Cardinality

Outline for Today

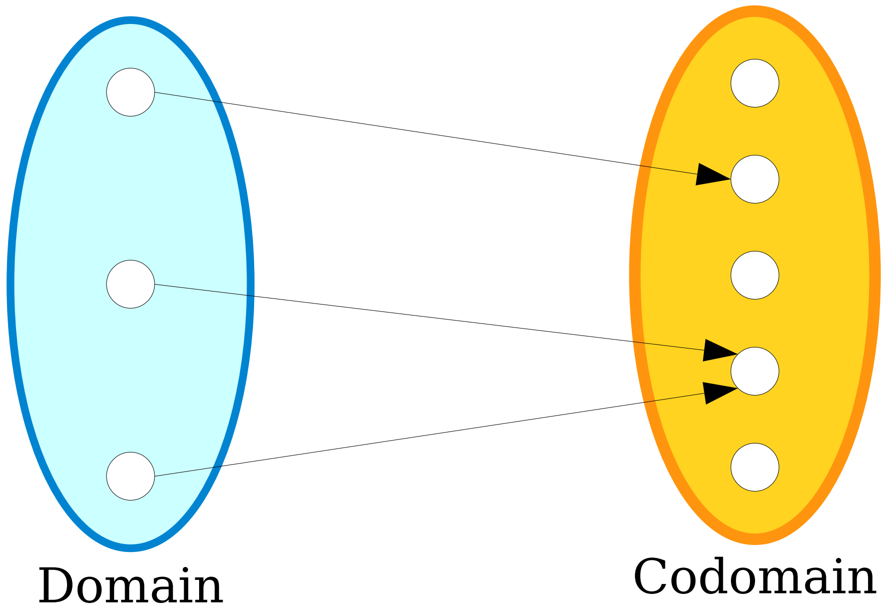
- ***Bijections***
  - A key and important class of functions.
- ***Cardinality, Formally***
  - What does it mean for two sets to have the same size?
- ***Cantor's Theorem, Formally***
  - Proving, indeed, that infinity is not infinity is not infinity.

Recap from Last Time

Domains and Codomains

- Every function *f* has two sets associated with it: its ***domain*** and its ***codomain***.
- A function *f* can only be applied to elements of its domain. For any *x* in the domain, *f*(*x*) belongs to the codomain.
- We write ***f* : *A* → *B*** to indicate that *f* is a function whose domain is *A* and whose codomain is *B*.

The function must be defined for each element of its domain.



The output of the function must always be in the codomain, but not all elements of the codomain need to be producable.

Function Composition

- If $f : A \to B$ and $g : B \to C$ are functions, the ***composition of f and g***, denoted $g \circ f$, is a function
  - whose domain is A,
  - whose codomain is C, and
  - which is evaluated as $(g \circ f)(x) = g(f(x))$.

Injective Functions

- A function $f : A \to B$ is called ***injective*** (or ***one-to-one***) if different inputs always map to different outputs.
  - A function with this property is called an ***injection***.
- Formally, $f : A \to B$ is an injection if this FOL statement is true:

$$\forall a_1 \in A.\ \forall a_2 \in A.\ (a_1 \neq a_2 \to f(a_1) \neq f(a_2))$$

(*"If the inputs are different, the outputs are different"*)

- Equivalently:

$$\forall a_1 \in A.\ \forall a_2 \in A.\ (f(a_1) = f(a_2) \to a_1 = a_2)$$

(*"If the outputs are the same, the inputs are the same"*)

- ***Theorem:*** The composition of two injections is an injection.

Surjective Functions

- A function $f : A \to B$ is called ***surjective*** (or ***onto***) if each element of the codomain is "covered" by at least one element of the domain.
  - A function with this property is called a ***surjection***.
- Formally, $f : A \to B$ is a surjection if this FOL statement is true:

$$\forall b \in B.\ \exists a \in A.\ f(a) = b$$

(*"For every possible output, there's at least one possible input that produces it"*)

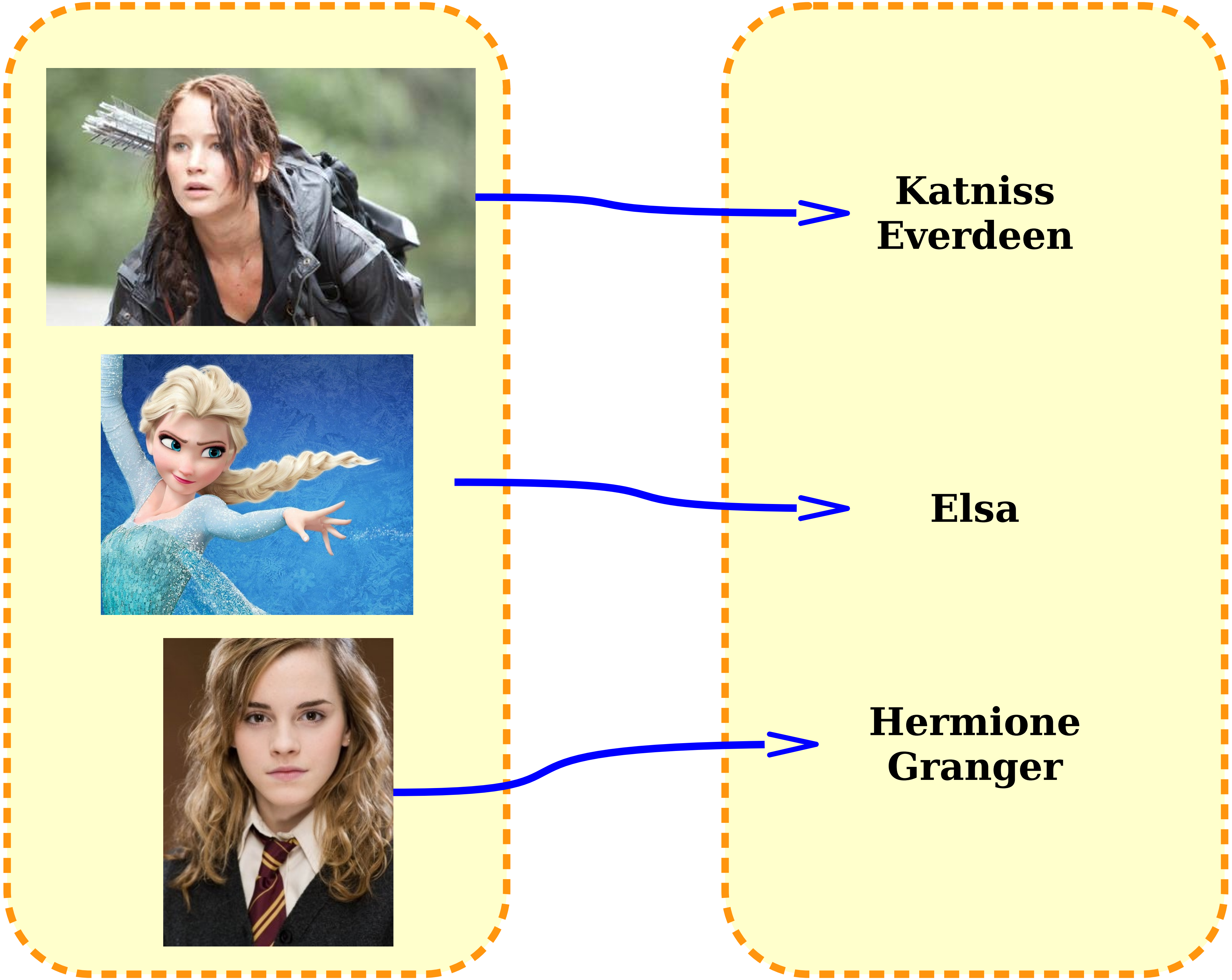
- ***Theorem:*** The composition of two surjections is a surjection.

New Stuff!

Bijections

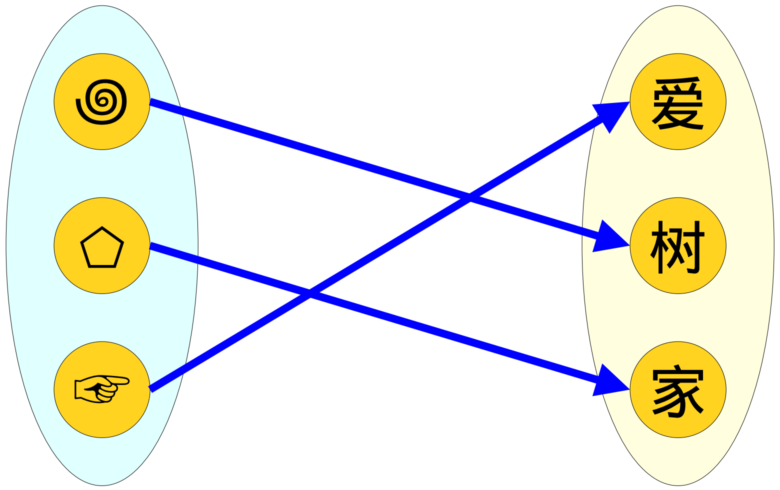
Injections and Surjections

- An injective function associates ***at most*** one element of the domain with each element of the codomain.
- A surjective function associates ***at least*** one element of the domain with each element of the codomain.
- What about functions that associate ***exactly one*** element of the domain with each element of the codomain?


Katniss Everdeen
Elsa
Hermione Granger

Bijections

- A ***bijection*** is a function that is both injective and surjective.
- Intuitively, if $f : A \to B$ is a bijection, then f represents a way of pairing off elements of A and elements of B.



Bijections

- Which of the following are bijections?
  - $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = x$. *Yep!*
  - $f : \mathbb{Z} \to \mathbb{R}$ defined as $f(x) = x$. *Nope!*
  - $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = 2x + 1$. *Yep!*
  - $f : \mathbb{Z} \to \mathbb{Z}$ defined as $f(x) = 2x + 1$. *Nope!*

A ***bijection*** is a function that is both injective and surjective.

