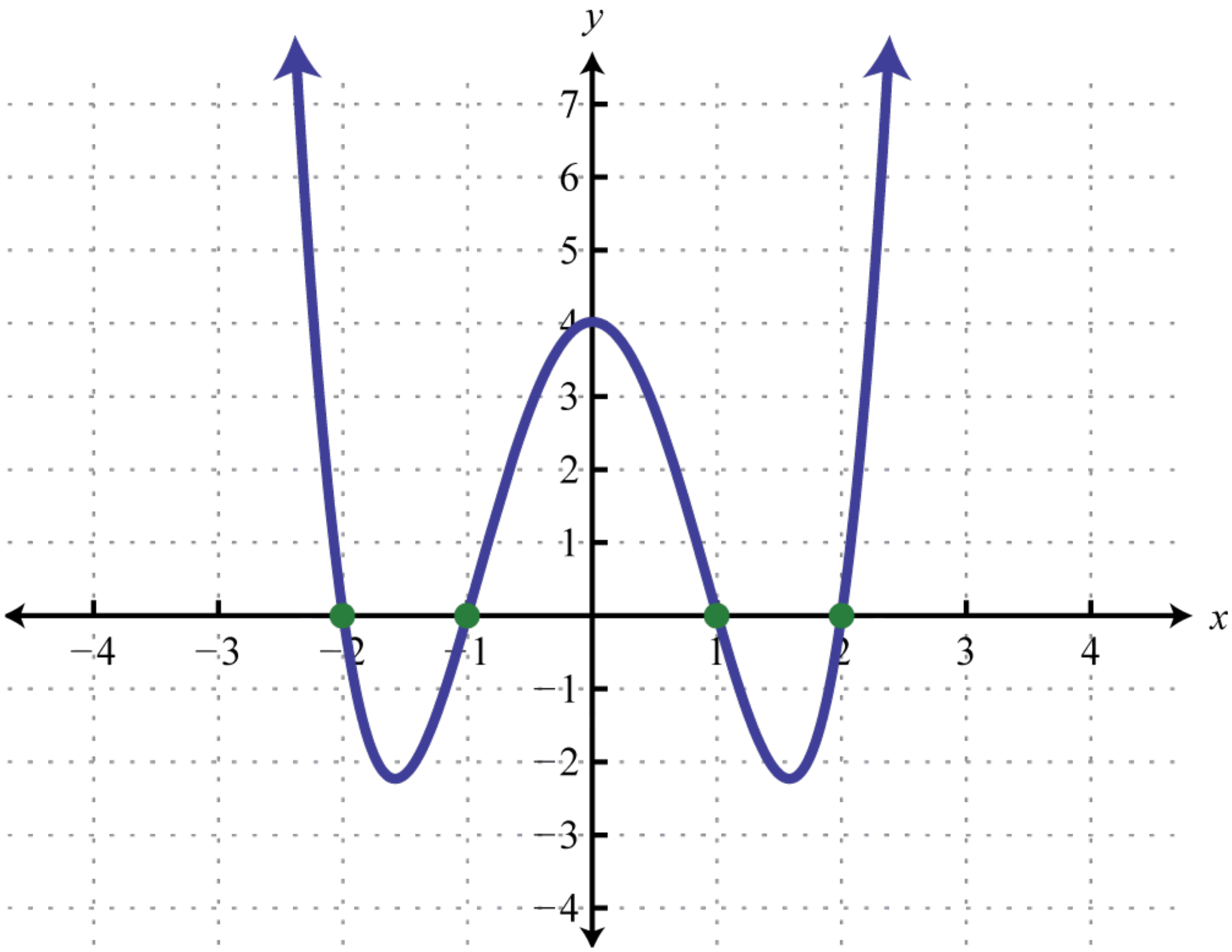
Functions

Outline for Today

- ***What is a Function?***
  - It’s more nuanced than you might expect.
- ***Domains and Codomains***
  - Where functions start, and where functions end.
- ***Defining a Function***
  - Expressing transformations compactly.
- ***Special Classes of Functions***
  - Useful types of functions you’ll encounter IRL.
- ***Proofs on First-Order Definitions***
  - A key skill.

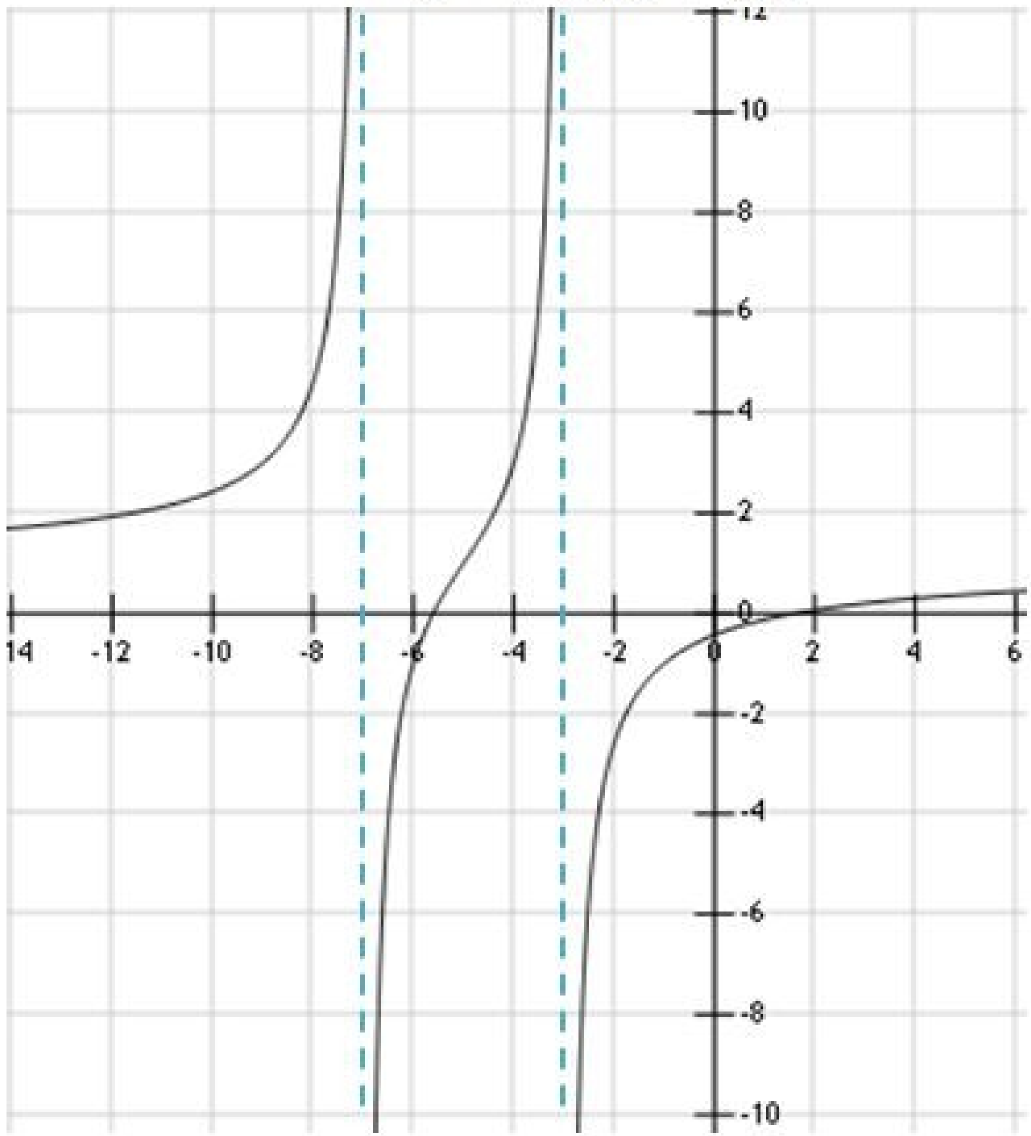
What is a function?

Functions, High-School Edition



$$f(x) = x^4 - 5x^2 + 4$$

$$f(x) = \frac{x^2 + 4x - 9}{x^2 + 10x + 21}$$



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Functions, High-School Edition

- In high school, functions are usually given as objects of the form

$$f(x) = \frac{x^3+3x^2+15x+7}{1-x^{137}}$$

- What does a function do?
  - It takes in as input a real number.
  - It outputs a real number
  - … except when there are vertical asymptotes or other discontinuities, in which case the function doesn't output anything.

Functions, CS Edition

```
int flipUntil(int n) {
  int numHeads = 0;
  int numTries = 0;

  while (numHeads < n) {
    if (randomBoolean()) {
      numHeads++;
    }
    numTries++;
  }

  return numTries;
}
```

Functions, CS Edition

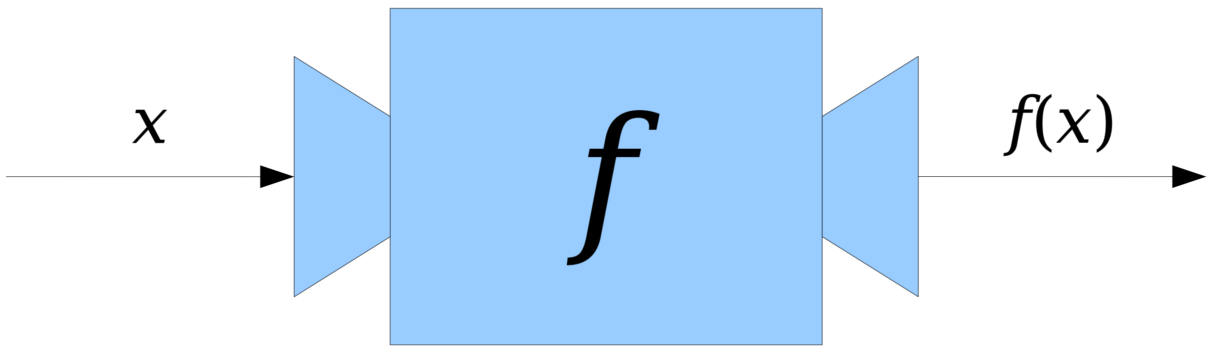
- In programming, functions
  - might take in inputs,
  - might return values,
  - might have side effects,
  - might never return anything,
  - might crash, and
  - might return different values when called multiple times.

What's Common?

- Although high-school math functions and CS functions are pretty different, they have two key aspects in common:
  - They take in inputs.
  - They produce outputs.
- In math, we like to keep things easy, so that's pretty much how we're going to define a function.

High-Level Intuition:

A function is an object f that takes in exactly one input x and produces exactly one output $f(x)$.



(This is not definition. It's just to help you build and intuition.)

