Functions

Part Two

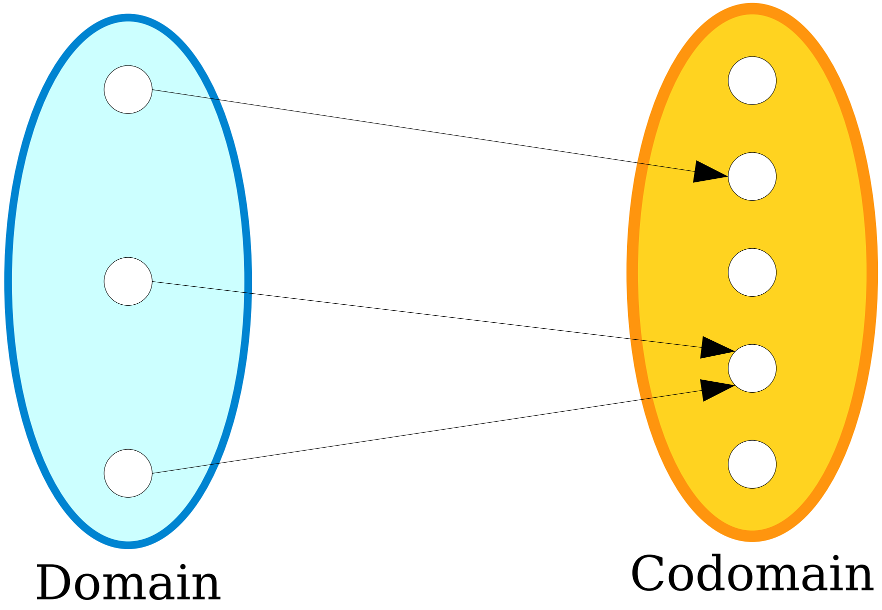
Outline for Today

- ***Recap from Last Time***
  - Where are we, again?
- ***A Proof About Birds***
  - Trust me, it’s relevant.
- ***Assuming vs Proving***
  - Two different roles to watch for.
- ***Connecting Function Types***
  - Relating the topics from last time.
- ***Function Composition***
  - Sequencing functions together.

Recap from Last Time

Domains and Codomains

- Every function *f* has two sets associated with it: its ***domain*** and its ***codomain***.
- A function *f* can only be applied to elements of its domain. For any *x* in the domain, *f*(*x*) belongs to the codomain.
- We write $\boldsymbol{f : A \to B}$ to indicate that *f* is a function whose domain is *A* and whose codomain is *B*.



The function must be defined for each element of its domain.

The output of the function must always be in the codomain, but not all elements of the codomain need to be producable.

Involutions

- A function $f : A \to A$ from a set back to itself is called an ***involution*** if the following first-order logic statement is true about f:

$$\boldsymbol{\forall x \in A.\ f(f(x)) = x.}$$

("Applying f twice is equivalent to not applying f at all.")

- For example, $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = -x$ is an involution.

Injective Functions

- A function $f : A \to B$ is called ***injective*** (or ***one-to-one***) if different inputs always map to different outputs.
    - A function with this property is called an ***injection***.
- Formally, $f : A \to B$ is an injection if this FOL statement is true:

$$\forall a_1 \in A.\ \forall a_2 \in A.\ (a_1 \neq a_2 \to f(a_1) \neq f(a_2))$$

(*"If the inputs are different, the outputs are different"*)

- Equivalently:

$$\forall a_1 \in A.\ \forall a_2 \in A.\ (f(a_1) = f(a_2) \to a_1 = a_2)$$

(*"If the outputs are the same, the inputs are the same"*)

Surjective Functions

- A function $f : A \to B$ is called ***surjective*** (or ***onto***) if each element of the codomain is "covered" by at least one element of the domain.
  - A function with this property is called a ***surjection***.
- Formally, $f : A \to B$ is a surjection if this FOL statement is true:

$$\forall b \in B.\ \exists a \in A.\ f(a) = b$$

(*"For every possible output, there's at least one possible input that produces it"*)

	To ***prove*** that this is true…	
$\forall x.\ A$	Have the reader pick an arbitrary x. We then prove A is true for that choice of x.	
$\exists x.\ A$	Find an x where A is true. Then prove that A is true for that specific choice of x.	
$A \to B$	Assume A is true, then prove B is true.	
$A \land B$	Prove A. Then prove B.	
$A \lor B$	Either prove $\neg A \to B$ or prove $\neg B \to A$. *(Why does this work?)*	
$A \leftrightarrow B$	Prove $A \to B$ and $B \to A$.	
$\neg A$	Simplify the negation, then consult this table on the result.	

New Stuff!

A Proof About Birds

Theorem: If all birds can fly,
then all herons can fly.

Theorem: If all birds can fly, then all herons can fly.

Given the predicates

$Bird(b)$, which says b is a bird;
$Heron(h)$, which says h is a heron; and
$CanFly(x)$, which says x can fly,

translate the theorem into first-order logic.

$$\underbrace{(\forall b.\ (Bird(b) \to CanFly(b)))}_{\text{All birds can fly}} \to \underbrace{(\forall h.\ (Heron(h) \to CanFly(h)))}_{\text{All herons can fly}}$$

	To ***prove*** that this is true…	
$\forall x.\ A$	Have the reader pick an arbitrary x. We then prove A is true for that choice of x.	
$\exists x.\ A$	Find an x where A is true. Then prove that A is true for that specific choice of x.	
$A \to B$	Assume A is true, then prove B is true.	
$A \land B$	Prove A. Then prove B.	
	Either prove $\neg A \to B$ or	

$$\underbrace{(\forall b.\ (Bird(b) \to CanFly(b)))}_{\text{All birds can fly}} \to \underbrace{(\forall h.\ (Heron(h) \to CanFly(h)))}_{\text{All herons can fly}}$$

Theorem: If all birds can fly, then all herons can fly.

Proof: Assume that all birds can fly. We will show that all herons can fly.

Which makes more sense as the next step in this proof?

1. Consider an arbitrary bird b.
2. Consider an arbitrary heron h.

$$\underbrace{(\forall b.\ (Bird(b) \to CanFly(b)))}_{\text{All birds can fly}} \to \underbrace{(\forall h.\ (Heron(h) \to CanFly(h)))}_{\text{All herons can fly}}$$

Theorem: If all birds can fly, then all herons can fly.

Proof: Assume that all birds can fly. We will show that all herons can fly.

Which makes more sense as the next step in this proof?

1. Consider an arbitrary bird b.
2. Consider an arbitrary heron h.

$$\underbrace{(\forall b.\ (Bird(b) \to CanFly(b)))}_{\text{All birds can fly}} \to \underbrace{(\forall h.\ (Heron(h) \to CanFly(h)))}_{\text{All herons can fly}}$$

Theorem: If all birds can fly, then all herons can fly.

Proof: Assume that all birds can fly. We will show that all herons can fly.

Consider an arbitrary bird b. Since b is a bird, b can fly. *[and now we're stuck! we are interested in herons, but b might not be one. It could be a hummingbird, for example!]*

$$\boxed{\underbrace{(\forall b.\ (Bird(b) \to CanFly(b)))}_{\text{All birds can fly}} \to \underbrace{(\forall h.\ (Heron(h) \to CanFly(h)))}_{\text{All herons can fly}}}$$

Theorem: If all birds can fly, then all herons can fly.

Proof: Assume that all birds can fly. We will show that all herons can fly.

Which makes more sense as the next step in this proof?

1. Consider an arbitrary bird b.
2. Consider an arbitrary heron h.

$$\underbrace{(\forall b. (Bird(b) \to CanFly(b)))}_{\text{All birds can fly}} \to \underbrace{(\forall h. (Heron(h) \to CanFly(h)))}_{\text{All herons can fly}}$$

Theorem: If all birds can fly, then all herons can fly.

Proof: Assume that all birds can fly. We will show that all herons can fly.