Cardinality Revisited

Cardinality

- Recall *(from our first lecture!)* that the ***cardinality*** of a set is the number of elements it contains.
- If S is a set, we denote its cardinality by $|S|$.
- For finite sets, cardinalities are natural numbers:
  - $|\{1, 2, 3\}| = 3$
  - $|\{100, 200\}| = 2$
- For infinite sets, we introduced ***infinite cardinals*** to denote the size of sets:

$$|\mathbb{N}| = \aleph_0$$

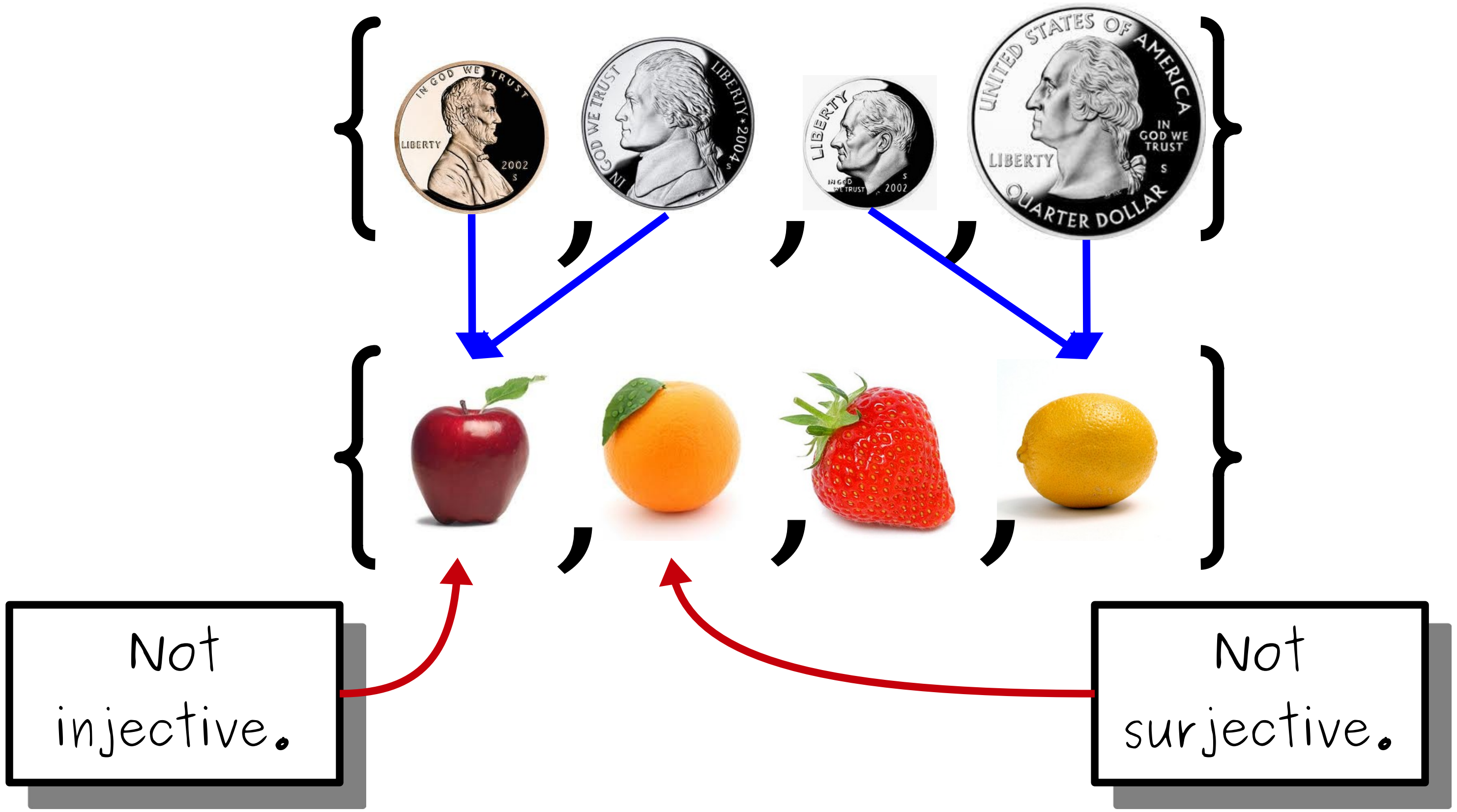
Defining Cardinality

- It is difficult to give a rigorous definition of what cardinalities actually are.
  - What is 4? What is $\aleph_0$?
  - (Take Math 161 for an answer!)
- Instead, we'll define cardinality as a *relation* between two sets rather than an absolute quantity.
- ***Intuition:*** Two sets have the same cardinality if there's a way to pair off their elements.

Comparing Cardinalities

- Here is the formal definition of what it means for two sets to have the same cardinality:

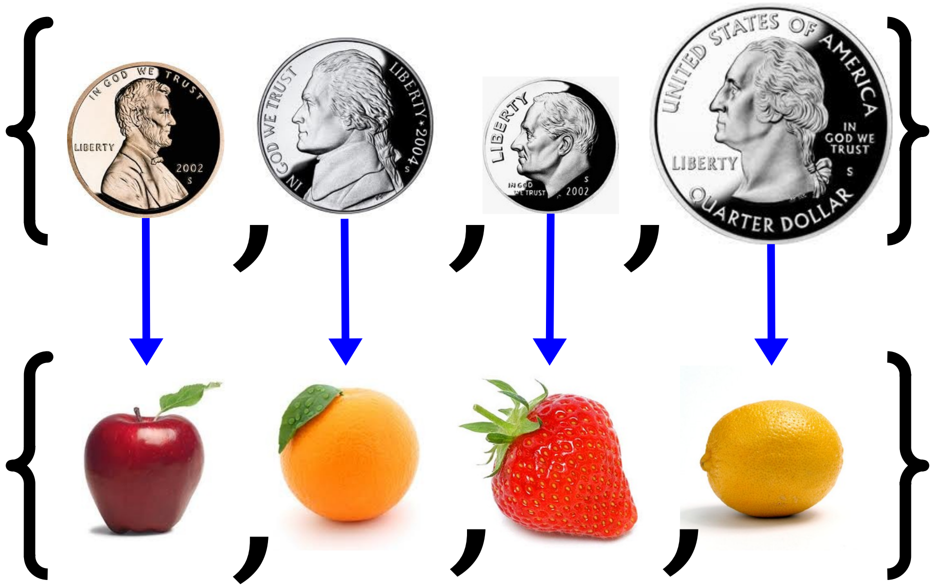
$|S| = |T|$ if there exists a *bijection* $f : S \to T$



Comparing Cardinalities

- Here is the formal definition of what it means for two sets to have the same cardinality:

$|S| = |T|$ if there exists a *bijection* $f : S \to T$



Fun with Cardinality

Terminology Refresher

- Let a and b be real numbers where $a \leq b$.
- The notation $\mathbf{[a, b]}$ denotes the set of all real numbers between a and b, inclusive.

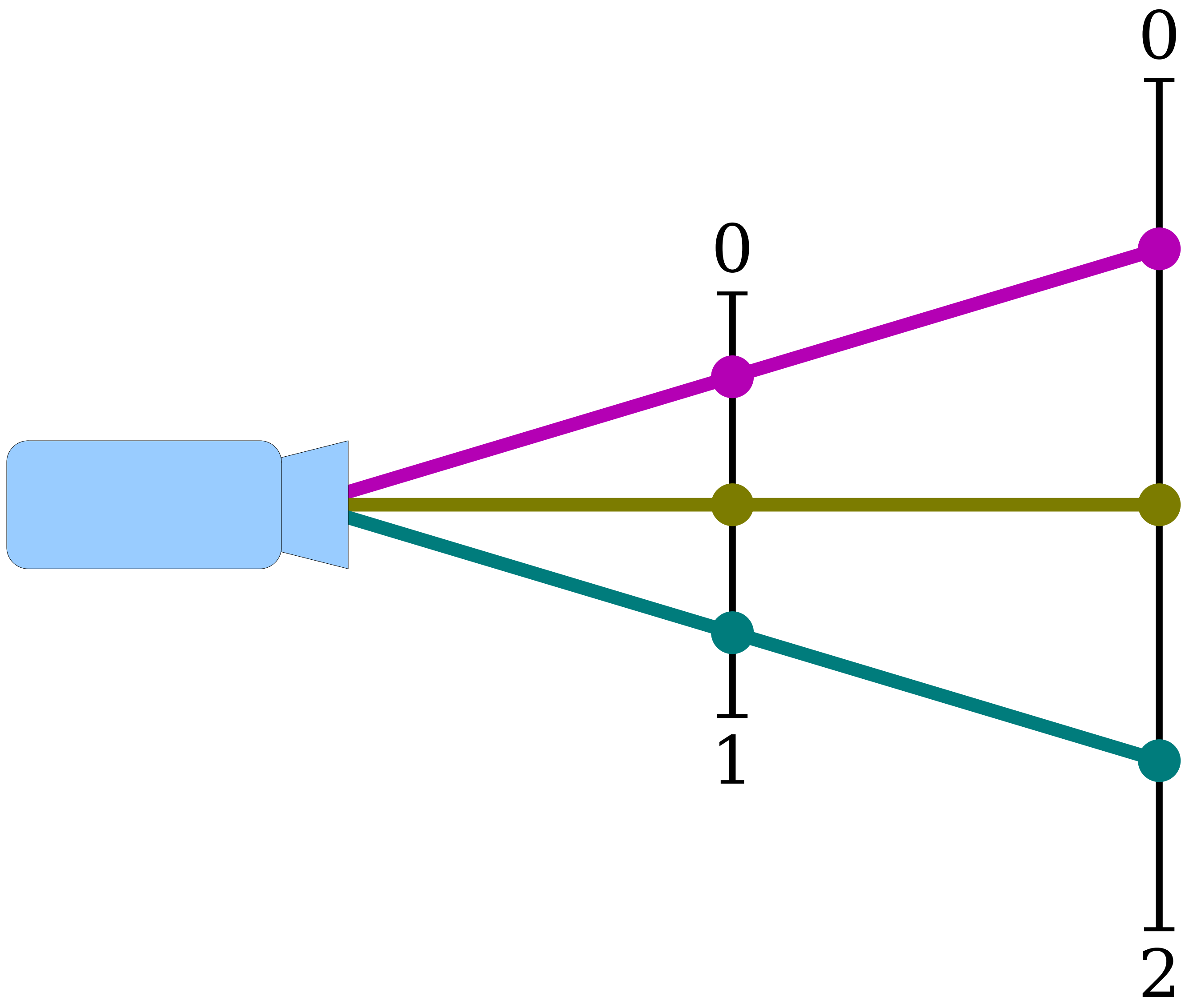
$$[a, b] = \{ x \in \mathbb{R} \mid a \leq x \leq b \}$$

- The notation $\mathbf{(a, b)}$ denotes the set of all real numbers between a and b, exclusive.

$$(a, b) = \{ x \in \mathbb{R} \mid a < x < b \}$$

Consider the sets $[0, 1]$ and $[0, 2]$.

How do their cardinalities compare?



$$f : [0, 1] \to [0, 2]$$

$$f(x) = 2x$$

Theorem: $|[0, 1]| = |[0, 2]|$

Proof: Consider the function $f : [0, 1] \to [0, 2]$ defined as $f(x) = 2x$. We will prove that f is a bijection.

First, we will show that f is a well-defined function. Choose any $x \in [0, 1]$. This means that $0 \le x \le 1$, so we know that $0 \le 2x \le 2$. Consequently, we see that $0 \le f(x) \le 2$, so $f(x) \in [0, 2]$.

Next, we'll show that f is injective. Pick any $x_1, x_2 \in [0, 1]$ where $f(x_1) = f(x_2)$. We will show that $x_1 = x_2$. To see this, notice that since $f(x_1) = f(x_2)$, we see that $2x_1 = 2x_2$, which in turn tells us that $x_1 = x_2$, as required.

Finally, we will show that f is surjective. To do so, consider any $y \in [0, 2]$. We'll show that there is some $x \in [0, 1]$ where $f(x) = y$.

Let $x = {}^{y}/_{2}$. Since $y \in [0, 2]$, we know $0 \le y \le 2$, and therefore that $0 \le {}^{y}/_{2} \le 1$. We picked $x = {}^{y}/_{2}$, so we know that $0 \le x \le 1$, which in turn means $x \in [0, 1]$. Moreover, notice that

$$f(x) = 2x = 2({}^{y}/_{2}) = y,$$

so $f(x) = y$, as required. ■

Theorem: $|[0, 1]| = |[0, 2]|$

Proof: Consider the function $f : [0, 1] \to [0, 2]$ defined as $f(x) = 2x$. We will prove that f is a bijection.