High School versus CS Functions

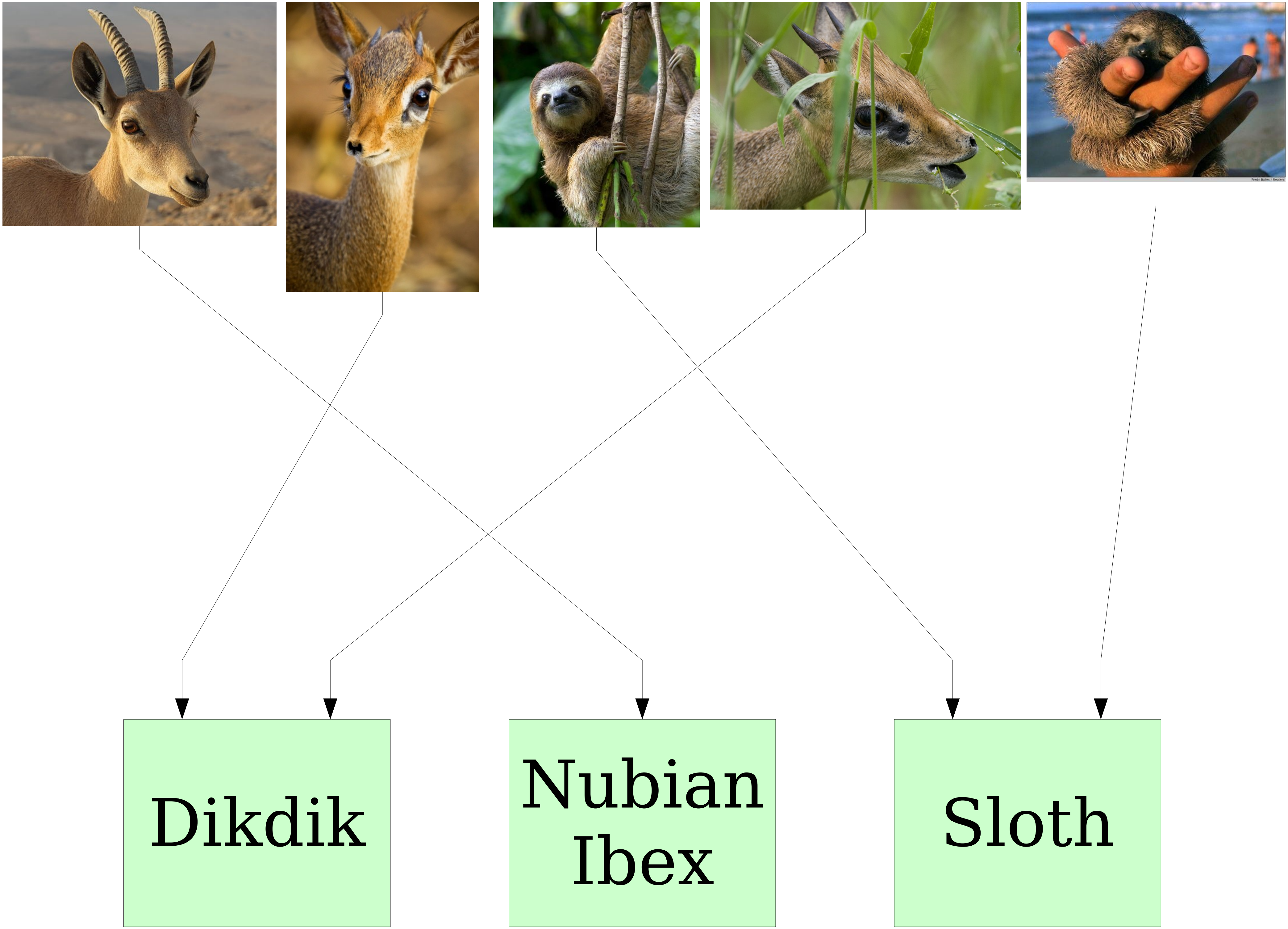
- In high school, functions usually were given by a rule:

$$f(x) = 4x + 15$$

- In CS, functions are usually given by code:

```
int factorial(int n) {
    int result = 1;
    for (int i = 1; i <= n; i++) {
        result *= i;
    }
    return result;
}
```

- What sorts of functions are we going to allow from a mathematical perspective?


Dikdik
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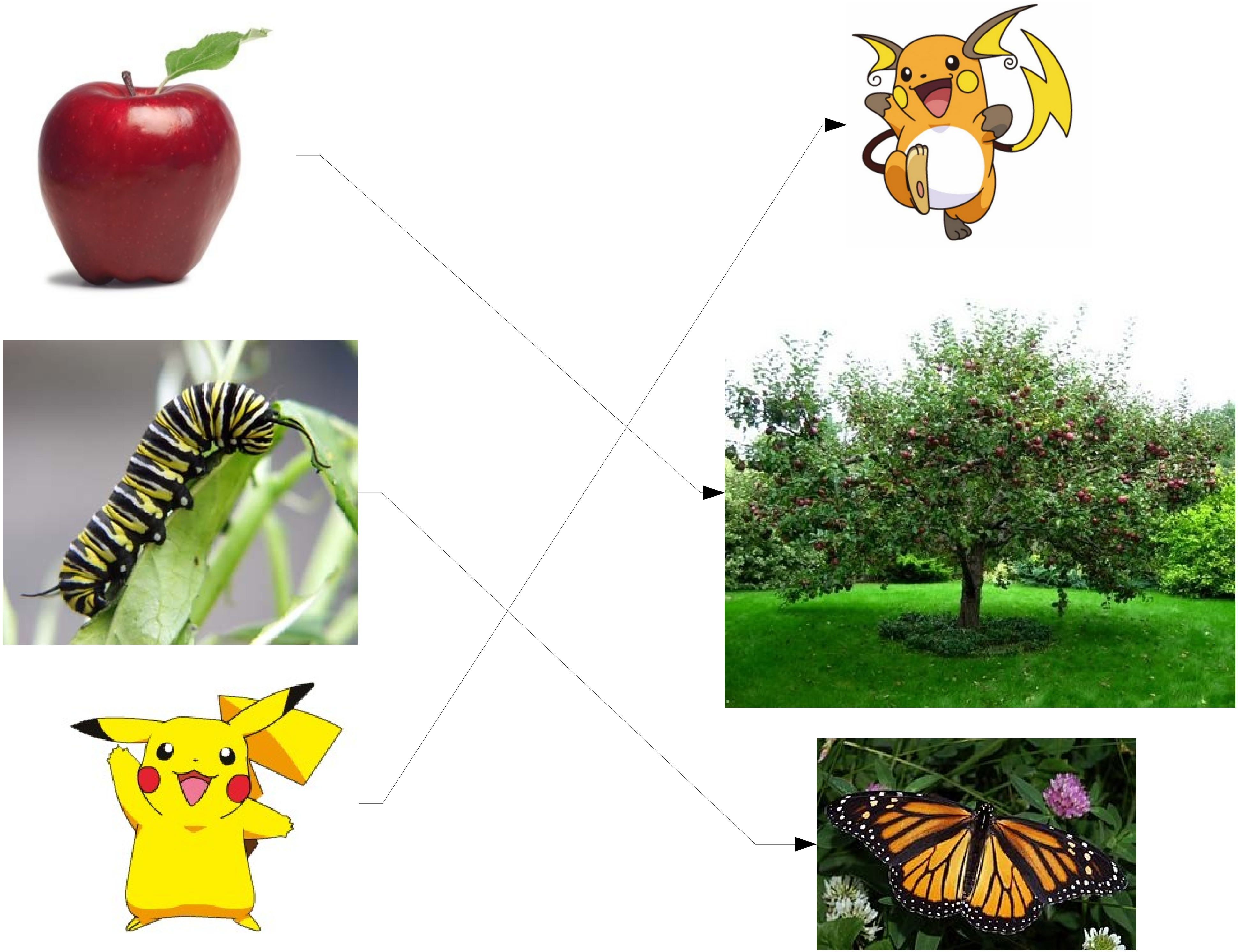
Sloth



… but also …

$$f(x) = x^2 + 3x - 15$$

In mathematics, functions are ***deterministic***.

That is, given the same input, a function must always produce the same output.

The following is a perfectly valid piece of C++ code, but it's not a valid function under our definition:

```
int randomNumber(int numOutcomes) {
    return rand() % numOutcomes;
}
```

One Challenge

$$f(x) = x^2 + 2x + 5$$

$$f(x) = x^2 + 2x + 5$$

$$f(\ 3\ ) = 3^2 + 3 \cdot 2 + 5 = 20$$

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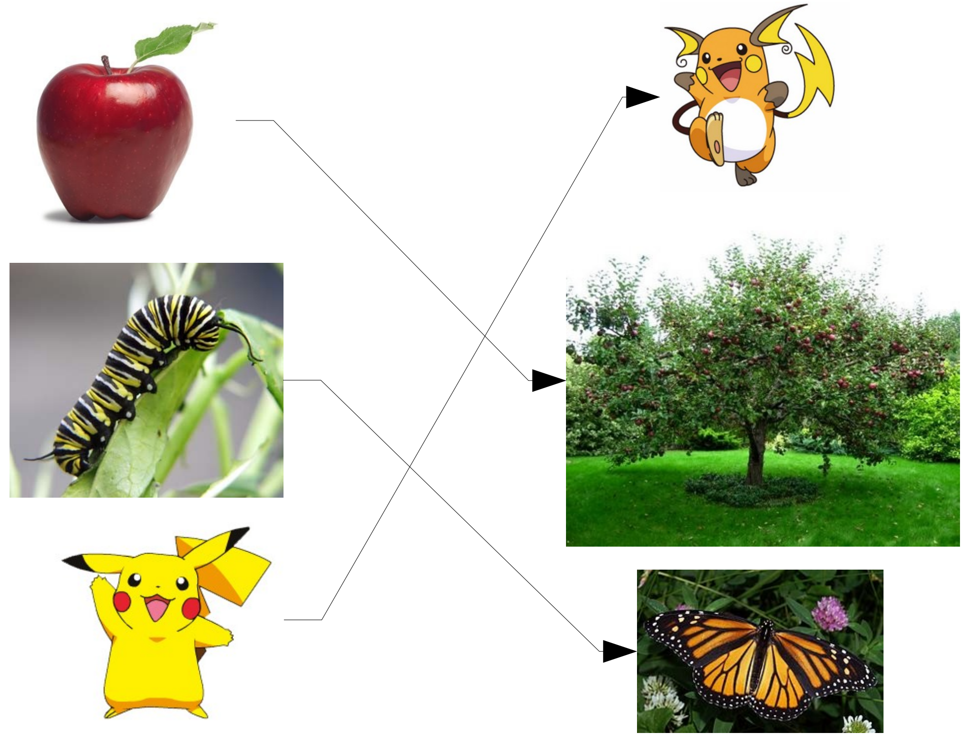
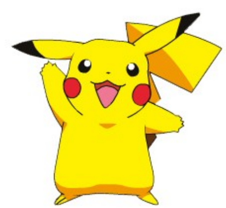
$f($  $) = \ldots\ ?$



$f($  $) =$ 

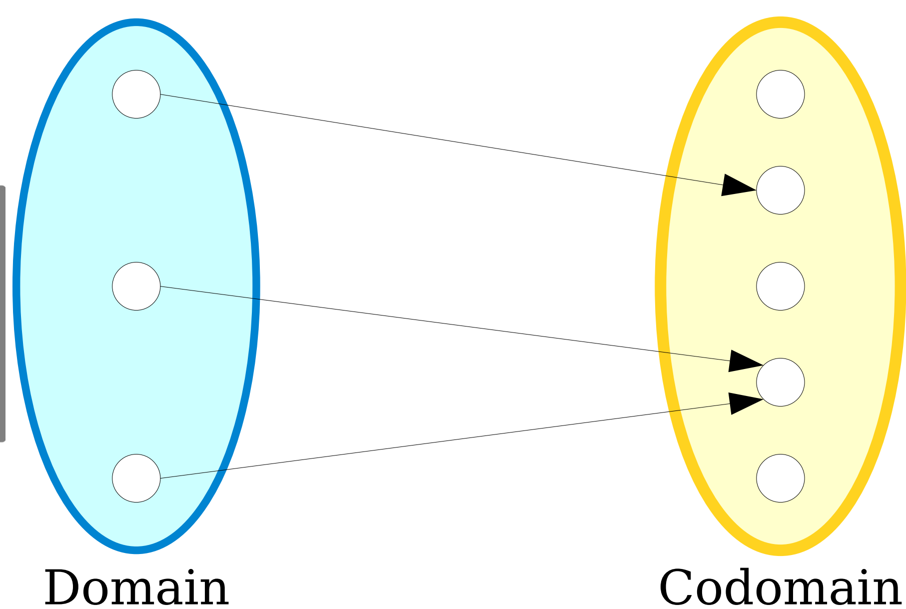
$f(137) = \ldots ?$

We need to make sure we can't apply functions to meaningless inputs.

Domains and Codomains

- Every function *f* has two sets associated with it: its ***domain*** and its ***codomain***.
- A function *f* can only be applied to elements of its domain. For any *x* in the domain, *f*(*x*) belongs to the codomain.

The function must be defined for every element of the domain.



The output of the function must always be in the codomain, but not all elements of the codomain must be produced as outputs.

