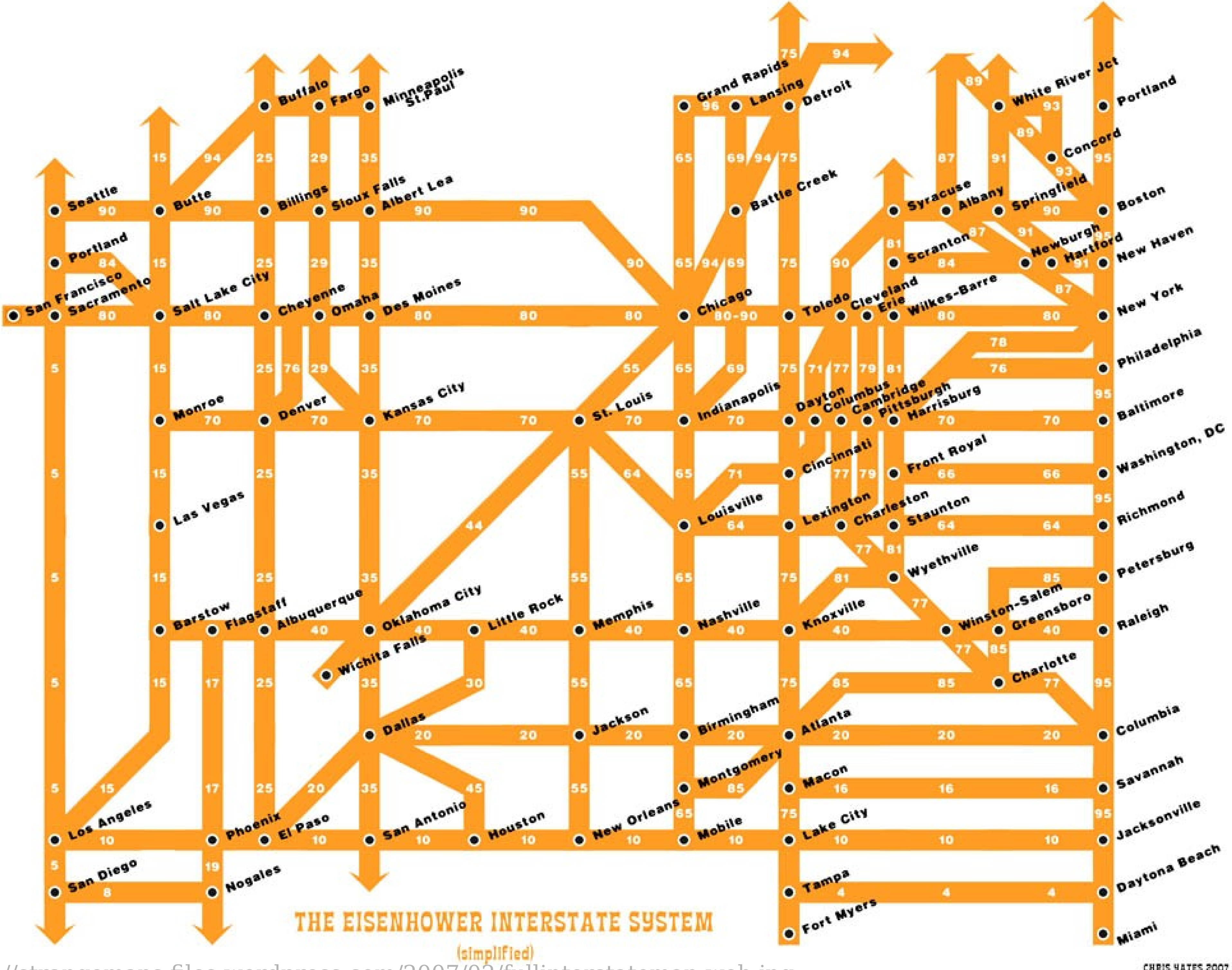
Graph Theory

Part One

Outline for Today

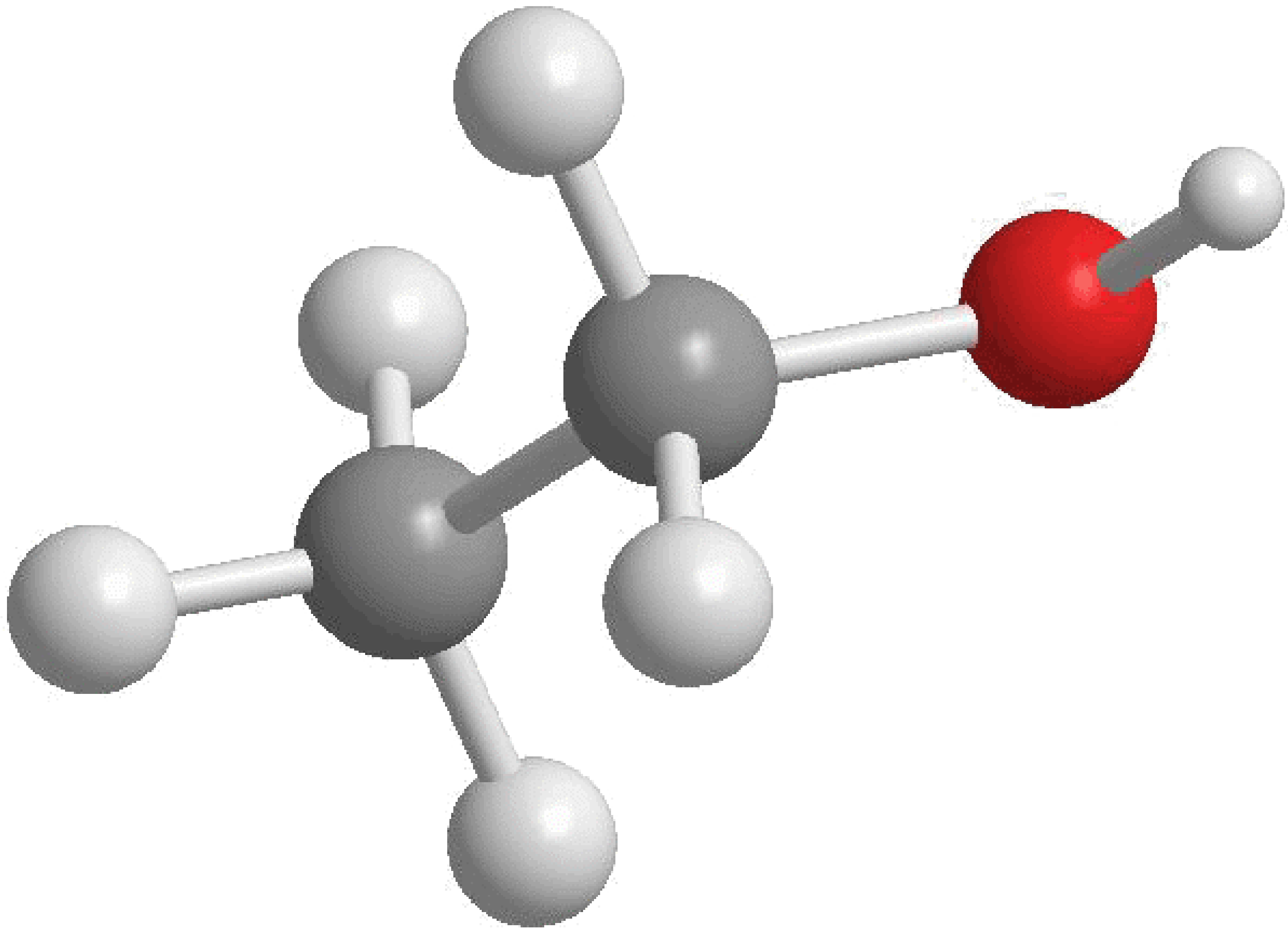
- ***Graphs and Digraphs***
  - Two fundamental mathematical structures.
- ***Graphs Meet FOL***
  - Building visual intuitions.
- ***Independent Sets and Vertex Covers***
  - Two structures in graphs.

Graphs and Digraphs



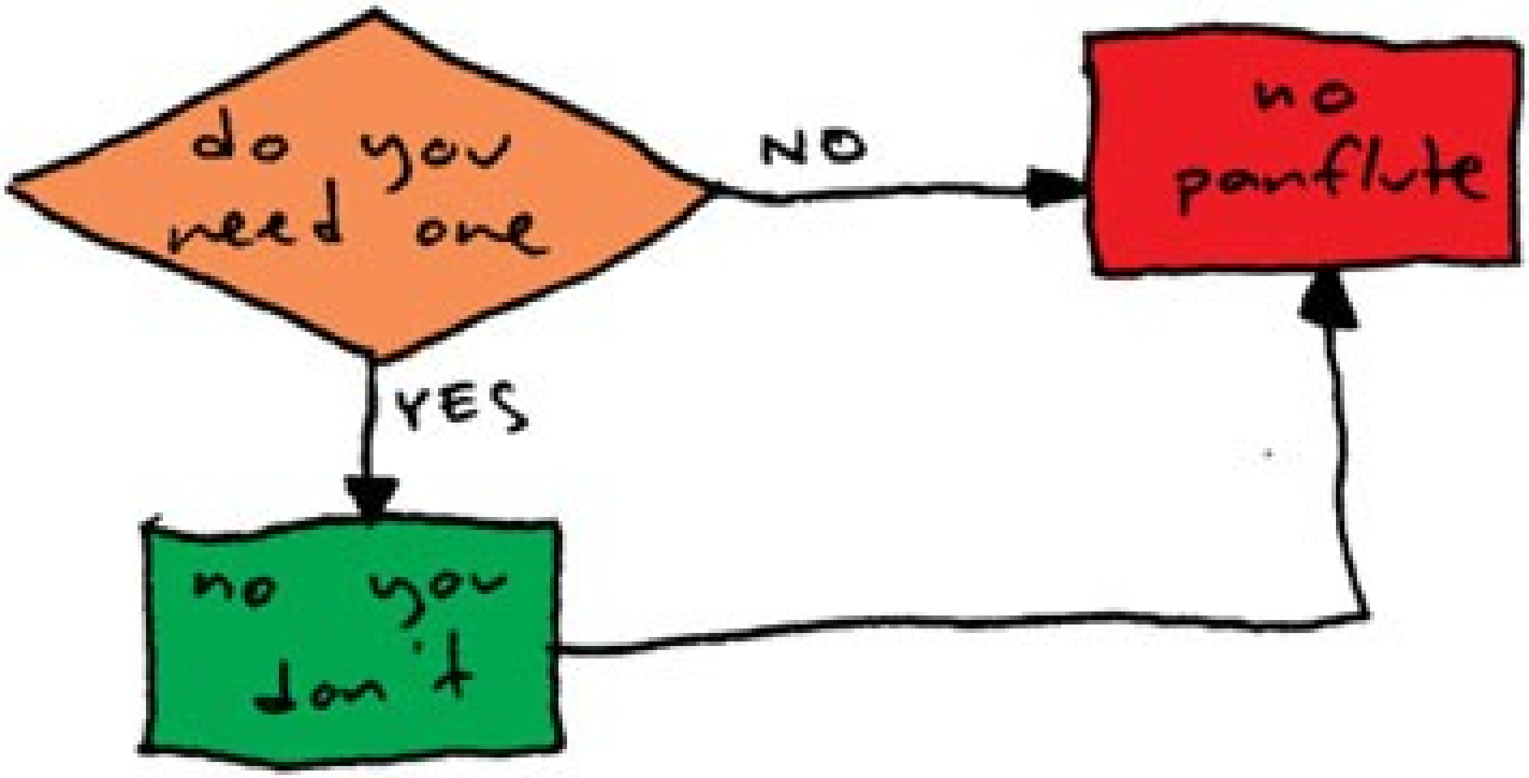
http://strangemaps.files.wordpress.com/2007/02/fullinterstatemap-web.jpg

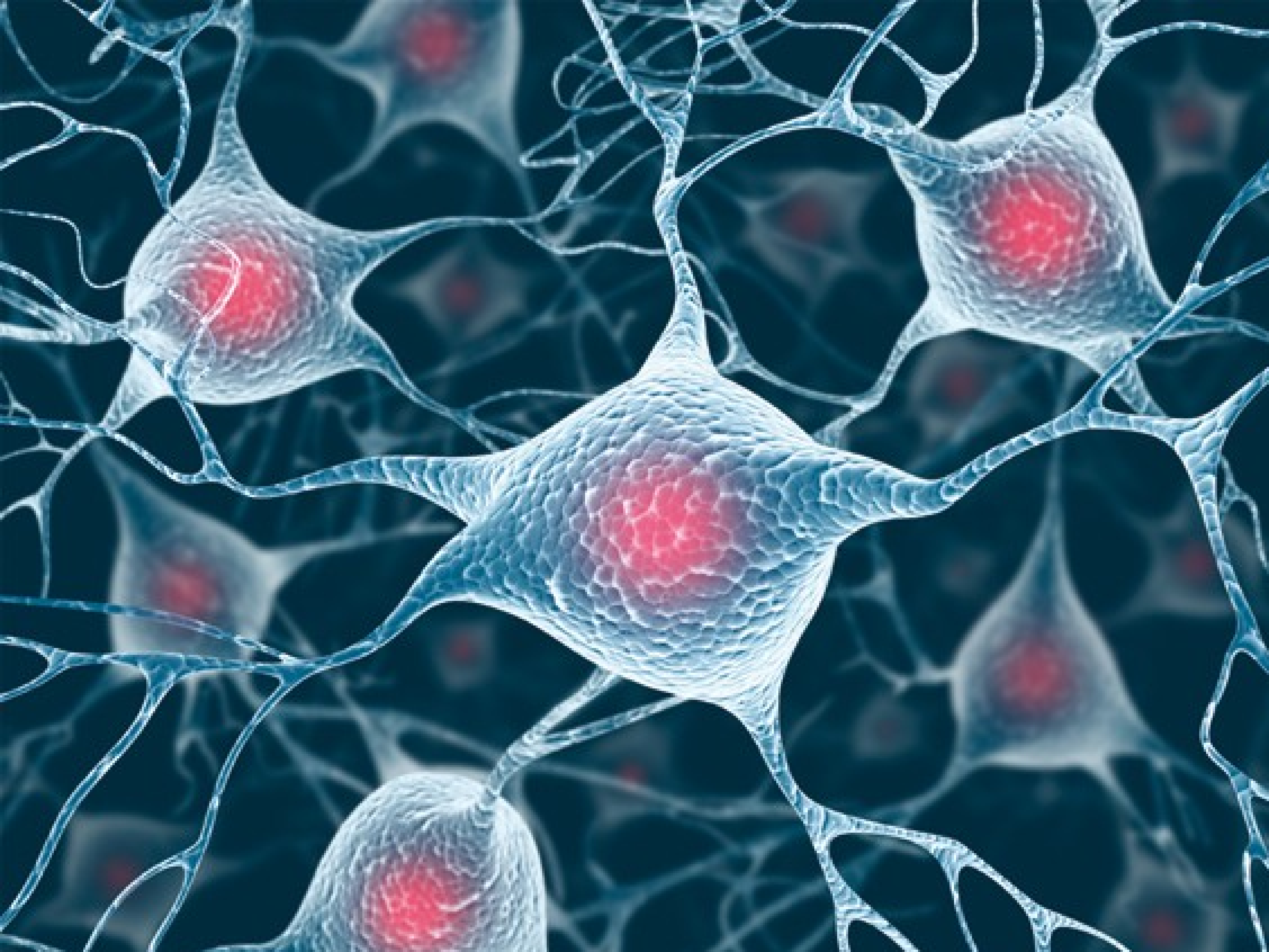
Chemical Bonds



http://4.bp.blogspot.com/-xCtBJ8lKHqA/Tjm0BONWBRI/AAAAAAAAAK4/-mHrbAUOHHg/s1600/

PANFLUTE FLOWCHART



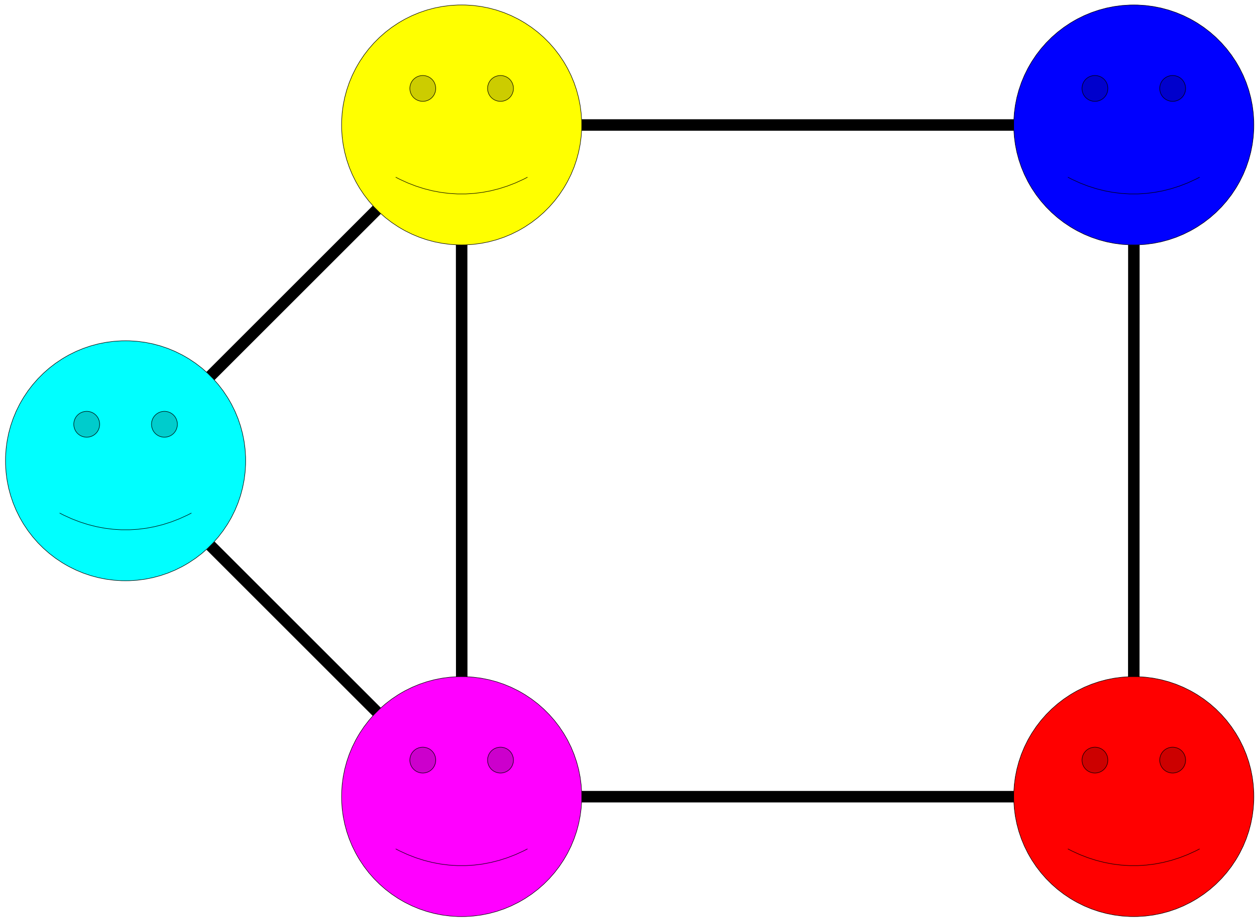


facebook®

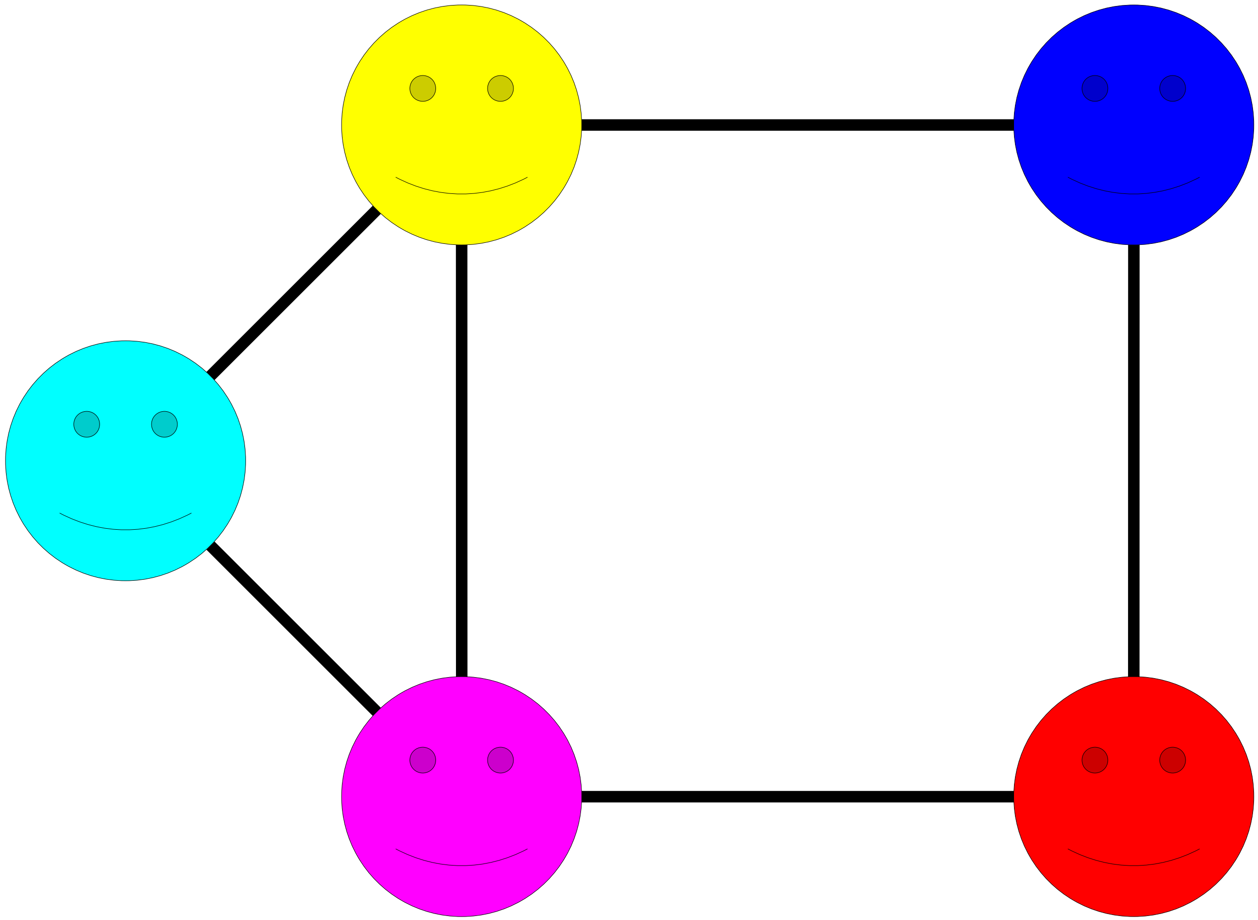
What's in Common

- Each of these structures consists of
  - a collection of objects and
  - links between those objects.
- ***Goal:*** find a general framework for describing these objects and their properties.