First, we will show that f is a well-defined function. Choose any $x \in [0, 1]$. This means that $0 \leq x \leq 1$, so we know that $0 \leq 2x \leq 2$. Consequently, we see that $0 \leq f(x) \leq 2$, so $f(x) \in [0, 2]$.

Next, we'll show that f is … ere $f(x_1) = f(x_2)$. We will show … at since $f(x_1) = f(x_2)$, we see … us that $x_1 = x_2$, as required.

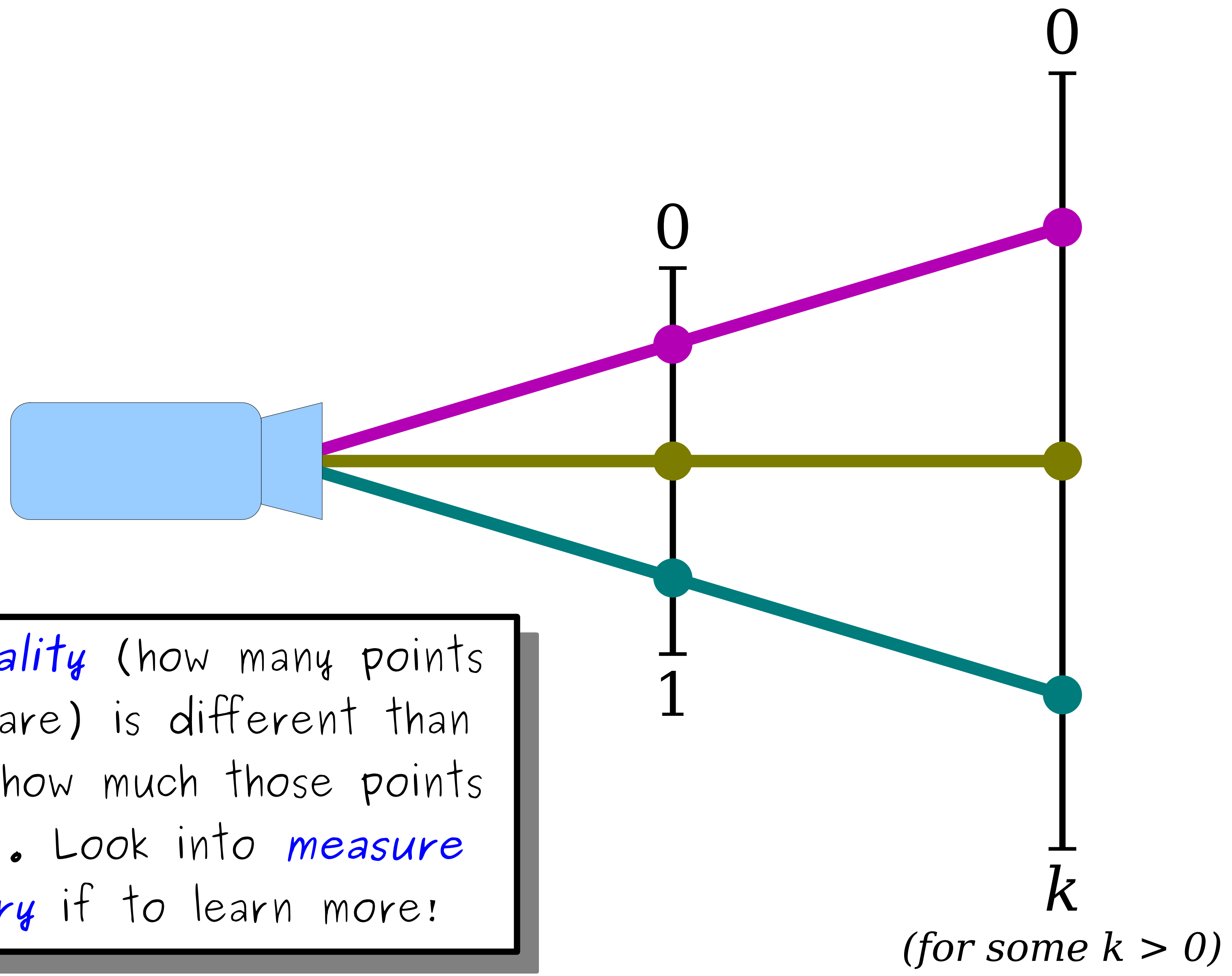
Finally, we will show tha … ny $y \in [0, 2]$. We'll show that … .

Let $x = {}^{y}/_{2}$. Since $y \in [0, 2$ … at $0 \leq {}^{y}/_{2} \leq 1$. We picked $x = {}^{y}/_{2}$, so we know that $0 \leq x \leq 1$, which in turn means $x \in [0, 1]$. Moreover, notice that

$$f(x) = 2x = 2({}^{y}/_{2}) = y,$$

so $f(x) = y$, as required. ■

When defining something we claim is a function, the convention is to prove that it obeys the domain/codomain rules. For whatever reason, there isn't a convention of showing that it's deterministic. Ah, tradition.



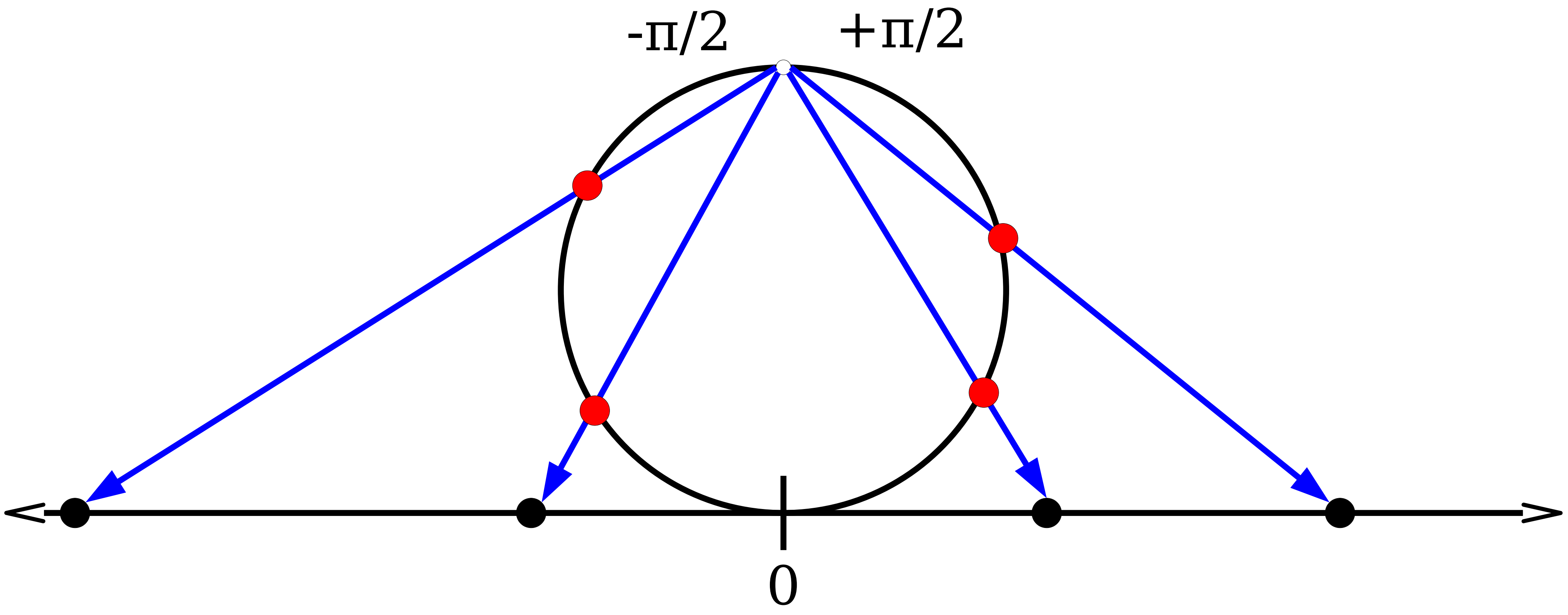
Cardinality (how many points there are) is different than *mass* (how much those points weigh). Look into *measure theory* if to learn more!

$$f : [0, 1] \to [0, k]$$

$$f(x) = kx$$

And one more example, just for funzies.

Put a Ring On It



$f : (-\pi/2, \pi/2) \to \mathbb{R}$

$f(x) = \tan x$

$|(-\pi/2, \pi/2)| = |\mathbb{R}|$

Some Properties of Cardinality

Theorem: For any set A, we have $|A| = |A|$.

Proof: Consider any set A, and let $f : A \to A$ be the function defined as $f(x) = x$. We will prove that f is a bijection.

First, we'll show that f is a well-defined function. To see this, note that for any $x \in A$, we have $f(x) = x \in A$, as needed.

Next, we'll show that f is injective. Pick any $x_1, x_2 \in A$ where $f(x_1) = f(x_2)$. We need to show that $x_1 = x_2$. Since $f(x_1) = f(x_2)$, we see by definition of f that $x_1 = x_2$, as required.

Finally, we'll show that f is surjective. Consider any $y \in A$. We will prove that there is some $x \in A$ where $f(x) = y$. Pick $x = y$. Then $x \in A$ (since $y \in A$) and $f(x) = x = y$, as required. ■

Theorem: If A, B, and C are sets where $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$.

Proof: Consider any sets A, B, and C where $|A| = |B|$ and $|B| = |C|$. We need to prove that $|A| = |C|$. To do so, we need to show that there is a bijection from A to C.

Since $|A| = |B|$, we know that there is a some bijection $f : A \to B$. Similarly, since $|B| = |C|$ we know that there is at least one bijection $g : B \to C$.

Consider the function $g \circ f : A \to C$. Since g and f are bijections and the composition of two bijections is a bijection, we see that $g \circ f$ is a bijection from A to C. Thus $|A| = |C|$, as required. ■

Great exercise: Prove that if A and B are sets where $|A| = |B|$, then $|B| = |A|$.

Time-Out for Announcements!

Problem Set Three

- Problem Set Two was due today at 2:30PM.
- Problem Set Three goes out today. It’s due next Friday at 2:30PM.
  - Play around with functions, set cardinality, and the nature of infinity!
- As always, ping us if you need help working on this one: post on EdStem or stop by office hours.

