# First-Order Logic

## Part Two

# Recap from Last Time

# What is First-Order Logic?

- ***First-order logic*** is a logical system for reasoning about properties of objects.
- Augments the logical connectives from propositional logic with
  - ***predicates*** that describe properties of objects,
  - ***functions*** that map objects to one another, and
  - ***quantifiers*** that allow us to reason about many objects at once.

Some muggle is intelligent.

$$\exists m.\ (\mathit{Muggle}(m) \land \mathit{Intelligent}(m))$$

∃ is the ***existential quantifier*** and says "for some choice of m, the following is true."

"For any natural number $n$,
$n$ is even if and only if $n^2$ is even"

$$\forall n.\ (n \in \mathbb{N} \to (Even(n) \leftrightarrow Even(n^2)))$$

∀ is the ***universal quantifier***
and says "for any choice of $n$,
the following is true."

**“All *A*'s are *B*'s”**

translates as

$$\forall x.\ (A(x) \to B(x))$$

***Useful Intuition:***

Universally-quantified statements are true unless there's a counterexample.

$$\forall x.\ (A(x) \rightarrow B(x))$$

If x is a counterexample, it *must* have property A but not have property B.

**“Some *A* is a *B*”**

translates as

$$\exists x.\ (A(x) \land B(x))$$

***Useful Intuition:***

Existentially-quantified statements are false unless there's a positive example.

$$\exists x.\ (A(x) \land B(x))$$

If x is an example, it *must* have property *A* on top of property *B*.

# New Stuff!

# The Aristotelian Forms

"All *A*s are *B*s"

$\forall x.\ (A(x) \to B(x))$

"Some *A*s are *B*s"

$\exists x.\ (A(x) \land B(x))$

"No *A*s are *B*s"

$\forall x.\ (A(x) \to \neg B(x))$

"Some *A*s aren't *B*s"

$\exists x.\ (A(x) \land \neg B(x))$

It is worth committing these patterns to memory. We'll be using them throughout the day and they form the backbone of many first-order logic translations.

# The Art of Translation

Using the predicates

- $Person(p)$, which states that $p$ is a person, and
- $Loves(x, y)$, which states that $x$ loves $y$,

write a sentence in first-order logic that means “every person loves someone else.”

*Every person loves someone else*

*Every person loves some other person*

*Every person p loves some other person*

$\forall p.\ (Person(p) \rightarrow$

*p loves some other person*

$)$

"All *A*s are *B*s"

$\boldsymbol{\forall x.\ (A(x) \rightarrow B(x))}$

$\forall p.\ (Person(p) \to$

*there is some other person that p loves*

$)$

$\forall p.\ (Person(p) \to$

*there is a person other than p that p loves*

$)$

$\forall p.\ (Person(p) \rightarrow$

*there is a person q, other than p, where p loves q*

$)$

$\forall p.\ (Person(p) \rightarrow$

*there is a person q, other than p, where*

*p loves q*

$)$

$\forall p.\ (Person(p) \rightarrow$

$\exists q.\ (Person(q) \land$, *other than p, where*

*p loves q*

$)$

$)$

“Some *A*s are *B*s”

$\exists x.\ (A(x) \land B(x))$

$\forall p.\ (Person(p) \rightarrow$

$\quad \exists q.\ (Person(q)\ \wedge$ *, other than p, where*

*p loves q*

$\quad )$

$)$

$$\forall p.\ (Person(p) \rightarrow$$
$$\quad \exists q.\ (Person(q) \land p \neq q \land$$
$$\qquad p \text{ loves } q$$
$$\quad )$$
$$)$$

$$
\begin{aligned}
&\forall p.\ (Person(p) \rightarrow \\
&\quad \exists q.\ (Person(q) \land p \neq q \land \\
&\qquad Loves(p, q) \\
&\quad ) \\
&)
\end{aligned}
$$

Using the predicates

- *Person*($p$), which states that $p$ is a person, and
- *Loves*($x$, $y$), which states that $x$ loves $y$,

write a sentence in first-order logic that means “there is a person that everyone else loves.”