A ***graph*** is a mathematical structure for representing relationships.

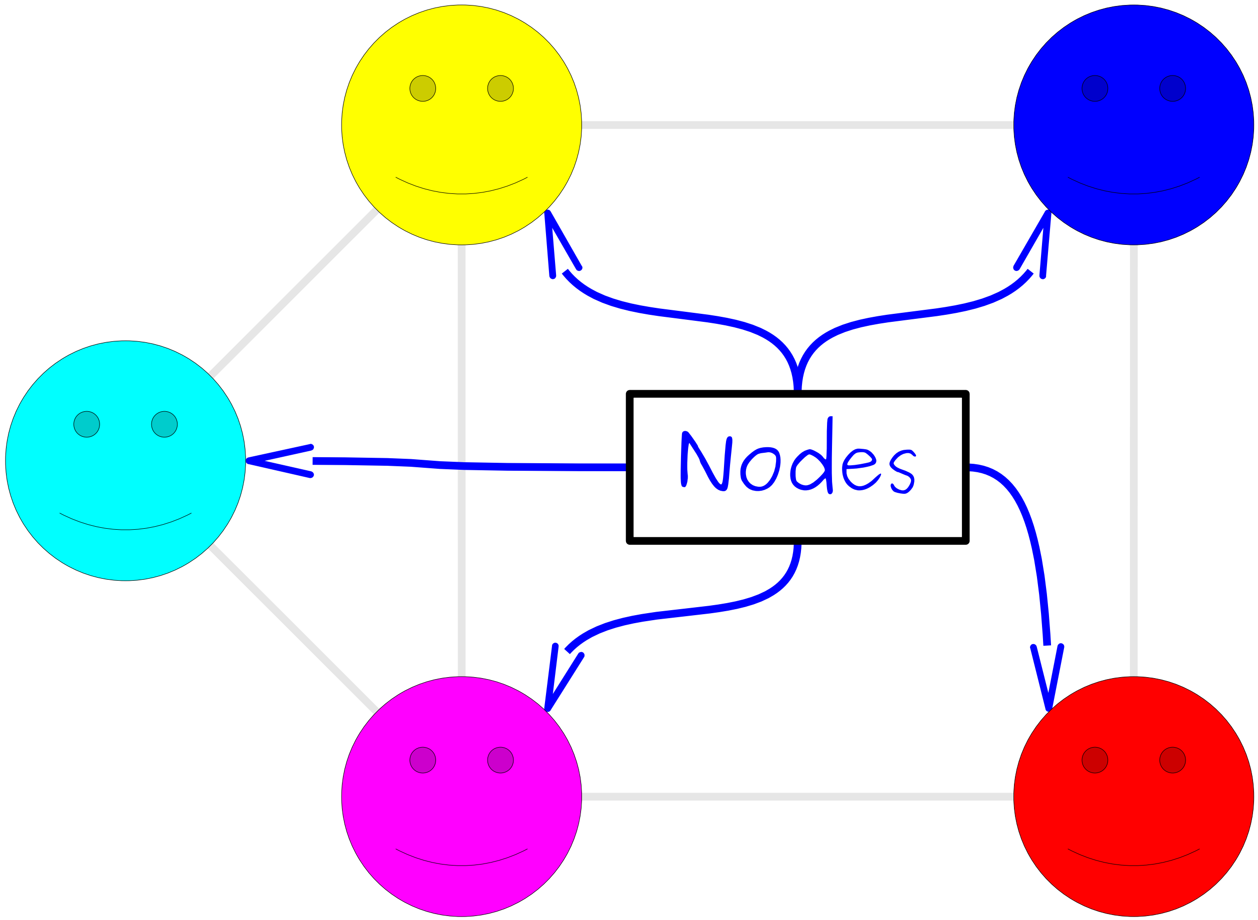


A ***graph*** is a mathematical structure for representing relationships.



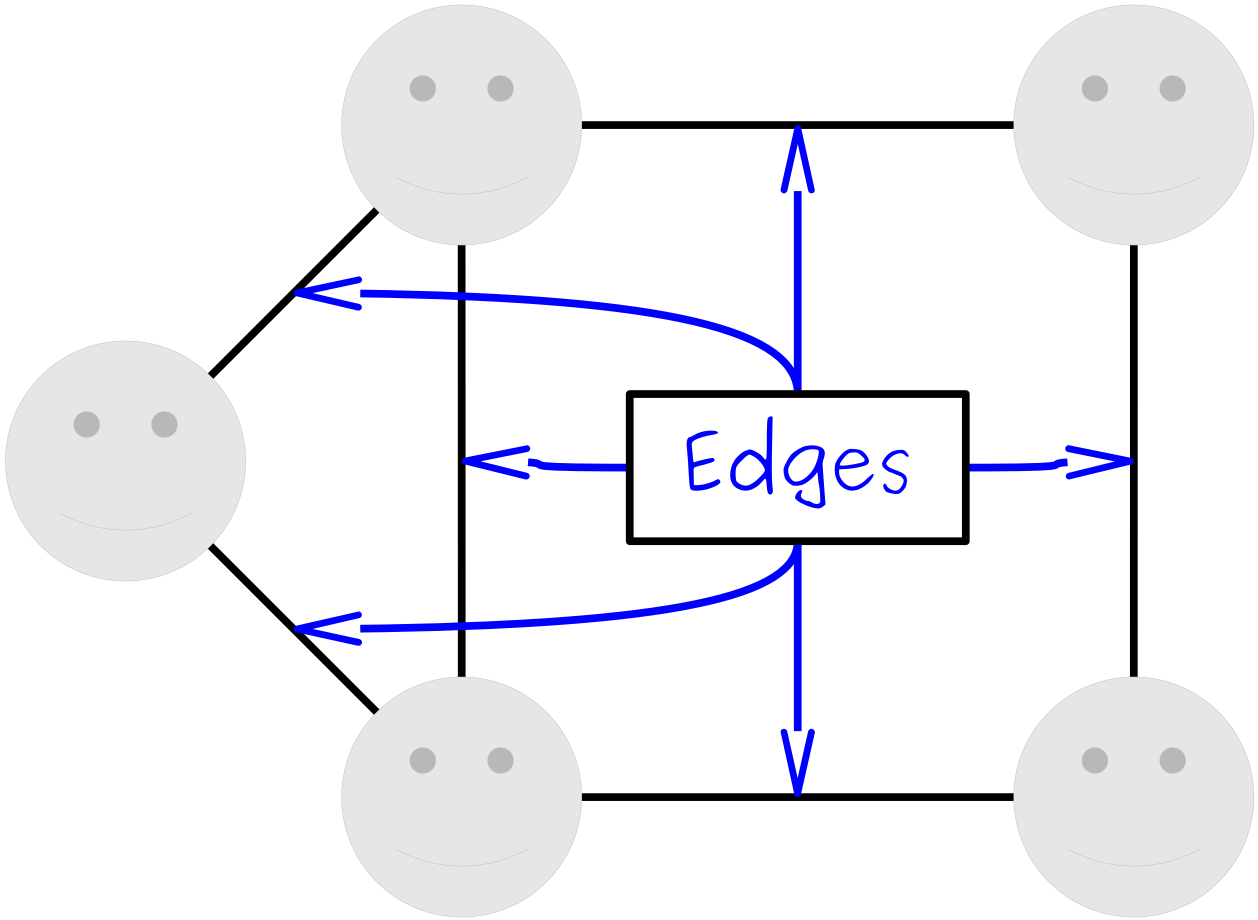
A graph consists of a set of ***nodes*** (or ***vertices***) connected by ***edges*** (or ***arcs***)

A ***graph*** is a mathematical structure for representing relationships.



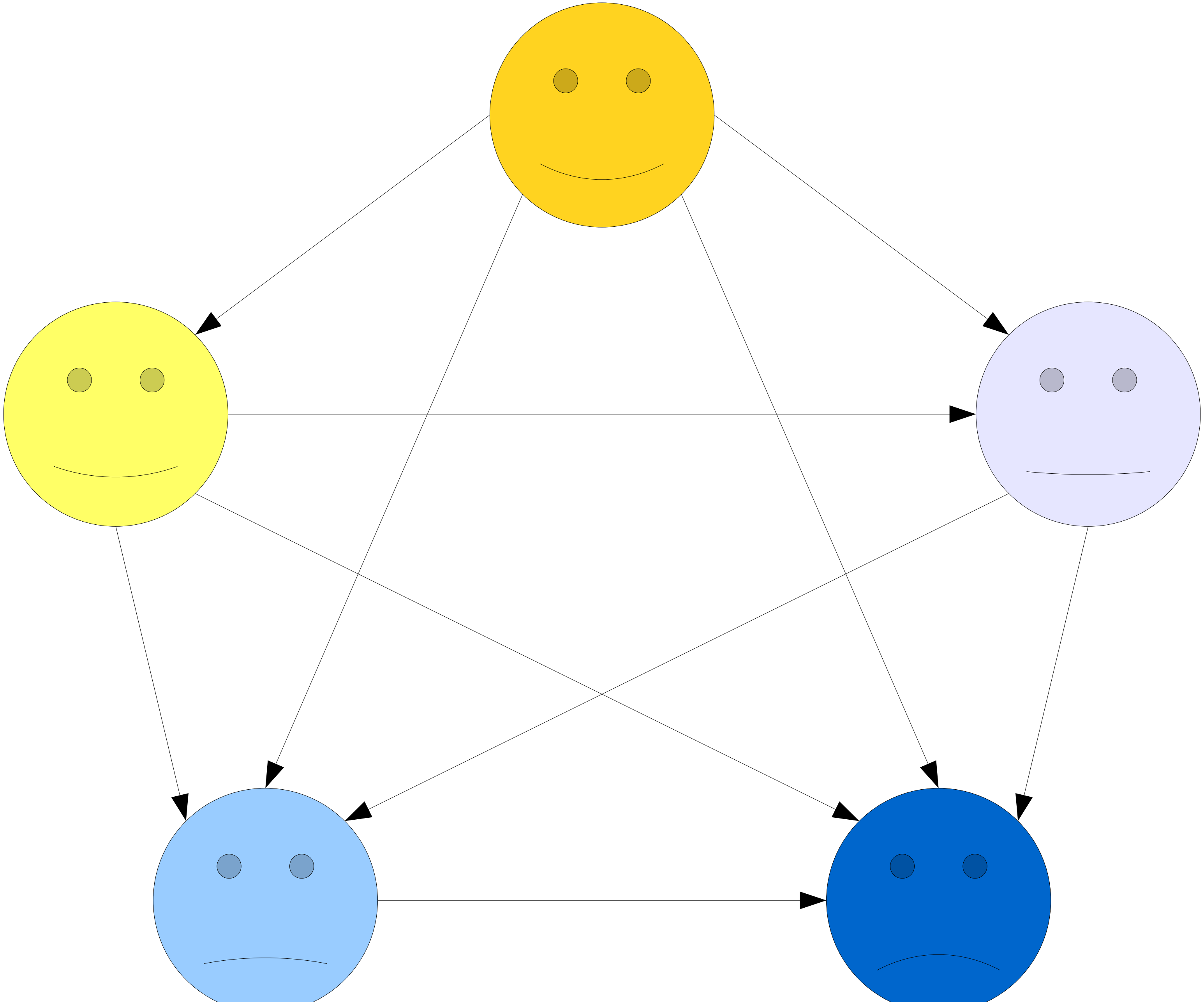
A graph consists of a set of ***nodes*** (or ***vertices***) connected by ***edges*** (or ***arcs***)

A ***graph*** is a mathematical structure for representing relationships.

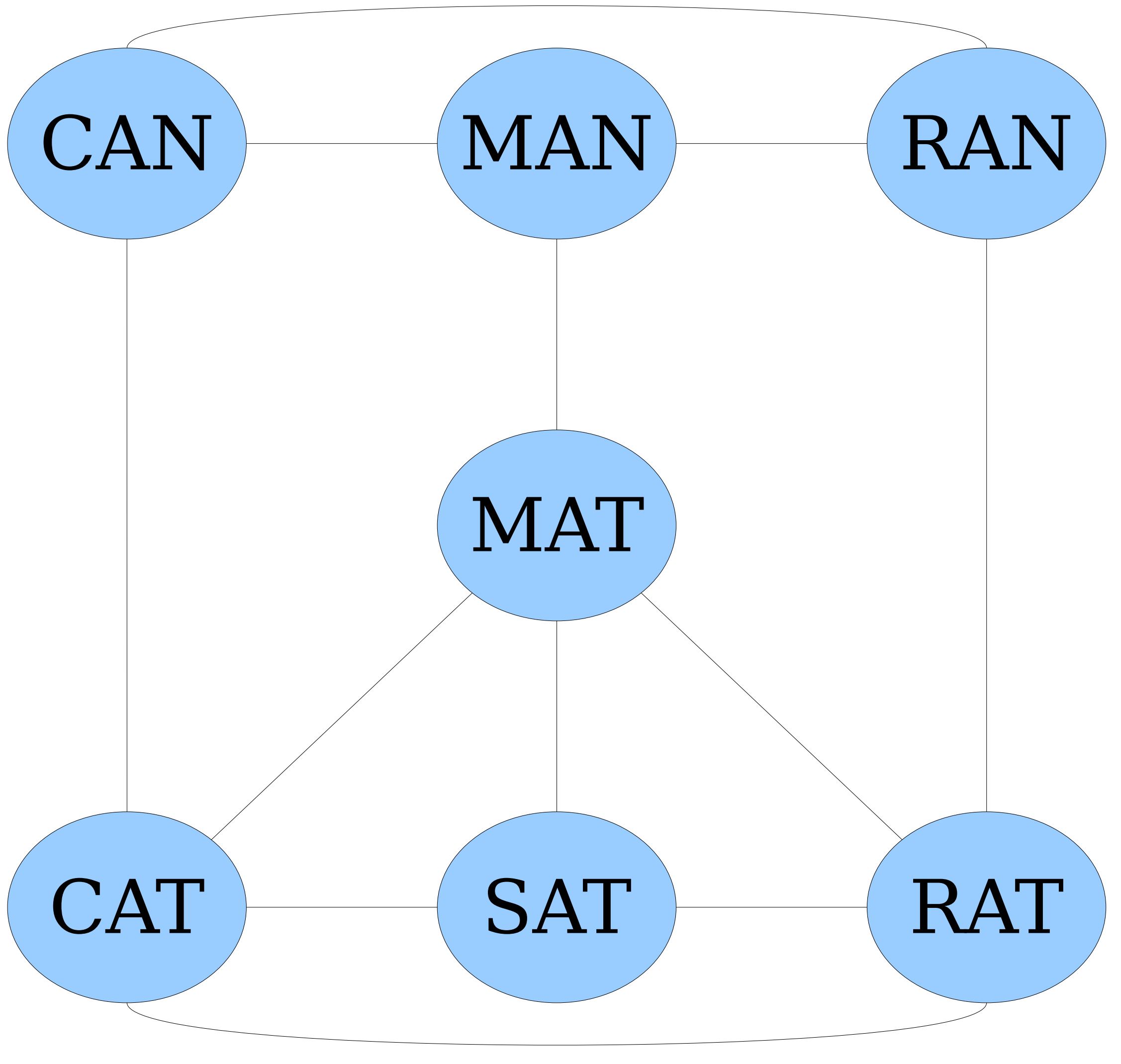


A graph consists of a set of ***nodes*** (or ***vertices***) connected by ***edges*** (or ***arcs***)

Some graphs are ***directed***.



Some graphs are ***undirected***.



Graphs and Digraphs

- An ***undirected graph*** is one where edges link nodes, with no endpoint preferred over the other.
- A ***directed graph*** (or ***digraph***) is one where edges have an associated direction.
- (There's something called a ***mixed graph*** that allows for both, but they're fairly uncommon and we won't talk about them.)
- Unless specified otherwise:

☞ ***"Graph" means "undirected graph"*** ☜

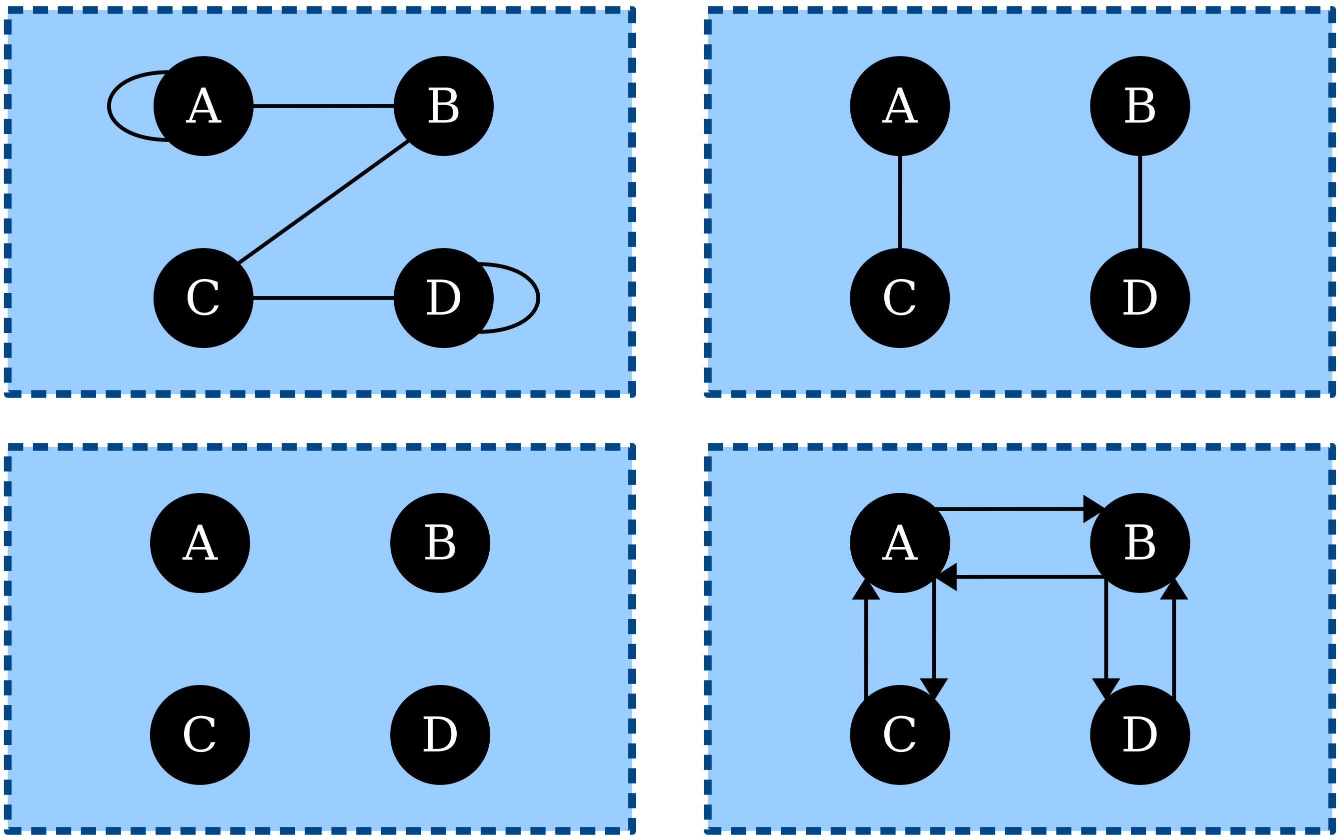
Formalizing Graphs

- How might we define a graph mathematically?
- We need to specify
  - what the nodes in the graph are, and
  - which edges are in the graph.
- The nodes can be pretty much anything.
- What about the edges?

Formalizing Graphs

- An ***unordered pair*** is a set $\{a, b\}$ of two elements $a \neq b$. (Remember that sets are unordered.)
  - For example, $\{0, 1\} = \{1, 0\}$
- An ***undirected graph*** is an ordered pair $G = (V, E)$, where
  - V is a set of nodes, which can be anything, and
  - E is a set of edges, which are *unordered* pairs of nodes drawn from V.
- A ***directed graph*** (or ***digraph***) is an ordered pair $G = (V, E)$, where
  - V is a set of nodes, which can be anything, and
  - E is a set of edges, which are *ordered* pairs of nodes drawn from V.

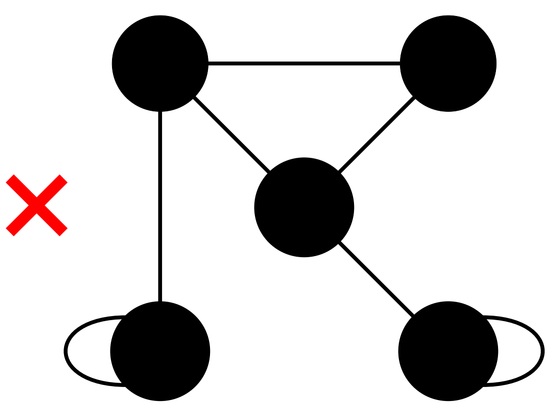
- An ***unordered pair*** is a set $\{a, b\}$ of two elements $a \neq b$.
- An ***undirected graph*** is an ordered pair $G = (V, E)$, where
  - V is a set of nodes, which can be anything, and
  - E is a set of edges, which are unordered pairs of nodes drawn from V.