Domains and Codomains

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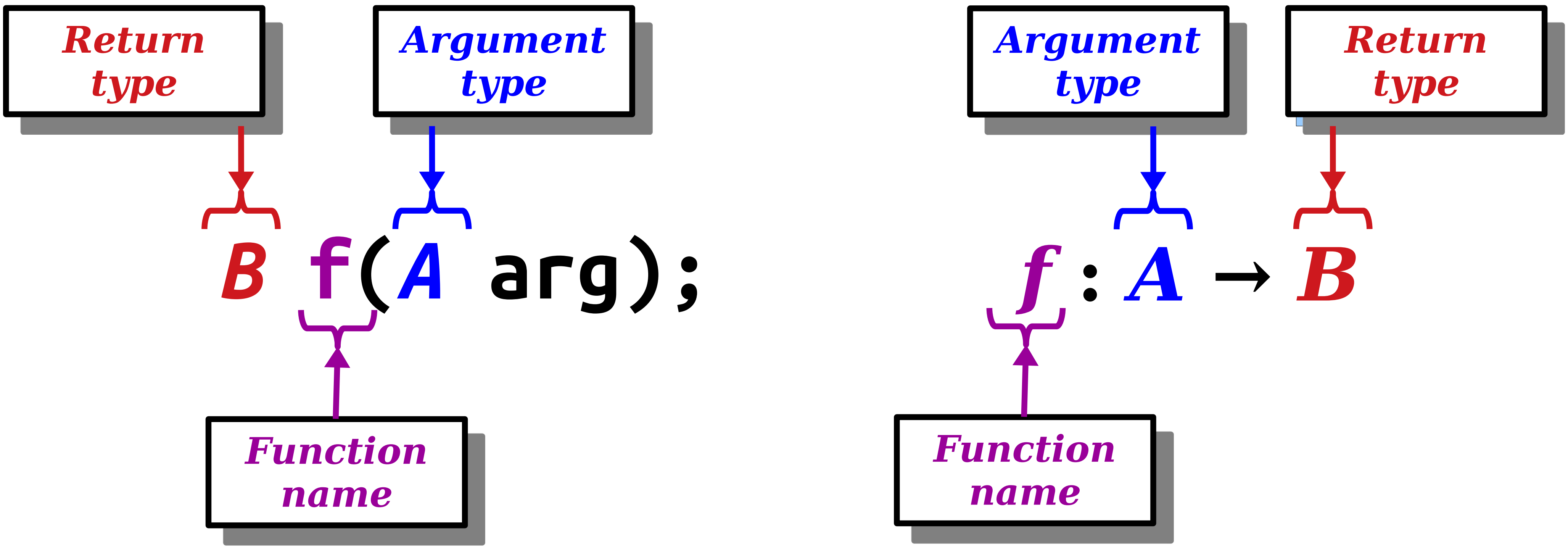
The ***domain*** of this function is ℝ. Any real number can be provided as input.

```
double absoluteValueOf(double x) {
    if (x >= 0) {
        return x;
    } else {
        return -x;
    }
}
```

The ***codomain*** of this function is ℝ. Everything produced is a real number, but not all real numbers can be produced.

Domains and Codomains

- If f is a function whose domain is A and whose codomain is B, we write $f : A \to B$.
- Think of this like a “function prototype” in C++.



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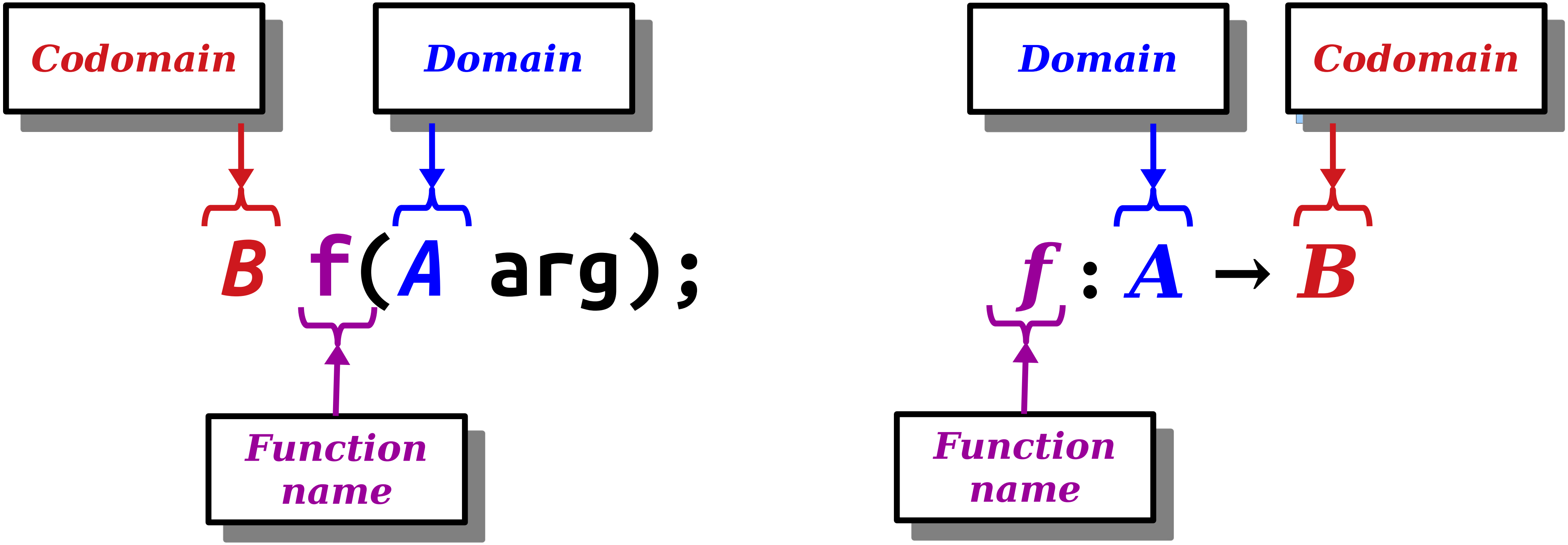


The Official Rules for Functions

- Formally speaking, we say that $f : A \to B$ if the following two rules hold.
- First, f must be obey its domain/codomain rules:

$$\forall a \in A.\ \exists b \in B.\ f(a) = b$$

(*"Every input in A maps to some output in B."*)

- Second, f must be deterministic:

$$\forall a_1 \in A.\ \forall a_2 \in A.\ (a_1 = a_2 \to f(a_1) = f(a_2))$$

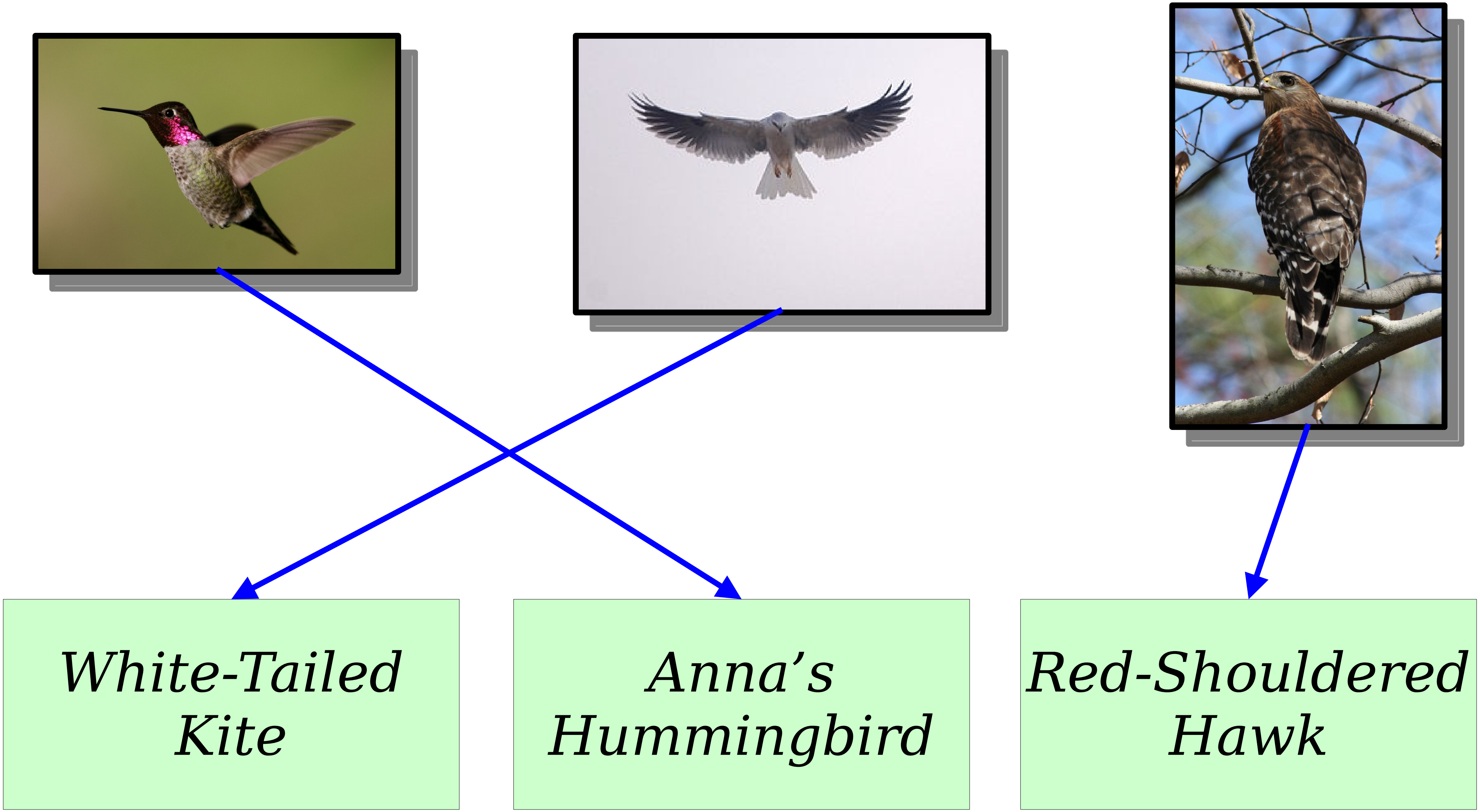
(*"Equal inputs produce equal outputs."*)

- If you're ever curious about whether something is a function, look back at these rules and check! For example:
  - Can a function have an empty domain?
  - Can a function have an empty codomain?

Defining Functions

Defining Functions

- To define a function, you need to
  - specify the domain,
  - specify the codomain, and
  - give a ***rule*** used to evaluate the function.
- All three pieces are necessary.
  - We need to domain to know what the function can be applied to.
  - We need to codomain to know what the output space is.
  - We need the rule to be able to evaluate the function.
- There are many ways to do this. Let’s go over a few examples.



Functions can be defined as a ***picture***.
Draw the domain and codomain explicitly.
Then, add arrows to show the outputs.

$$f : \mathbb{Z} \to \mathbb{Z}, \text{ where}$$

$$f(x) = x^2 + 3x - 15$$

Functions can be defined as a ***rule***.
Be sure to explicitly state what the domain and codomain are!

$f : \mathbb{Z} \to \mathbb{N}$, where

$$f(n) = \begin{cases} n & \text{if } n \geq 0 \\ -n & \text{if } n \leq 0 \end{cases}$$

Some rules are given ***piecewise***. We select which rule to apply based on the conditions on the right. (Just make sure at least one condition applies and that all applicable conditions give the same result!)