*There is a person that everyone else loves*

*There is a person p where everyone else loves p*

$\exists p. \ (Person(p) \ \land$

*everyone else loves p*

)

“Some *A*s are *B*s”

$\boldsymbol{\exists x. \ (A(x) \ \land \ B(x))}$

$\exists p.\ (Person(p)\ \land$

*every other person q loves p*

$)$

$\exists p.\ (Person(p)\ \wedge$

*every person q, other than p, loves p*

$)$

$\exists p.\ (Person(p)\ \land$

$\quad \forall q.\ (Person(q)\ \land\ p \neq q\ \rightarrow$

$\quad\quad$ *q loves p*

$\quad )$

$)$

“All *A*s are *B*s”

$\boldsymbol{\forall x.\ (A(x) \rightarrow B(x))}$

$$
\begin{aligned}
&\exists p.\ (Person(p)\ \land \\
&\quad \forall q.\ (Person(q)\ \land\ p \neq q \rightarrow \\
&\qquad Loves(q, p) \\
&\quad ) \\
&)
\end{aligned}
$$

# Combining Quantifiers

- Most interesting statements in first-order logic require a combination of quantifiers.
- Example: “Every person loves someone else”

For every person… $\forall p.\ (Person(p) \to$

… there is another person … $\exists q.\ (Person(q) \land p \neq q \land$

… they love $Loves(p, q)$

$)$

$)$

# Combining Quantifiers

- Most interesting statements in first-order logic require a combination of quantifiers.
- Example: “There is someone everyone else loves.”

There is a person… $\exists p.\ (Person(p) \land$

… that everyone else … $\forall q.\ (Person(q) \land p \neq q \to$

… loves. $Loves(q, p)$

$)$

$)$

# For Comparison

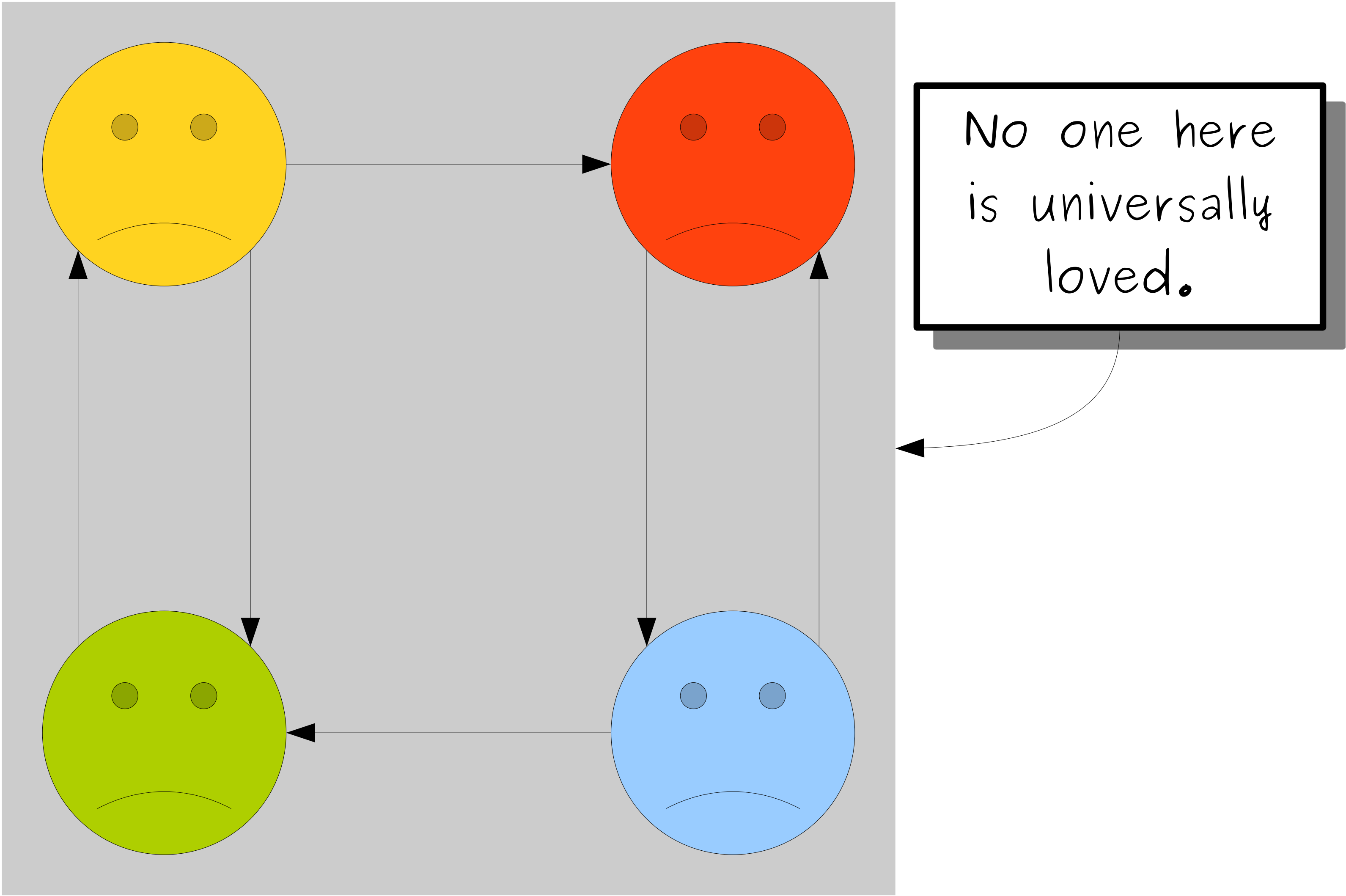
For every person…
… there is another person …
… they love