Consider an arbitrary heron h. We will show that h can fly. To do so, note that since h is a heron we know h is a bird. Therefore, by our earlier assumption, h can fly. ■

$$\underbrace{(\forall b. (Bird(b) \to CanFly(b)))}_{\text{All birds can fly}} \to \underbrace{(\forall h. (Heron(h) \to CanFly(h)))}_{\text{All herons can fly}}$$

Theorem: If all birds can fly, then all herons can fly.

Proof: Assume that all birds can fly. We will show that all herons can fly.

Consider an arbitrary heron h. We will show that h can fly. To do so, note that since h is a heron we know h is a bird. Therefore, by our earlier assumption, h can fly. ■

$$\underbrace{(\forall b.\ (Bird(b) \to CanFly(b)))}_{\text{We never introduce a variable } b.} \to \underbrace{(\forall h.\ (Heron(h) \to CanFly(h)))}_{\text{We introduce a variable } h \text{ almost immediately.}}$$

Proving vs. Assuming

- In the context of a proof, you will need to assume some statements and prove others.
  - Here, we ***assumed*** all birds can fly.
  - Here, we ***proved*** all herons can fly.
- Statements behave differently based on whether you're assuming or proving them.

$$\underbrace{(\forall b.\ (\mathit{Bird}(b) \to \mathit{CanFly}(b)))}_{\text{We never introduce a variable } b.} \to \underbrace{(\forall h.\ (\mathit{Heron}(h) \to \mathit{CanFly}(h)))}_{\text{We introduce a variable } h \text{ almost immediately.}}$$

Proving vs. Assuming

- To ***prove*** the universally-quantified statement

  $$\forall x.\ P(x)$$

  we introduce a new variable x representing some arbitrarily-chosen value.
- Then, we prove that $P(x)$ is true for that variable x.
- That's why we introduced a variable h in this proof representing a heron.

$$\underbrace{(\forall b.\ (Bird(b) \to CanFly(b)))}_{\text{We never introduce a variable } b.} \to \underbrace{(\forall h.\ (Heron(h) \to CanFly(h)))}_{\text{We introduce a variable } h \text{ almost immediately.}}$$

Proving vs. Assuming

- If we ***assume*** the statement

$$\forall x.\ P(x)$$

  we ***do not*** introduce a variable x.
- Rather, if we find a relevant value z somewhere else in the proof, we can conclude that $P(z)$ is true.
- That's why we didn't introduce a variable b in our proof, and why we concluded that h, our heron, can fly.

$$\underbrace{(\forall b.\ (Bird(b) \to CanFly(b)))}_{\text{We never introduce a variable } b\text{.}} \to \underbrace{(\forall h.\ (Heron(h) \to CanFly(h)))}_{\text{We introduce a variable } h \text{ almost immediately.}}$$

	To ***prove*** that this is true…	If you ***assume*** this is true…
$\forall x.\ A$	Have the reader pick an arbitrary x. We then prove A is true for that choice of x.	Initially, ***do nothing***. Once you find a z through other means, you can state it has property A.
$\exists x.\ A$	Find an x where A is true. Then prove that A is true for that specific choice of x.	Introduce a variable x into your proof that has property A.
$A \to B$	Assume A is true, then prove B is true.	Initially, ***do nothing***. Once you know A is true, you can conclude B is also true.
$A \land B$	Prove A. Then prove B.	Assume A. Then assume B.
$A \lor B$	Either prove $\neg A \to B$ or prove $\neg B \to A$. *(Why does this work?)*	Consider two cases. Case 1: A is true. Case 2: B is true.
$A \leftrightarrow B$	Prove $A \to B$ and $B \to A$.	Assume $A \to B$ and $B \to A$.
$\neg A$	Simplify the negation, then consult this table on the result.	Simplify the negation, then consult this table on the result.

Connecting Function Types

Types of Functions

- Last time, we saw three special types of functions:
  - ***involutions***, functions that undo themselves;
  - ***injections***, functions where different inputs go to different outputs; and
  - ***surjections***, functions that cover their whole codomain.
- ***Question:*** How do these three classes of functions relate to one another?

Theorem: For any function $f : A \to A$, if f is an involution, then f is surjective.

$$\underbrace{(\forall x \in A.\ f(f(x)) = x)}_{\substack{f \text{ is an} \\ \text{involution.}}} \quad \to \quad \underbrace{(\forall b \in A.\ \exists a \in A.\ f(a) = b)}_{\substack{f \text{ is} \\ \text{surjective.}}}$$

Theorem: For any function $f : A \to A$,
if f is an involution, then f is surjective.

$$\underbrace{(\forall x \in A.\ f(f(x)) = x)}_{\text{Assume this.}} \to \underbrace{(\forall b \in A.\ \exists a \in A.\ f(a) = b)}_{\text{Prove this.}}$$

$$\boxed{\underbrace{(\forall b.\ (Bird(b) \to CanFly(b)))}_{\text{Assume this.}} \to \underbrace{(\forall h.\ (Heron(h) \to CanFly(h)))}_{\text{Prove this.}}}$$

Theorem: For any function $f : A \to A$, if f is an involution, then f is surjective.

$$\underbrace{(\forall x \in A.\ f(f(x)) = x)}_{\uparrow} \to \underbrace{(\forall b \in A.\ \exists a \in A.\ f(a) = b)}_{\uparrow}$$

Assume this.

Prove this.

Since we're *assuming* this, we aren't going to pick a specific choice of x right now. Instead, we're going to keep an eye out for something to apply this fact to.

Proof Outline

1. Assume f is an involution.

Theorem: For any function $f : A \to A$, if f is an involution, then f is surjective.

$$\underbrace{(\forall x \in A.\ f(f(x)) = x)}_{\text{Ass}} \to \underbrace{(\forall b \in A.\ \exists a \in A.\ f(a) = b)}_{\text{Prove this.}}$$

We've said that we need to prove this statement. How do we do that?

Proof Outline

1. Assume f is an involution.

Theorem: For any function $f : A \to A$, if f is an involution, then f is surjective.

$$\underbrace{(\forall x \in A.\ f(f(x)) = x)}_{\text{Ass}} \to \underbrace{(\forall b \in A.\ \exists a \in A.\ f(a) = b)}_{\text{Prove this.}}$$

There's a universal quantifier up front. Since we're *proving* this, we'll pick an arbitrary $b \in A$.

Proof Outline

1. Assume f is an involution.
2. Pick an arbitrary $b \in A$.

Theorem: For any function $f : A \to A$, if f is an involution, then f is surjective.

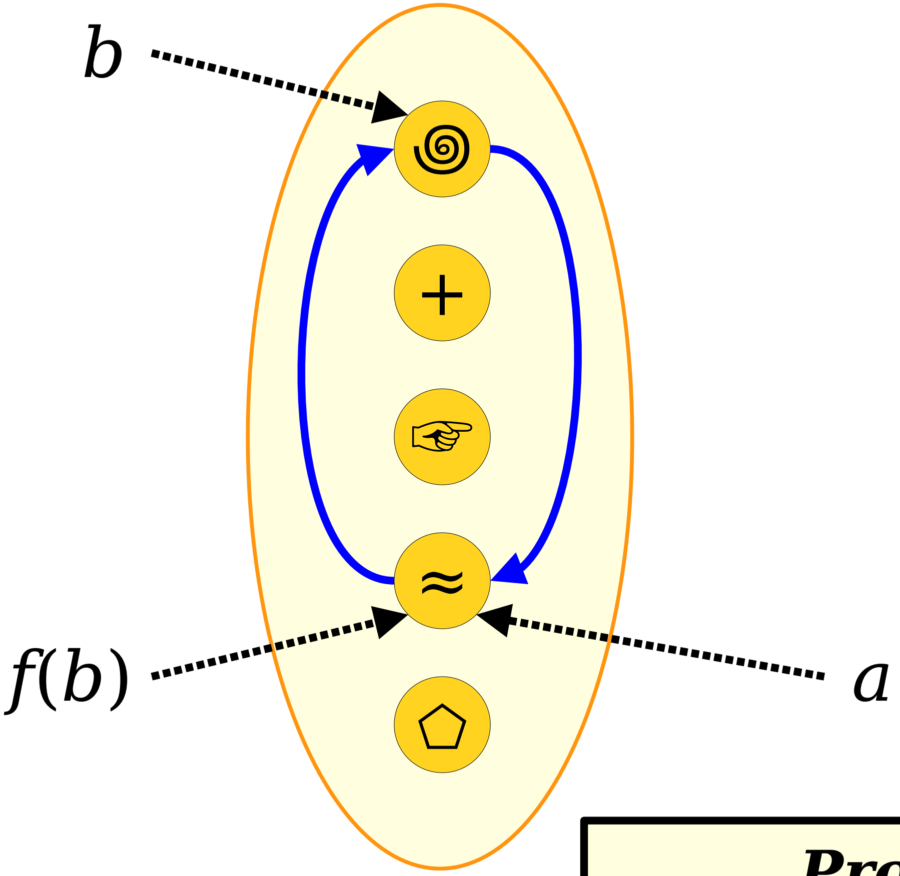
$$\underbrace{(\forall x \in A.\ f(f(x)) = x)} \quad \to \quad \underbrace{(\forall b \in A.\ \exists a \in A.\ f(a) = b)}_{\text{Prove this.}}$$