Some Nuances

$$f(x) = \frac{x+2}{x+1}$$

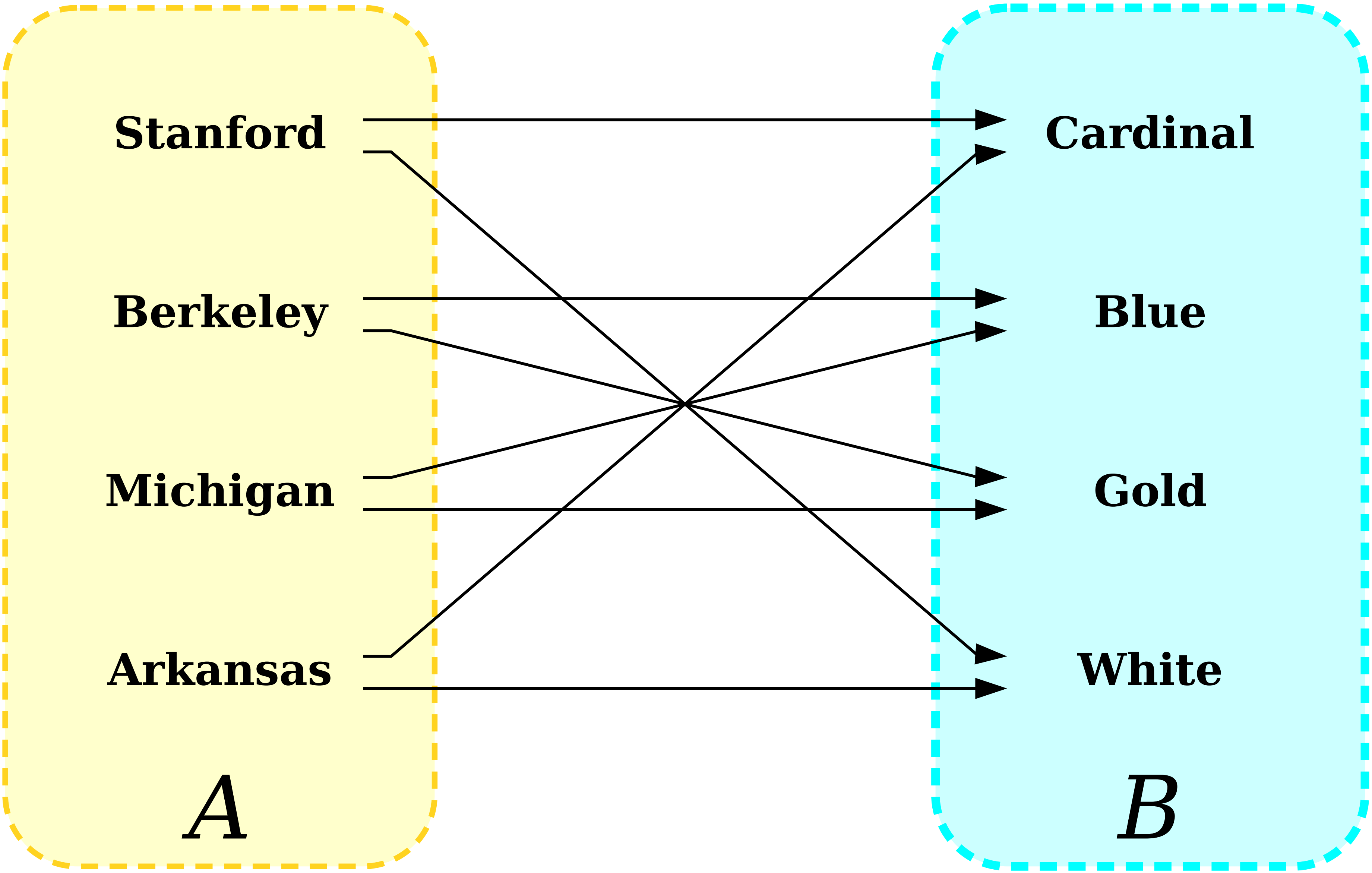
This expression isn't defined when x = -1, so f isn't defined over its full domain. We therefore don't consider it to be a function.

Is this a function from ℝ to ℝ?

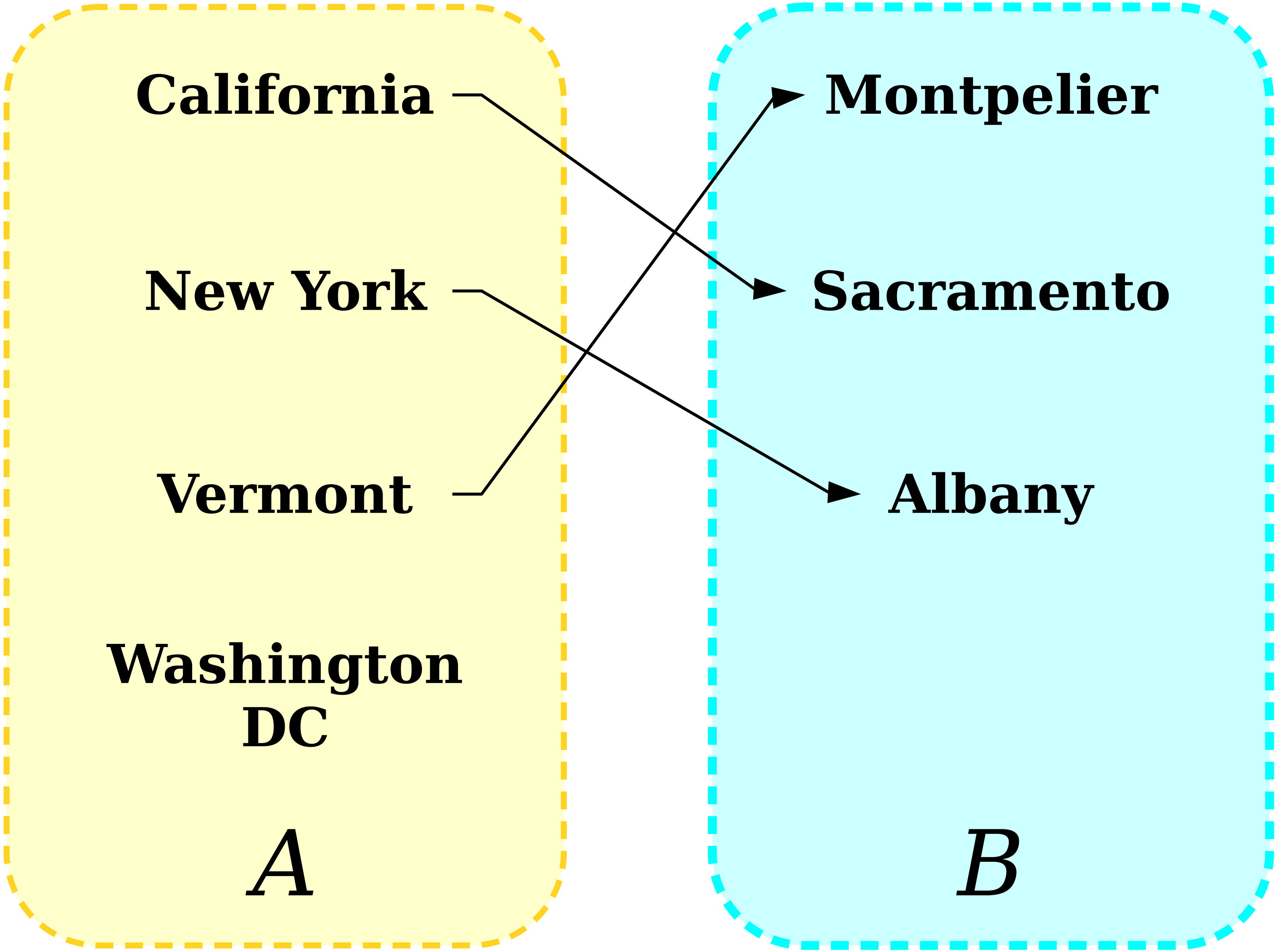
$$f(x) = \frac{x+2}{x+1}$$

Yep, it's a function! Every natural number maps to some real number.

Is this a function from $\mathbb{N}$ to $\mathbb{R}$?



Is this a function from A to B?



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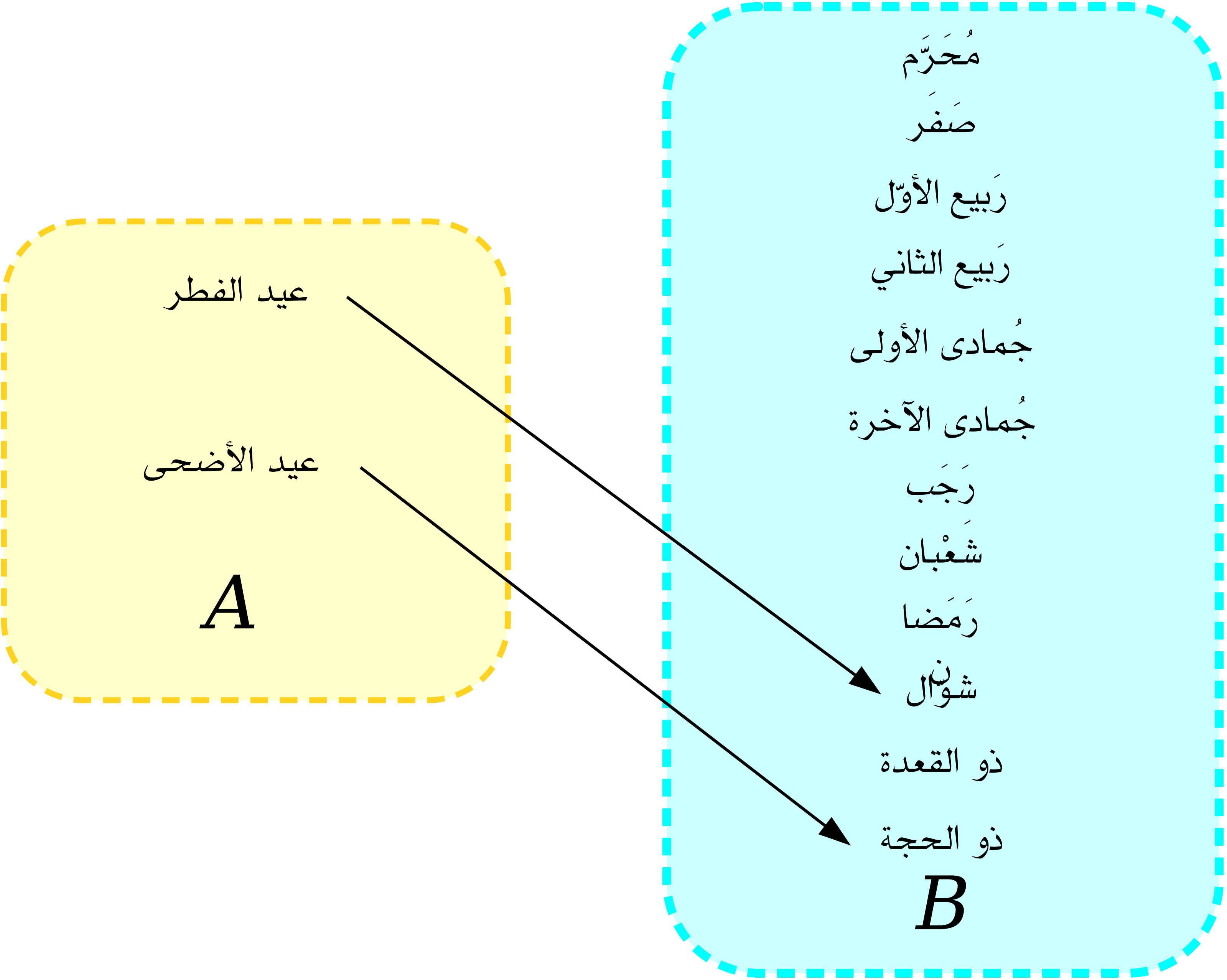

Is this a function from A to B?

```
int squigglebah(int input) {
    if (randomCoinTossIsHeads()) {
        return input;
    } else {
        return -input;
    }
}
```

This piece of code is not ***deterministic***. Calling `squigglebah(137)` multiple times might give back different values. It's therefore not a function in the mathematical sense.

Is this a function from $\mathbb{Z}$ to $\mathbb{Z}$?

```
int pizkwat(int input) {
    int steps = 0;
    while (input != 0) {
        input -= 2;
        steps++;
    }
    return steps;
}
```

This code never produces a value when called on the input 137. It's therefore not defined for all elements of the domain, so it's not a function in the mathematical sense.

Is this a function from $\mathbb{Z}$ to $\mathbb{Z}$?

Time-Out for Announcements!

EdStem

- We’ve set up threads on EdStem associated with each problem on the problem set.
- Have a question on a specific problem? Check that thread, and feel free to post your question there!

{w}

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ACTIVITIES FAIR: SEPT 28, 4-7 PM @ WHITE PLAZA
INFO SESSION: SEPT 30, 3:30 PM @ BIT.LY/WICSINFO21

Code the Change

- Code the Change is a group that builds software for nonprofit groups.
- They've been around for over a decade and have received a bunch of awards and press coverage.
- Interested in joining? There's an info session tonight from 7PM – 8PM at this link:

  http://tinyurl.com/ctcinfo21
- You can also fill out this form to get added to their Slack group:

  https://forms.gle/DFM786dDjexZY8cd7

Your Questions

“What are some classes to look into if you want to get into data science?”

A lot of data science is applied probability, statistics, and linear algebra, so courses in those spaces are great to look into. Check out Math 51 and CS109 as launching points. You might also want to take a class in optimization, like MS&E 111.

“Why did you pick teaching over industry?”

By the time I graduated I had a mix of experience with teaching, research, and industry. Of the three, teaching seemed to be the most fun, so I figured I'd give it a try knowing that the other options were always there if I changed my mind. Turns out, a decade later, It's still remarkably rewarding and fulfilling, so I'm still here!