$$\forall p. \; (Person(p) \to \\ \quad \exists q. \; (Person(q) \land p \neq q \land \\ \qquad Loves(p, q) \\ \quad ) \\ )$$

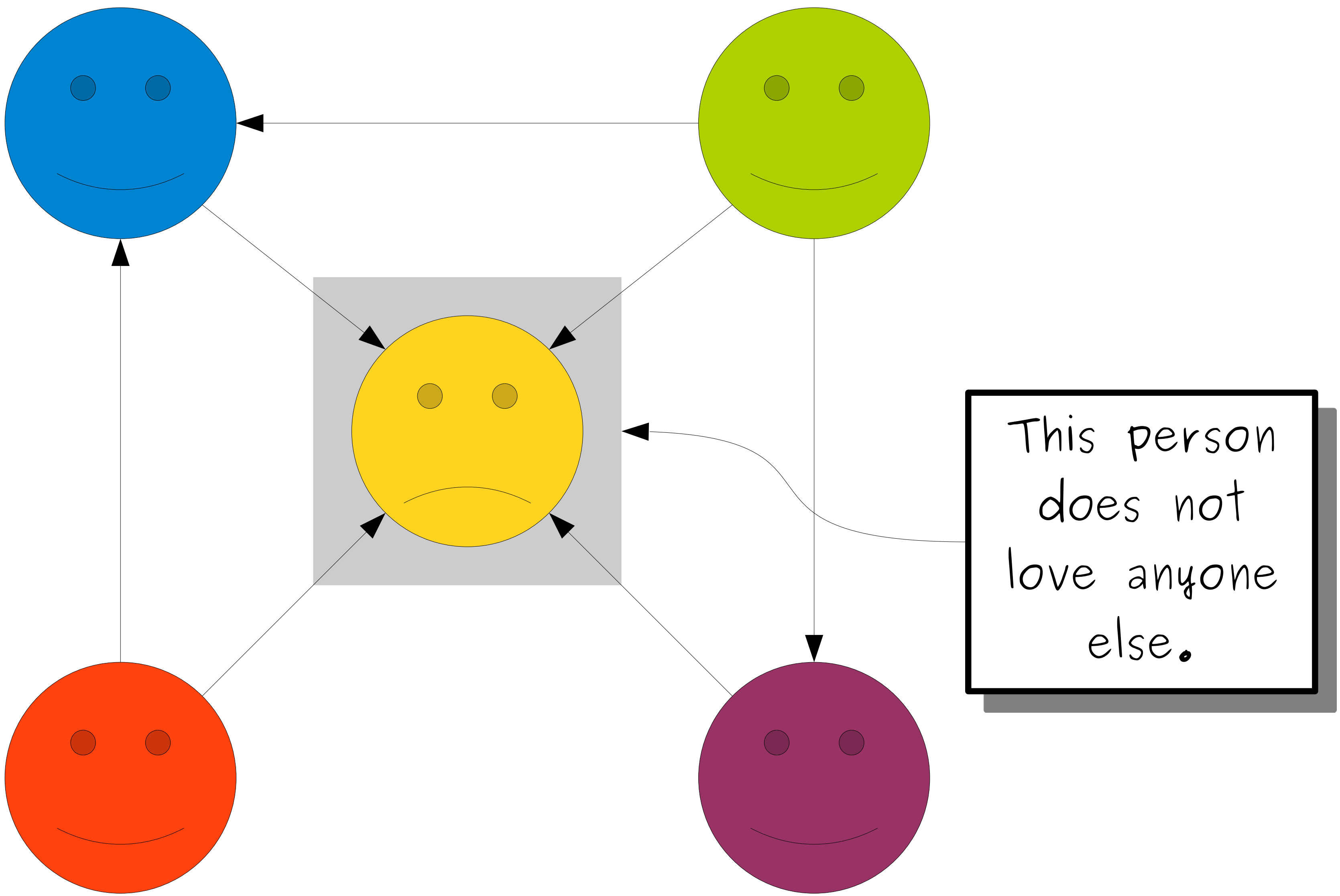
There is a person…
… that everyone else …
… loves.