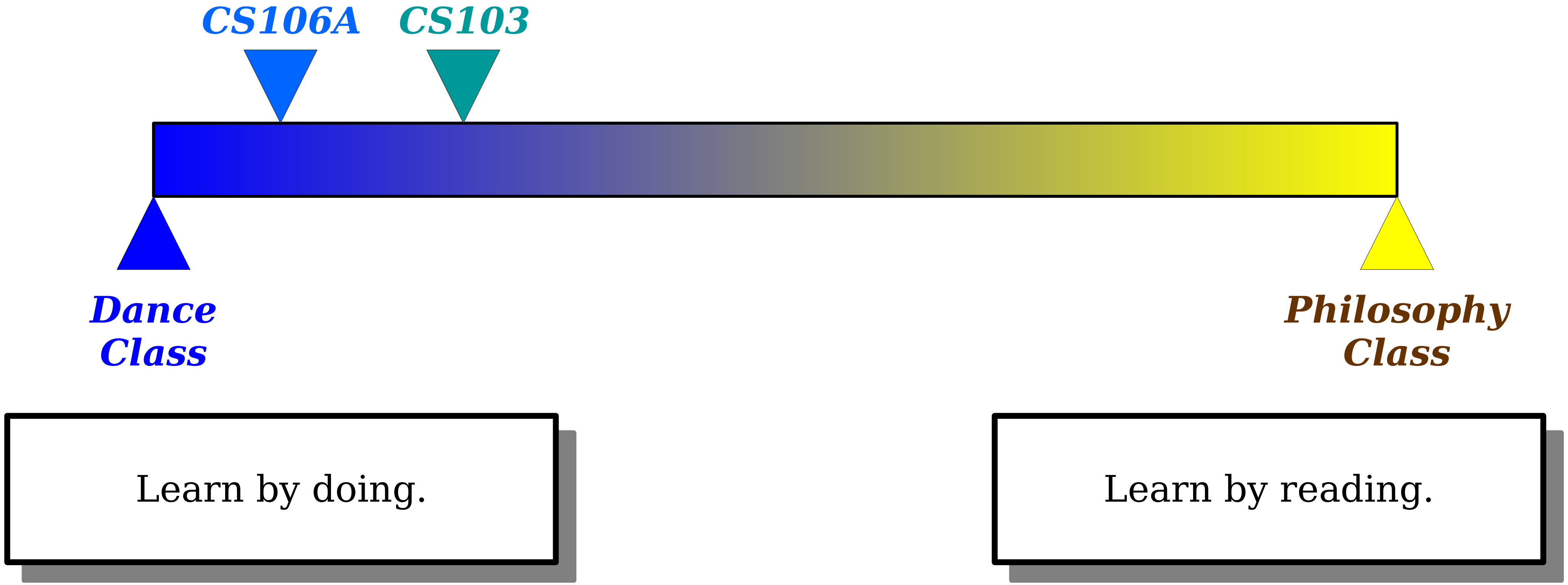
Midterm Exam Logistics

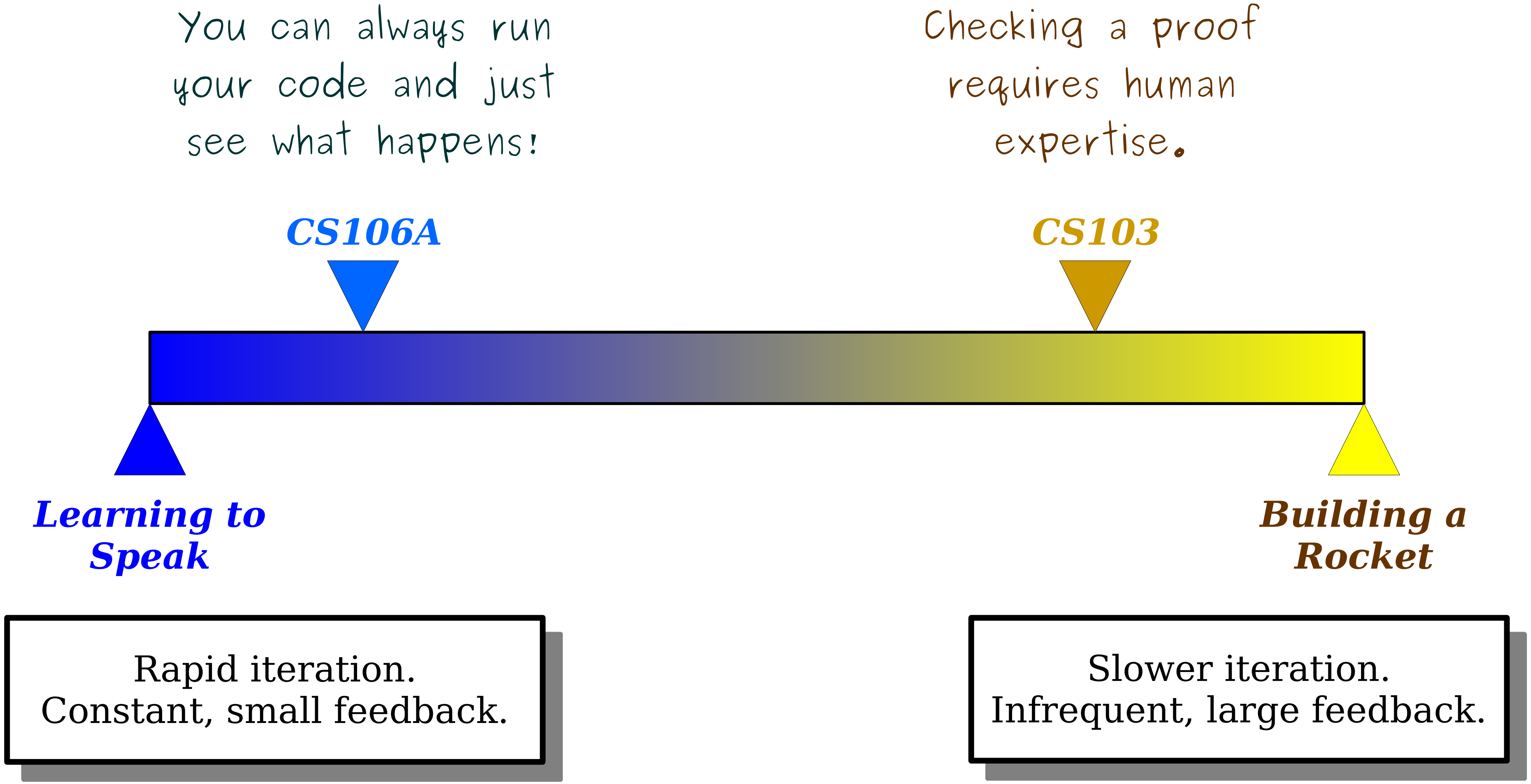
- Our first 48-hour take-home midterm exam runs from 2:30PM next Friday, October 15th to 2:30PM next Sunday, October 17th.
- The exam format is similar to the problem sets: the questions will be online, you'll download any relevant starter files, and submit everything through GradeScope.
- You have the full 48 hours to work on the midterm. It's designed to take about three hours to finish.
- You're responsible for Lectures 00 – 05 and topics from PS1 – PS2. Later lectures (functions onward) and problem sets (PS3) won't be tested here. Exam problems may build on the written or coding components from the problem sets.
- The exam is open-book, open-note, and closed-other-humans. You must not communicate with other humans about the exam.

Midterm Exam

- ***We want you to do well on this exam.*** We're not trying to weed out weak students. We're not trying to enforce a curve where there isn't one. We want you to show what you've learned up to this point so that you get a sense for where you stand and where you can improve.
- The purpose of this midterm is to give you a chance to show what you've learned in the past few weeks. It is not designed to assess your “mathematical potential” or “innate mathematical ability.”

Preparing for the Exam


CS106A
CS103
Dance Class
Philosophy Class
Learn by doing.
Learn by reading.


You can always run your code and just see what happens!
Checking a proof requires human expertise.
CS106A
CS103
Learning to Speak
Building a Rocket
Rapid iteration.
Constant, small feedback.
Slower iteration.
Infrequent, large feedback.

Extra Practice Problems

- Up on the course website, you'll find Extra Practice Problems 1, a set of eighteen practice problems on the topics covered by the upcoming midterm.
- Many of these are old midterm questions. Some are just really fun problems we thought you might enjoy working through.
- Take the time to work through some of these problems. This is, perhaps, the best way to study.

Doing Practice Problems

- As you work through practice problems, ***keep other humans in the loop!***
- Ask your problem set partner to review your answers and offer feedback – and volunteer to do the same!
- Post your answers as private questions on EdStem and ask for TA feedback!
- ***Feedback loops are key to improving!***

Preparing for the Exam

- We’ve posted a “Preparing for the Exam” page on the course website with full details and logistics.
- It also includes advice from former CS103 students about how to do well here.
- Check it out – there are tons of goodies there!

Your Questions

“I don't here a lot of talk about the CS graphics track, especially when compared to tracks like AI, information, etc. What are your thoughts on it? What can you do with it?”

Graphics is a very cool track to pursue. You can work with what we think of as traditional graphics (better techniques for rendering and displaying things on screen), but also things like photography, artwork, and physics simulation. The snow from “Frozen” was developed here, for example. The graphics group has something like five academy awards and one Turing award.

Back to CS103!

Unequal Cardinalities

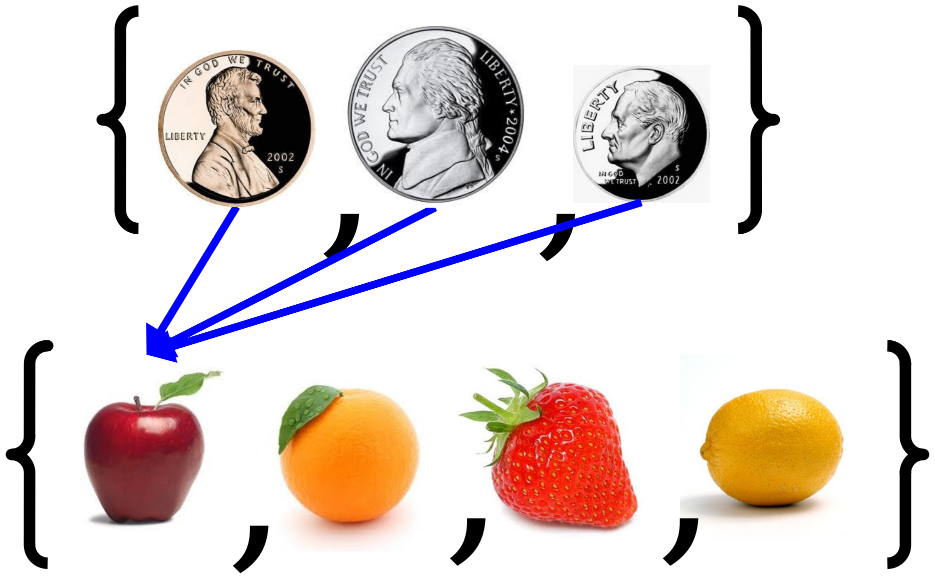
- Recall: $|A| = |B|$ if the following statement is true:

  There exists a bijection $f : A \to B$

- What does it mean for $|A| \neq |B|$ to be true?

  Every function $f : A \to B$ is not a bijection.

- This is a strong statement! To prove $|A| \neq |B|$, we need to show that *no possible function* from A to B can be injective and surjective.



Cantor’s Theorem Revisited

Cantor's Theorem

- In our very first lecture, we sketched out a proof of ***Cantor's theorem***, which says that

If S is a set, then $|S| < |\wp(S)|$.

- That proof was visual and pretty hand-wavy. Let's see if we can go back and formalize it!

Where We're Going

- Today, we're going to formally prove the following result:

If S is a set, then $|S| \neq |\wp(S)|$.

- We've released an online Guide to Cantor's Theorem, which will go into *way* more depth than what we're going to see here.
- The goal for today will be to see how to start with our picture and turn it into something rigorous.
- On the problem set, you'll explore the proof in more depth and see some other applications.