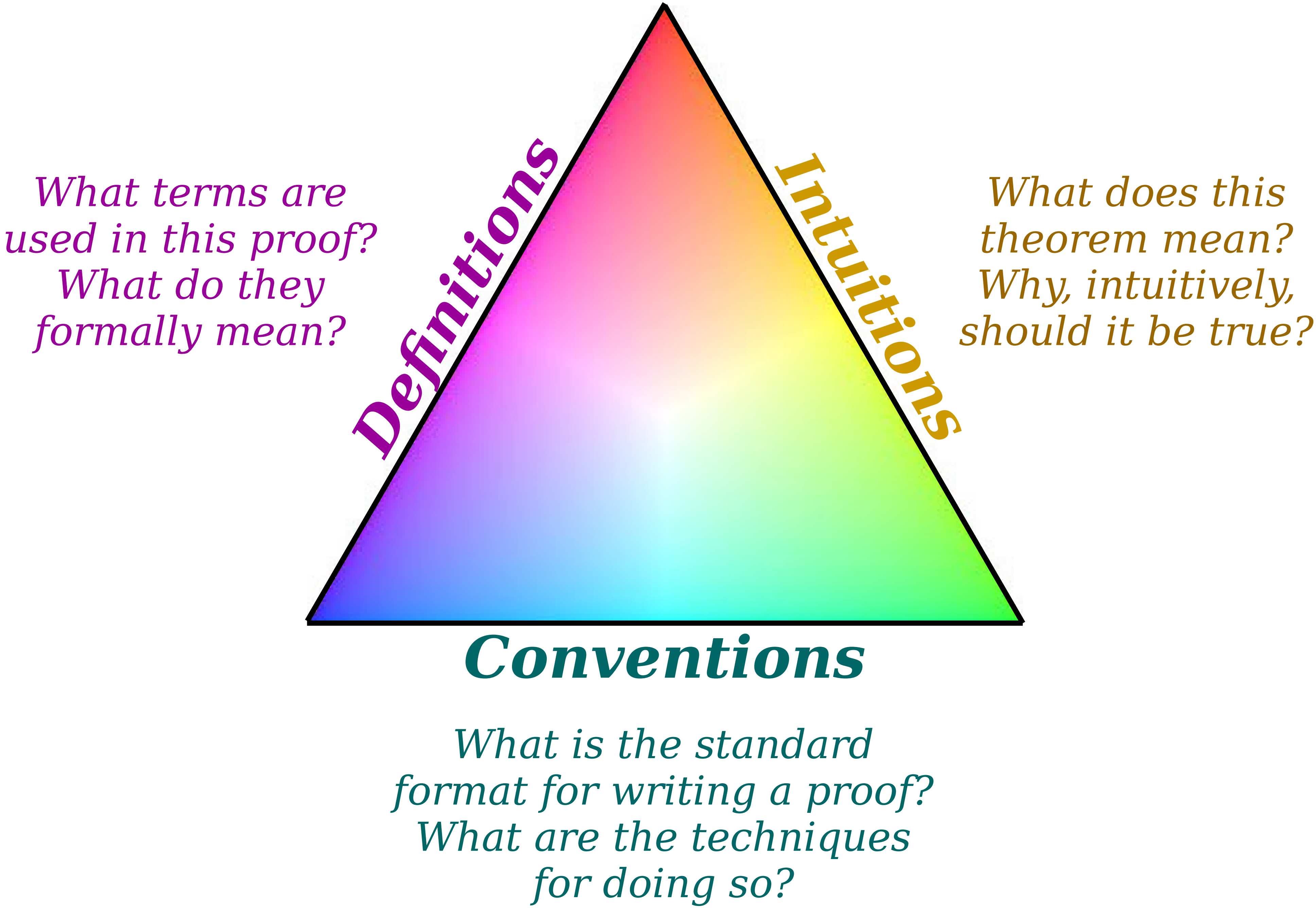
“Did you minor in anything, and would you recommend doing so?”

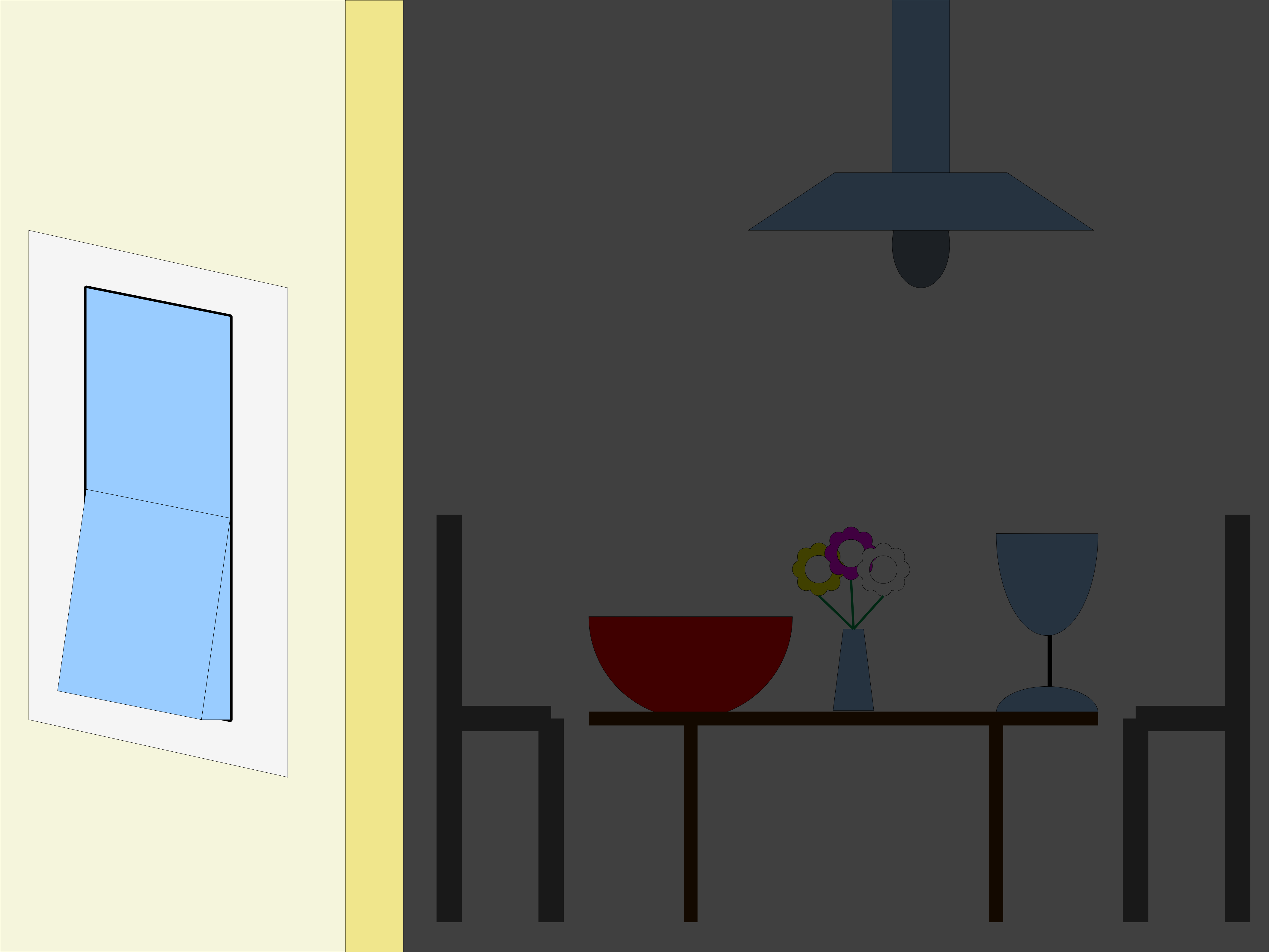
I did a math minor. It was definitely not something I'd planned from the start – in my junior year I realized I was close to having all the requirements and just needed to take a couple more classes.

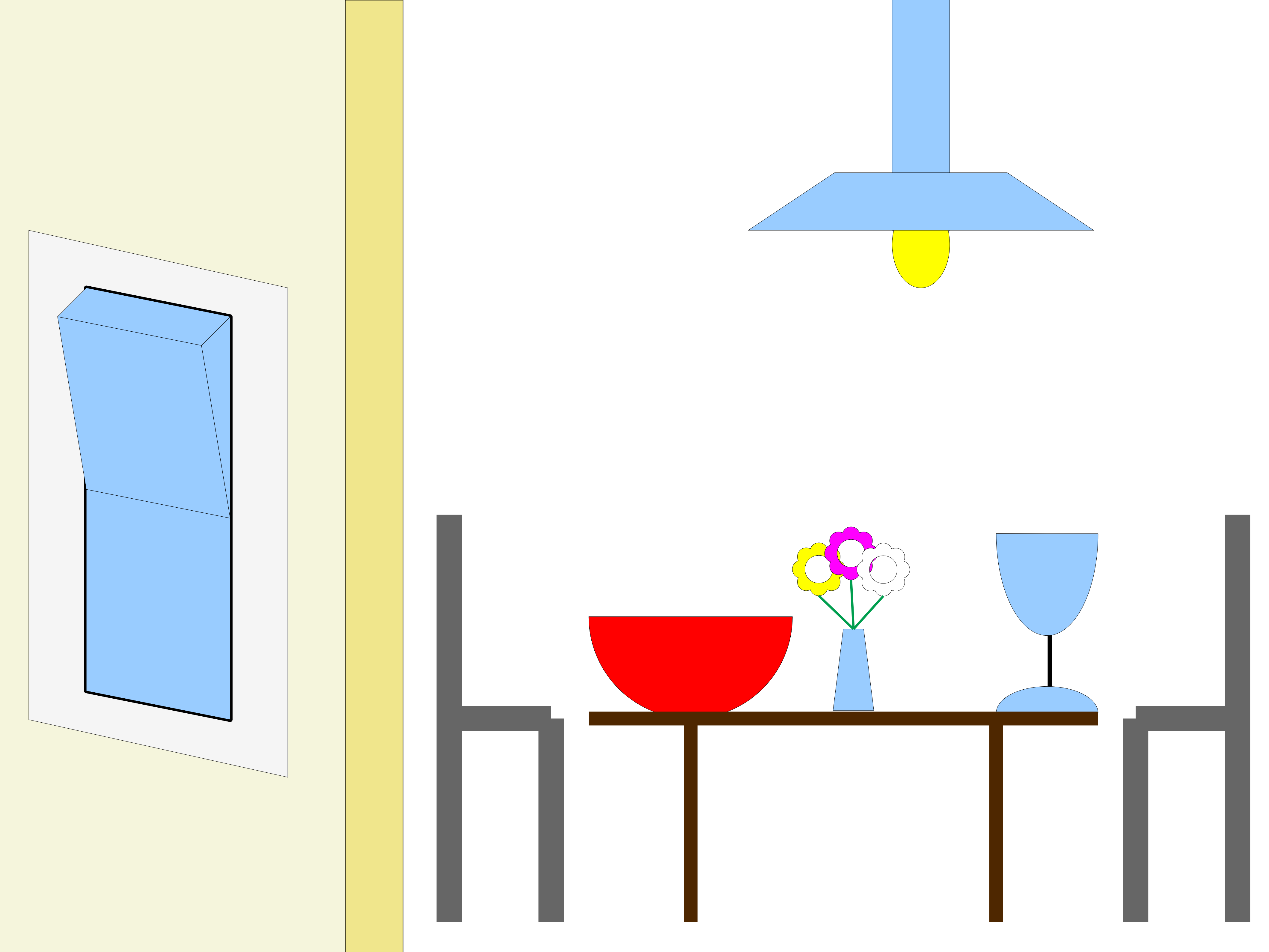
I definitely recommend exploring a bunch of different areas to see what you're interested in. Whether to do a minor or not is largely a question about whether reaching that goal intrinsically means something to you and whether you're interested in filling out the requirements. There are lots of other ways to explore two areas (major/coterm, major/minor, and even major/“I took a bunch of classes in this are because that's something that appealed to me”).

Back to CS103!

Special Types of Functions


What terms are used in this proof? What do they formally mean?
Definitions
Intuitions
What does this theorem mean? Why, intuitively, should it be true?
Conventions
What is the standard format for writing a proof? What are the techniques for doing so?





Undoing by Doing Again

- Some operations invert themselves. For example:
  - Flipping a switch twice is the same as not flipping it at all.
  - In first-order logic, $\neg\neg A$ is equivalent to A.
  - In algebra, $-(-x) = x$.
  - In set theory, $(A \Delta B) \Delta B = A$. *(Yes, really!)*
- Operations with these properties are surprisingly useful in CS theory and come up in a bunch of contexts.
  - Storing compressed approximations of sets (XOR filters).
  - Theoretically unbreakable encryption (one-time pads).
  - Transmitting a large file to multiple receivers (fountain codes).