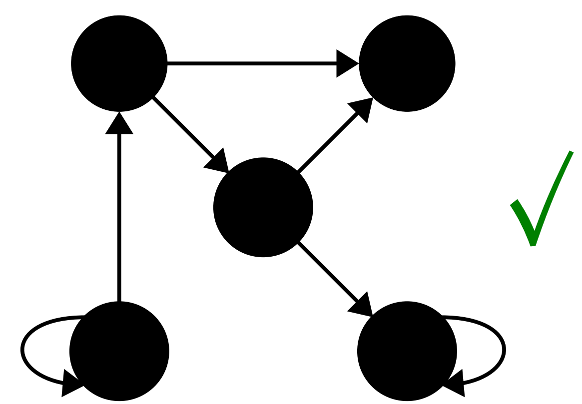


How many of these drawings are of valid undirected graphs?

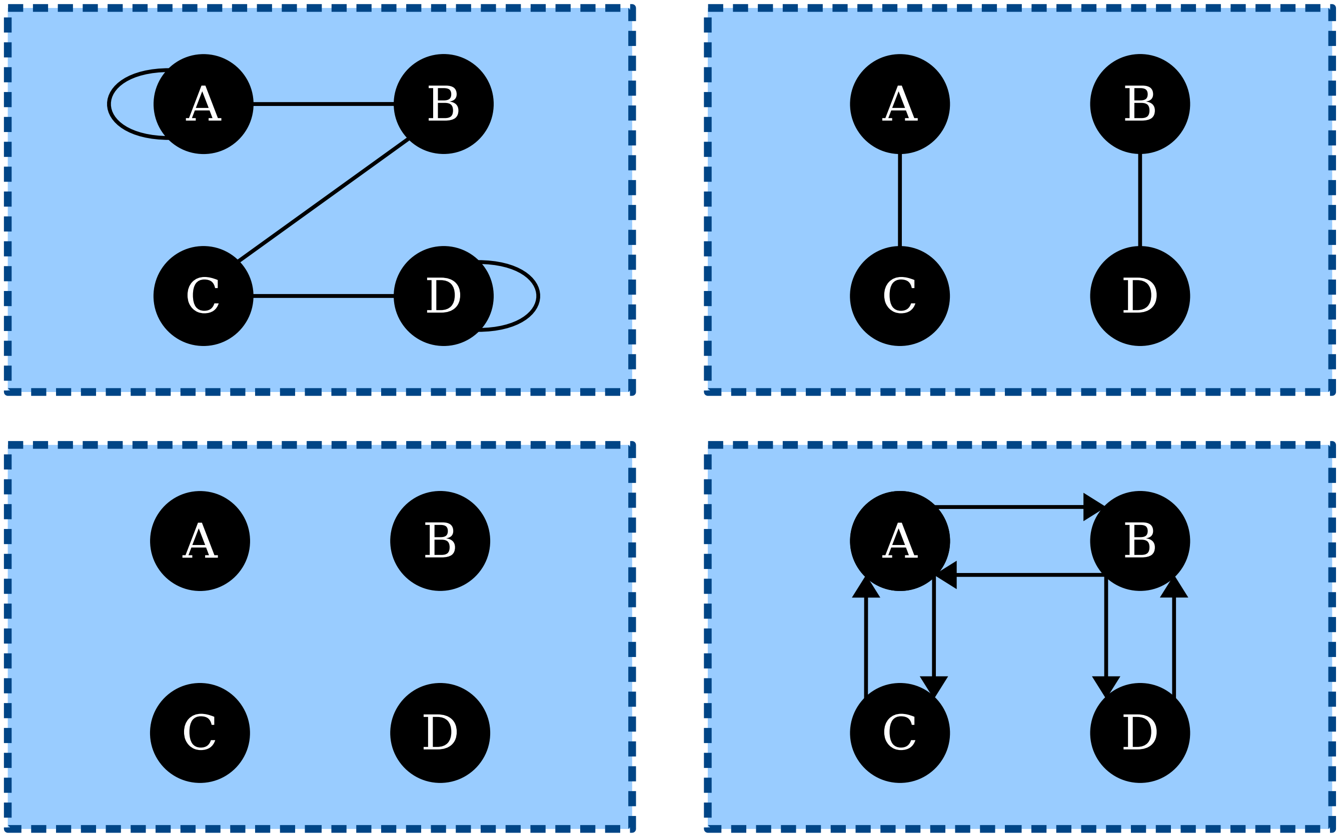
Self-Loops

- An edge from a node to itself is called a ***self-loop***.
- In (undirected) graphs, self-loops are generally not allowed.
  - Can you see how this follows from the definition?
- In digraphs, self-loops are generally allowed unless specified otherwise.



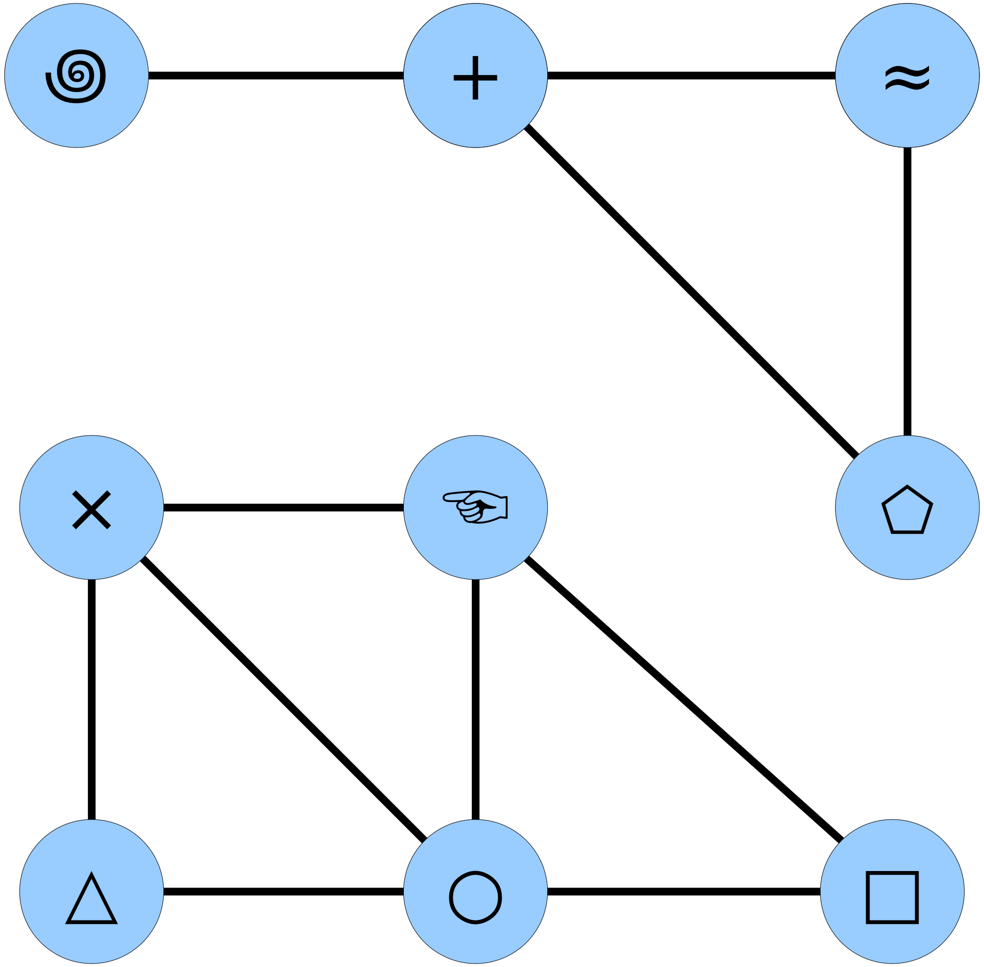


- An ***unordered pair*** is a set $\{a, b\}$ of two elements $a \neq b$.
- An ***undirected graph*** is an ordered pair $G = (V, E)$, where
  - V is a set of nodes, which can be anything, and
  - E is a set of edges, which are unordered pairs of nodes drawn from V.



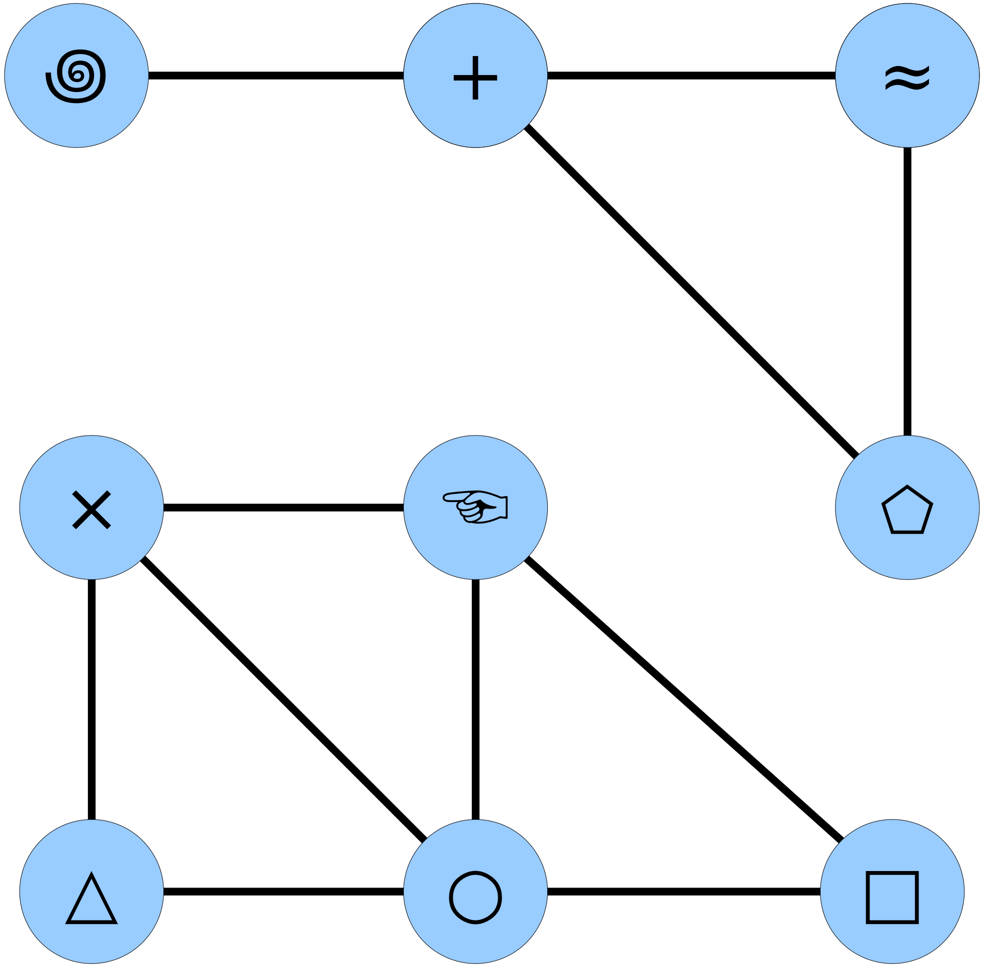
How many of these drawings are of valid undirected graphs?

The Great Graph Gallery



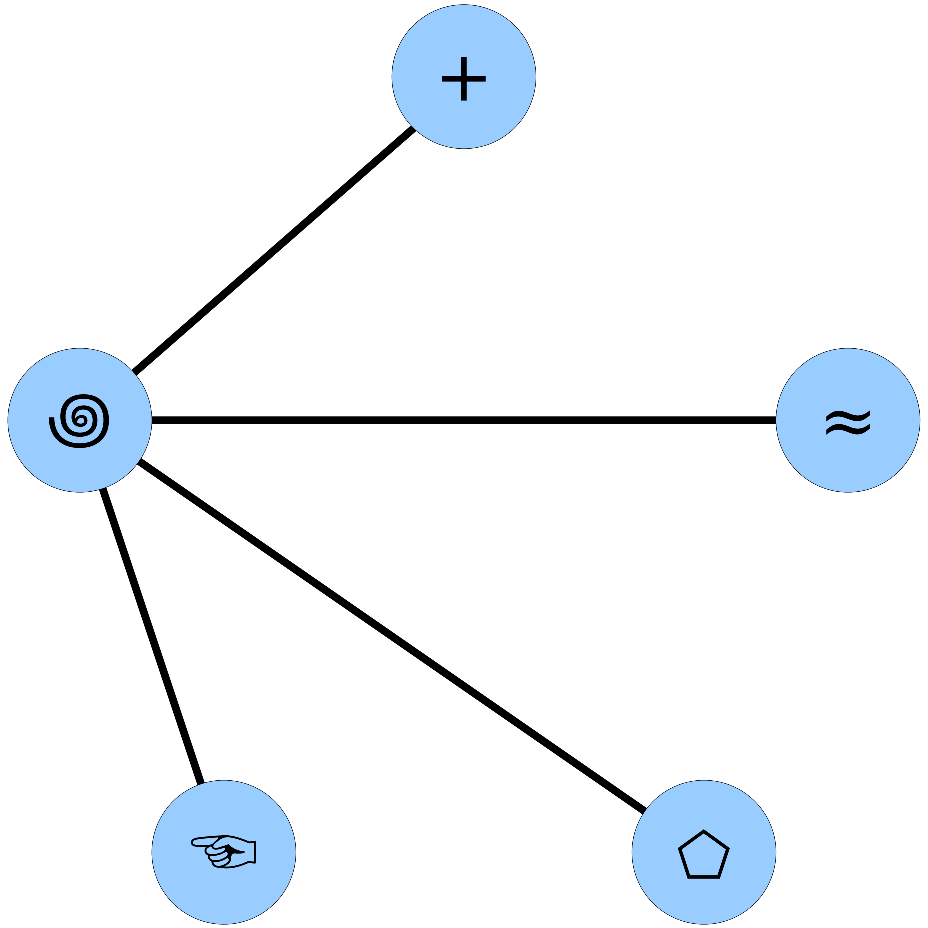
Is this formula true about this graph?

$$\forall u \in V.\ \exists v \in V.\ \{u, v\} \in E$$



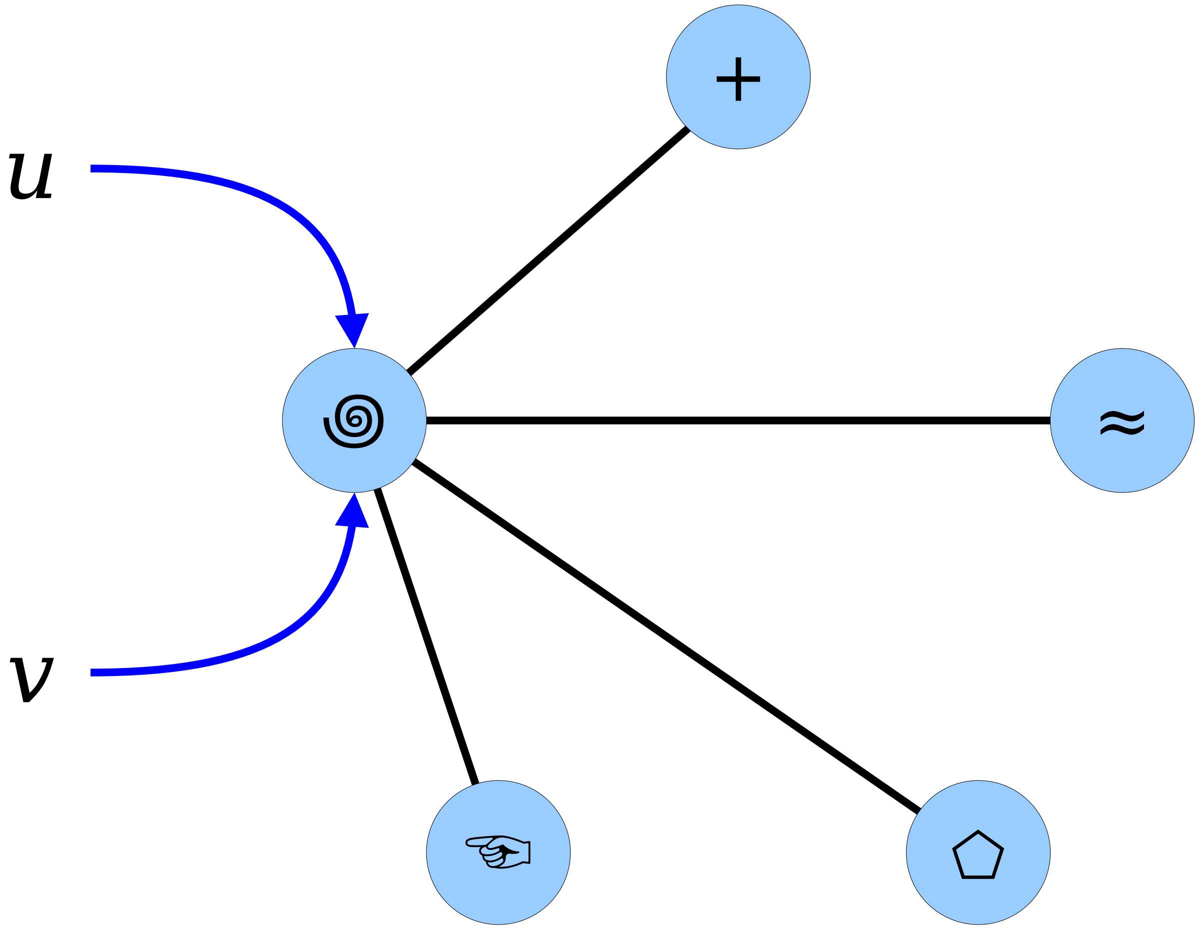
Is this formula true about this graph?

$$\exists u \in V.\ \forall v \in V.\ \{u, v\} \in E$$



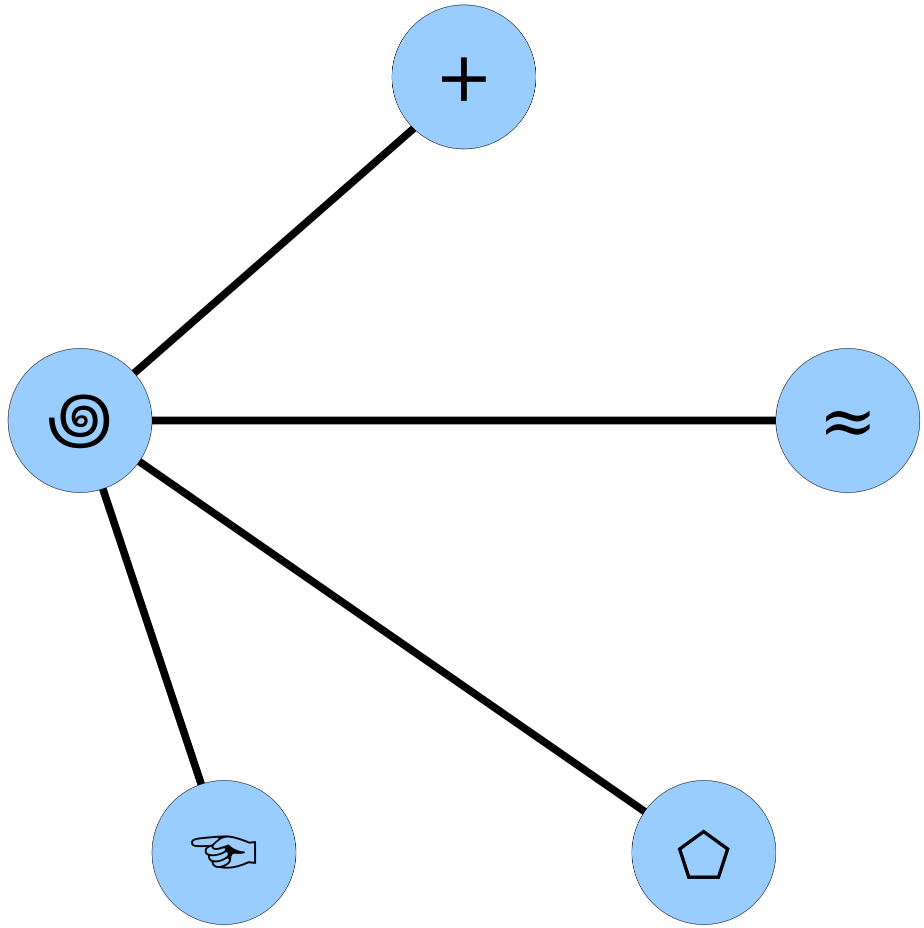
Is this formula true about this graph?

$$\exists u \in V.\ \forall v \in V.\ \{u, v\} \in E$$



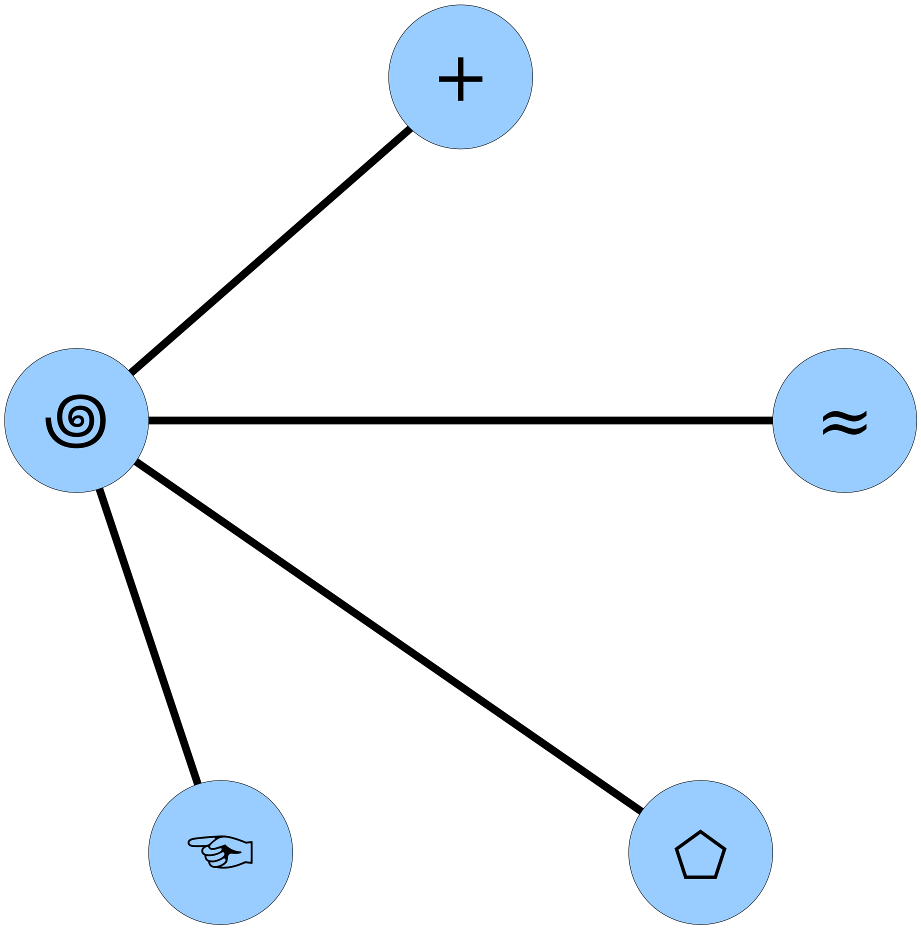
Is this formula true about this graph?