Now, we hit an existential quantifier. Since we're *proving* this, we need to find a choice of $a \in A$ where this is true.

Proof Outline

1. Assume f is an involution.
2. Pick an arbitrary $b \in A$.
3. Give a choice of $a \in A$ where $f(a) = b$.

Theorem: For any function $f : A \to A$, if f is an involution, then f is surjective.



Proof Outline

1. Assume f is an involution.
2. Pick an arbitrary $b \in A$.
3. Give a choice of $a \in A$ where $f(a) = b$.

Theorem: For any function $f : A \to A$,
if f is an involution, then f is surjective.

Theorem: For any function $f : A \to A$, if f is an involution, then f is surjective.

Proof: Pick any involution $f : A \to A$. We will prove that f is surjective. To do so, pick an arbitrary $b \in A$. We need to show that there is an $a \in A$ where $f(a) = b$.

Specifically, pick $a = f(b)$. This means that $f(a) = f(f(b))$, and since f is an involution we know that $f(f(b)) = b$. Putting this together, we see that $f(a) = b$, which is what we needed to show. ■

Proof Outline

1. Assume f is an involution.
2. Pick an arbitrary $b \in A$.
3. Give a choice of $a \in A$ where $f(a) = b$.

Theorem: For any function $f : A \to A$, if f is an involution, then f is injective.

$$\underbrace{(\forall x \in A.\, f(f(x)) = x)}_{\substack{f \text{ is an} \\ \text{involution.}}} \to \underbrace{(\forall a_1 \in A.\, \forall a_2 \in A.\, (a_1 \neq a_2 \to f(a_1) \neq f(a_2))}_{\substack{f \text{ is} \\ \text{injective.}}}$$

Theorem: For any function $f : A \to A$,
if f is an involution, then f is injective.

$$\underbrace{(\forall x \in A.\ f(f(x)) = x)}_{\text{Assume this.}} \to \underbrace{(\forall a_1 \in A.\ \forall a_2 \in A.\ (a_1 \neq a_2 \to f(a_1) \neq f(a_2)))}_{\text{Prove this.}}$$

$$\boxed{\underbrace{(\forall b.\ (\mathit{Bird}(b) \to \mathit{CanFly}(b)))}_{\text{Assume this.}} \to \underbrace{(\forall h.\ (\mathit{Heron}(h) \to \mathit{CanFly}(h)))}_{\text{Prove this.}}}$$

Theorem: For any function $f : A \to A$, if f is an involution, then f is injective.

$$\underbrace{(\forall x \in A. f(f(x)) = x)}_{\text{Assume this.}} \to \underbrace{(\forall a_1 \in A. \forall a_2 \in A. (a_1 \neq a_2 \to f(a_1) \neq f(a_2)))}_{\text{Prove this.}}$$

Since we're *assuming* this, we aren't going to pick a specific choice of x right now. Instead, we're going to keep an eye out for something to apply this fact to.

Proof Outline

1. Assume f is an involution.

Theorem: For any function $f : A \to A$, if f is an involution, then f is injective.

$$\underbrace{(\forall x \in A.\ f(f(x)) = x)} \to \underbrace{(\forall a_1 \in A.\ \forall a_2 \in A.\ (a_1 \neq a_2 \to f(a_1) \neq f(a_2)))}$$

We need to *prove* this part.
What does that mean?

Prove
this.

Proof Outline

1. Assume f is an involution.

Theorem: For any function $f : A \to A$,
if f is an involution, then f is injective.

$$\underbrace{(\forall x \in A.\ f(f(x)) = x)} \to \underbrace{(\forall a_1 \in A.\ \forall a_2 \in A.\ (a_1 \neq a_2 \to f(a_1) \neq f(a_2)))}_{\text{Prove this.}}$$

Since we're *proving* something universally-quantified, we'll pick some values arbitrarily.

Proof Outline

1. Assume f is an involution.
2. Pick arbitrary a_1, $a_2 \in A$.

Theorem: For any function $f : A \to A$, if f is an involution, then f is injective.

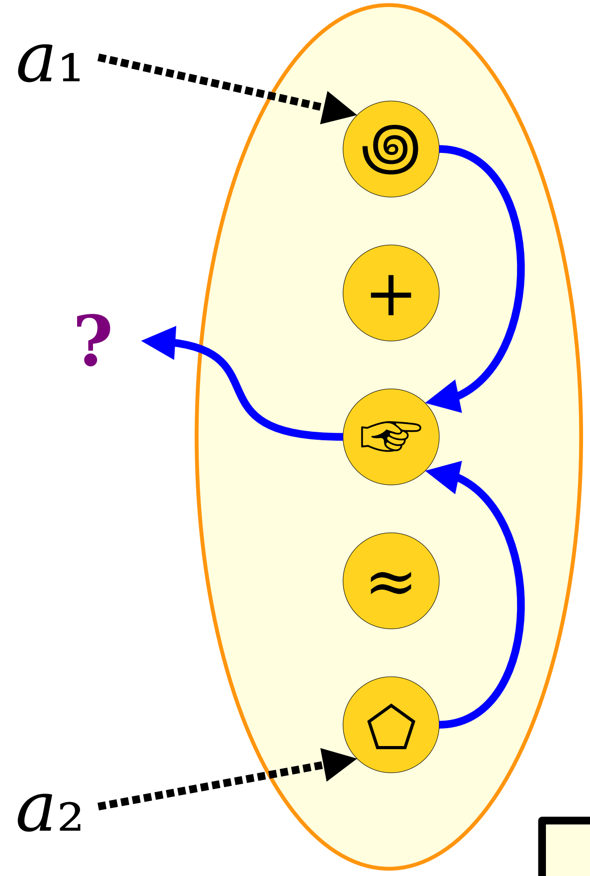
$$\underbrace{(\forall x \in A.\ f(f(x)) = x)}_{} \to \underbrace{(\forall a_1 \in A.\ \forall a_2 \in A.\ (a_1 \neq a_2 \to f(a_1) \neq f(a_2)))}_{\text{Prove this.}}$$

We now need to *prove* this implication. But we know how to do that! We assume the antecedent and prove the consequent.

Proof Outline

1. Assume f is an involution.
2. Pick arbitrary $a_1, a_2 \in A$ such that $a_1 \neq a_2$.
3. Prove $f(a_1) \neq f(a_2)$.

Theorem: For any function $f : A \to A$, if f is an involution, then f is injective.



Proof Outline

1. Assume f is an involution.
2. Pick arbitrary a_1, $a_2 \in A$ such that $a_1 \neq a_2$.
3. Prove $f(a_1) \neq f(a_2)$.

Theorem: For any function $f : A \to A$, if f is an involution, then f is injective.

Theorem: For any function $f : A \to A$, if f is an involution, then f is injective.