$$\exists p. \; (Person(p) \land \\ \quad \forall q. \; (Person(q) \land p \neq q \to \\ \qquad Loves(q, p) \\ \quad ) \\ )$$

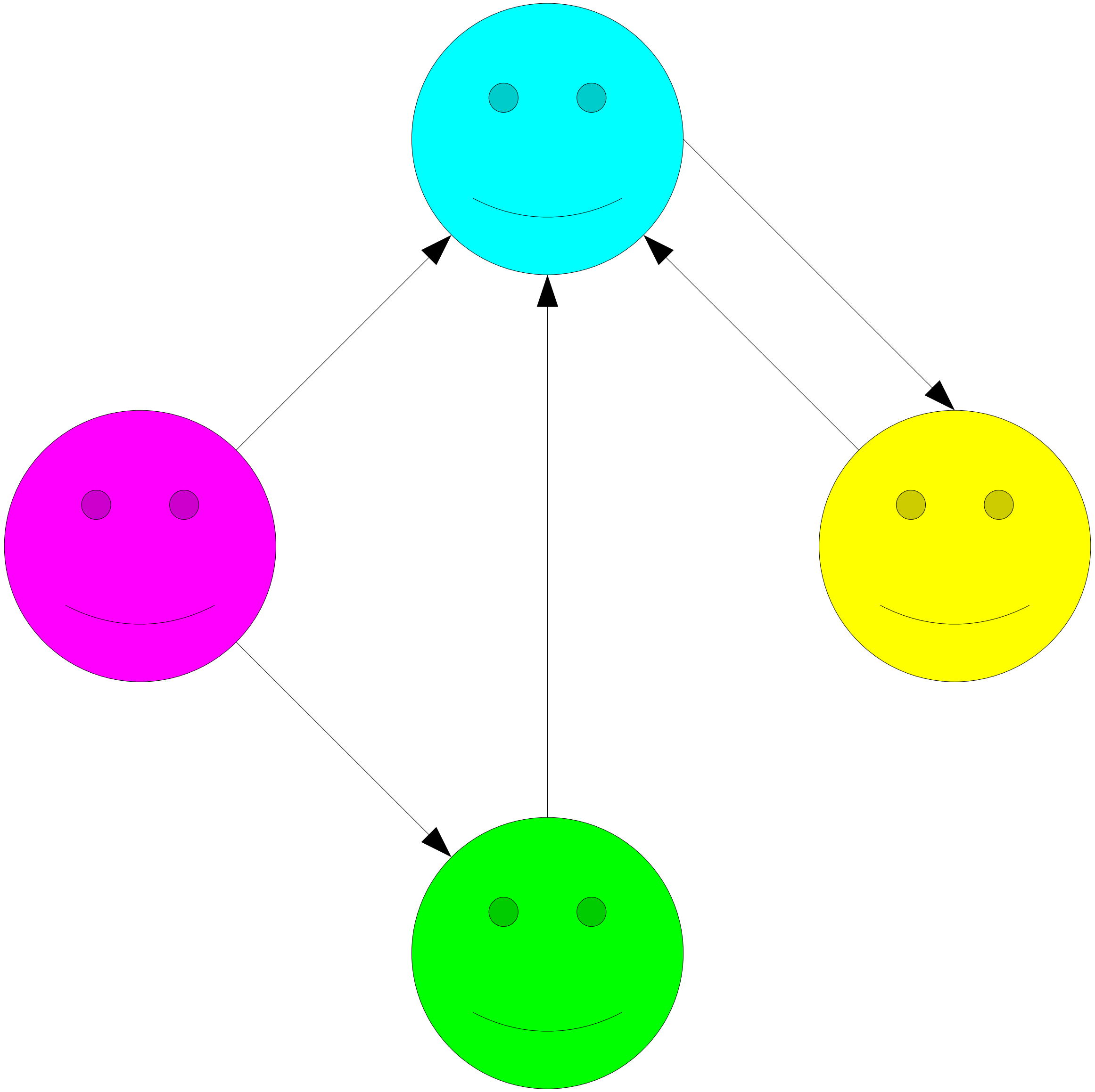
# Every Person Loves Someone Else



# There is Someone Everyone Else Loves



# Every Person Loves Someone Else ***and*** There is Someone Everyone Else Loves



For every person… $\forall p.\ (Person(p) \rightarrow$

… there is another person … $\exists q.\ (Person(q) \land p \neq q \land$

… they love $Loves(p, q)$

$)$

$)$

and $\land$

There is a person… $\exists p.\ (Person(p) \land$

… that everyone else … $\forall q.\ (Person(q) \land p \neq q \rightarrow$

… loves. $Loves(q, p)$

$)$

$)$

# Quantifier Ordering

- The statement

$$\forall x.\ \exists y.\ P(x, y)$$

  means "for any choice of $x$, there's some choice of $y$ where $P(x, y)$ is true."
- The choice of $y$ can be different every time and can depend on $x$.

# Quantifier Ordering

- The statement

  $$\exists x.\ \forall y.\ P(x, y)$$

  means "there is some $x$ where for any choice of $y$, we get that $P(x, y)$ is true."
- Since the inner part has to work for any choice of $y$, this places a lot of constraints on what $x$ can be.

***Order matters*** when mixing existential and universal quantifiers!

Time-Out for Announcements!

# Problem Set Two

- Problem Set One was due today at 2:30PM.
  - Didn’t submit by then? Ping us ASAP.
- Problem Set Two goes out today. It’s due next Friday at 2:30PM.
  - Explore first-order logic, and expand your proofwriting repertoire.
- We have some online readings for this problem set.
  - Check out the ***Guide to Logic Translations*** for more on how to convert from English to FOL.
  - Check out the ***Guide to Negations*** for information about how to negate formulas.
  - Check out the ***First-Order Translation Checklist*** for details on how to check your work.

# Your Questions

“What was your most embarrassing moment in college?”

I went to a conference. I packed my suit and forgot my dress shoes. Hilarity ensued.

# “Tips for CS co-term recommendations if you've only taken large lecture classes & don't know professors personally?”

Two things:

1. Come talk to us! One of the best parts of this job is getting to meet people. So don't be a stranger – chat with me after class, send me emails, etc.

2. Specifically for coterm rec letters: it's totally fine to ask someone for a DWIC letter (“Did Well In Class.”) After all, with the coterm, you're signing up to take more CS courses, so a rec like that actually provides a good signal. That's especially true for pandemic classes where the instructor can then give more detailed feedback.

Back to CS103!

# Set Translations

Using the predicates

- $Set(S)$, which states that $S$ is a set, and
- $x \in y$, which states that $x$ is an element of $y$,

write a sentence in first-order logic that means “the empty set exists.”

> First-order logic doesn't have set operators or symbols “built in.” If we only have the predicates given above, how might we describe this?