$$\exists u \in V.\ \forall v \in V.\ \{u, v\} \in E$$



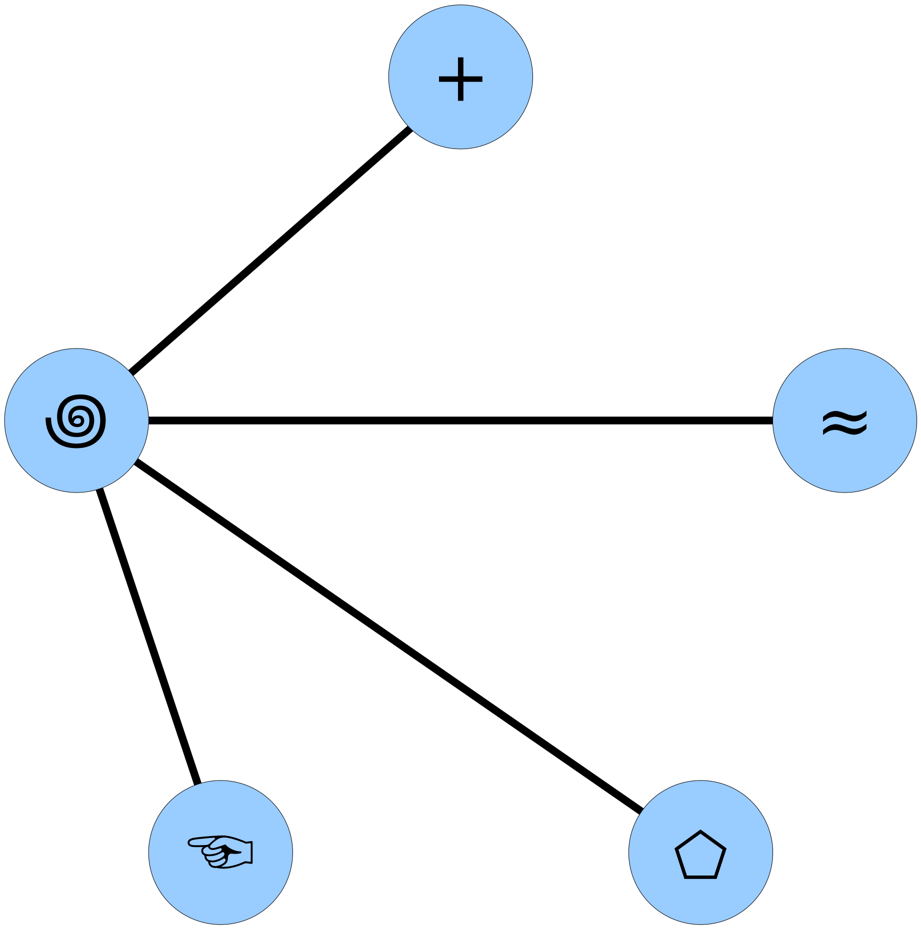
Let’s look at the negation!

$$\exists u \in V.\ \forall v \in V.\ \{u, v\} \in E$$



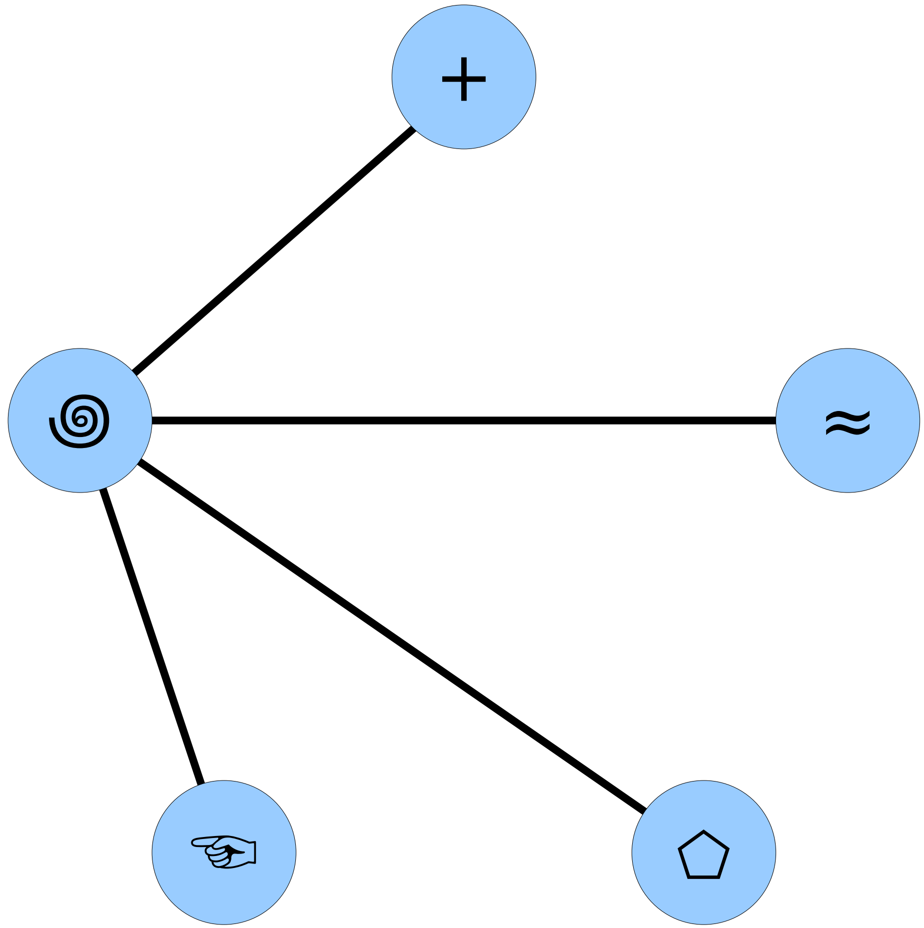
Let's look at the negation!

$$\neg \exists u \in V.\ \forall v \in V.\ \{u, v\} \in E$$



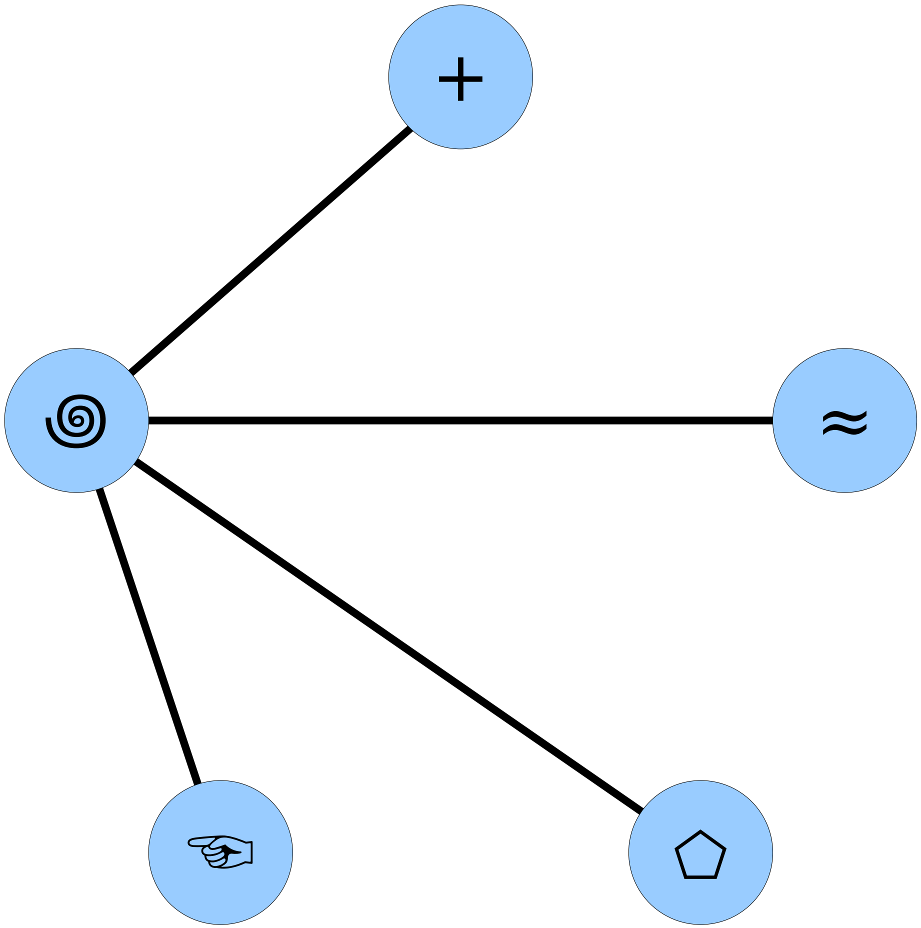
Let’s look at the negation!

$$\forall u \in V.\ \neg\forall v \in V.\ \{u, v\} \in E$$



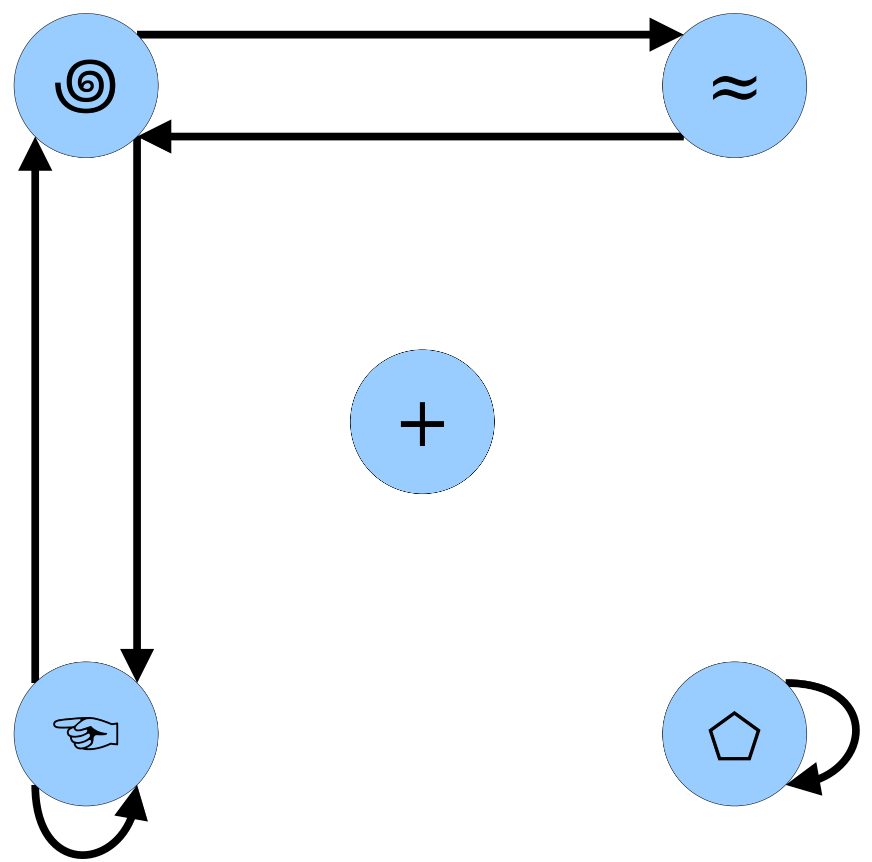
Let’s look at the negation!

$$\forall u \in V.\ \exists v \in V.\ \neg(\{u, v\} \in E)$$



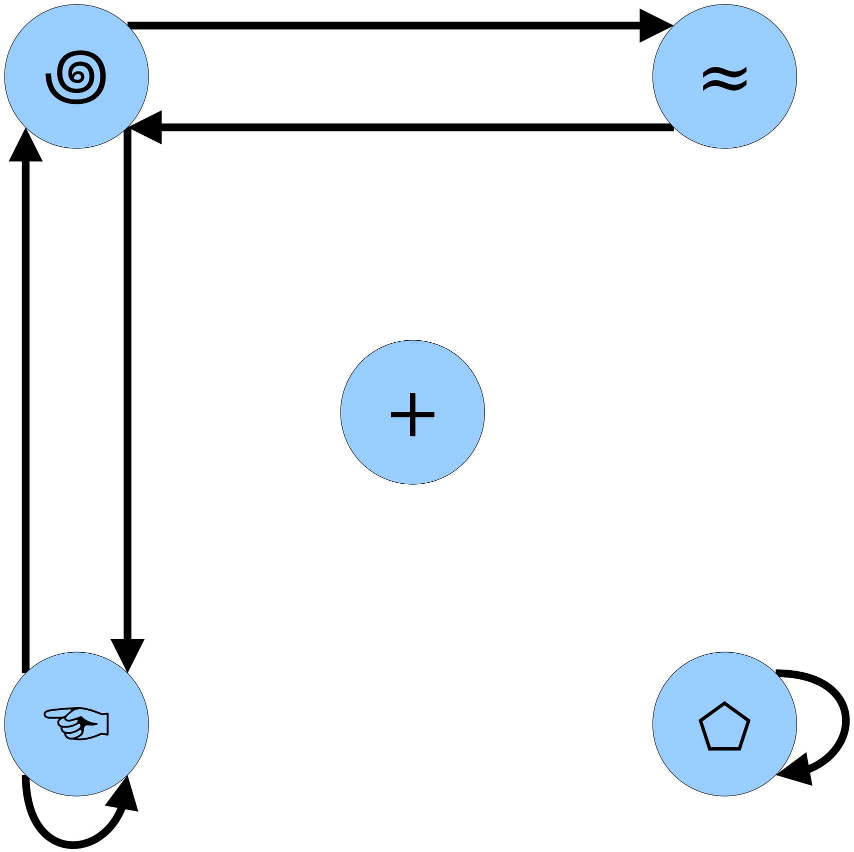
Let’s look at the negation!

$$\forall u \in V.\ \exists v \in V.\ \{u, v\} \notin E$$



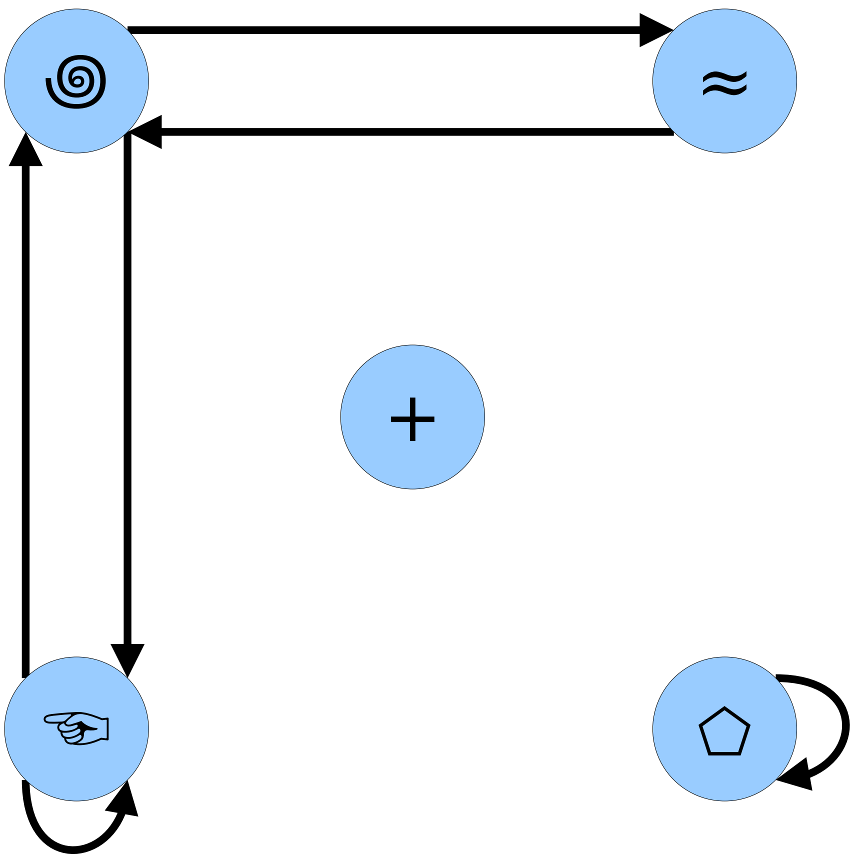
Is this formula true about this digraph?

$$\forall u \in V.\ \forall v \in V.\ ((u, v) \in E \rightarrow (v, u) \in E)$$



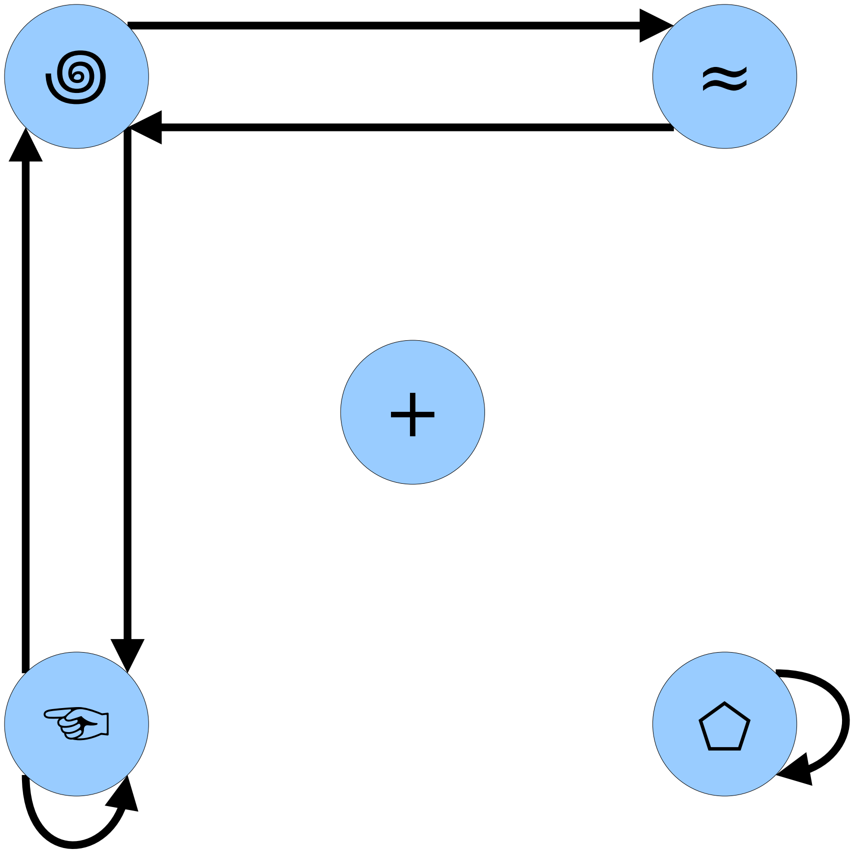
What is the negation of this statement?

$$\forall u \in V.\ \forall v \in V.\ ((u, v) \in E \rightarrow (v, u) \in E)$$



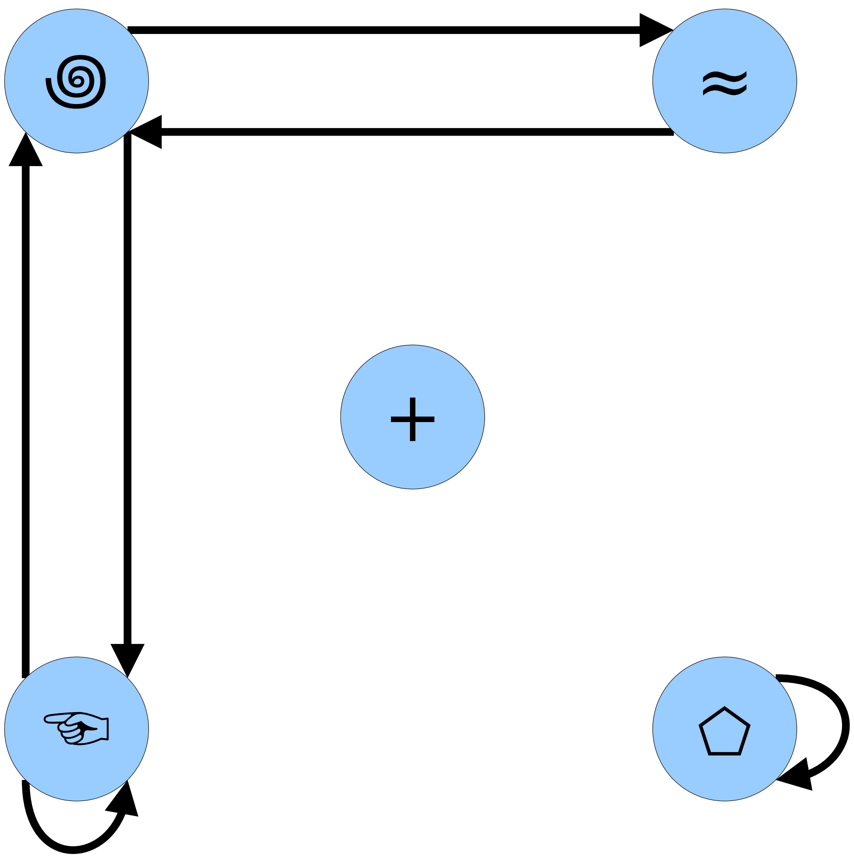
What is the negation of this statement?