Proof: Consider any function $f : A \to A$ that's an involution. We will prove that f is injective. To do so, choose any $a_1, a_2 \in A$ where $a_1 \neq a_2$. We need to show that $f(a_1) \neq f(a_2)$.

We'll proceed by contradiction. Suppose that $f(a_1) = f(a_2)$. This means $f(f(a_1)) = f(f(a_2))$, which in turn tells us $a_1 = a_2$ because f is an involution. But that's impossible, since $a_1 \neq a_2$.

We've reached a contradiction, so our assumption was wrong. Therefore, we see that $f(a_1) \neq f(a_2)$, as required. ■

Proof Outline

1. Assume f is an involution.
2. Pick arbitrary $a_1, a_2 \in A$ such that $a_1 \neq a_2$.
3. Prove $f(a_1) \neq f(a_2)$.

Time-Out for Announcements!

STANFORD
WICS

coffee chat

with Prof. Chris Piech

CHAT WITH A STANFORD CS PROFESSOR
MEET OTHER WICS MEMBERS
IN-PERSON AT STANFORD
FREE STARBUCKS ON US!

TUESDAY, OCT 12TH, 2021
4:00 - 4:45PM PST
RSVP: BIT.LY/CHRISCOFFEECHAT

Gradescope Tagging

- When you upload a PDF to Gradescope, please make sure to tag the pages that have your problem answers on them.
- The ***altruistic*** reason: if you don’t do this, the TAs have to do it for you, and across 170 submissions that adds up to hours of extra work.
- The ***selfish*** reason: if you don’t tag the page containing a problem, Gradescope marks it as though you didn’t submit it, and the TAs might give you no points because they thought you didn’t submit anything.
- You can tag pages after you submit, so if you submit and then realize you forgot to tag things you can always go back and fix it.

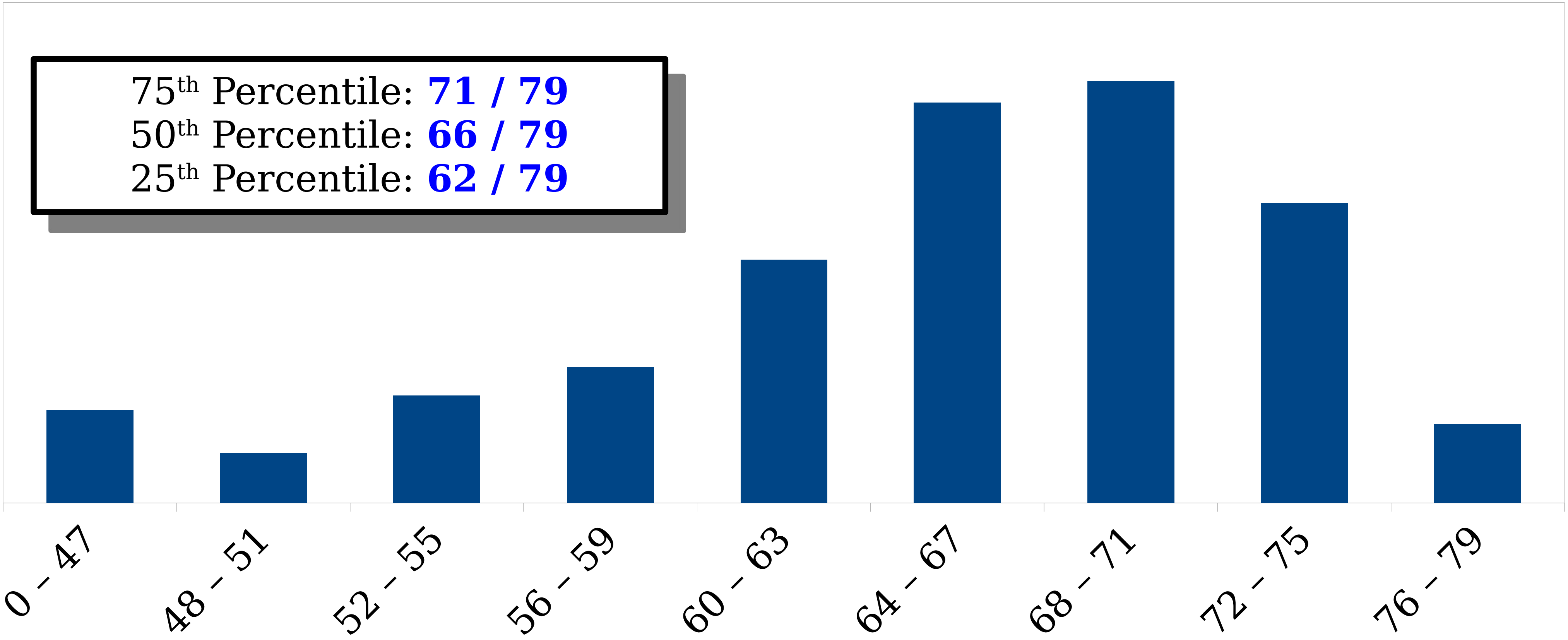
Partner Searches

- Looking for a problem set partner? Feel free to post on EdStem to let people know you’re looking.
- Use the “Partners” tag so people can filter down to posts on that topic.

Problem Set One Graded

- Your wonderful TAs have finished grading Problem Set One.
- Grades and feedback are up on the Gradescope.
- Solutions are available online on the course website (visit the page for PS1 to get the link).
- Regrades will open up this Friday at noon and close next Wednesday at noon.

Problem Set One Graded

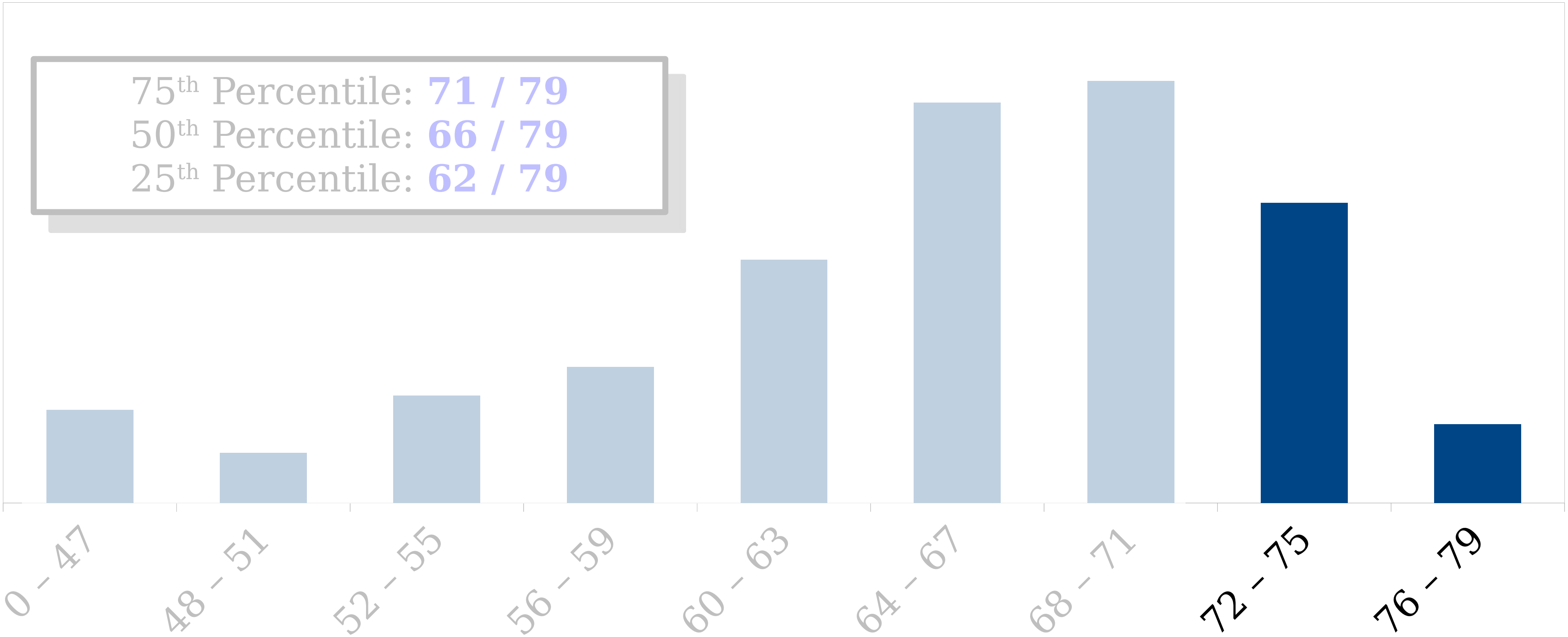


75th Percentile: **71 / 79**
50th Percentile: **66 / 79**
25th Percentile: **62 / 79**

Pro tips when reading a grading distribution:

1. Standard deviations are ***malicious lies***. Ignore them.
2. The average score is a ***malicious lie***. Ignore it.
3. Raw scores are ***malicious lies***. Ignore them.