Involutions

- A function $f : A \to A$ from a set back to itself is called an ***involution*** if the following first-order logic statement is true about f:

$$\forall x \in A.\ f(f(x)) = x.$$

("Applying f twice is equivalent to not applying f at all.")

- Involutions have lots of interesting properties. Let's explore them and see what we can find.

Involutions

- Which of the following are involutions?
  - $f : \mathbb{Z} \to \mathbb{Z}$ defined as $f(x) = x$.
  - $f : \mathbb{Z} \to \mathbb{Z}$ defined as $f(x) = -x$.
  - $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = 1/x$.
  - $f : \mathbb{N} \to \mathbb{N}$ defined as follows:

$$f(n) = \begin{cases} n+1 & \text{if } n \text{ is even} \\ n-1 & \text{if } n \text{ is odd} \end{cases}$$

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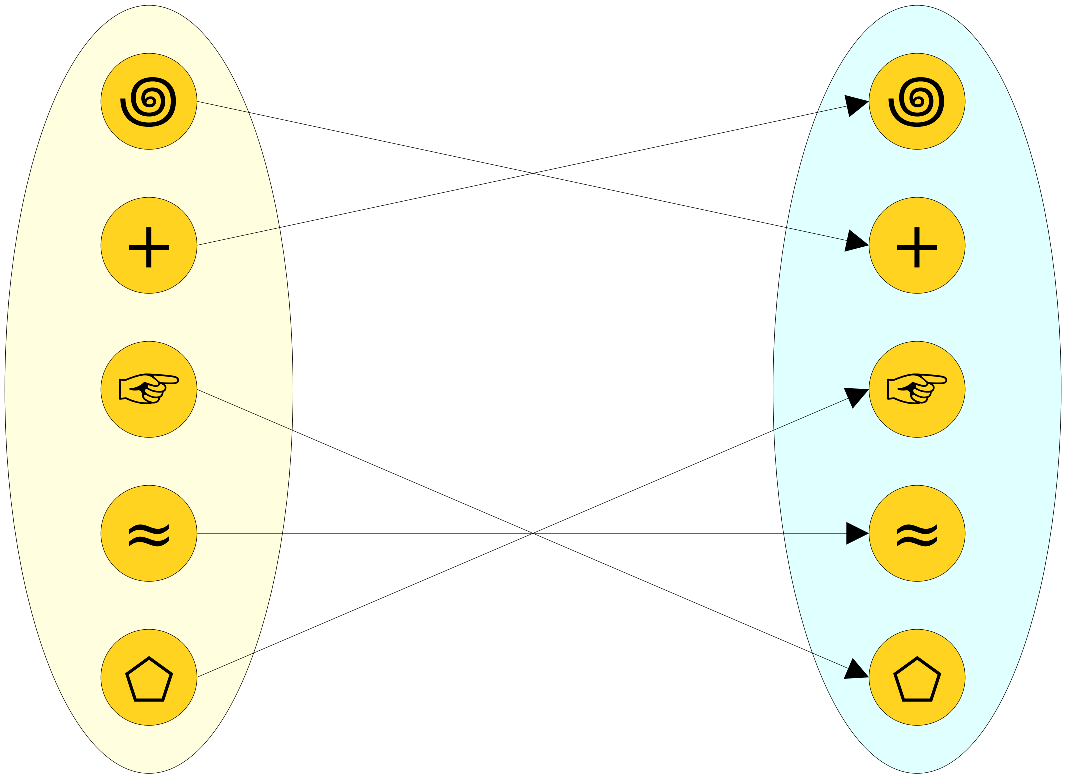
Involutions, Visually



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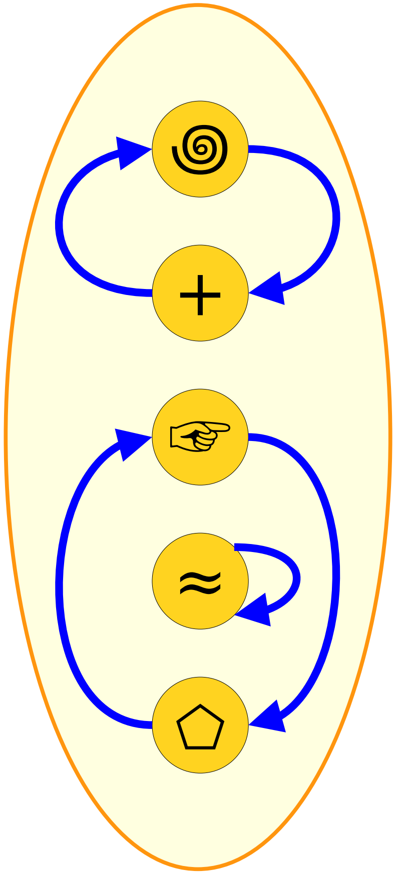

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Proofs on Involutions

Theorem: The function $f : \mathbb{Z} \to \mathbb{Z}$ defined as

$$f(n) = \begin{cases} n+1 & \text{if } n \text{ is even} \\ n-1 & \text{if } n \text{ is odd} \end{cases}$$

is an involution.

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Proof:

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What does it mean for f to be an involution?

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This proof contains no first-order logic syntax (quantifiers, connectives, etc.). It's written in plain English, just as usual.

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What is the negation of this statement?

$$\neg\forall n \in \mathbb{N}.\ f(f(n)) = n$$
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Pick $n = 2$. Then

$$\begin{aligned} f\,(f(n)) &= f(f(2)) \\ &= f(4) \\ &= 16, \end{aligned}$$

which means that $f(f(n)) \neq 2$, as required. ■

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$\forall x.\ A$	Have the reader pick an arbitrary x. We then prove A is true for that choice of x.	
$\exists x.\ A$	Find an x where A is true. Then prove that A is true for that specific choice of x.	
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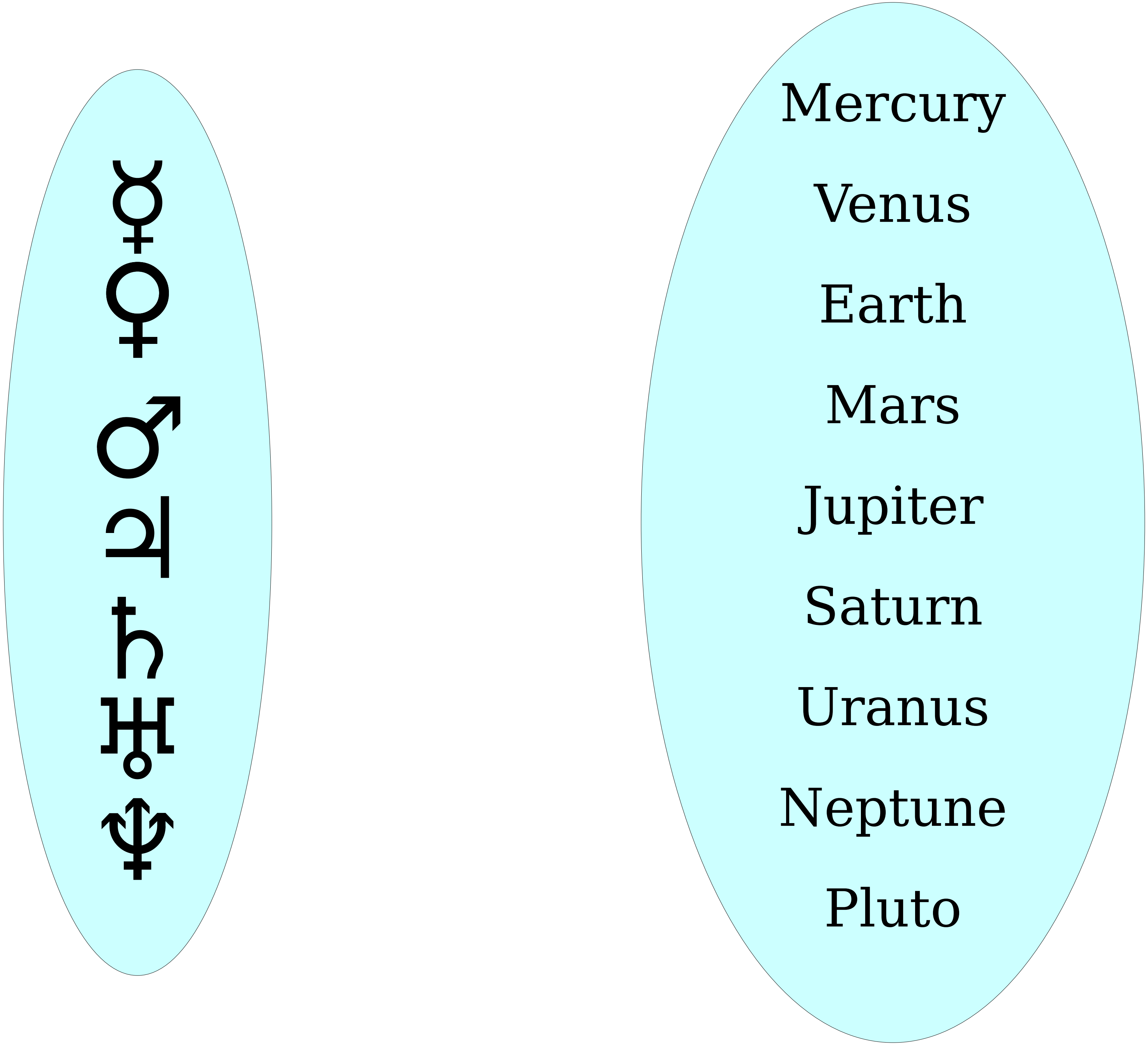
Another Class of Functions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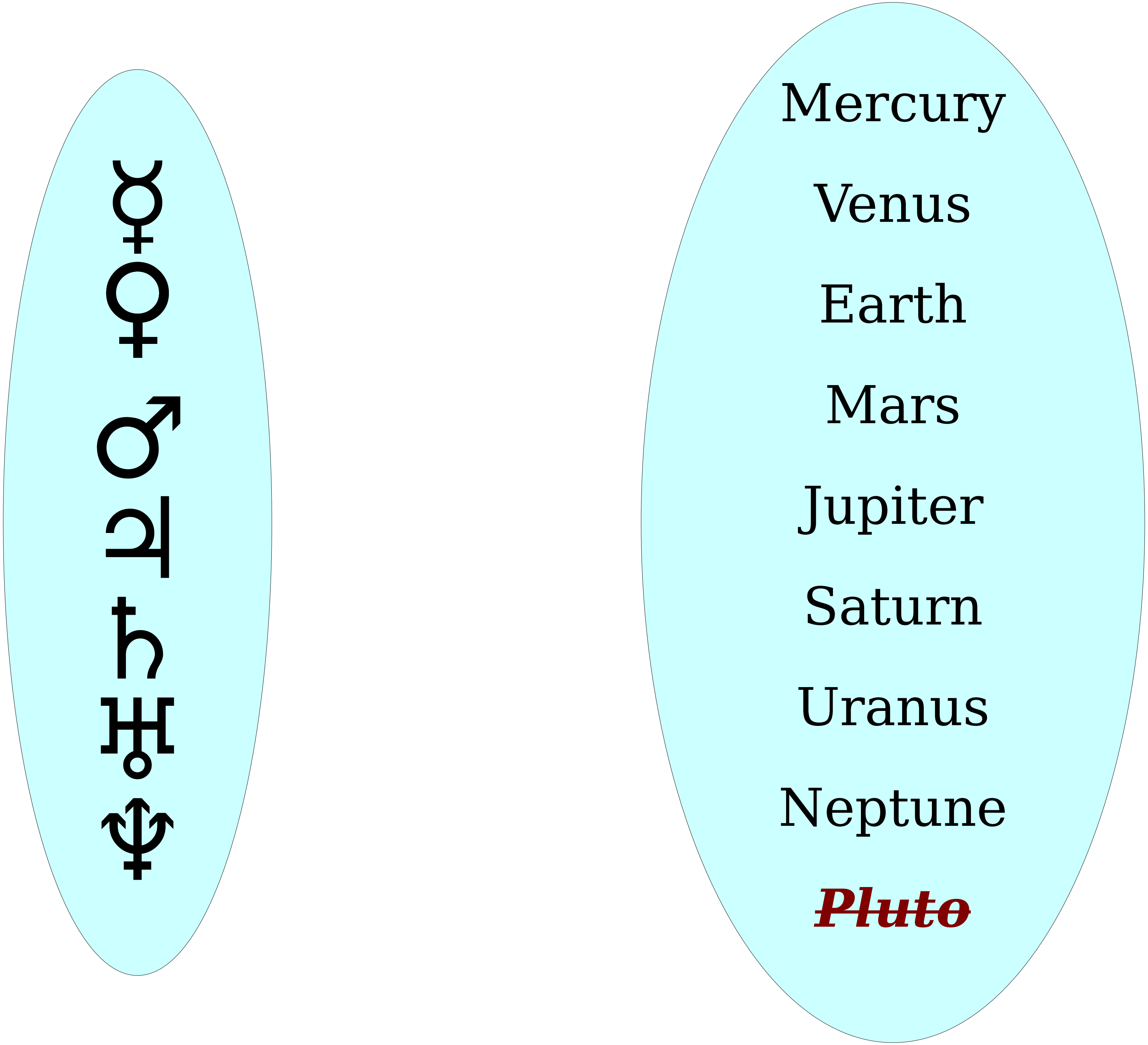