$$\neg \forall u \in V.\ \forall v \in V.\ ((u, v) \in E \rightarrow (v, u) \in E)$$



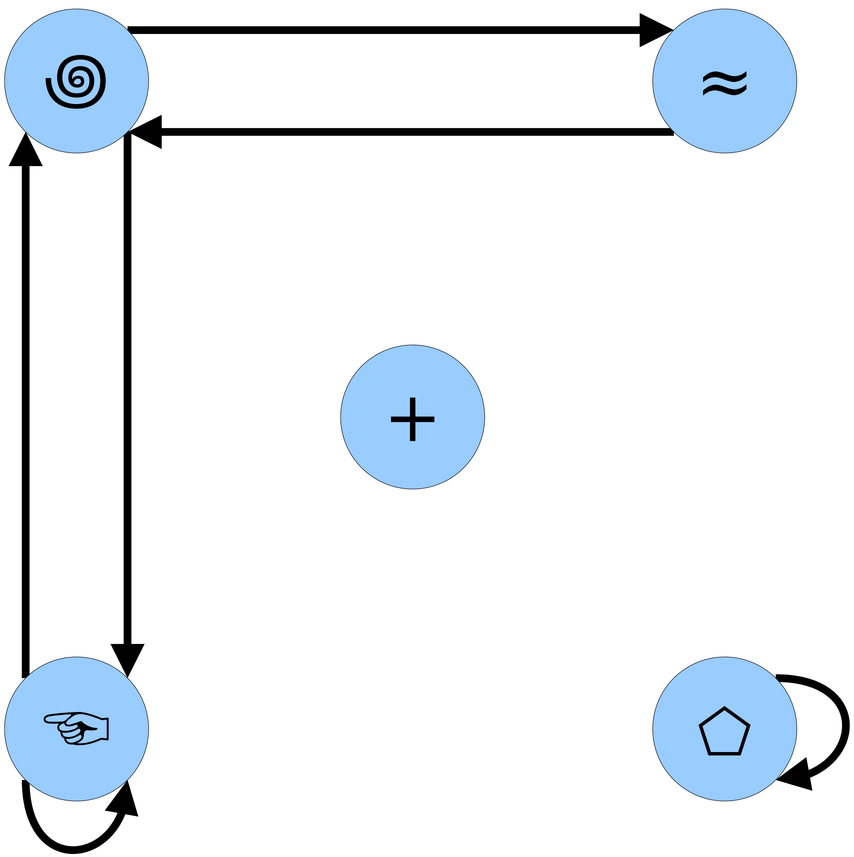
What is the negation of this statement?

$$\exists u \in V. \neg \forall v \in V. ((u, v) \in E \to (v, u) \in E)$$



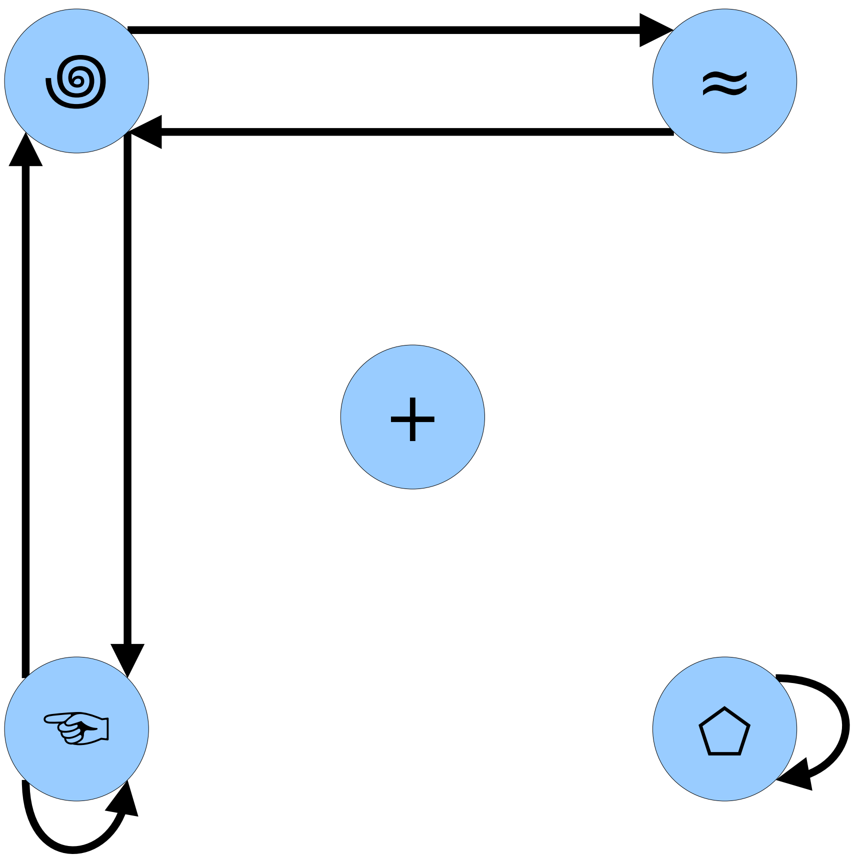
What is the negation of this statement?

$$\exists u \in V.\ \exists v \in V.\ \neg((u, v) \in E \rightarrow (v, u) \in E)$$



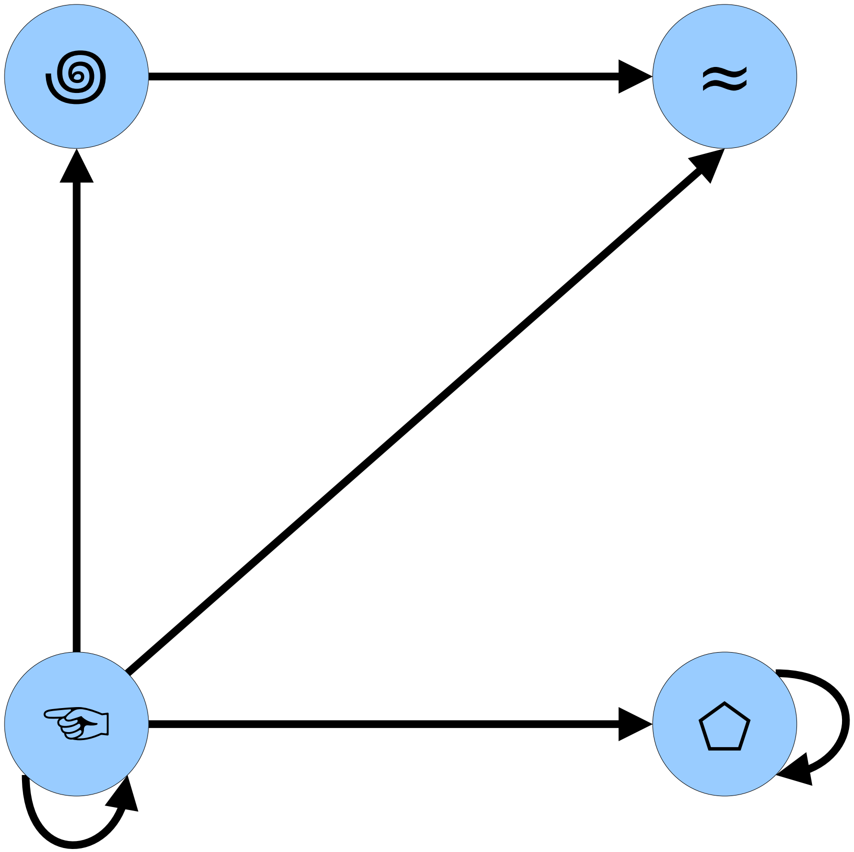
What is the negation of this statement?

$$\exists u \in V.\ \exists v \in V.\ ((u, v) \in E \land (v, u) \notin E)$$



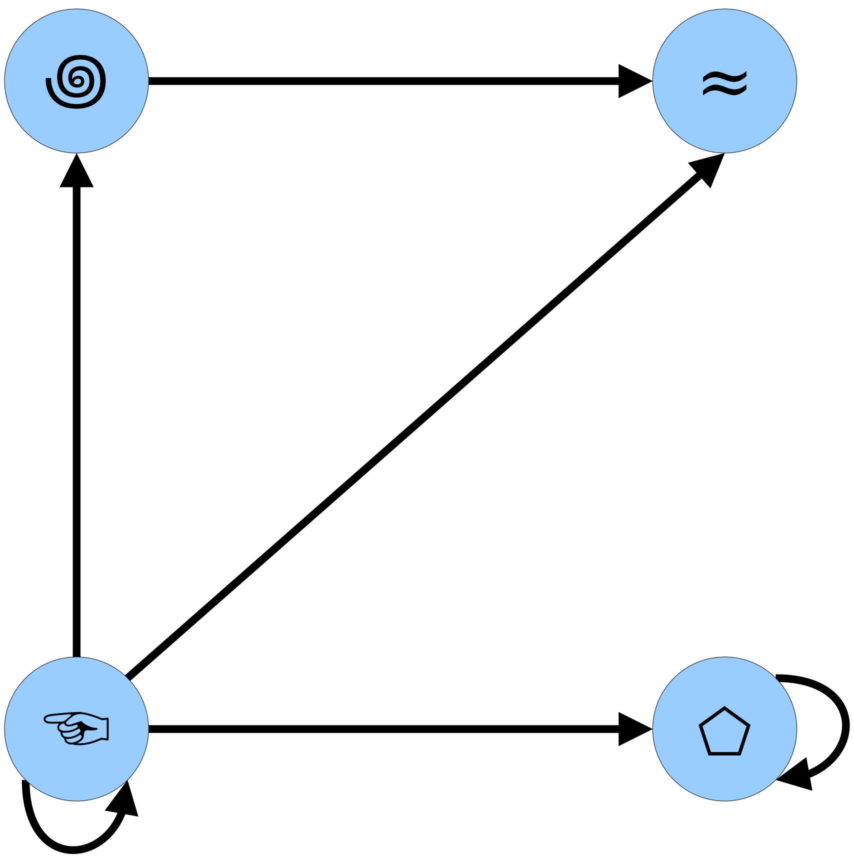
Is the negation true?

$$\exists u \in V.\ \exists v \in V.\ ((u, v) \in E \wedge (v, u) \notin E)$$



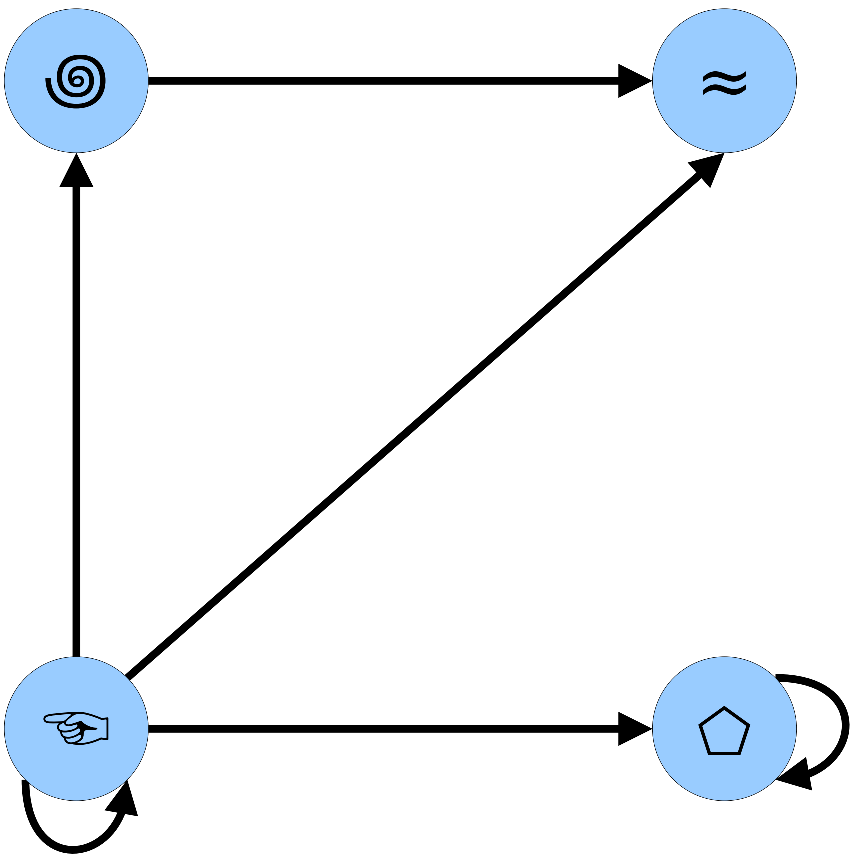
Is this formula true about this digraph?

$$\forall x \in V. \forall y \in V. \forall z \in V. \; ((x, y) \in E \land (y, z) \in E \rightarrow (x, z) \in E)$$



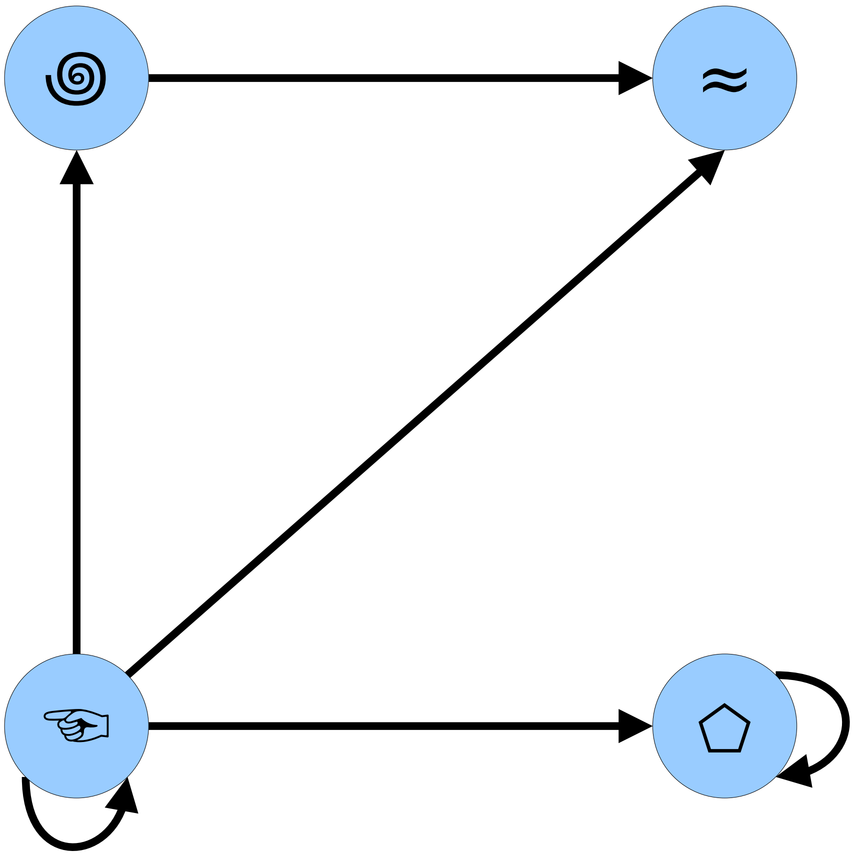
What is the negation of this statement?

$$\forall x \in V.\ \forall y \in V.\ \forall z \in V.\ (((x, y) \in E \land (y, z) \in E) \to (x, z) \in E)$$



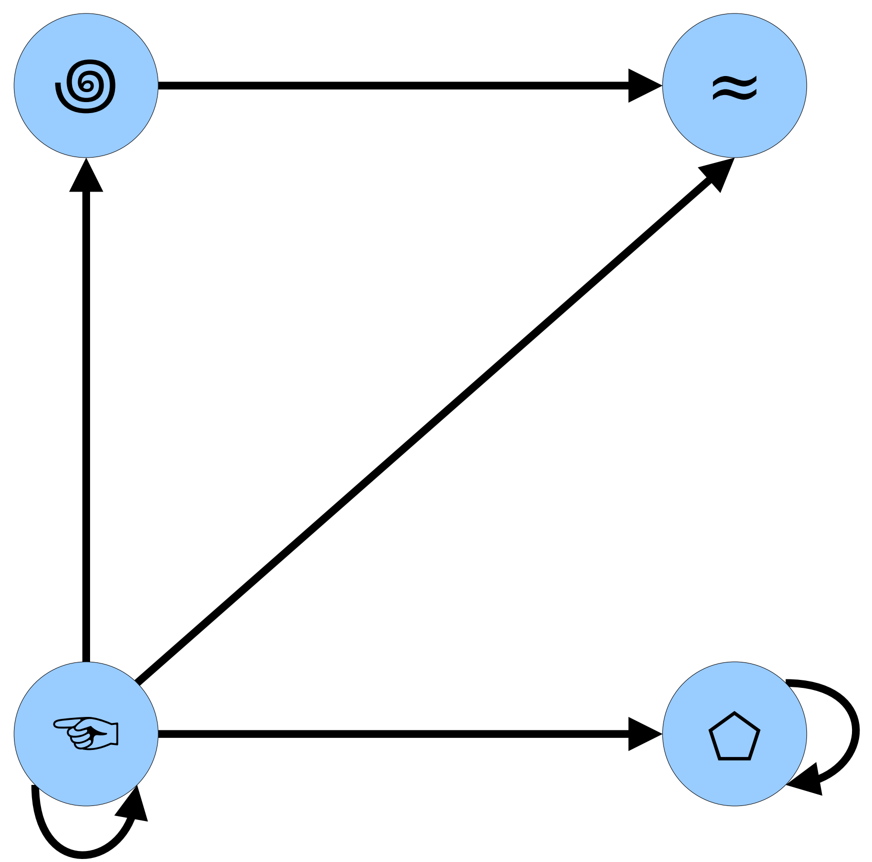
What is the negation of this statement?

$$\exists x \in V.\ \exists y \in V.\ \exists z \in V.\ ((x, y) \in E \land (y, z) \in E \land (x, z) \notin E)$$



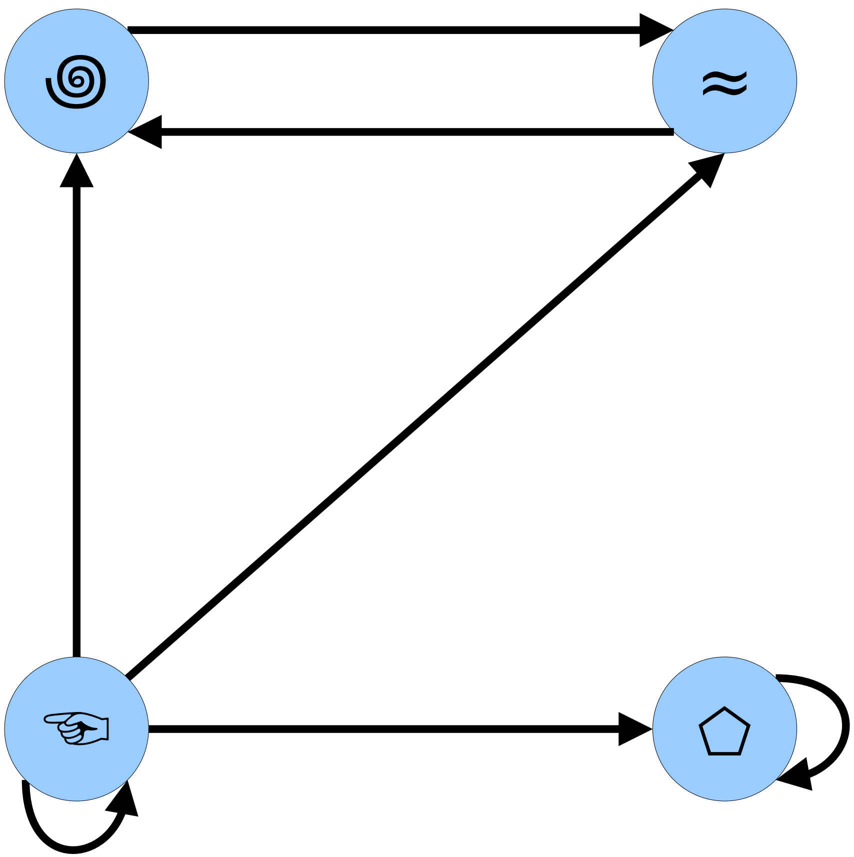
Is this negation true?