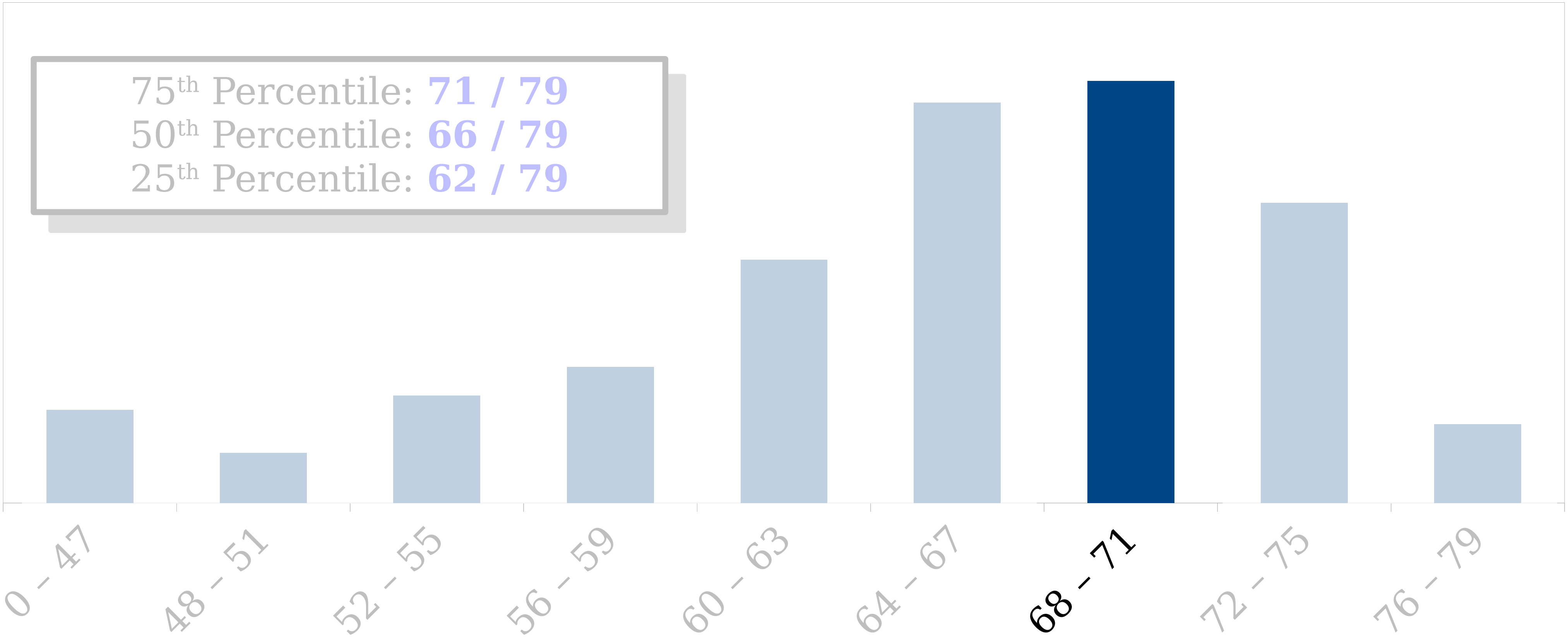
Problem Set One Graded



75th Percentile: **71 / 79**
50th Percentile: **66 / 79**
25th Percentile: **62 / 79**

"Great job! Look over your feedback for some tips on how to tweak things for next time."

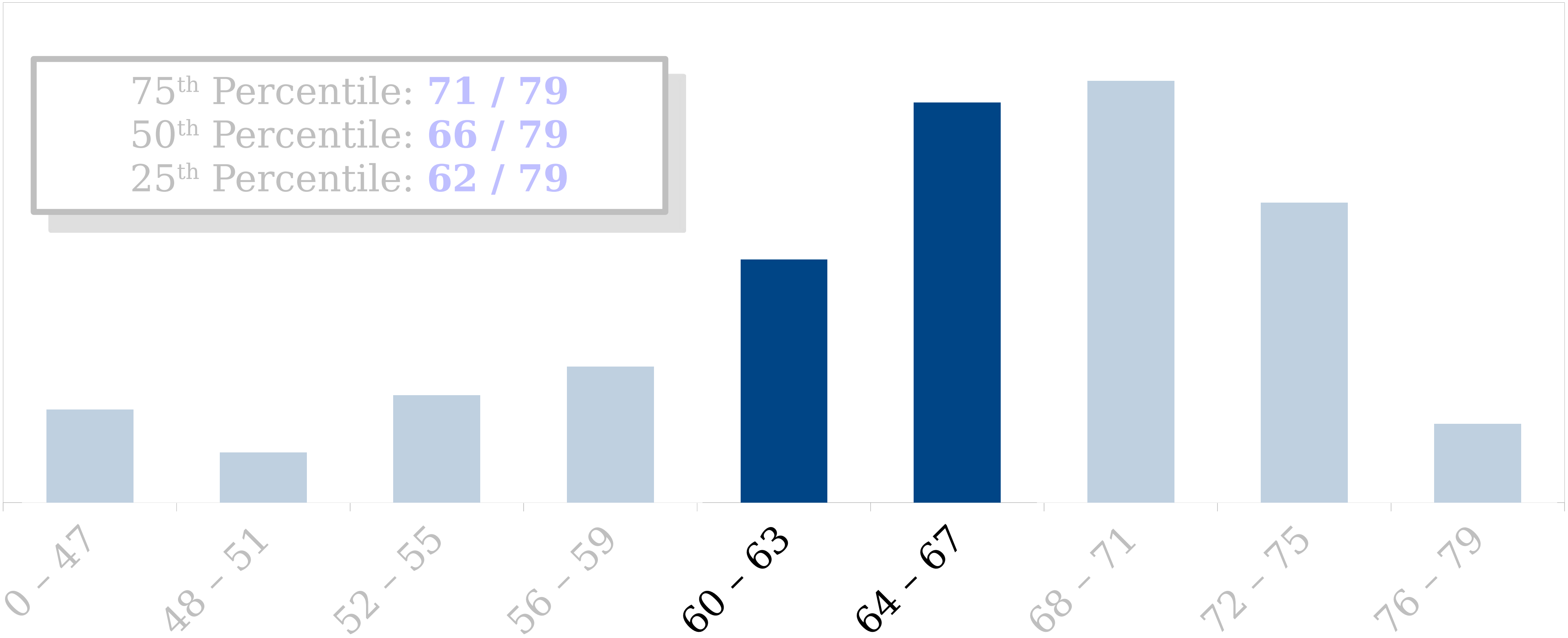
Problem Set One Graded



75th Percentile: **71 / 79**
50th Percentile: **66 / 79**
25th Percentile: **62 / 79**

"You're almost there! Review the feedback on your submission and see what to focus on for next time."