The Roadmap

- We're going to prove this statement:

$$\text{If } S \text{ is a set, then } |S| \neq |\wp(S)|.$$

- Here's how this will work:
  - Pick an arbitrary set S.
  - Pick an arbitrary function $f : S \to \wp(S)$.
  - Show that f is not surjective using a diagonal argument.
  - Conclude that there are no bijections from S to $\wp(S)$.
  - Conclude that $|S| \neq |\wp(S)|$.

This is a drawing of our function $f : S \to \wp(S)$.

$$x_0 \leftrightarrow \{\ x_0,\ x_2,\ x_4,\ \dots\ \}$$

$$x_1 \leftrightarrow \{\ x_3,\ x_5,\ \dots\ \}$$

$$x_2 \leftrightarrow \{\ x_0,\ x_1,\ x_2,\ x_5,\ \dots\ \}$$

$$x_3 \leftrightarrow \{\ x_1,\ x_4,\ \dots\ \}$$

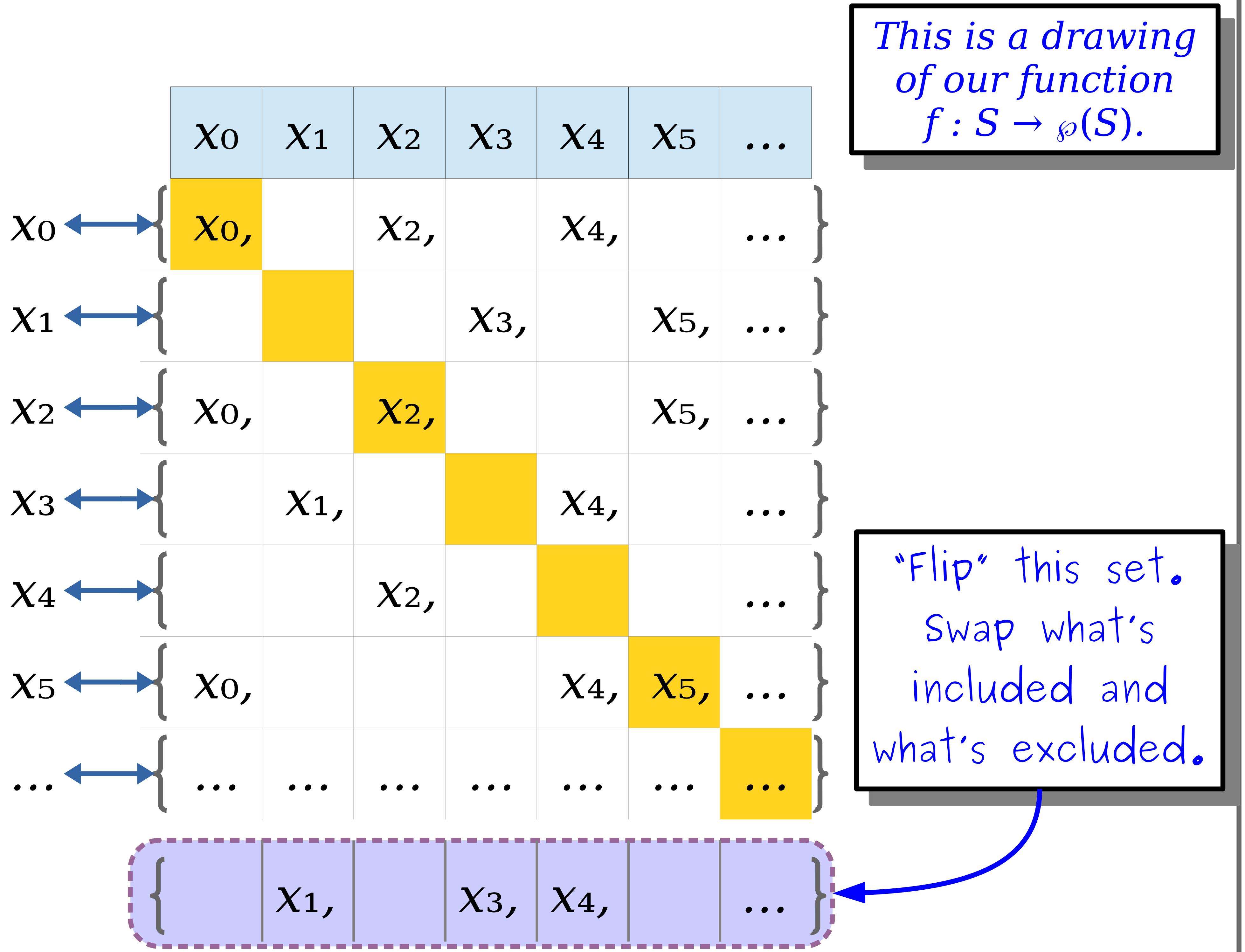
$$x_4 \leftrightarrow \{\ x_2,\ \dots\ \}$$

$$x_5 \leftrightarrow \{\ x_0,\ x_4,\ x_5,\ \dots\ \}$$

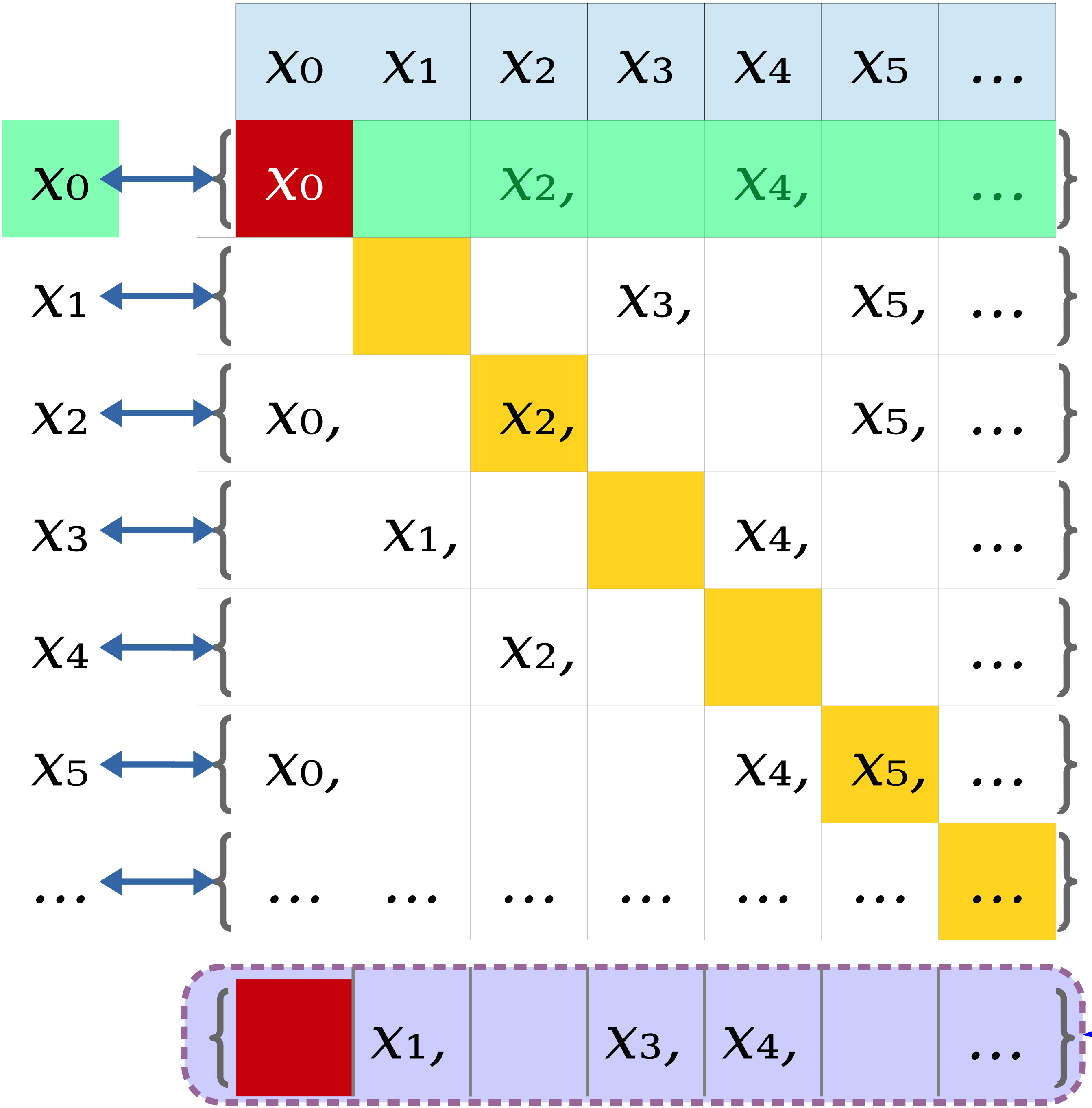
$$\dots \leftrightarrow \{\ \dots\ \}$$

This is a drawing of our function $f : S \to \wp(S)$.

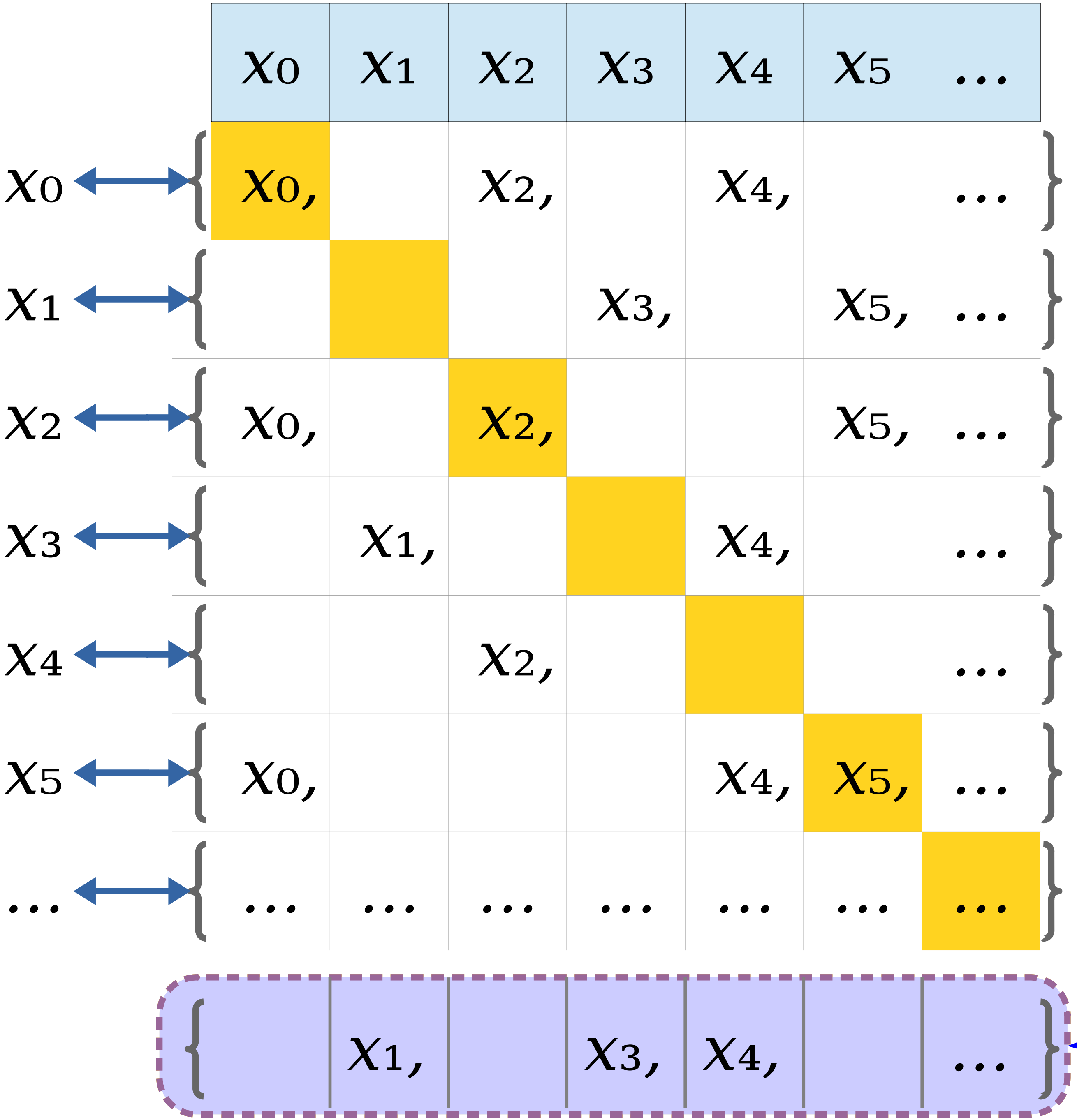
		x_0	x_1	x_2	x_3	x_4	x_5	…	
x_0	⟷ {	x_0,		x_2,		x_4,		…	}
x_1	⟷ {				x_3,		x_5,	…	}
x_2	⟷ {	x_0,		x_2,			x_5,	…	}
x_3	⟷ {		x_1,			x_4,		…	}
x_4	⟷ {			x_2,				…	}
x_5	⟷ {	x_0,				x_4,	x_5,	…	}
…	⟷ {	…	…	…	…	…	…	…	}