$\exists x \in V.\ \exists y \in V.\ \exists z \in V.\ ((x, y) \in E \land (y, z) \in E \land (x, z) \notin E)$



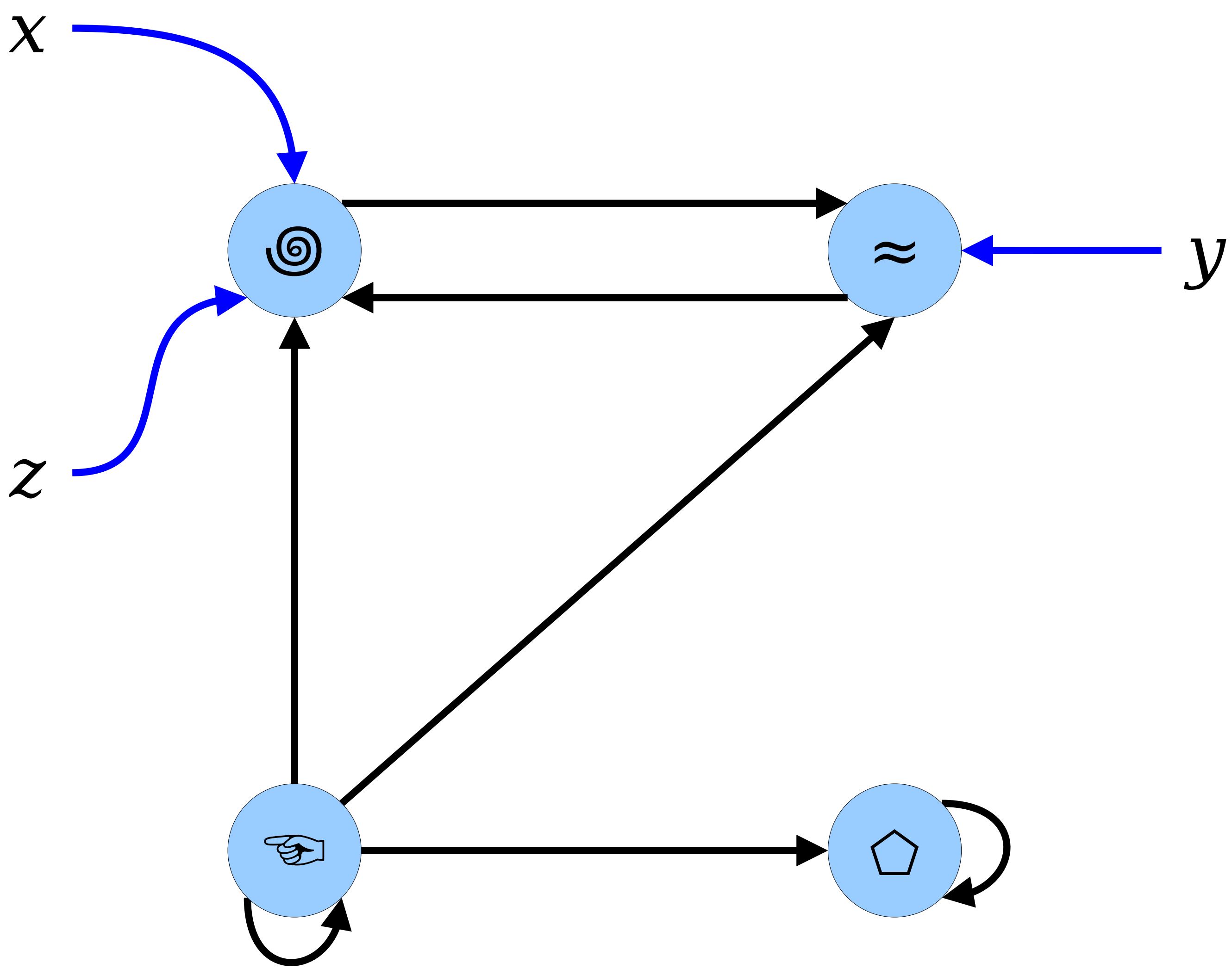
Is this formula true about this digraph?

$$\forall x \in V.\ \forall y \in V.\ \forall z \in V.\ (((x, y) \in E \land (y, z) \in E \to (x, z) \in E)$$



Is this formula true about this digraph?

$$\forall x \in V.\ \forall y \in V.\ \forall z \in V.\ (((x, y) \in E \land (y, z) \in E \rightarrow (x, z) \in E)$$



Is this formula true about this digraph?

$\forall x \in V.\ \forall y \in V.\ \forall z \in V.\ (((x, y) \in E \wedge (y, z) \in E \rightarrow (x, z) \in E)$

Time-Out for Announcements!

PS2 Solutions Released

- Solutions to Problem Set Two are now available on the course website.
- We generally don't release solutions to autograded problems. If you have any questions about those, please ping us privately over EdStem or come talk to us in our office hours.
- PS3 is due this Friday at 2:30PM, right before the take-home midterm starts.

Research Mentorship

- The CS department is launching an undergraduate research mentorship program.
- This seems like an amazing opportunity. Check your email for more details.
- Apply online here by October 15:

  https://forms.gle/tZwSVUuWEfdA6BJ57

Your Questions

"How do you stop worrying so much about grades and focus more on learning in classes? I feel like people always talk about how "grades don't matter" when it comes to working in the industry, but it's hard to ignore the pressure of maintaining high marks (especially when your goals might be grad school)."

This hits at a broader question – what do you want to do with your time here at Stanford and how do you want to grow? There will always be more available to do than you'll be able to fit into your schedule (classes, student groups, music, exploration, research, work, etc.), and ultimately your goal is to figure out how to partition your time across things that matter. That's deeply personal, and something you'll need to think about.

Take charge of your learning and get out of your classes what you want to get out of them. If your goal is to learn new concepts, always focus first on the learning and skill acquisition and put in the time to get additional practice and ask questions. If your goal is to get a sense of what's out there, put in as much work as you think is necessary to achieve that.

As for grad school – for PhD programs your research experience probably carries more weight than your GPA. For MS programs your GPA matters, but probably more in the vein of "get more As than Bs" than "nail a 4.0."

“Is 137 your favorite number? If so, why? If not, why do you always use it as an example?”

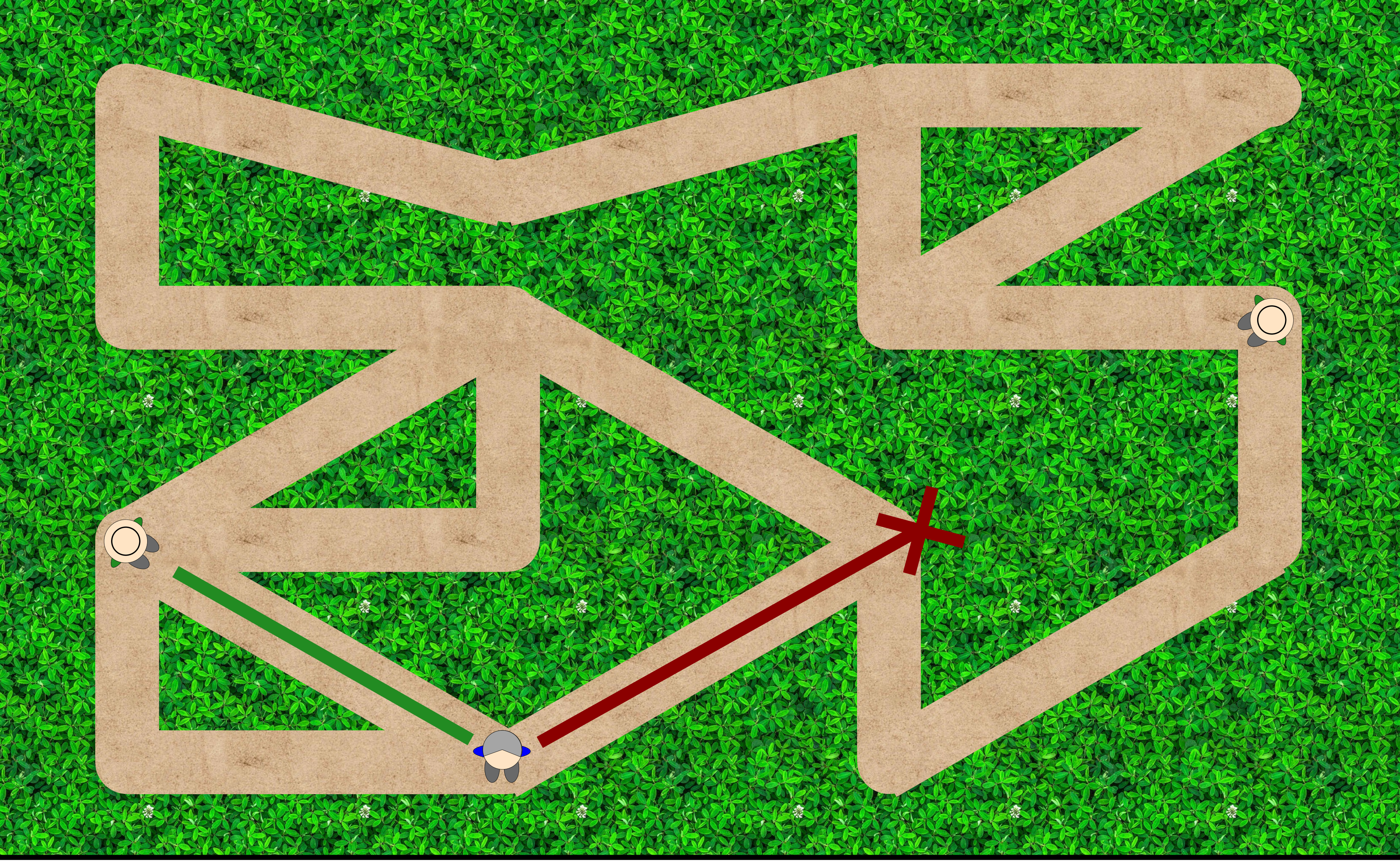
It's the reciprocal of the fine structure constant, rounded to the nearest integer. Throw a bunch of fundamental constants into a formula and this number pops out. There are a lot of interesting stories behind it.

It's also a great “nothing-up-my-sleeve” number. (That's a real term in CS, by the way – it's used in crypto to mean “we needed to pick a number, and we picked one because it's so clearly not something anyone could manipulate that we're confident it wasn't chosen with some subterfuge in mind.)

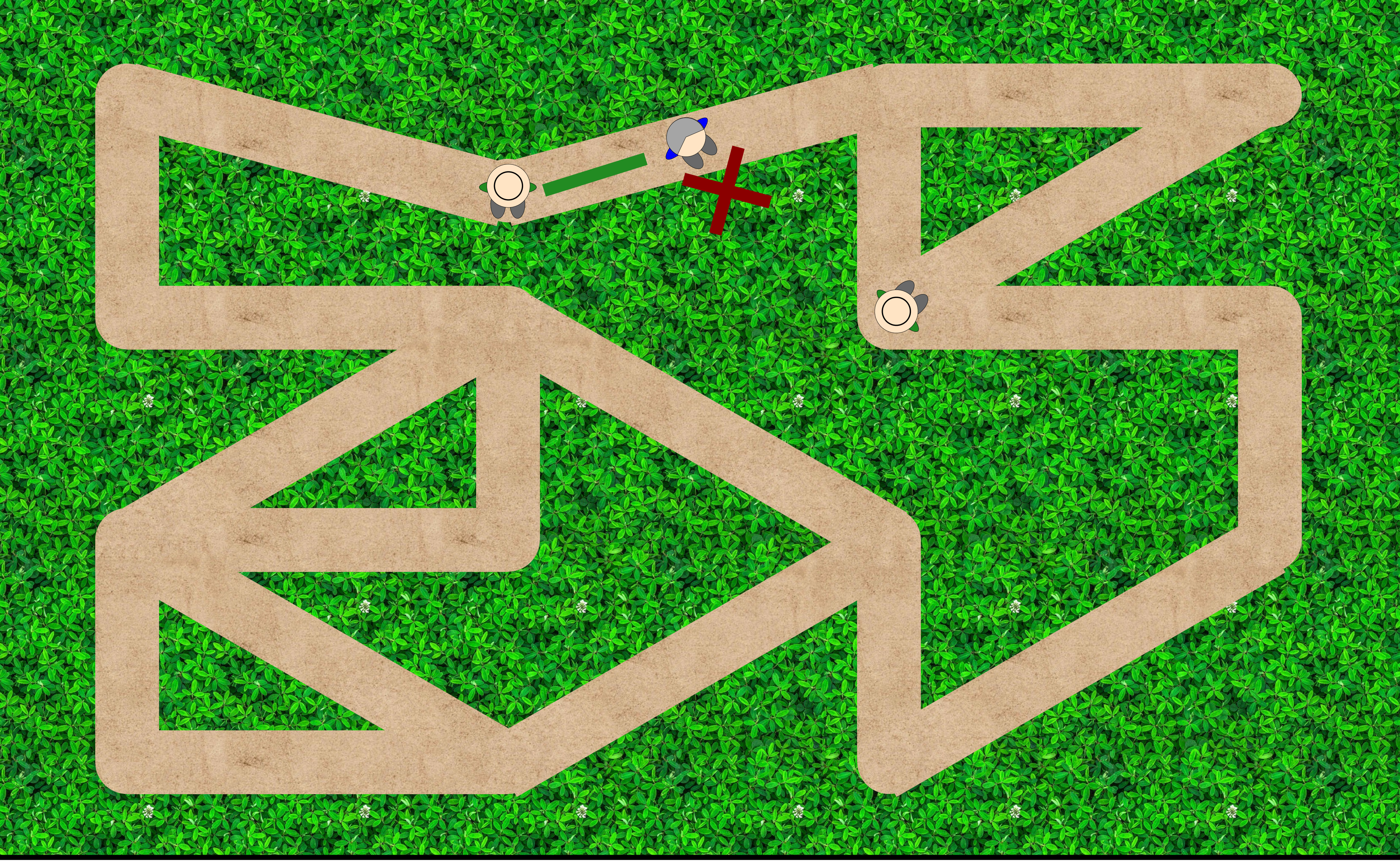
Back to CS103!

Independent Sets and Vertex Covers

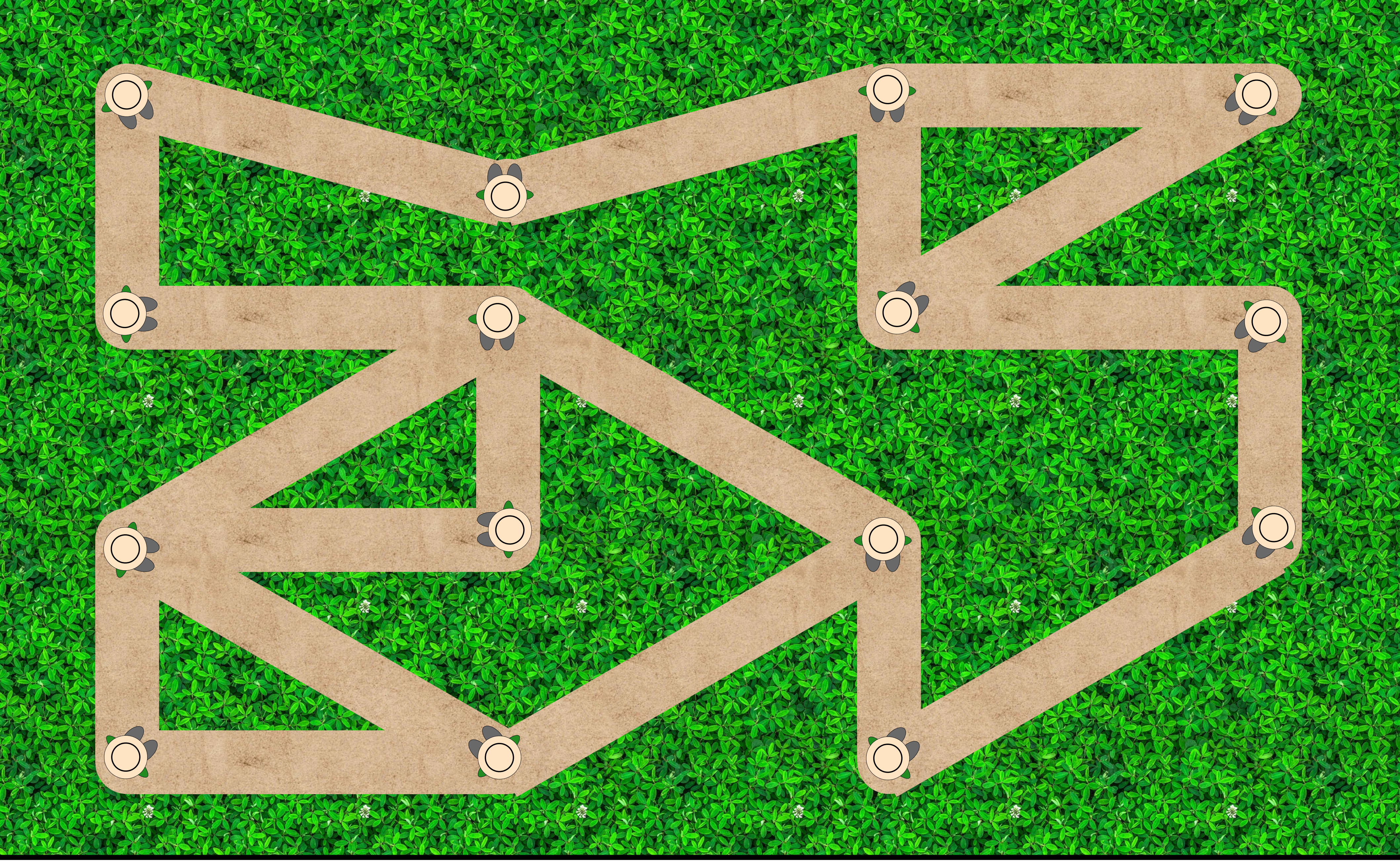
Two Motivating Problems



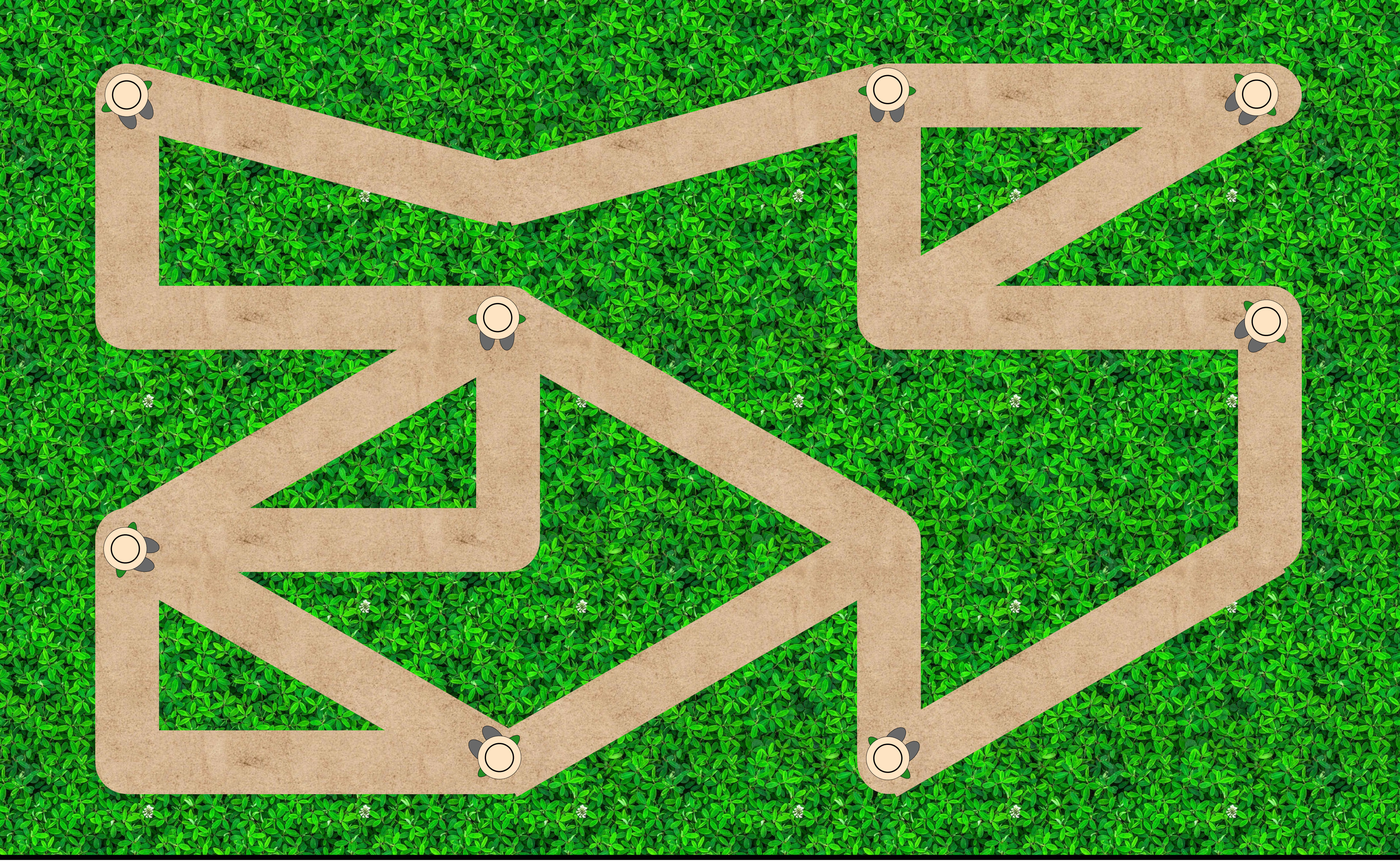
Place park rangers in these forest trails so that a hiker anywhere on a trail can see a park ranger.



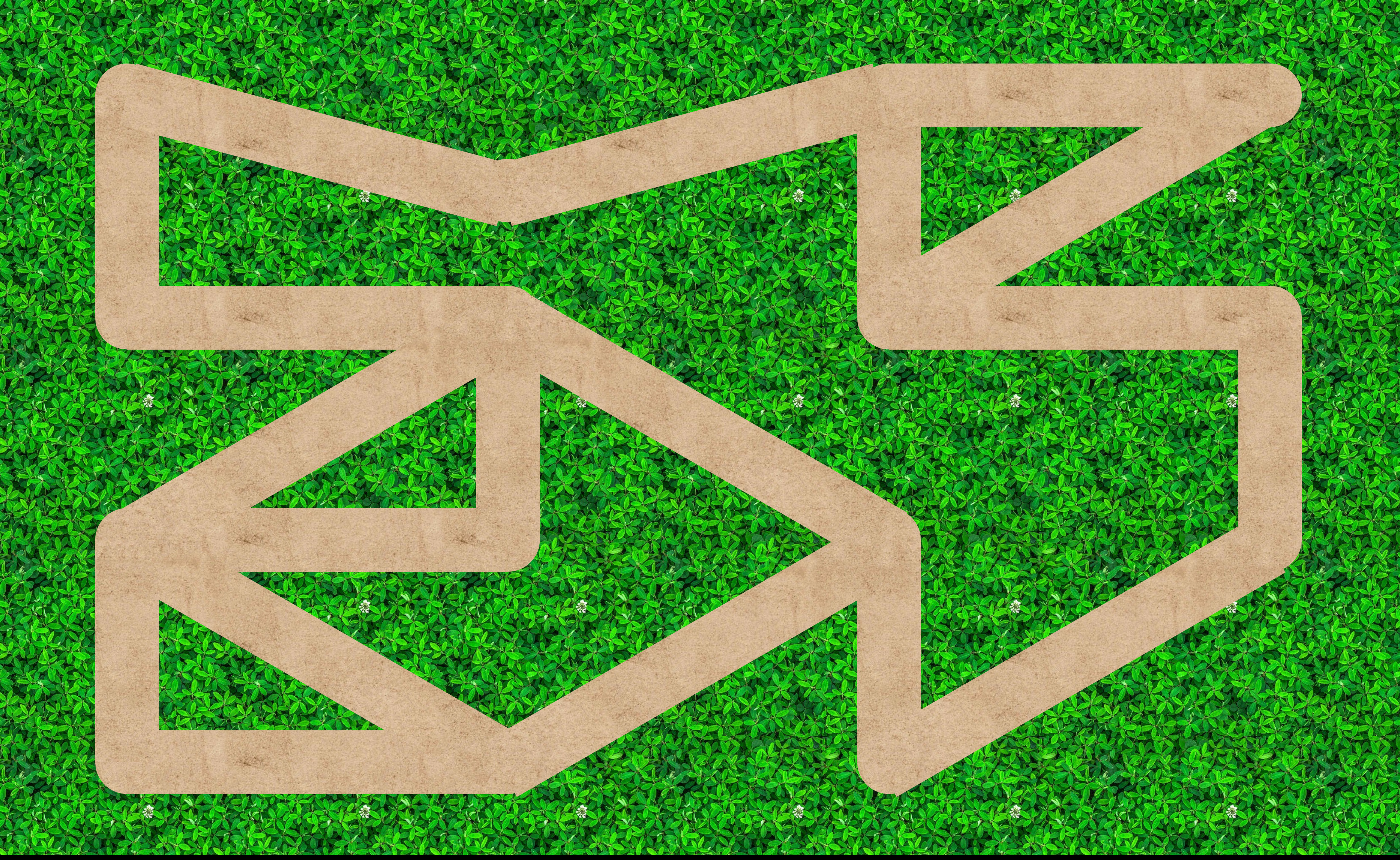
Place park rangers in these forest trails so that
a hiker anywhere on a trail can see a park ranger.