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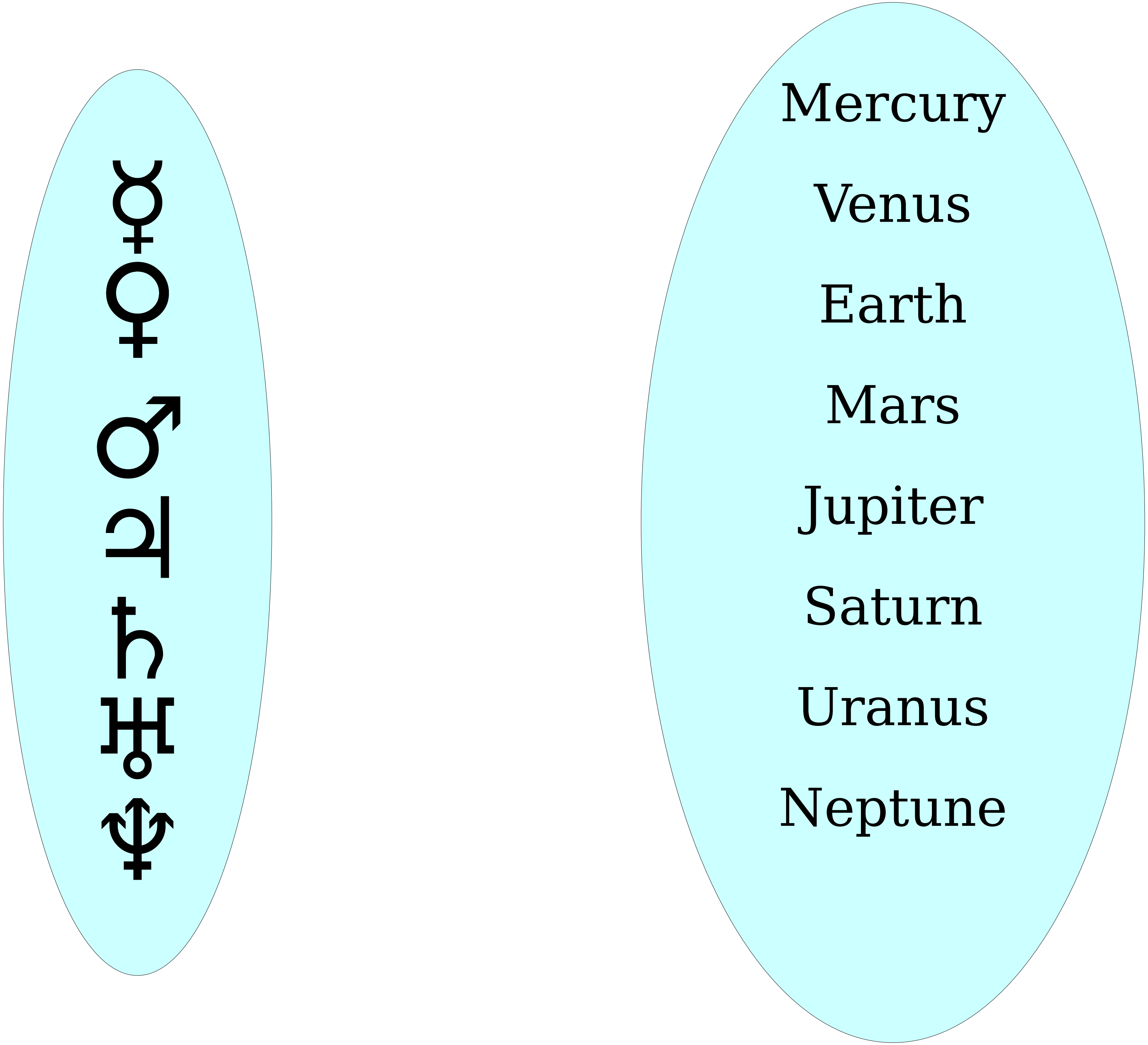
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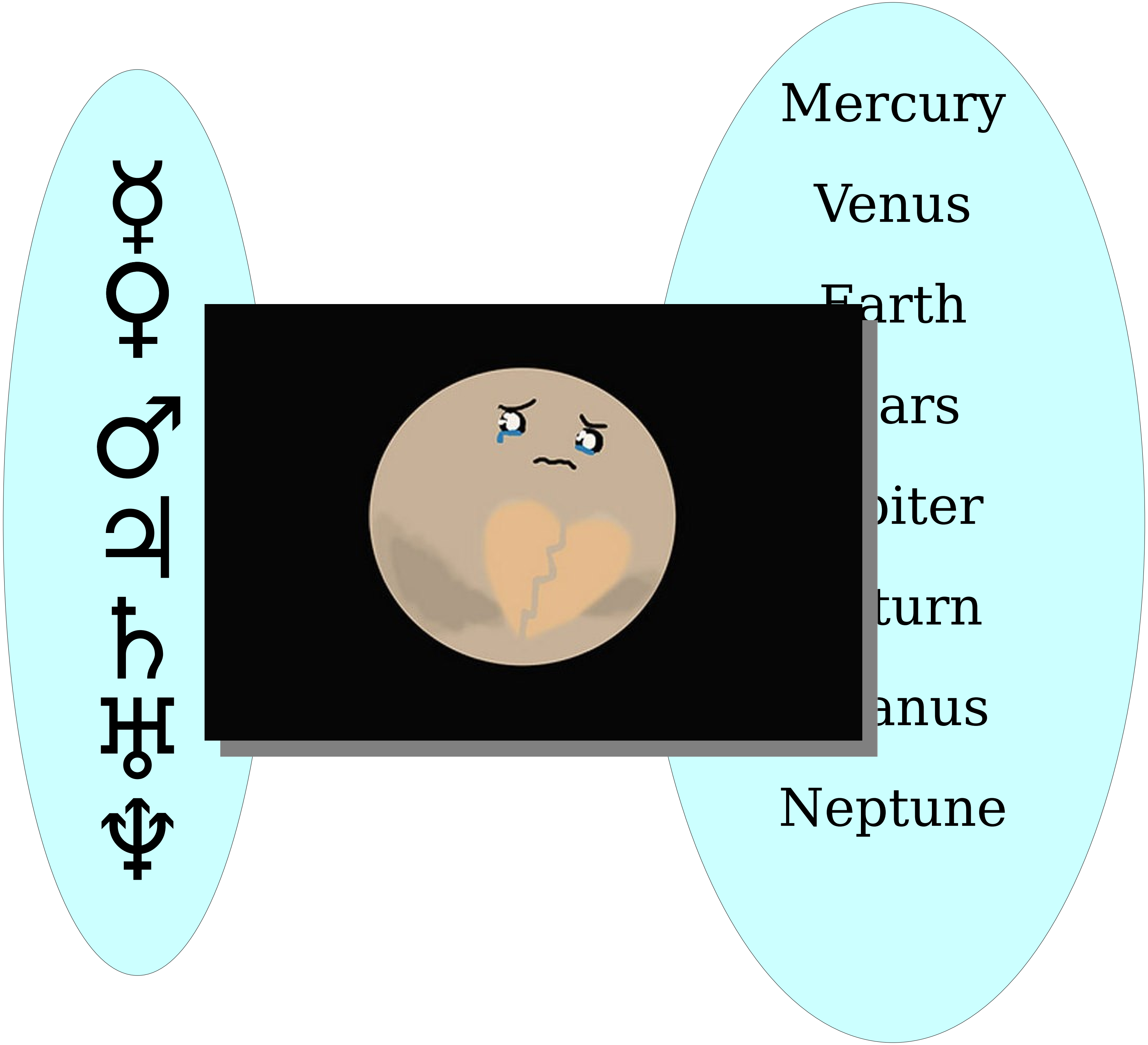
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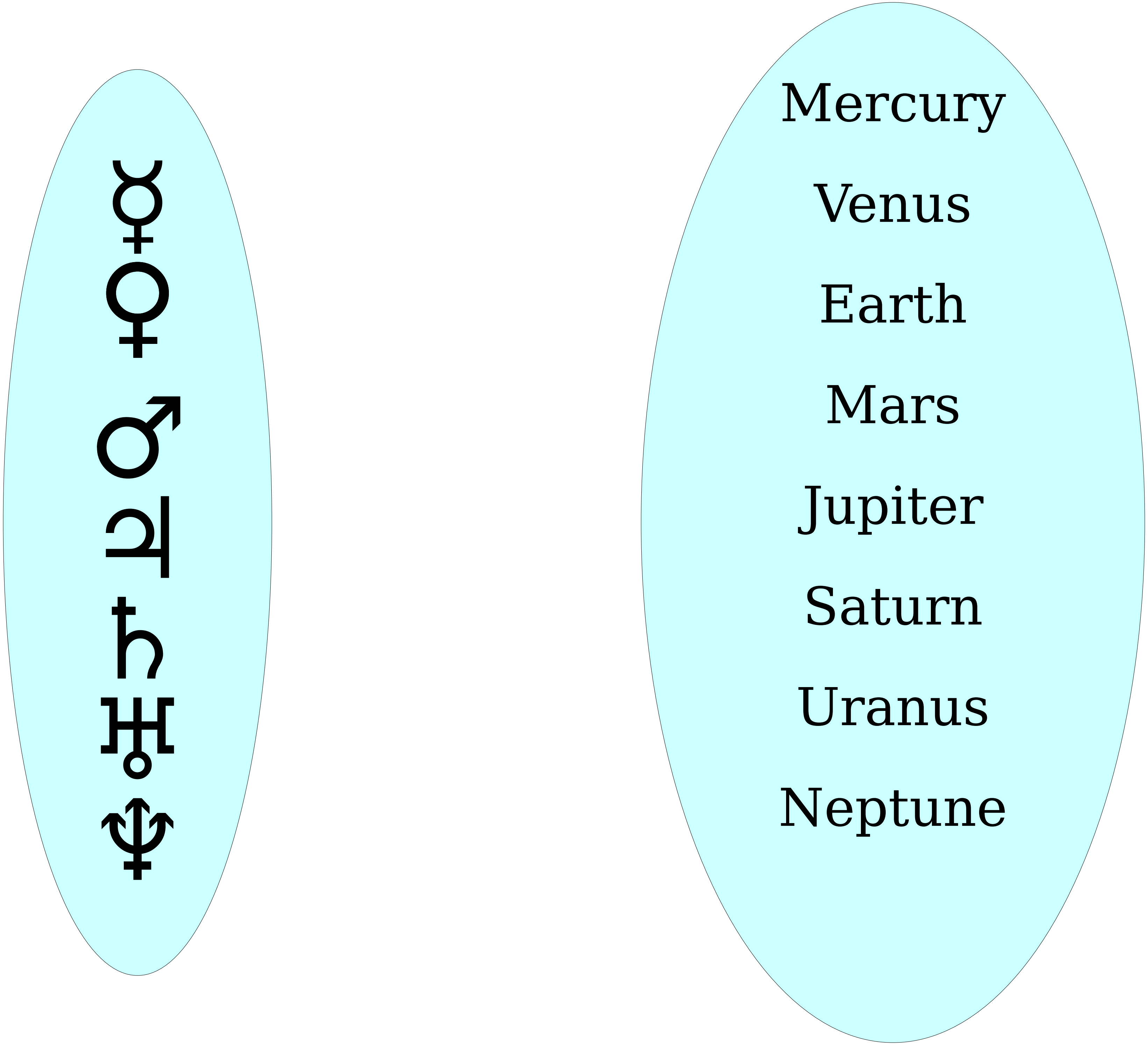
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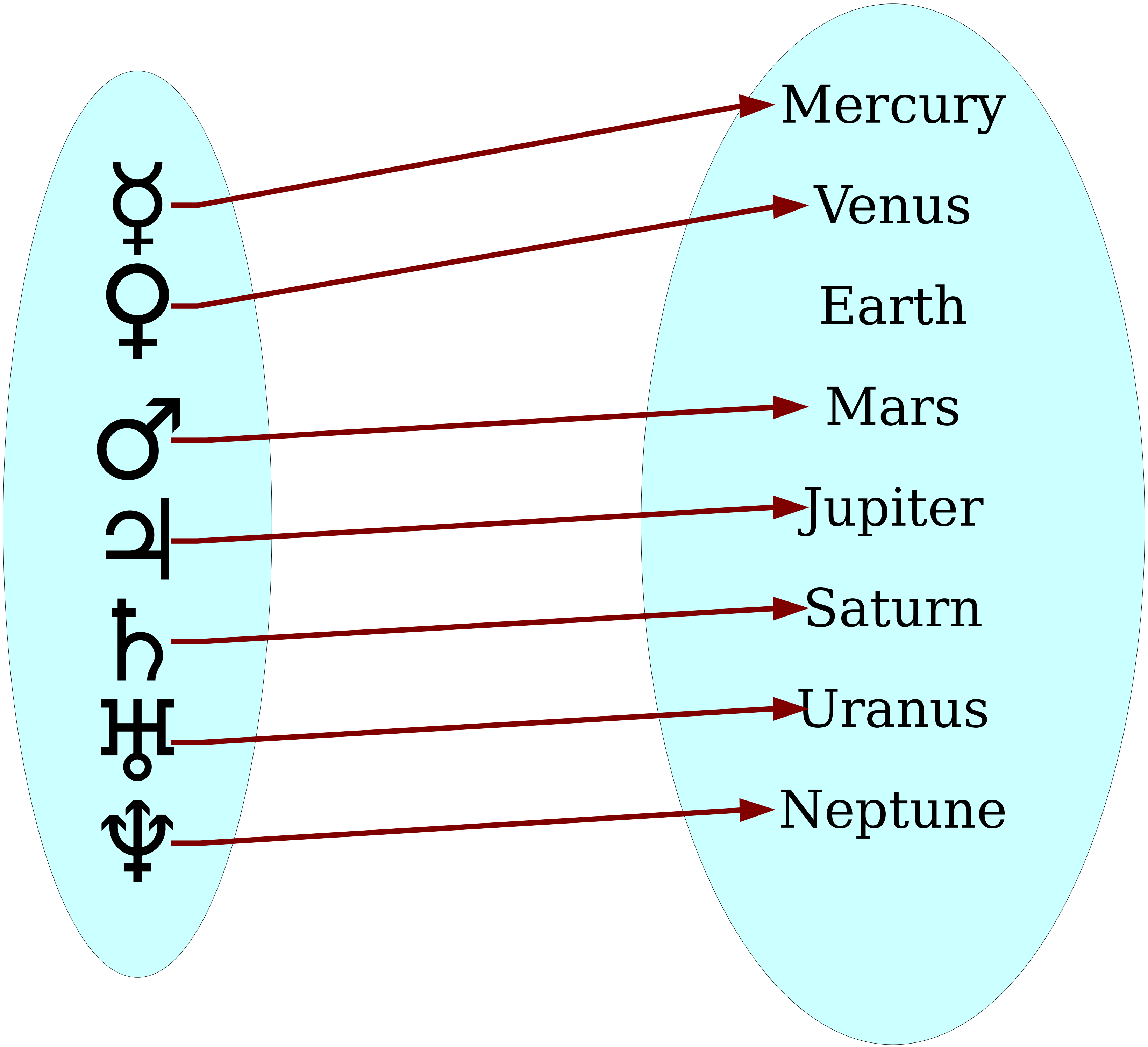

