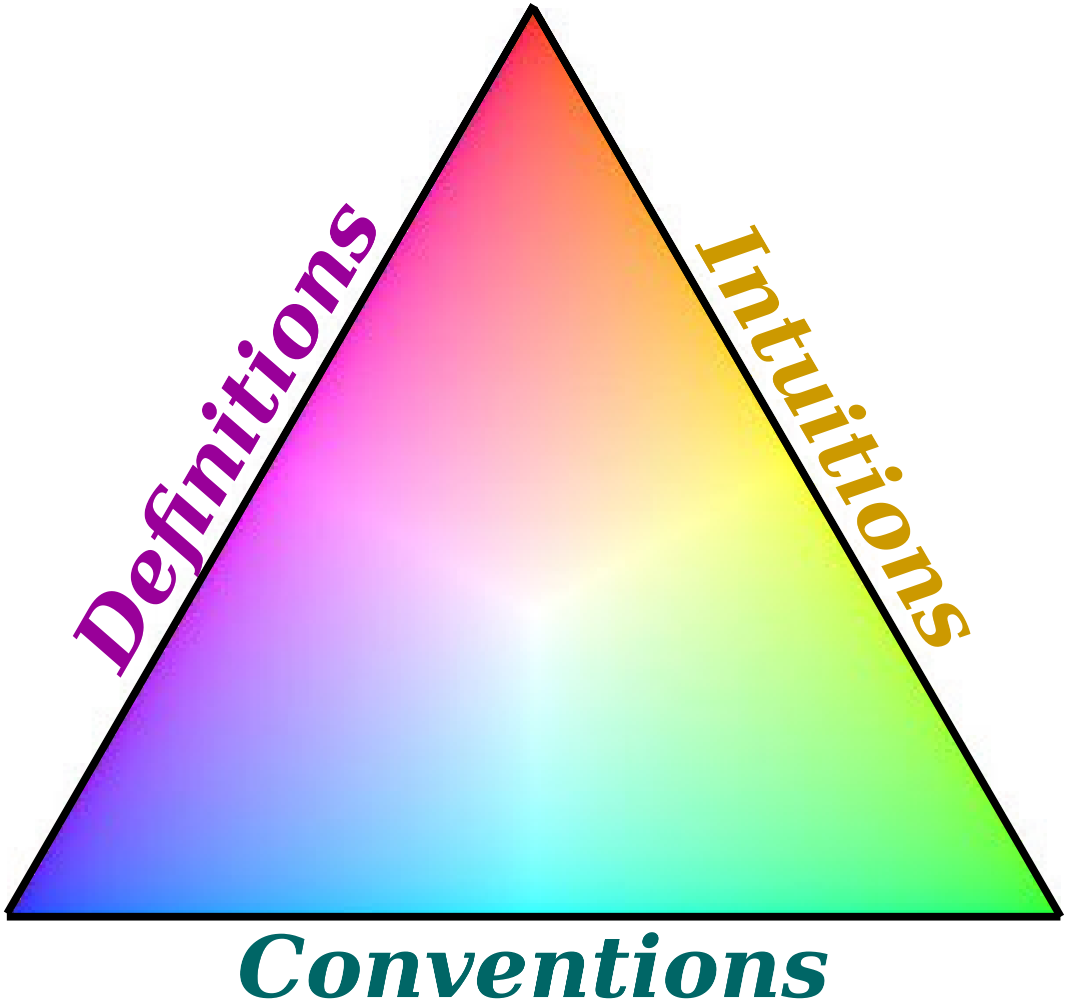
Proof Reader

Be mindful of who owns what variable.

Don’t change something you don’t own.

You don’t always need to name things, especially if they already have a name.

To Recap



Writing a good proof requires a blend of definitions, intuitions, and conventions.

An integer n is ***even*** if there is an integer k where $n = 2k$.

An integer n is ***odd*** if there is an integer k where $n = 2k+1$.

Definitions tell us what we need to do in a proof.
Many proofs directly reference these definitions.

Let's Draw Some Pictures!

Let's Do Some Math!

Let's Try Some Examples!

Building intuition for results requires creativity, trial, and error.

- Prove universal statements by making arbitrary choices.
- Prove existential statements by making concrete choices.
- Prove "If P, then Q" by assuming P and proving Q.
- Write in complete sentences.
- Number sub-formulas when referring to them.
- Summarize what was shown in proofs by cases.
- Articulate your start and end points.

Mathematical proofs have established conventions that increase rigor and readability.

Your Action Items

- ***Read "Guide to ∈ and ⊆."***
  - You'll want to have a handle on how these concepts are related, and on how they differ.
- ***Read "Guide to Proofs."***
  - This resource covers proofwriting strategies and conventions and is an essential complement to this lecture.
- ***Read "Guide to Partners."***
  - It's all about how to work effectively in pairs. Mull this over so you're ready to go for Problem Set 1.
- ***Finish and submit Problem Set 0.***
  - Don't put this off until the last minute!

Next Time

- ***Indirect Proofs***
  - How do you prove something without actually proving it?
- ***Mathematical Implications***
  - What exactly does "if *P*, then *Q*" mean?
- ***Proof by Contrapositive***
  - A helpful technique for proving implications.
- ***Proof by Contradiction***
  - Proving something is true by showing it can't be false.