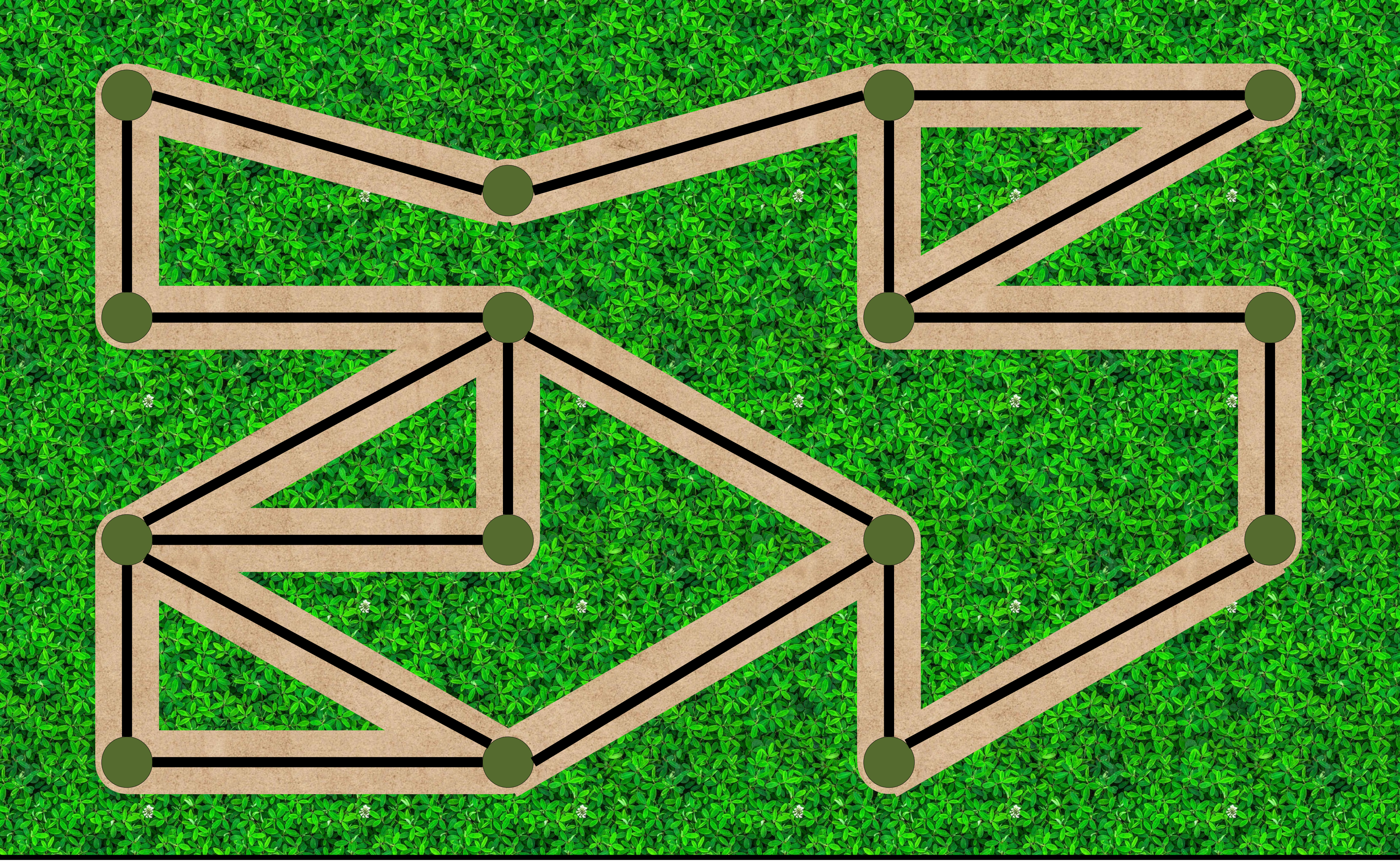
Place park rangers in these forest trails so that a hiker anywhere on a trail can see a park ranger.



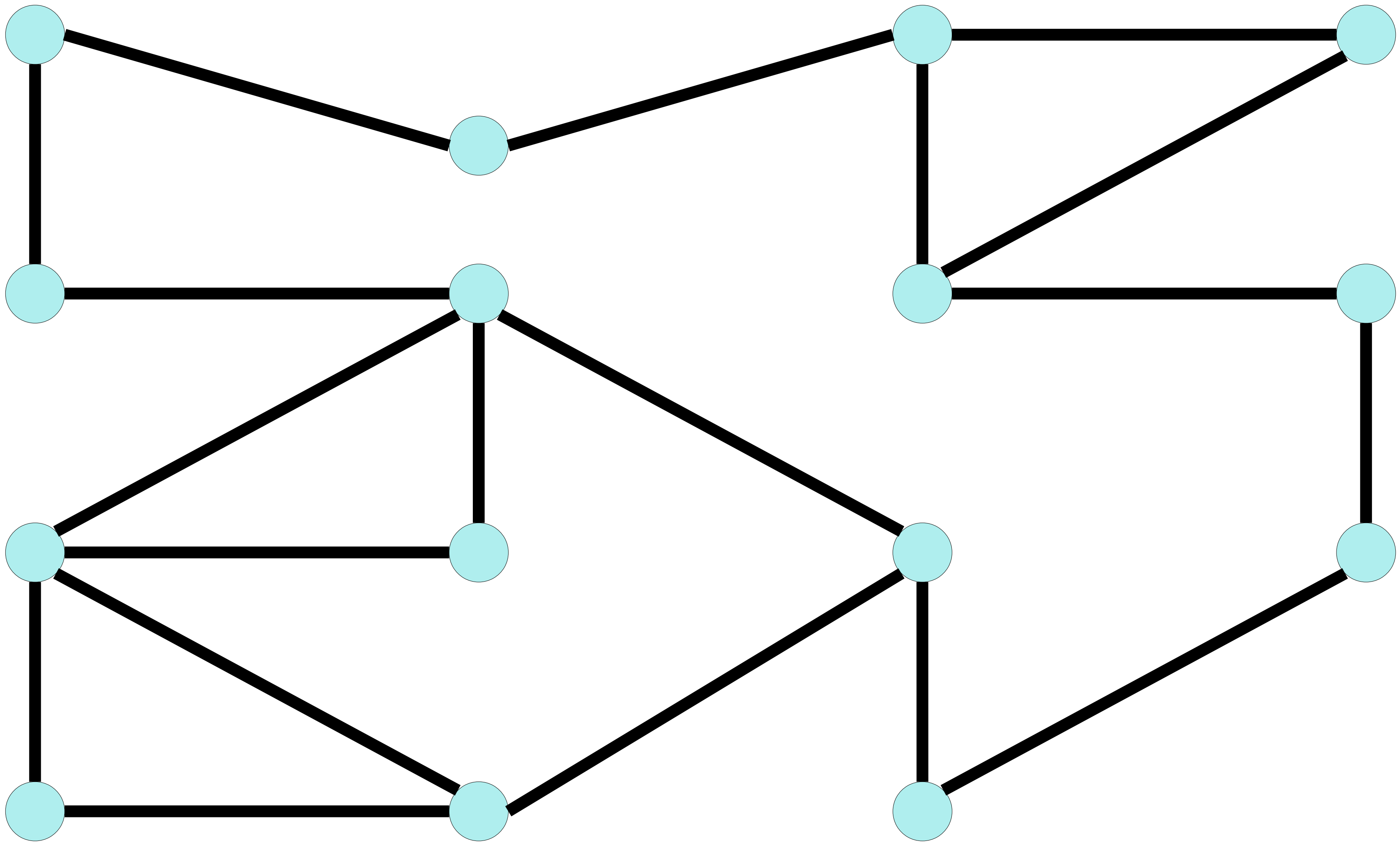
Place park rangers in these forest trails so that
a hiker anywhere on a trail can see a park ranger.



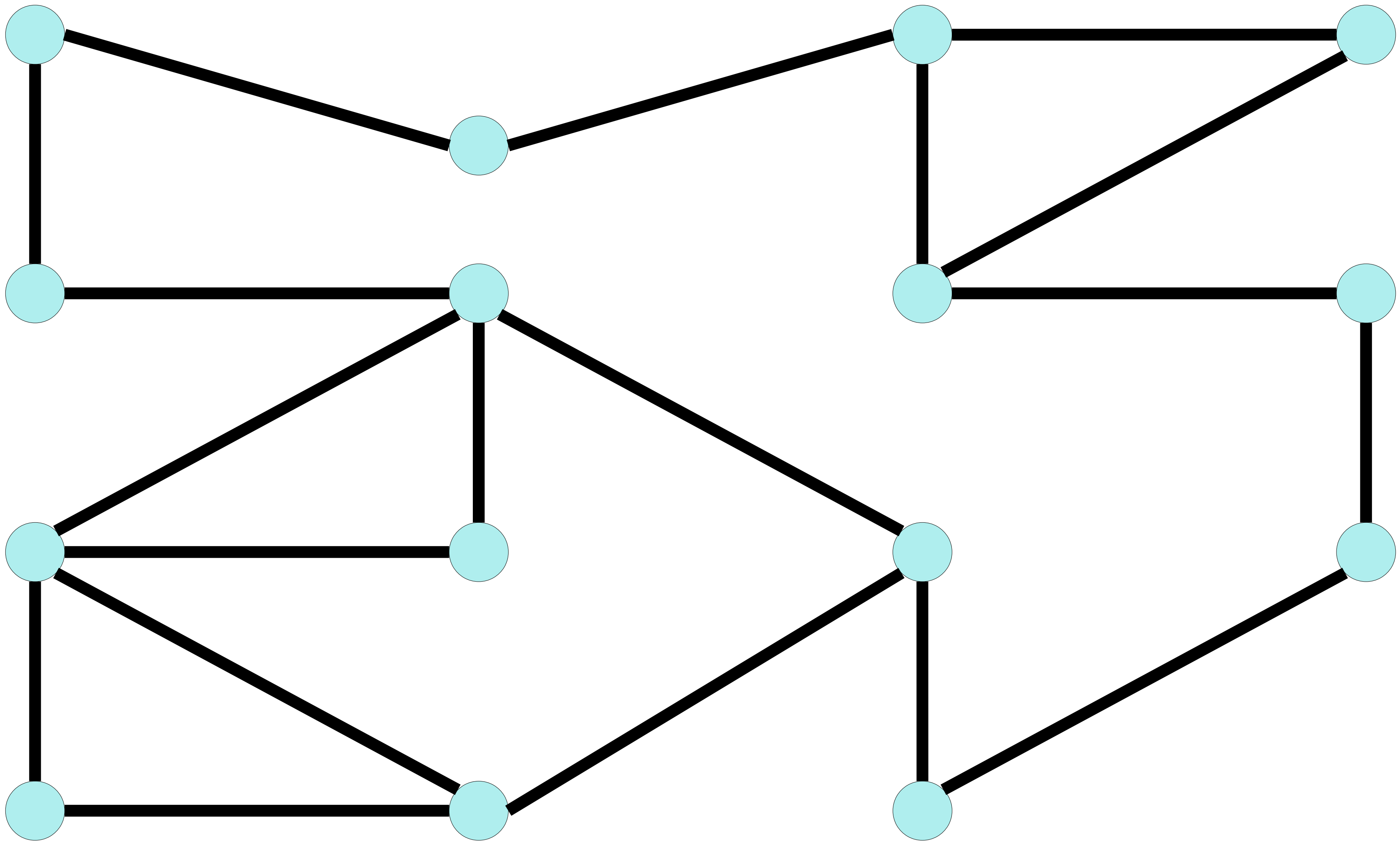
Place park rangers in these forest trails so that
a hiker anywhere on a trail can see a park ranger.



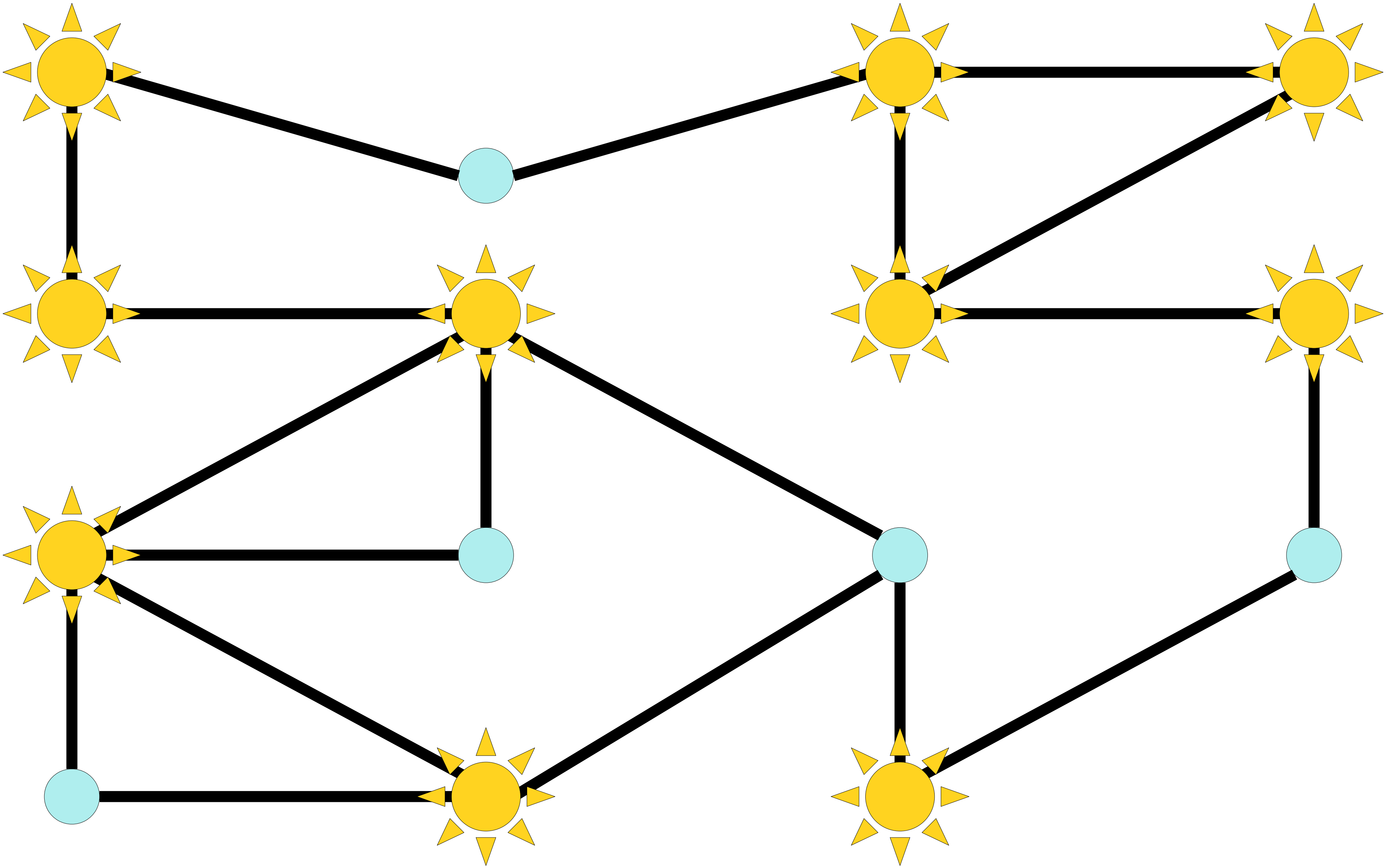
Place park rangers in these forest trails so that a hiker anywhere on a trail can see a park ranger.



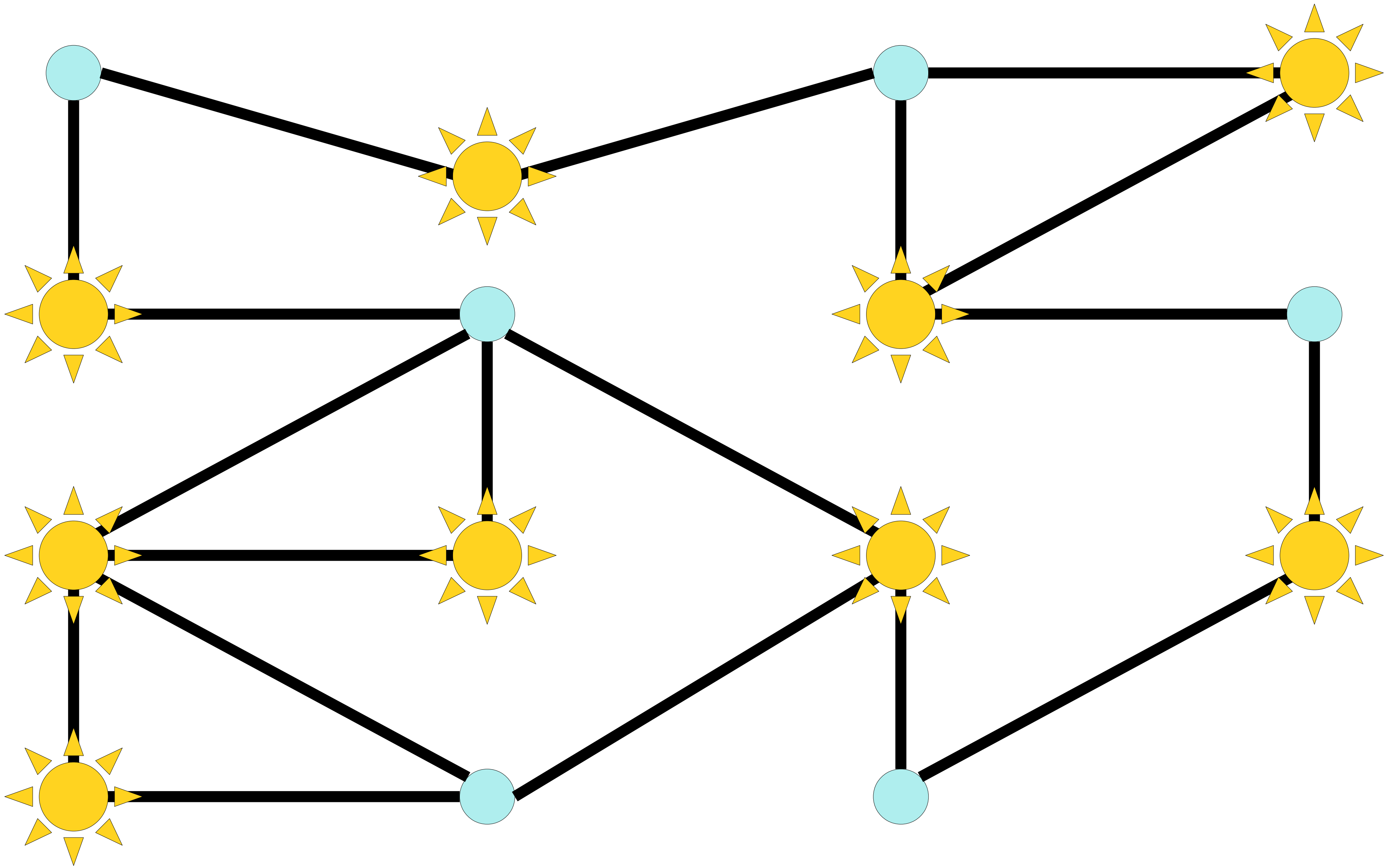
Place park rangers in these forest trails so that
a hiker anywhere on a trail can see a park ranger.