*The empty set exists.*

*There is some set S that is empty.*

$\exists S. \; (Set(S) \; \wedge$

*S is empty.*

$)$

$\exists S. \ (Set(S) \ \land$

*there are no elements in S*

$)$

$\exists S.\ (Set(S)\ \land$

$\neg$*there is an element in S*

$)$

$\exists S. \ (Set(S) \ \wedge$

$\neg$*there is an element x in S*

)

$$
\begin{aligned}
&\exists S.\ (Set(S)\ \land \\
&\quad \neg\exists x.\ x \in S \\
&)
\end{aligned}
$$

$\exists S.\ (Set(S) \land \neg\exists x.\ x \in S)$

$\exists S.\ (Set(S) \land$

*there are no elements in S*

$)$

$\exists S.\ (Set(S) \land \neg \exists x.\ x \in S)$

$\exists S.\ (Set(S) \land$

*every object does not belong to S*

$)$

$\exists S.\ (Set(S) \land \neg \exists x.\ x \in S)$

$\exists S.\ (Set(S) \land$

*every object x does not belong to S*

$)$

$$\exists S.\ (Set(S) \land \neg\exists x.\ x \in S)$$

$$\exists S.\ (Set(S) \land \forall x.\ x \notin S)$$

Both of these translations are correct. Just like in propositional logic, there are many different equivalent ways of expressing the same statement in first-order logic.

$$\exists S.\ (Set(S) \land \neg\exists x.\ x \in S)$$

$$\exists S.\ (Set(S) \land \forall x.\ x \notin S)$$

Why can we switch which quantifier we're using here?

# Mechanics: Negating Statements

# An Extremely Important Table

| | When is this true? | When is this false? |
|---|---|---|
| $\forall x.\ P(x)$ | For all choices of $x$, $P(x)$ | For some choice of $x$, $\neg P(x)$ |
| $\exists x.\ P(x)$ | For some choice of $x$, $P(x)$ | For all choices of $x$, $\neg P(x)$ |
| $\forall x.\ \neg P(x)$ | For all choices of $x$, $\neg P(x)$ | For some choice of $x$, $P(x)$ |
| $\exists x.\ \neg P(x)$ | For some choice of $x$, $\neg P(x)$ | For all choices of $x$, $P(x)$ |

# An Extremely Important Table

| | When is this true? | When is this false? |
|---|---|---|
| $\forall x.\ P(x)$ | For all choices of $x$, $P(x)$ | $\exists x.\ \neg P(x)$ |
| $\exists x.\ P(x)$ | For some choice of $x$, $P(x)$ | $\forall x.\ \neg P(x)$ |
| $\forall x.\ \neg P(x)$ | For all choices of $x$, $\neg P(x)$ | $\exists x.\ P(x)$ |
| $\exists x.\ \neg P(x)$ | For some choice of $x$, $\neg P(x)$ | $\forall x.\ P(x)$ |

# Negating First-Order Statements

- Use the equivalences

  $$\neg\forall x.\ A \quad \textit{is equivalent to} \quad \exists x.\ \neg A$$

  $$\neg\exists x.\ A \quad \textit{is equivalent to} \quad \forall x.\ \neg A$$

  to negate quantifiers.
- Mechanically:
  - Push the negation across the quantifier.
  - Change the quantifier from ∀ to ∃ or vice-versa.
- Use techniques from propositional logic to negate connectives.

# Taking a Negation

$$\forall x.\ \exists y.\ Loves(x, y)$$

*("Everyone loves someone.")*

$$\neg\forall x.\ \exists y.\ Loves(x, y)$$

$$\exists x.\ \neg\exists y.\ Loves(x, y)$$

$$\exists x.\ \forall y.\ \neg Loves(x, y)$$

*("There's someone who doesn't love anyone.")*

# Two Useful Equivalences

- The following equivalences are useful when negating statements in first-order logic:

$$\neg(p \land q) \quad \textit{is equivalent to} \quad p \to \neg q$$

$$\neg(p \to q) \quad \textit{is equivalent to} \quad p \land \neg q$$

- These identities are useful when negating statements involving quantifiers.
    - $\land$ is used in existentially-quantified statements.
    - $\to$ is used in universally-quantified statements.
- When pushing negations across quantifiers, we *strongly recommend* using the above equivalences to keep $\to$ with $\forall$ and $\land$ with $\exists$.