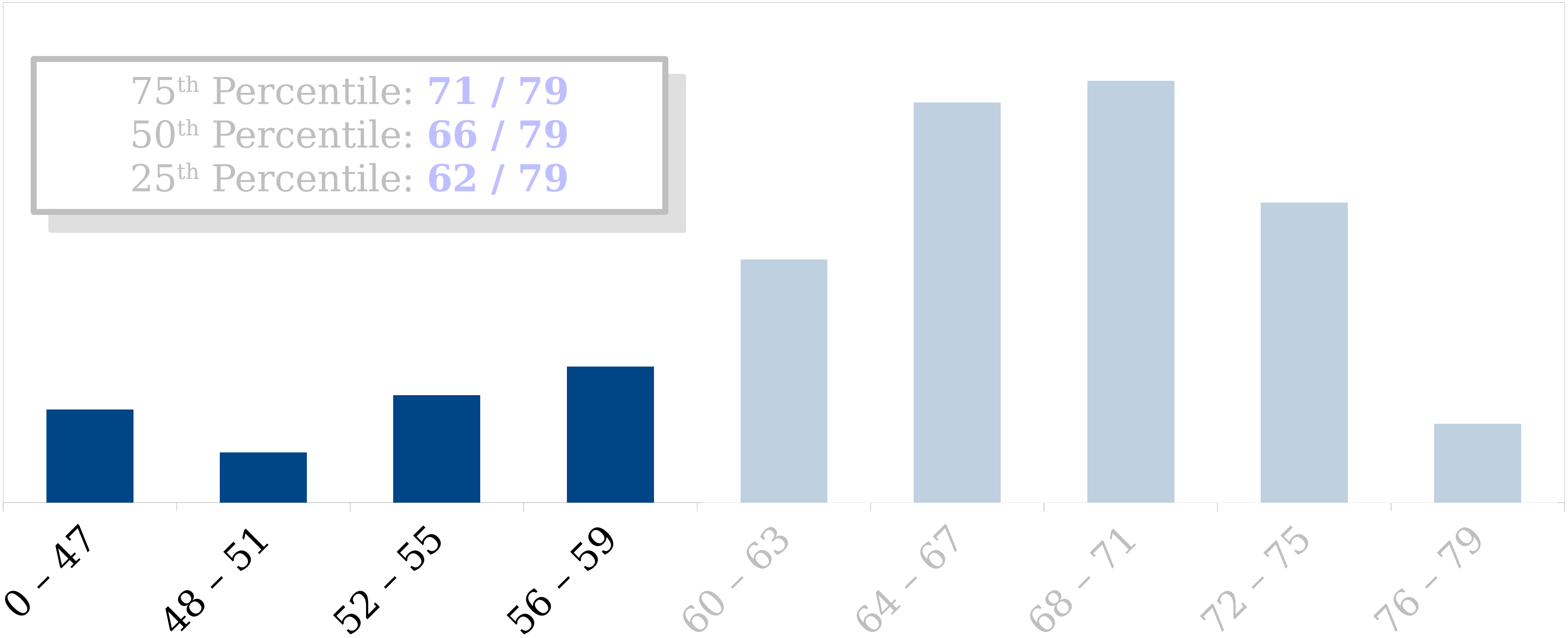
Problem Set One Graded



75th Percentile: **71 / 79**
50th Percentile: **66 / 79**
25th Percentile: **62 / 79**

"You're on the right track, but there are some areas where you need to improve. Review your feedback and ask us questions when you have them."

Problem Set One Graded



75th Percentile: **71 / 79**
50th Percentile: **66 / 79**
25th Percentile: **62 / 79**

"Looks like something hasn't quite clicked yet.
Get in touch with us and stop by office hours
to get some extra feedback and advice.
Don't get discouraged – you can do this!"

What Not to Think

- “Well, I guess I’m just not good at math.”
  - For most of you, this is your first time doing any rigorous proof-based math.
  - Don’t judge your future performance based on a single data point.
  - Life advice: think about download times.
  - Life advice: have a growth mindset!
- “Hey, I did above the median. That’s good enough.”
  - Unless you literally earned every single point on this problem set – which no one did – there’s some area where the course staff thinks you can improve. ***Take the time to see what that is***.

Your Questions

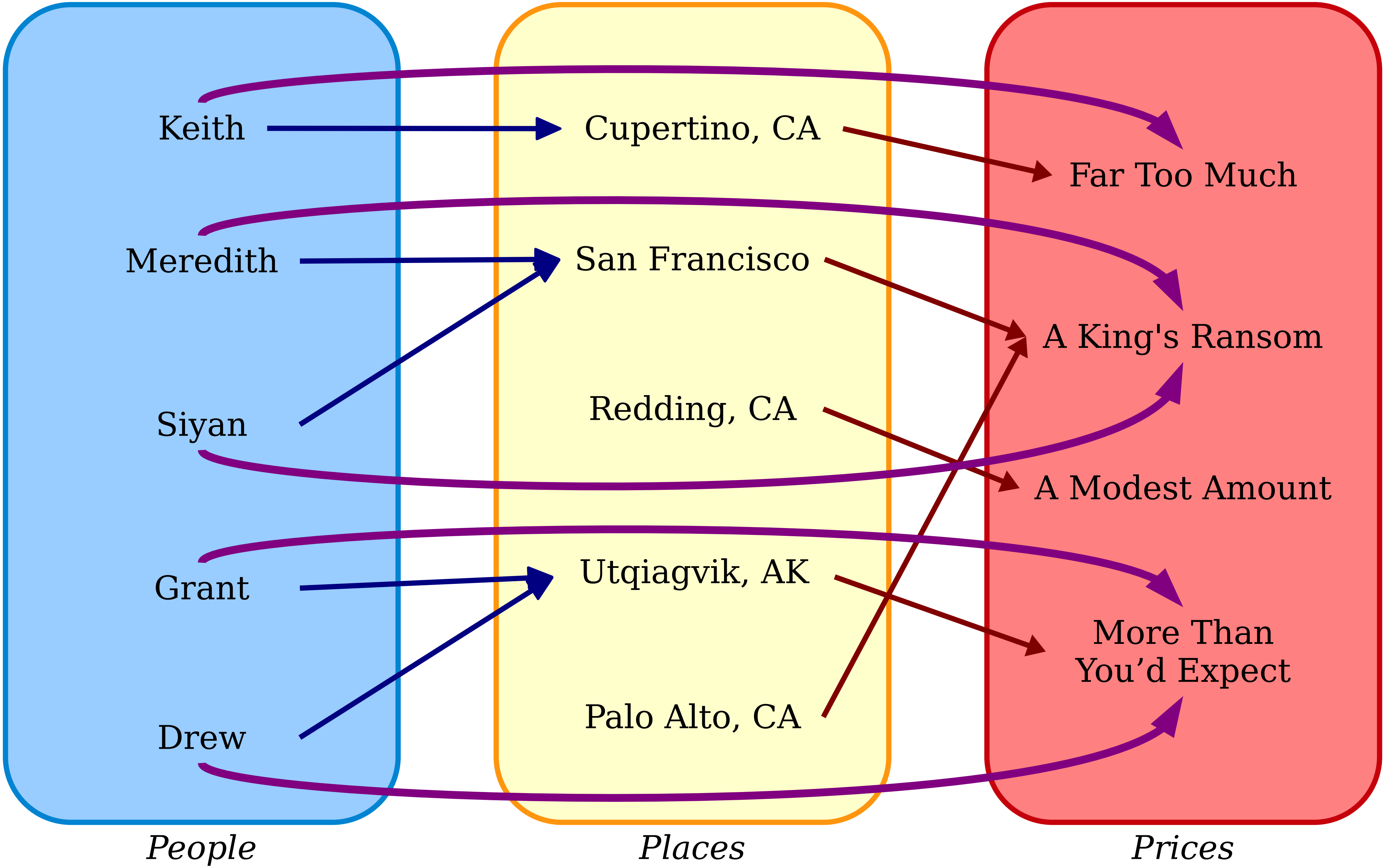
"Who is the gnarliest person you ever met during your time at Stanford and how do you handle impostor syndrome/feeling intimidated when you meet gnarly people?"