This is a drawing of our function $f : S \to \wp(S)$.
x_0 x_1 x_2 x_3 x_4 x_5 …
x_0 ↔ { x_0, x_2, x_4, … }
x_1 ↔ { x_3, x_5, … }
x_2 ↔ { x_0, x_2, x_5, … }
x_3 ↔ { x_1, x_4, … }
x_4 ↔ { x_2, … }
x_5 ↔ { x_0, x_4, x_5, … }
… ↔ { … … … … … … … }
{ x_1, x_3, x_4, … }
"Flip" this set. Swap what's included and what's excluded.

This is a drawing of our function $f : S \to \wp(S)$.

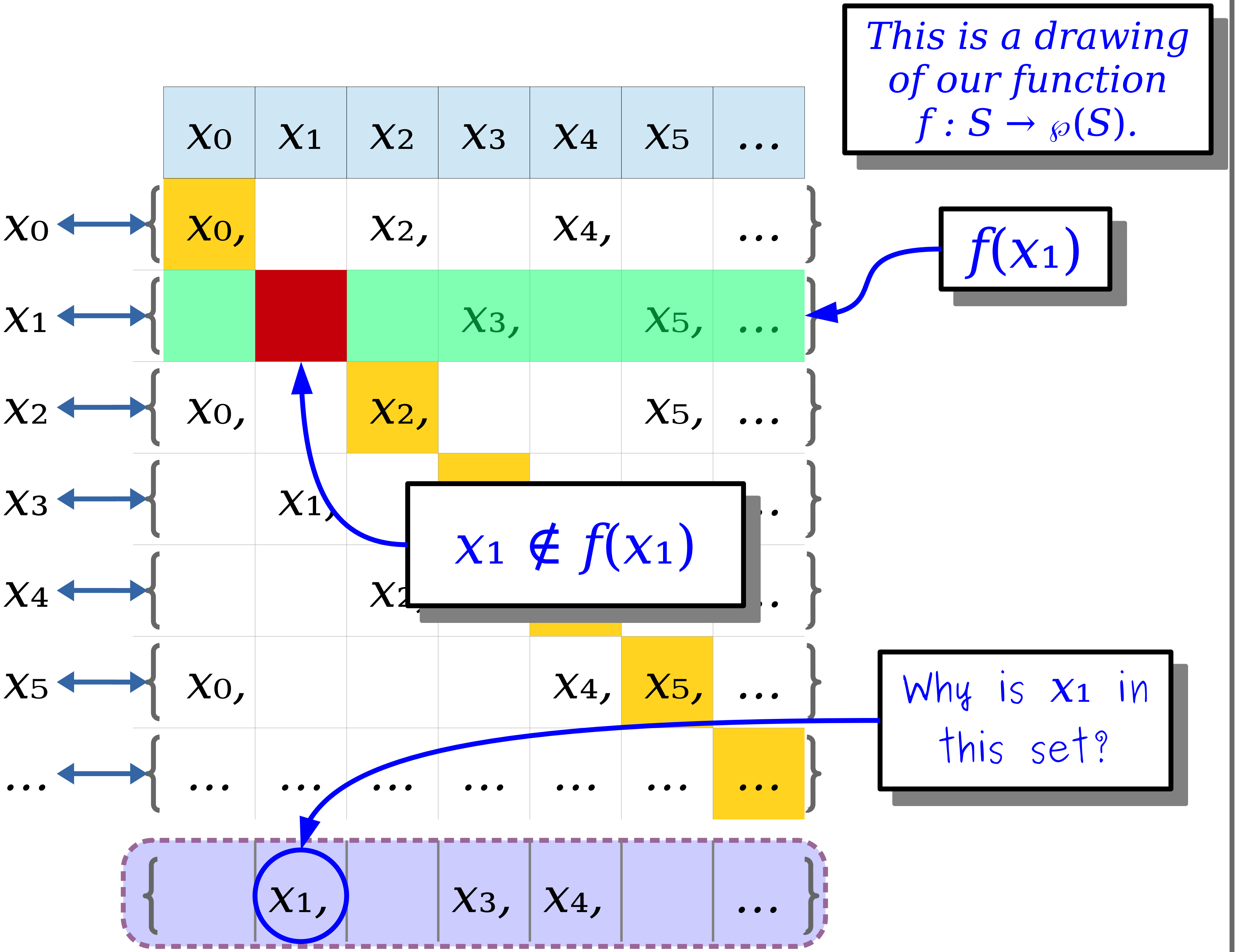


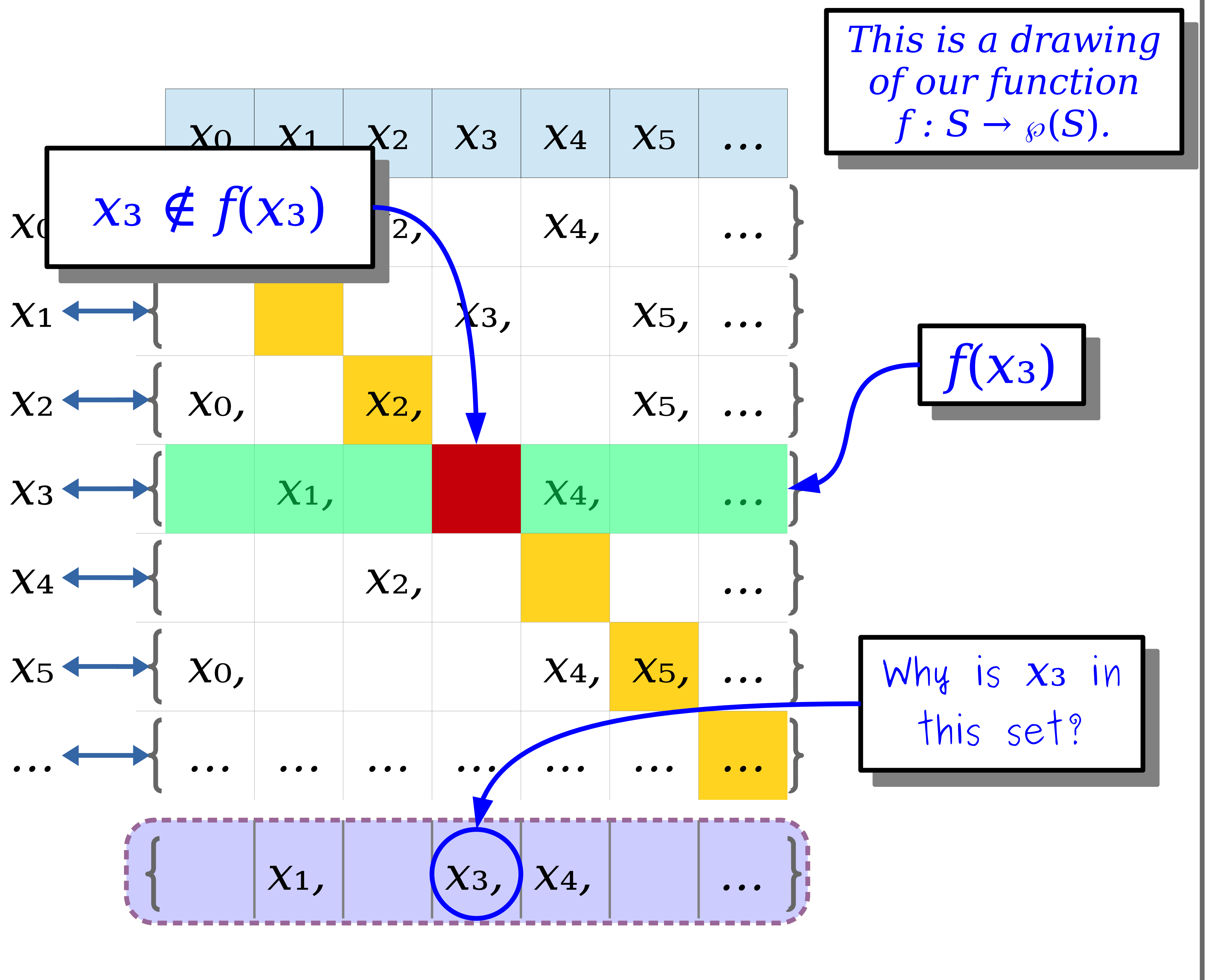
Which element is paired with this set?