# Negating Quantifiers

- What is the negation of the following statement, which says "there is a cute puppy"?

$$\boldsymbol{\exists x. \, (Puppy(x) \land Cute(x))}$$

- We can obtain it as follows:

$$\neg\exists x. \, (Puppy(x) \land Cute(x))$$

$$\forall x. \, \neg(Puppy(x) \land Cute(x))$$

$$\forall x. \, (Puppy(x) \to \neg Cute(x))$$

- This says "no puppy is cute."
- Do you see why this is the negation of the original statement from both an intuitive and formal perspective?

$$\exists S.\ (Set(S) \land \forall x.\ \neg(x \in S))$$

*("There is a set with no elements.")*

$$\neg\exists S.\ (Set(S) \land \forall x.\ \neg(x \in S))$$
$$\forall S.\ \neg(Set(S) \land \forall x.\ \neg(x \in S))$$
$$\forall S.\ (Set(S) \to \neg\forall x.\ \neg(x \in S))$$
$$\forall S.\ (Set(S) \to \exists x.\ \neg\neg(x \in S))$$
$$\forall S.\ (Set(S) \to \exists x.\ x \in S)$$

*("Every set contains at least one element.")*

# Restricted Quantifiers

# Quantifying Over Sets

- The notation

$$\forall x \in S.\ P(x)$$

  means "for any element $x$ of set $S$, $P(x)$ holds." (It's vacuously true if $S$ is empty.)

- The notation

$$\exists x \in S.\ P(x)$$

  means "there is an element $x$ of set $S$ where $P(x)$ holds." (It's false if $S$ is empty.)

# Quantifying Over Sets

- The syntax

$$\forall x \in S.\ P(x)$$

$$\exists x \in S.\ P(x)$$

  is allowed for quantifying over sets.
- In CS103, feel free to use these restricted quantifiers, but please do not use variants of this syntax.
- For example, don't do things like this:

$$\forall x \text{ with } P(x).\ Q(x)$$

$$\forall y \text{ such that } P(y) \land Q(y).\ R(y).$$

$$\exists P(x).\ Q(x)$$

# Expressing Uniqueness

Using the predicate

- $WayToFindOut(w)$, which states that $w$ is a way to find out,

write a sentence in first-order logic that means “there is only one way to find out.”

*There is only one way to find out.*

*Something is a way to find out, and nothing else is.*

*Some thing w is a way to find out, and nothing else is.*

*Some thing w is a way to find out, and nothing besides w is a way to find out*

$\exists w. \, (WayToFindOut(w) \land$

*nothing besides w is way to find out*

$)$

$\exists w.\ (WayToFindOut(w)\ \wedge$

*anything that isn't w isn't a way to find out*

$)$

$\exists w.\ (WayToFindOut(w)\ \wedge$

*any thing x that isn't w isn't a way to find out*

$)$

$\exists w.\ (WayToFindOut(w)\ \land$
$\quad \forall x.\ (x \neq w \rightarrow$ *x isn't a way to find out*$)$
$)$

$$
\begin{aligned}
&\exists w.\ (\mathit{WayToFindOut}(w) \land \\
&\quad \forall x.\ (x \neq w \rightarrow \neg \mathit{WayToFindOut}(x)) \\
&)
\end{aligned}
$$

$$
\begin{aligned}
&\exists w.\ (\mathit{WayToFindOut}(w)\ \wedge \\
&\quad \forall x.\ (\mathit{WayToFindOut}(x) \rightarrow x = w) \\
&)
\end{aligned}
$$

# Expressing Uniqueness

- To express the idea that there is exactly one object with some property, we write that
  - there exists at least one object with that property, and that
  - there are no other objects with that property.
- You sometimes see a special “uniqueness quantifier” used to express this:

$$\exists! x.\ P(x)$$

- For the purposes of CS103, please do not use this quantifier. We want to give you more practice using the regular ∀ and ∃ quantifiers.

# Next Time

- ***Functions***
  - How do we model transformations and pairings?
- ***First-Order Definitions***
  - Where does first-order logic come into all of this?
- ***Proofs with Definitions***
  - How does first-order logic interact with proofs?