1. Don't mistake talent and experience.
2. Don't confuse unions and intersections.
3. Remember you're always growing.

Back to CS103!

Function Composition

f* : *People* → *Places　　***g* : *Places* → *Prices***

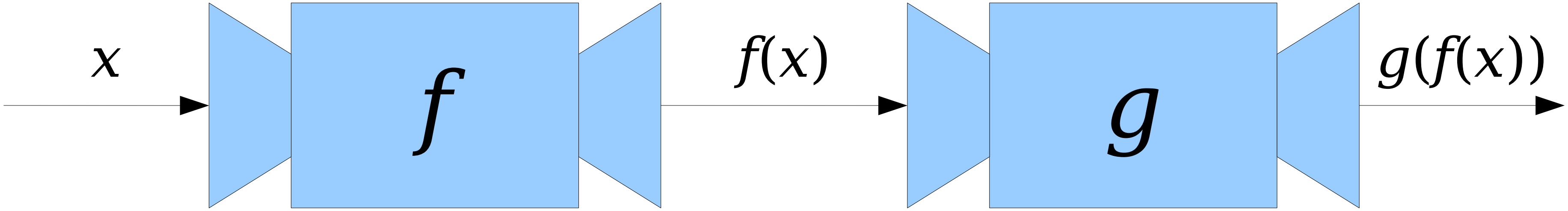


h* : *People* → *Prices

$$h(x) = g(f(x))$$

Function Composition

- Suppose that we have two functions $f : A \to B$ and $g : B \to C$.
- Notice that the codomain of f is the domain of g. This means that we can use outputs from f as inputs to g.



Function Composition

- Suppose that we have two functions $f : A \to B$ and $g : B \to C$.
- The ***composition of f and g***, denoted $\boldsymbol{g \circ f}$, is a function where
  - $g \circ f : A \to C$, and
  - $(g \circ f)(x) = g(f(x))$.

The name of the function is $g \circ f$. When we apply it to an input x, we write $(g \circ f)(x)$. I don't know why, but that's what we do.

- A few things to notice:
  - The domain of $g \circ f$ is the domain of f. Its codomain is the codomain of g.
  - Even though the composition is written $g \circ f$, when evaluating $(g \circ f)(x)$, the function f is evaluated first.

Properties of Composition

Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is an injection.

Organizing Our Thoughts

Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is an injection.

What We're Assuming

$f : A \to B$ is an injection.

$$\forall x \in A.\ \forall y \in A.\ (x \neq y \to$$
$$f(x) \neq f(y)$$
$$)$$

$g : B \to C$ is an injection.

$$\forall x \in B.\ \forall y \in B.\ (x \neq y \to$$
$$g(x) \neq g(y)$$
$$)$$

We're *assuming* these universally-quantified statements, so we won't introduce any variables for what's here.

What We Need to Prove

$g \circ f$ is an injection.

$$\forall a_1 \in A.\ \forall a_2 \in A.\ (a_1 \neq a_2 \to$$
$$(g \circ f)(a_1) \neq (g \circ f)(a_2)$$
$$)$$

We need to ***prove*** this universally-quantified statement. So let's introduce arbitrarily-chosen values.

Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is an injection.

What We're Assuming

$f : A \to B$ is an injection.

$$\begin{aligned}&\forall x \in A.\ \forall y \in A.\ (x \neq y \to\\&\qquad f(x) \neq f(y)\\&)\end{aligned}$$

$g : B \to C$ is an injection.

$$\begin{aligned}&\forall x \in B.\ \forall y \in B.\ (x \neq y \to\\&\qquad g(x) \neq g(y)\\&)\end{aligned}$$

$a_1 \in A$ is arbitrarily-chosen.

$a_2 \in A$ is arbitrarily-chosen.

What We Need to Prove

$g \circ f$ is an injection.

$$\begin{aligned}&\forall a_1 \in A.\ \forall a_2 \in A.\ (a_1 \neq a_2 \to\\&\qquad (g \circ f)(a_1) \neq (g \circ f)(a_2)\\&)\end{aligned}$$

We need to *prove* this universally-quantified statement. So let's introduce arbitrarily-chosen values.

Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is an injection.

What We're Assuming

$f : A \to B$ is an injection.

$$\forall x \in A.\ \forall y \in A.\ (x \neq y \to f(x) \neq f(y))$$

$g : B \to C$ is an injection.

$$\forall x \in B.\ \forall y \in B.\ (x \neq y \to g(x) \neq g(y))$$

$a_1 \in A$ is arbitrarily-chosen.

$a_2 \in A$ is arbitrarily-chosen.

$a_1 \neq a_2$

What We Need to Prove

$g \circ f$ is an injection.

$$\forall a_1 \in A.\ \forall a_2 \in A.\ (a_1 \neq a_2 \to (g \circ f)(a_1) \neq (g \circ f)(a_2))$$

Now we're looking at an implication. Let's *assume* the antecedent and *prove* the consequent.

Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is an injection.

What We're Assuming

$f : A \to B$ is an injection.

$$\forall x \in A.\ \forall y \in A.\ (x \neq y \to f(x) \neq f(y))$$

$g : B \to C$ is an injection.

$$\forall x \in B.\ \forall y \in B.\ (x \neq y \to g(x) \neq g(y))$$

$a_1 \in A$ is arbitrarily-chosen.

$a_2 \in A$ is arbitrarily-chosen.

$a_1 \neq a_2$

What We Need to Prove

$g \circ f$ is an injection.

$$\forall a_1 \in A.\ \forall a_2 \in A.\ (a_1 \neq a_2 \to (g \circ f)(a_1) \neq (g \circ f)(a_2))$$

Let's write this out separately and simplify things a bit.

Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is an injection.

What We're Assuming

$f : A \to B$ is an injection.

$$\forall x \in A.\ \forall y \in A.\ (x \neq y \to$$
$$f(x) \neq f(y)$$
$$)$$

$g : B \to C$ is an injection.

$$\forall x \in B.\ \forall y \in B.\ (x \neq y \to$$
$$g(x) \neq g(y)$$
$$)$$

$a_1 \in A$ is arbitrarily-chosen.

$a_2 \in A$ is arbitrarily-chosen.

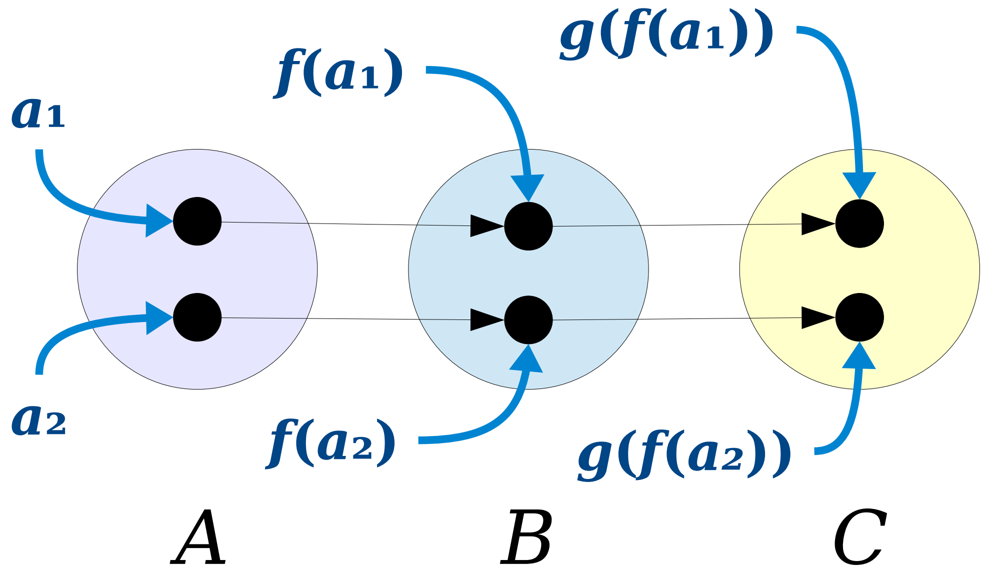
$a_1 \neq a_2$

What We Need to Prove

$g \circ f$ is an injection.