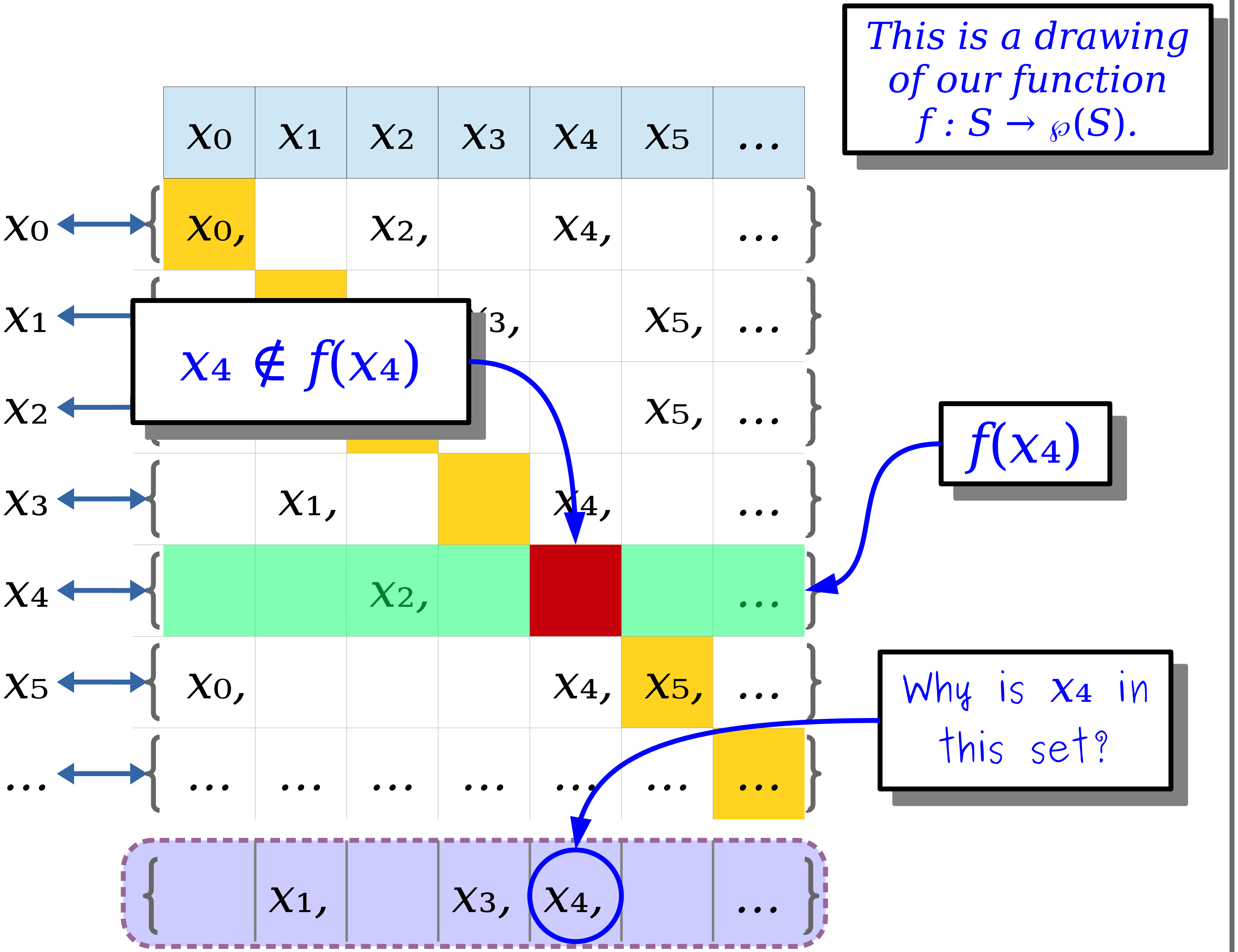
This is a drawing of our function $f : S \to \wp(S)$.

What set is this?

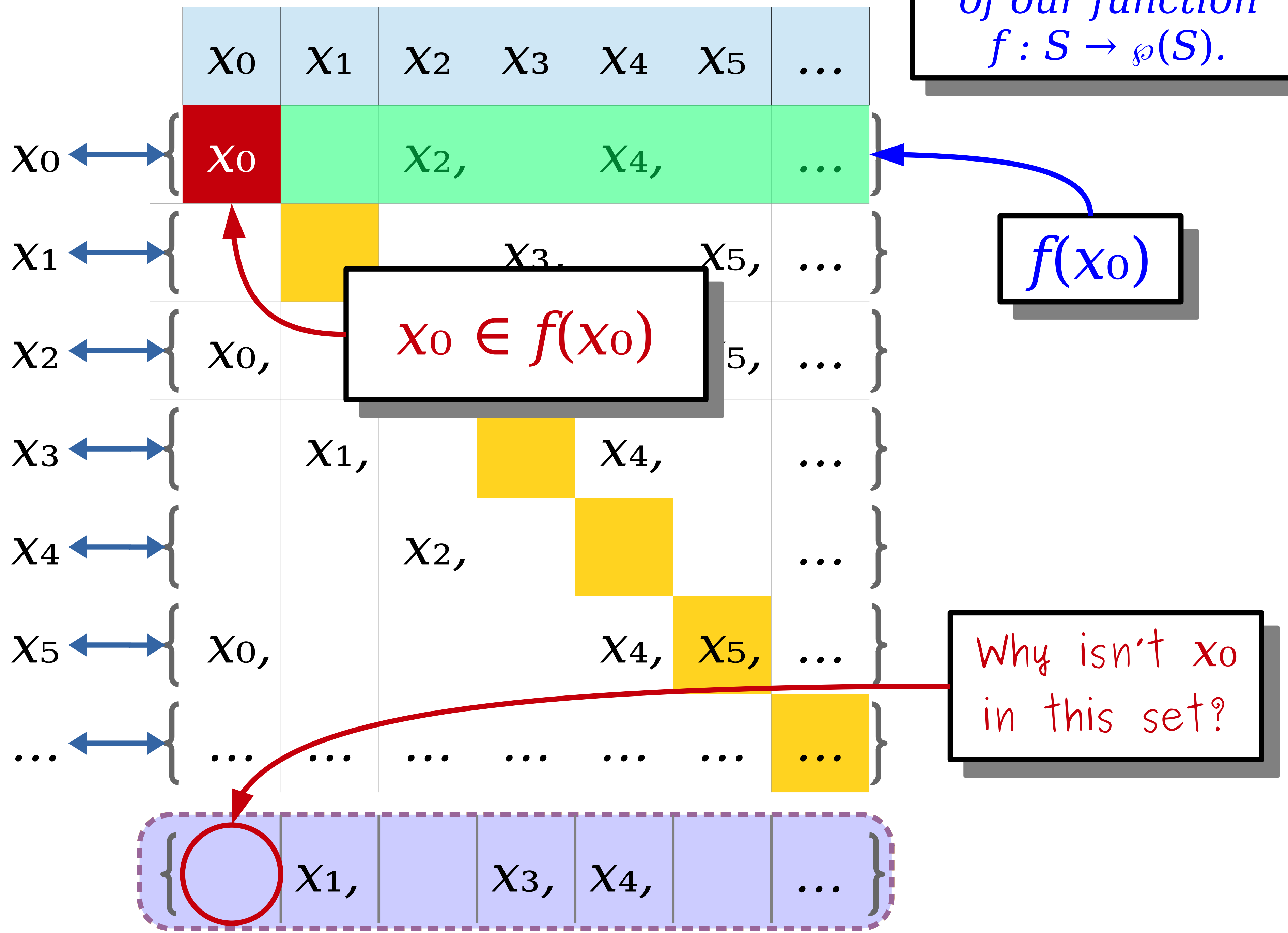

This is a drawing of our function $f : S \to \wp(S)$.
$f(x_1)$
$x_1 \notin f(x_1)$
Why is x_1 in this set?