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Injective Functions

- A function $f : A \to B$ is called ***injective*** (or ***one-to-one***) if the following statement is true about f:

$$\forall a_1 \in A.\ \forall a_2 \in A.\ (a_1 \neq a_2 \to f(a_1) \neq f(a_2))$$

(*"If the inputs are different, the outputs are different."*)

- The following first-order definition is equivalent *(why?)* and is often useful in proofs.

$$\forall a_1 \in A.\ \forall a_2 \in A.\ (f(a_1) = f(a_2) \to a_1 = a_2)$$

(*"If the outputs are the same, the inputs are the same."*)

- A function with this property is called an ***injection***.
- How does this compare to our second rule for functions?

Injections

- Let   be the set of all CS103 students. Which of the following are injective?
  - f :   → ℕ where $f(x)$ is x's Stanford ID number.
  - f :   → , where   is the set of all countries and $f(x)$ is x's country of birth.
  - f :   → , where   is the set of all given (first) names, where $f(x)$ is x's given (first) name.

A function $f : A \to B$ is ***injective*** if either statement is true:

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Proof: Consider any $n_1, n_2 \in \mathbb{N}$ where $f(n_1) = f(n_2)$. We will prove that $n_1 = n_2$.

Since $f(n_1) = f(n_2)$, we see that

$$2n_1 + 7 = 2n_2 + 7.$$

This in turn means that

$$2n_1 = 2n_2,$$

so $n_1 = n_2$, as required. ■

Good exercise: Repeat this proof using the other definition of injectivity!

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What does it mean for f to be injective?

$$\forall x_1 \in \mathbb{Z}.\ \forall x_2 \in \mathbb{Z}.\ (x_1 \neq x_2 \to f(x_1) \neq f(x_2))$$

What is the negation of this statement?

$$\neg\forall x_1 \in \mathbb{Z}.\ \forall x_2 \in \mathbb{Z}.\ (x_1 \neq x_2 \to f(x_1) \neq f(x_2))$$
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Therefore, we need to find $x_1, x_2 \in \mathbb{Z}$ such that $x_1 \neq x_2$, but $f(x_1) = f(x_2)$. Can we do that?

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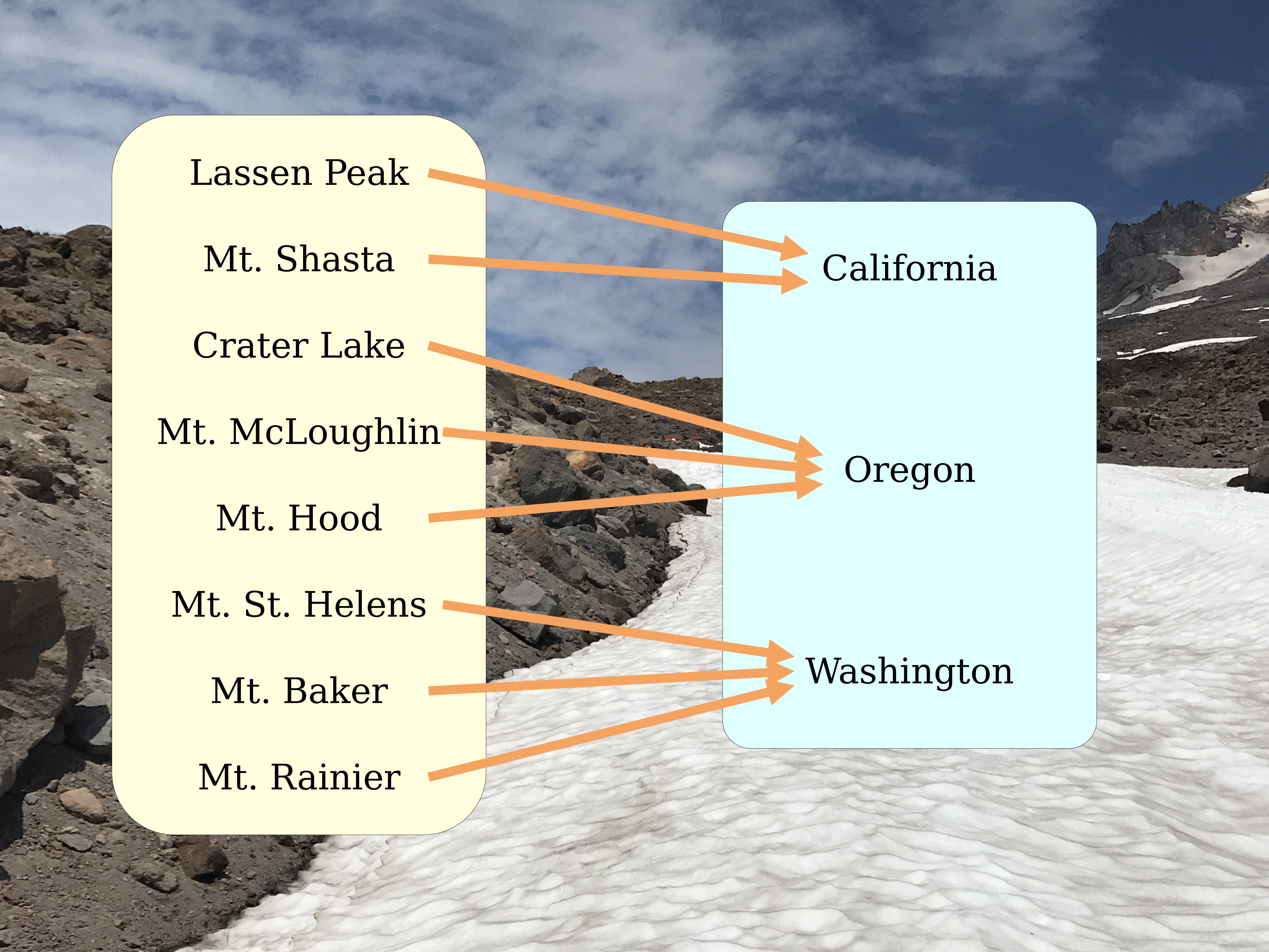
Another Class of Functions


Lassen Peak
Mt. Shasta
Crater Lake
Mt. McLoughlin
Mt. Hood
Mt. St. Helens
Mt. Baker
Mt. Rainier
California
Oregon
Washington

Surjective Functions

- A function $f : A \to B$ is called ***surjective*** (or ***onto***) if this first-order logic statement is true about f:

$$\forall b \in B.\ \exists a \in A.\ f(a) = b$$

(*"For every output, there's an input that produces it."*)

- A function with this property is called a ***surjection***.
- How does this compare to our first rule of functions?

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Theorem: Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = 2x$. Then $f(x)$ is surjective.

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What does it mean for f to be surjective?

$$\forall y \in \mathbb{R}.\ \exists x \in \mathbb{R}.\ f(x) = y$$

Therefore, we'll choose an arbitrary $y \in \mathbb{R}$, then prove that there is some $x \in \mathbb{R}$ where $f(x) = y$.

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Let $x = y / 2$.

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What does it mean for g to be surjective?

$$\forall n \in \mathbb{N}.\ \exists m \in \mathbb{N}.\ g(m) = n$$

What is the negation of the above statement?

$$\neg\forall n \in \mathbb{N}.\ \exists m \in \mathbb{N}.\ g(m) = n$$
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Therefore, we need to find a natural number n where, regardless of which m we pick, we have $g(m) \neq n$.

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Our overall goal is to prove

$$\exists n \in \mathbb{N}.\ \forall m \in \mathbb{N}.\ g(m) \neq n.$$

We just made our choice of n. Therefore, we need to prove

$$\forall m \in \mathbb{N}.\ g(m) \neq n.$$

We'll therefore pick an arbitrary $m \in \mathbb{N}$, then prove that $g(m) \neq n$.

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Notice that $g(m) = 2m$ is even, while 137 is odd.

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Recap from Today

- A ***function*** takes in an element of a ***domain*** and maps it to an element of a ***codomain***. Functions must be deterministic.
- Definitions are often given in first-order logic, and the structure of a first-order logic statement dictates the structure of a proof.
- ***Involutions***, ***injections***, and ***surjections*** are specific classes of functions that have nice properties.

Next Time

- ***First-Order Assumptions***
  - The difference between assuming something is true and proving something is true.
- ***Connecting Function Types***
  - Involutions, injections, and surjections are related to one another. How?
- ***Function Composition***
  - Sequencing functions together.