$$\forall a_1 \in A.\ \forall a_2 \in A.\ (a_1 \neq a_2 \to$$
$$(g \circ f)(a_1) \neq (g \circ f)(a_2)$$
$$)$$

$g(f(a_1)) \neq g(f(a_2))$

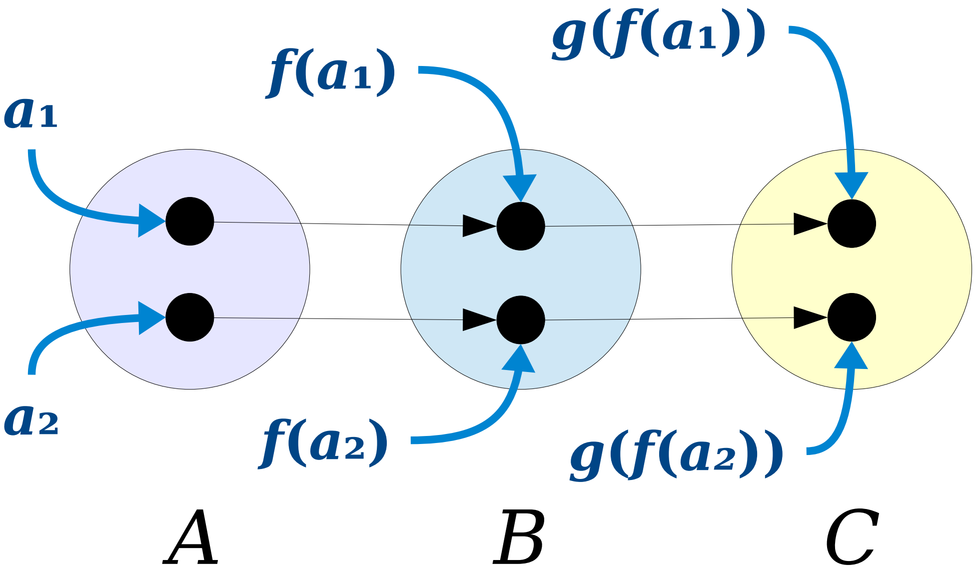


Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is also an injection.

Proof: Let $f : A \to B$ and $g : B \to C$ be arbitrary injections. We will prove that the function $g \circ f : A \to C$ is also injective. To do so, consider any $a_1, a_2 \in A$ where $a_1 \neq a_2$. We will prove that $(g \circ f)(a_1) \neq (g \circ f)(a_2)$. Equivalently, we need to show that $g(f(a_1)) \neq g(f(a_2))$.

Since f is injective and $a_1 \neq a_2$, we see that $f(a_1) \neq f(a_2)$. Then, since g is injective and $f(a_1) \neq f(a_2)$, we see that $g(f(a_1)) \neq g(f(a_2))$, as required. ■

Great exercise: Repeat this proof using the other definition of injectivity.



Theorem: If $f : A \to B$ is a surjection and $g : B \to C$ is a surjection, then the function $g \circ f : A \to C$ is a surjection.

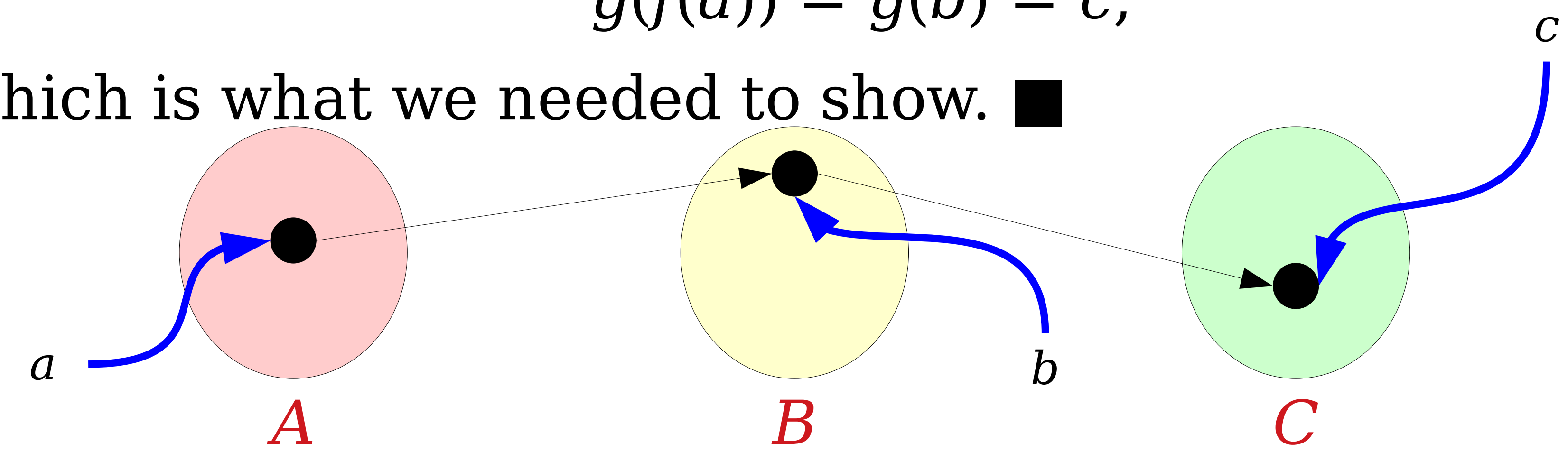
Theorem: If $f : A \to B$ is surjective and $g : B \to C$ is surjective, then $g \circ f : A \to C$ is also surjective.

Proof: Let $f : A \to B$ and $g : B \to C$ be arbitrary surjections. We will prove that the function $g \circ f : A \to C$ is also surjective. To do so, we will prove that for any $c \in C$, there is some $a \in A$ such that $(g \circ f)(a) = c$. Equivalently, we will prove that for any $c \in C$, there is some $a \in A$ such that $g(f(a)) = c$.

Consider any $c \in C$. Since $g : B \to C$ is surjective, there is some $b \in B$ such that $g(b) = c$. Similarly, since $f : A \to B$ is surjective, there is some $a \in A$ such that $f(a) = b$. Then we see that

$$g(f(a)) = g(b) = c,$$

which is what we needed to show. ■



Major Ideas From Today

- Statements behave differently based on whether you’re ***assuming*** or ***proving*** them.
- When you ***assume*** a universally-quantified statement, initially, do nothing. Instead, keep an eye out for a place to apply the statement more specifically.
- When you ***prove*** a universally-quantified statement, pick an arbitrary value and try to prove it has the needed property.
- As always: try concrete examples, draw pictures, etc. before you dive into writing a proof.

	To ***prove*** that this is true…	If you ***assume*** this is true…
$\forall x.\ A$	Have the reader pick an arbitrary x. We then prove A is true for that choice of x.	Initially, ***do nothing***. Once you find a z through other means, you can state it has property A.
$\exists x.\ A$	Find an x where A is true. Then prove that A is true for that specific choice of x.	Introduce a variable x into your proof that has property A.
$A \to B$	Assume A is true, then prove B is true.	Initially, ***do nothing***. Once you know A is true, you can conclude B is also true.
$A \land B$	Prove A. Then prove B.	Assume A. Then assume B.
$A \lor B$	Either prove $\neg A \to B$ or prove $\neg B \to A$. *(Why does this work?)*	Consider two cases. Case 1: A is true. Case 2: B is true.
$A \leftrightarrow B$	Prove $A \to B$ and $B \to A$.	Assume $A \to B$ and $B \to A$.
$\neg A$	Simplify the negation, then consult this table on the result.	Simplify the negation, then consult this table on the result.

Next Time

- ***Cardinality Revisited***
  - Formalizing our definitions.
- ***The Nature of Infinity***
  - Infinity is more interesting than it looks!
- ***Cantor's Theorem Revisited***
  - Formally proving a major result.