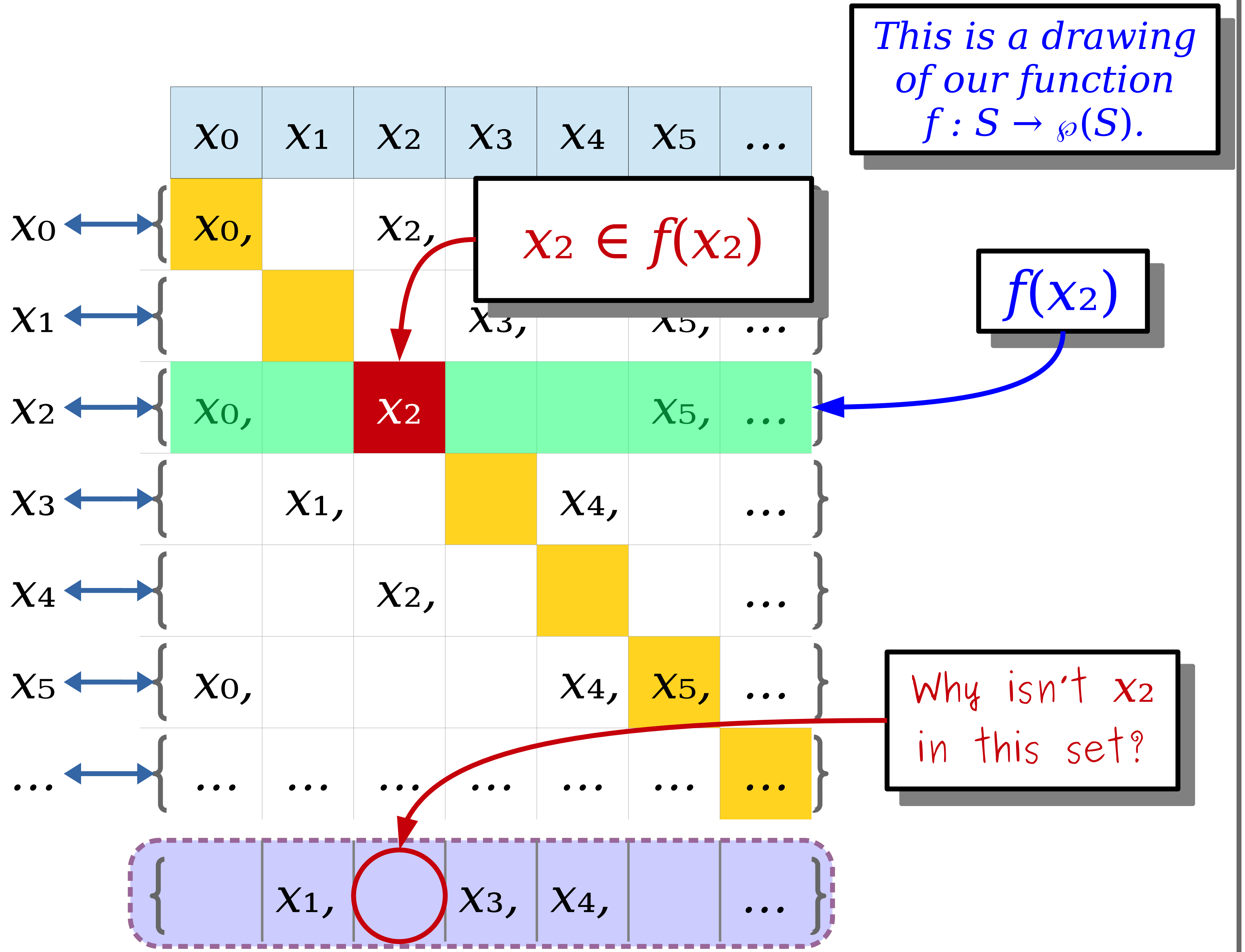


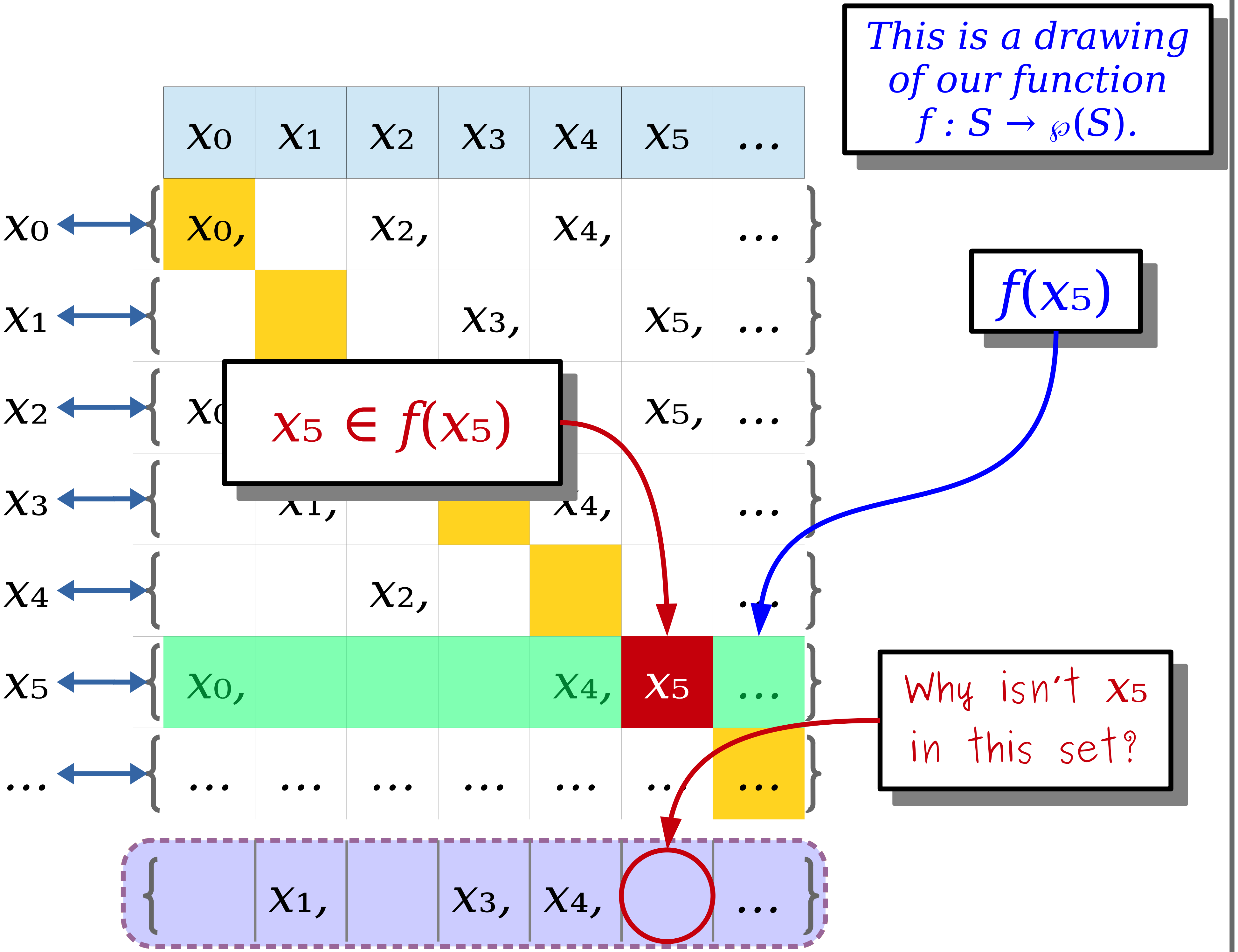
This is a drawing of our function $f : S \to \wp(S)$.
$x_3 \notin f(x_3)$
$f(x_3)$
Why is x_3 in this set?
x_0 x_1 x_2 x_3 x_4 x_5 ...
$\{x_1, x_3, x_4, \ldots\}$


This is a drawing of our function $f : S \to \wp(S)$.
x_0 x_1 x_2 x_3 x_4 x_5 …
x_0, x_2, x_4, …
x_5, …
x_5, …
x_1, x_4, …
x_2, …
x_0, x_4, x_5, …
$x_4 \notin f(x_4)$
$f(x_4)$
Why is x_4 in this set?
x_1, x_3, x_4, …

This is a drawing of our function $f : S \to \wp(S)$.

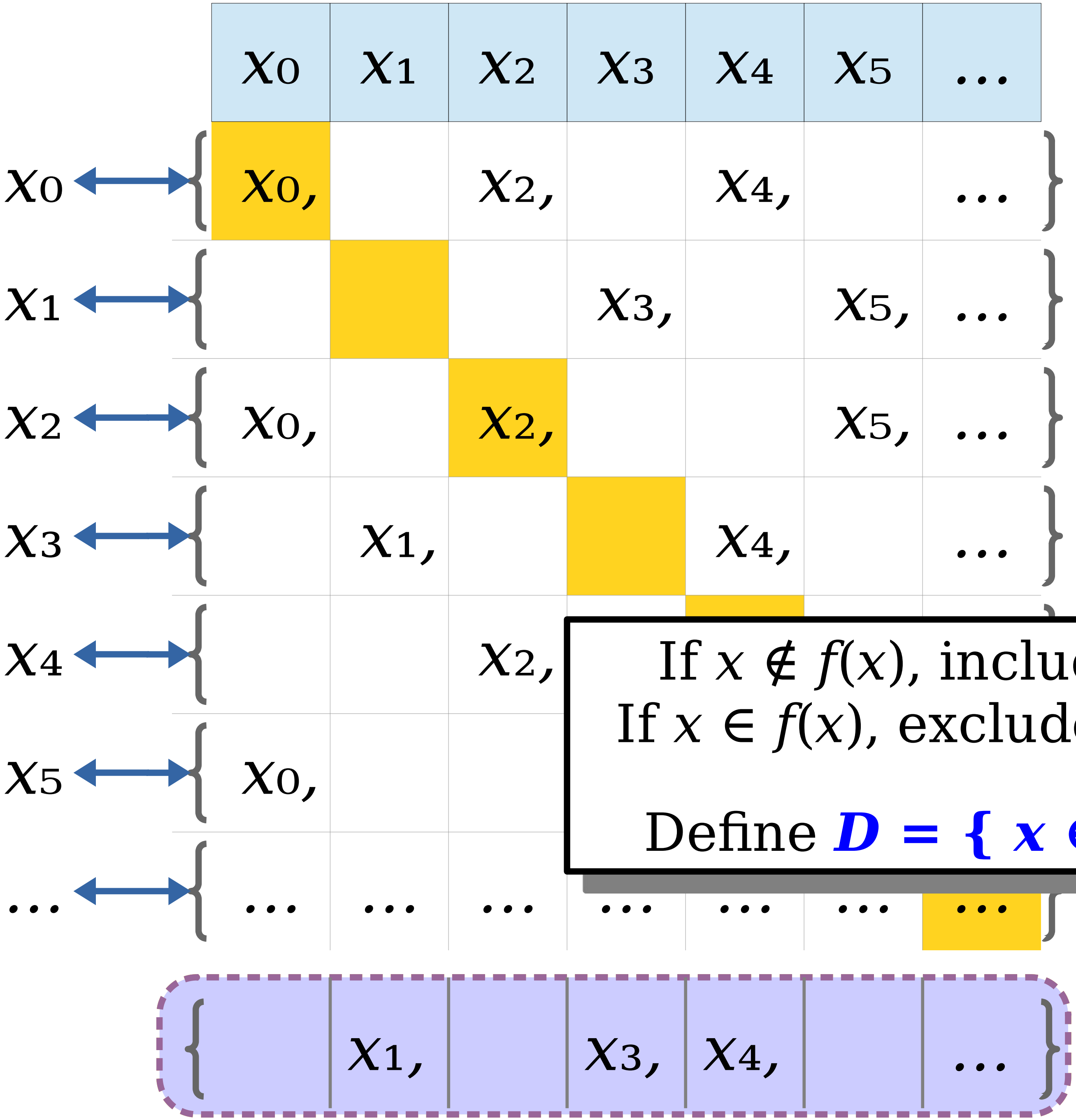



This is a drawing of our function $f : S \to \wp(S)$.
x_0 x_1 x_2 x_3 x_4 x_5 …
x_0 { x_0, x_2, … }
x_1 { x_3, x_5, … }
x_2 { x_0, x_2 x_5, … }
x_3 { x_1, x_4, … }
x_4 { x_2, … }
x_5 { x_0, x_4, x_5, … }
… { … … … … … … … }
$x_2 \in f(x_2)$
$f(x_2)$
Why isn't x_2 in this set?
{ x_1, x_3, x_4, … }



This is a drawing of our function $f : S \to \wp(S)$.
x_0 x_1 x_2 x_3 x_4 x_5 ...
x_0 { x_0, x_2, x_4, ... }
x_1 { x_3, x_5, ... }
x_2 { x_5, ... }
x_3 { x_4, ... }
x_4 { x_2, ... }
x_5 { x_0, x_4, x_5 ... }
... { ... ... }
{ x_1, x_3, x_4, ... }
$f(x_5)$
$x_5 \in f(x_5)$
Why isn't x_5 in this set?

This is a drawing of our function $f : S \to \wp(S)$.



If $x \notin f(x)$, include x in the set.
If $x \in f(x)$, exclude x from the set.

Define $\boldsymbol{D = \{ x \in S \mid x \notin f(x) \}}$

The Diagonal Set

- For any set S and function $f : S \to \wp(S)$, we can define a set D as follows:

$$\boldsymbol{D = \{ x \in S \mid x \notin f(x) \}}$$

("The set of all elements x where x is not an element of the set f(x).")

- This is a formalization of the set we found in the previous picture.
- Using this choice of D, we can formally prove that no function $f : S \to \wp(S)$ is a bijection.

Theorem: If S is a set, then $|S| \neq |\wp(S)|$.

Proof: Let S be an arbitrary set. We will prove that $|S| \neq |\wp(S)|$ by showing that there are no bijections from S to $\wp(S)$. To do so, choose an arbitrary function $f : S \to \wp(S)$. We will prove that f is not surjective.

Starting with f, we define the set

$$D = \{ x \in S \mid x \notin f(x) \}. \qquad (1)$$

We will show that there is no $y \in S$ such that $f(y) = D$. To do so, we proceed by contradiction. Suppose that there is some $y \in S$ such that $f(y) = D$. By the definition of D, we know that

$$y \in D \text{ if and only if } y \notin f(y). \qquad (2)$$

By assumption, $f(y) = D$. Combined with (2), this tells us

$$y \in D \text{ if and only if } y \notin D. \qquad (3)$$

This is impossible. We have reached a contradiction, so our assumption must have been wrong. Therefore, there is no $y \in S$ such that $f(y) = D$, so f is not surjective. This means that f is not a bijection, and since our choice of f was arbitrary, we conclude that there are no bijections between S and $\wp(S)$. Thus $|S| \neq |\wp(S)|$, as required. ■

The Big Recap

- We define equal cardinality in terms of bijections between sets.
- Lots of different sets of infinite size have the same cardinality.
- Cardinality acts like an equivalence relation – but only because we can prove specific properties of how it behaves by relying on properties of function.
- Cantor’s theorem can be formalized in terms of surjectivity.

Next Time

- ***Graphs***
  - A ubiquitous, expressive, and flexible abstraction!
- ***Properties of Graphs***
  - Building high-level structures out of lower-level ones!