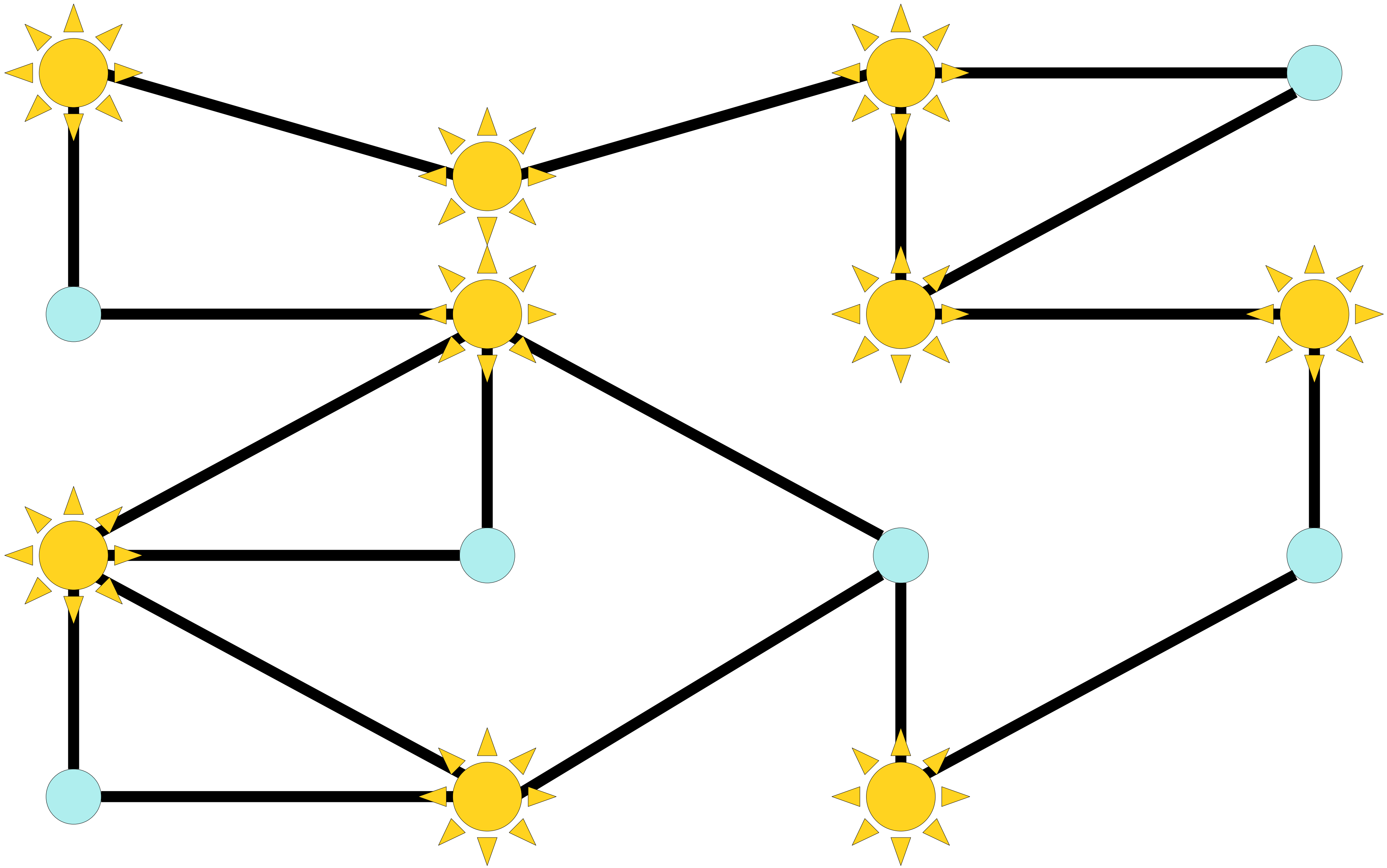
Choose at least one endpoint of each edge.



Choose at least one endpoint of each edge.



Choose at least one endpoint of each edge.



Choose at least one endpoint of each edge.

Vertex Covers

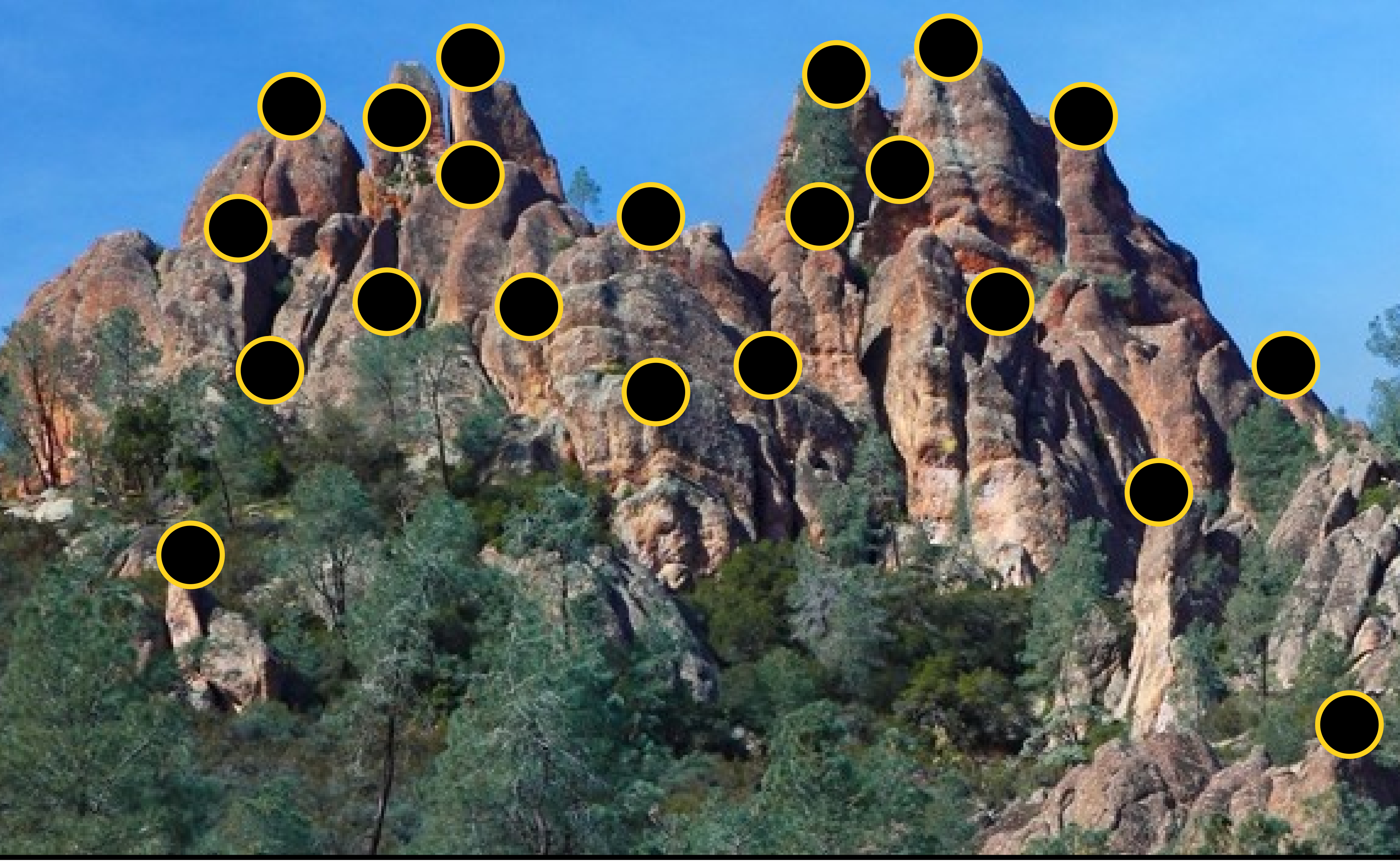
- Let $G = (V, E)$ be an undirected graph. A ***vertex cover*** of G is a set $C \subseteq V$ such that the following statement is true:

$$\forall x \in V.\ \forall y \in V.\ (\{x, y\} \in E \to (x \in C \lor y \in C))$$

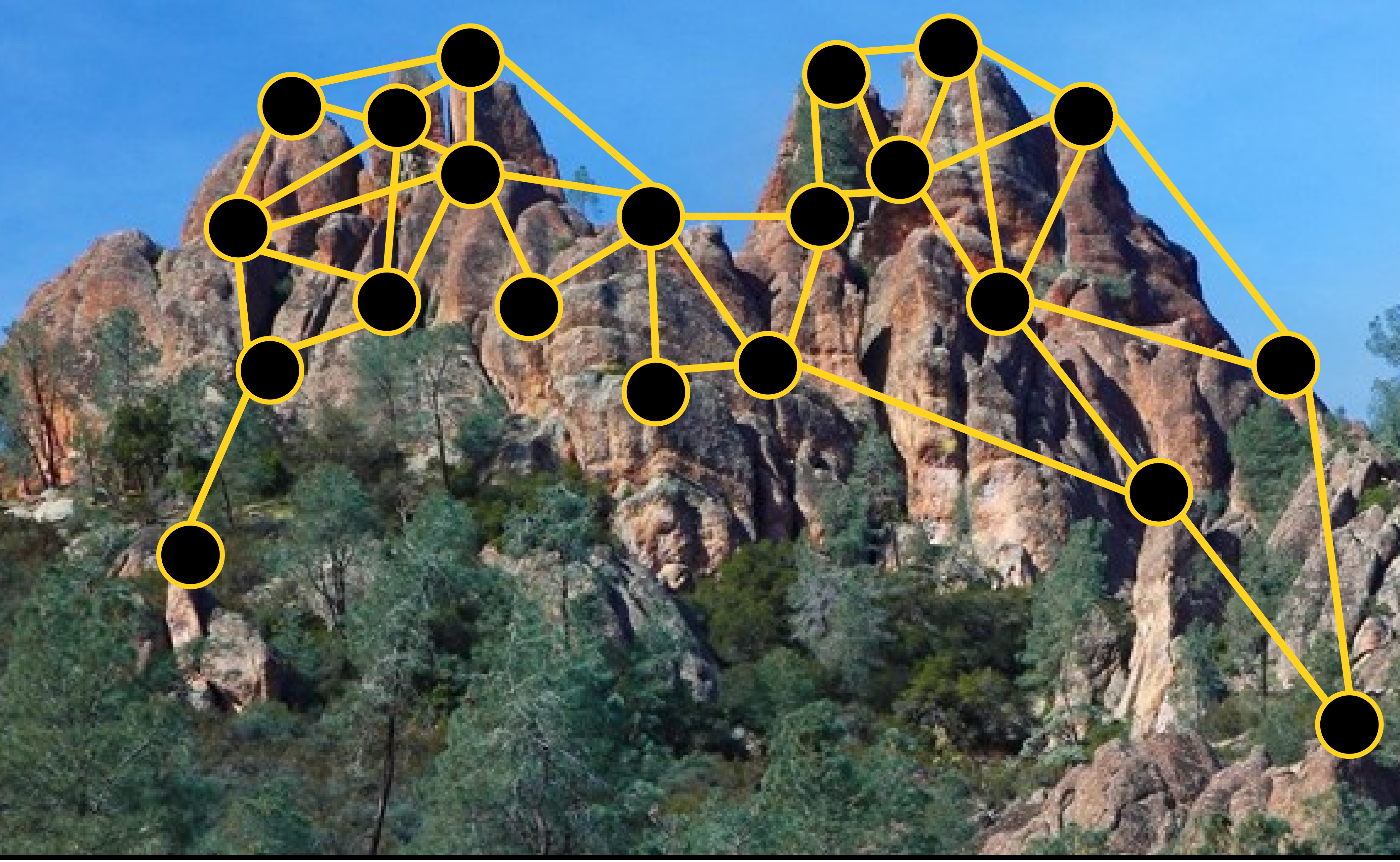
("Every edge has at least one endpoint in C.")

- Intuitively speaking, a vertex cover is a set formed by picking at least one endpoint of each edge in the graph.
- Vertex covers have applications to placing streetlights/benches/security guards, as well as in gene sequencing, machine learning, and combinatorics.

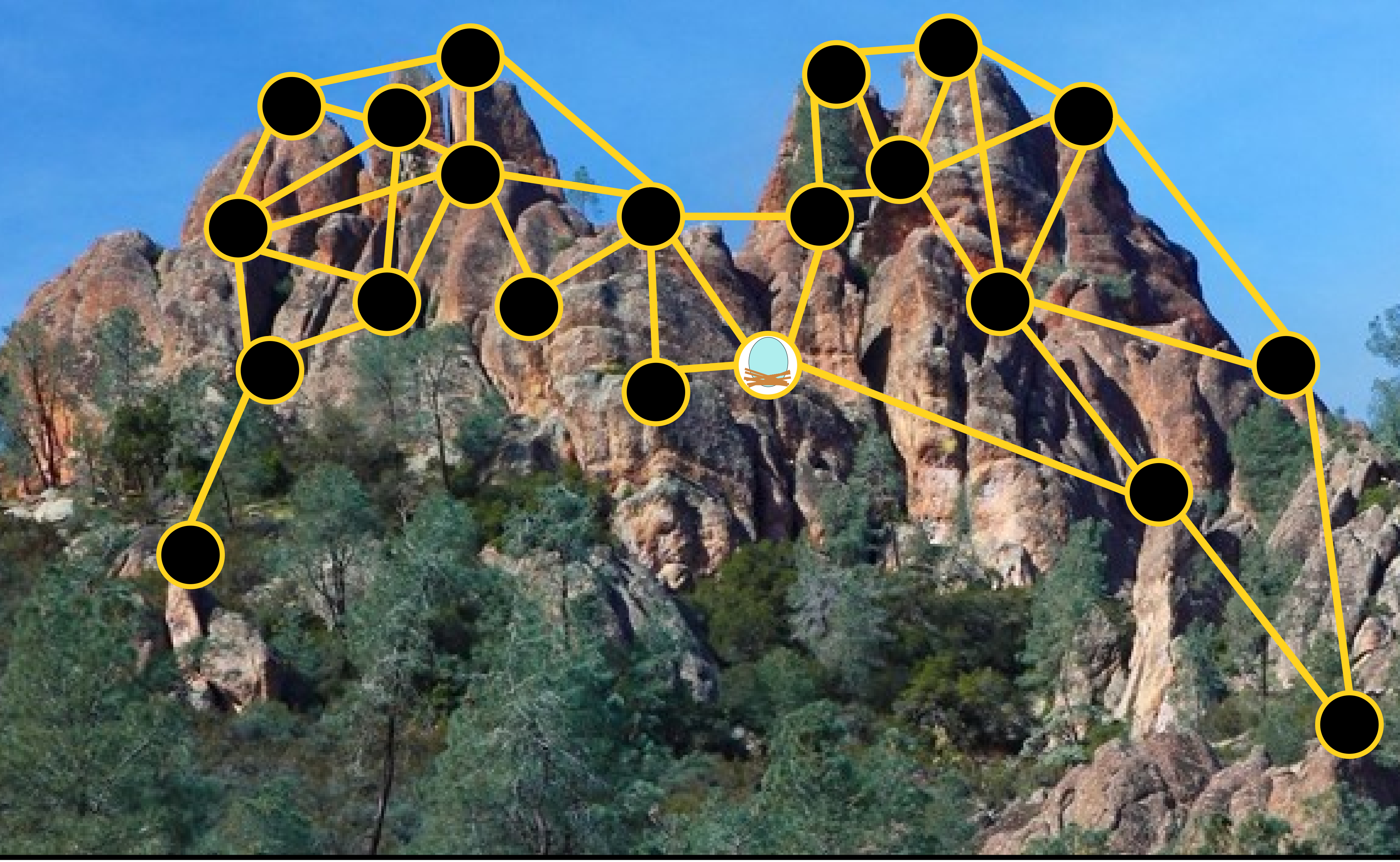
Set up nests for the California condor. Condors are territorial and won’t nest if they can see other condors.



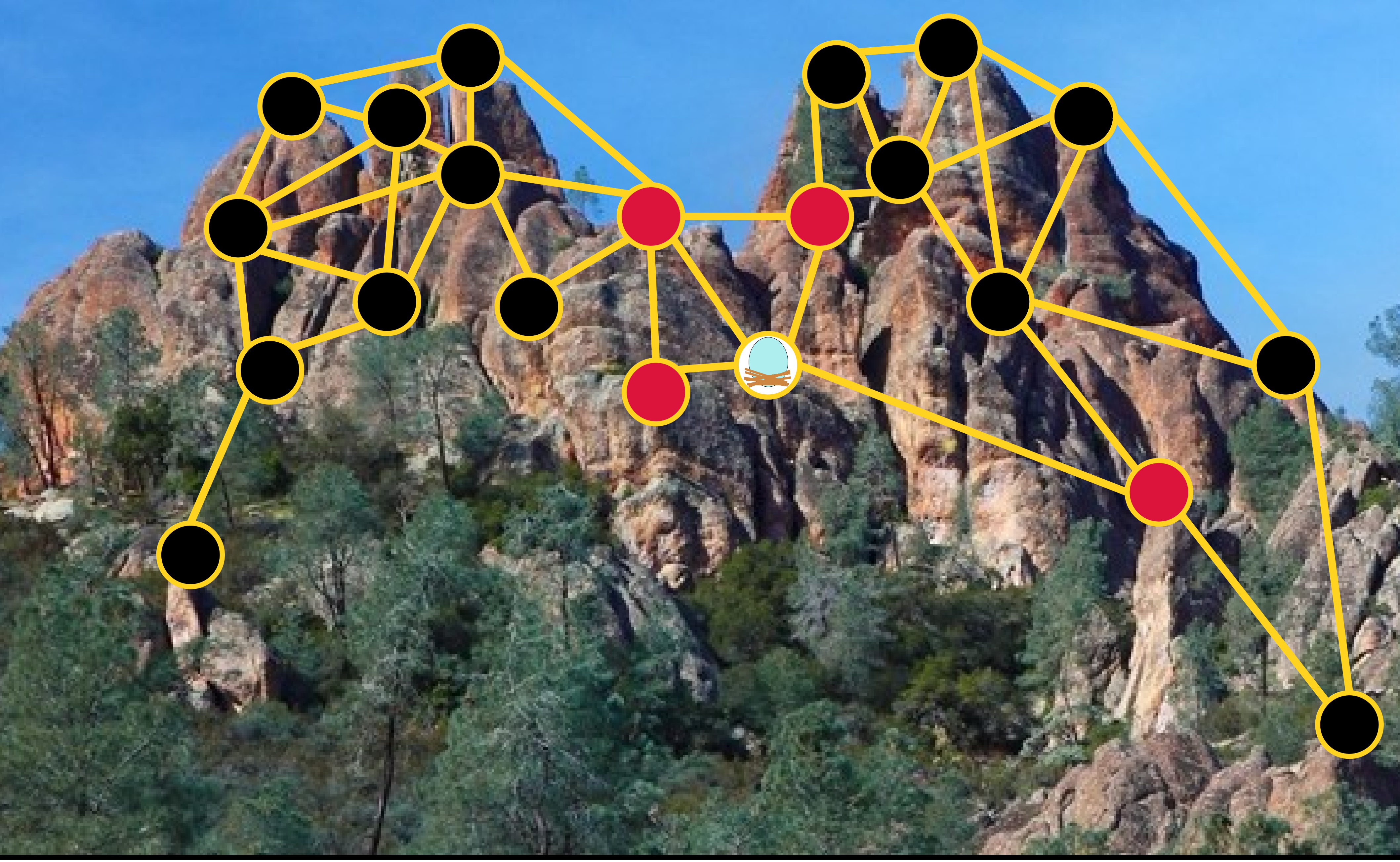
Set up nests for the California condor. Condors are territorial and won’t nest if they can see other condors.



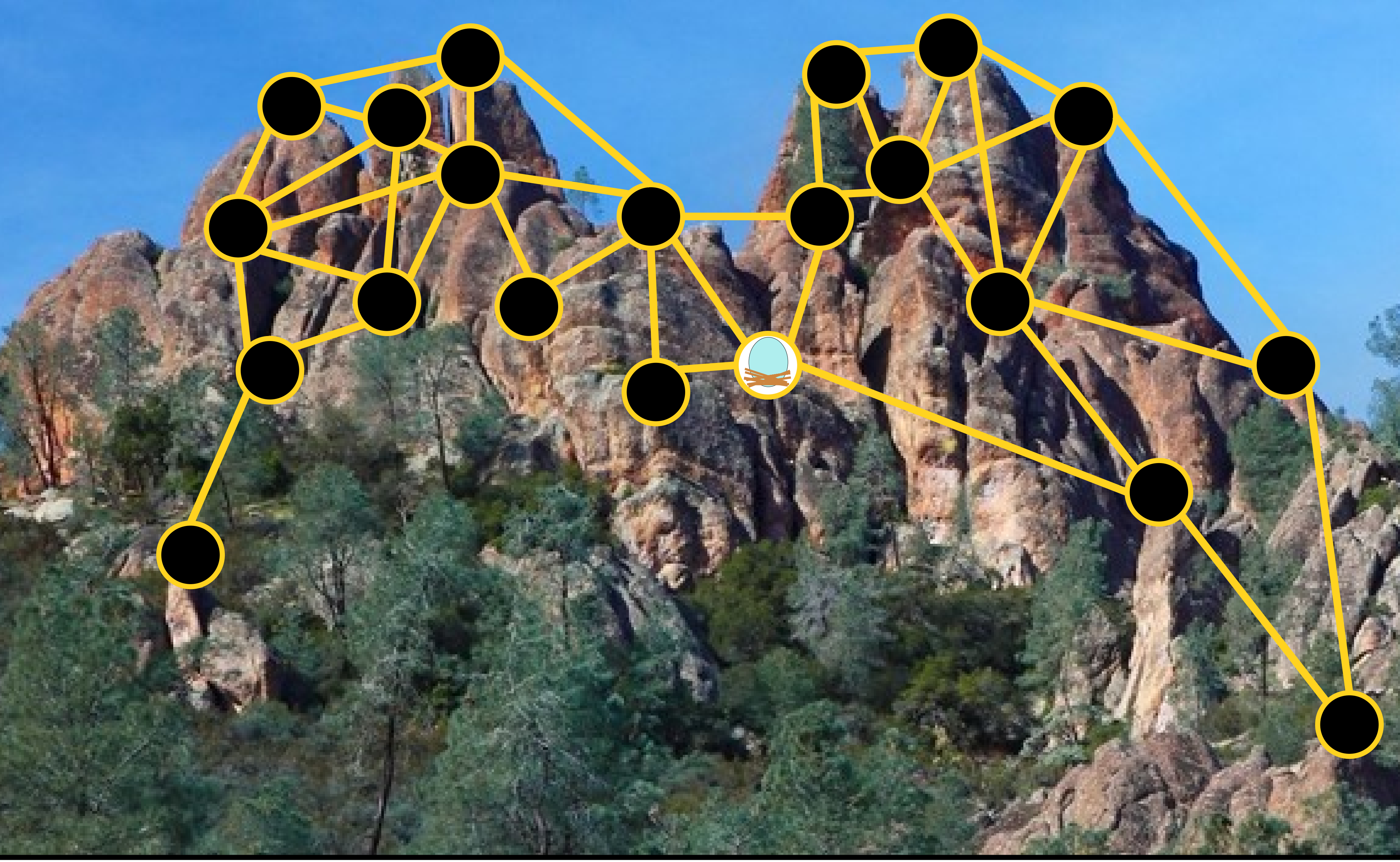
Set up nests for the California condor. Condors are territorial and won’t nest if they can see other condors.



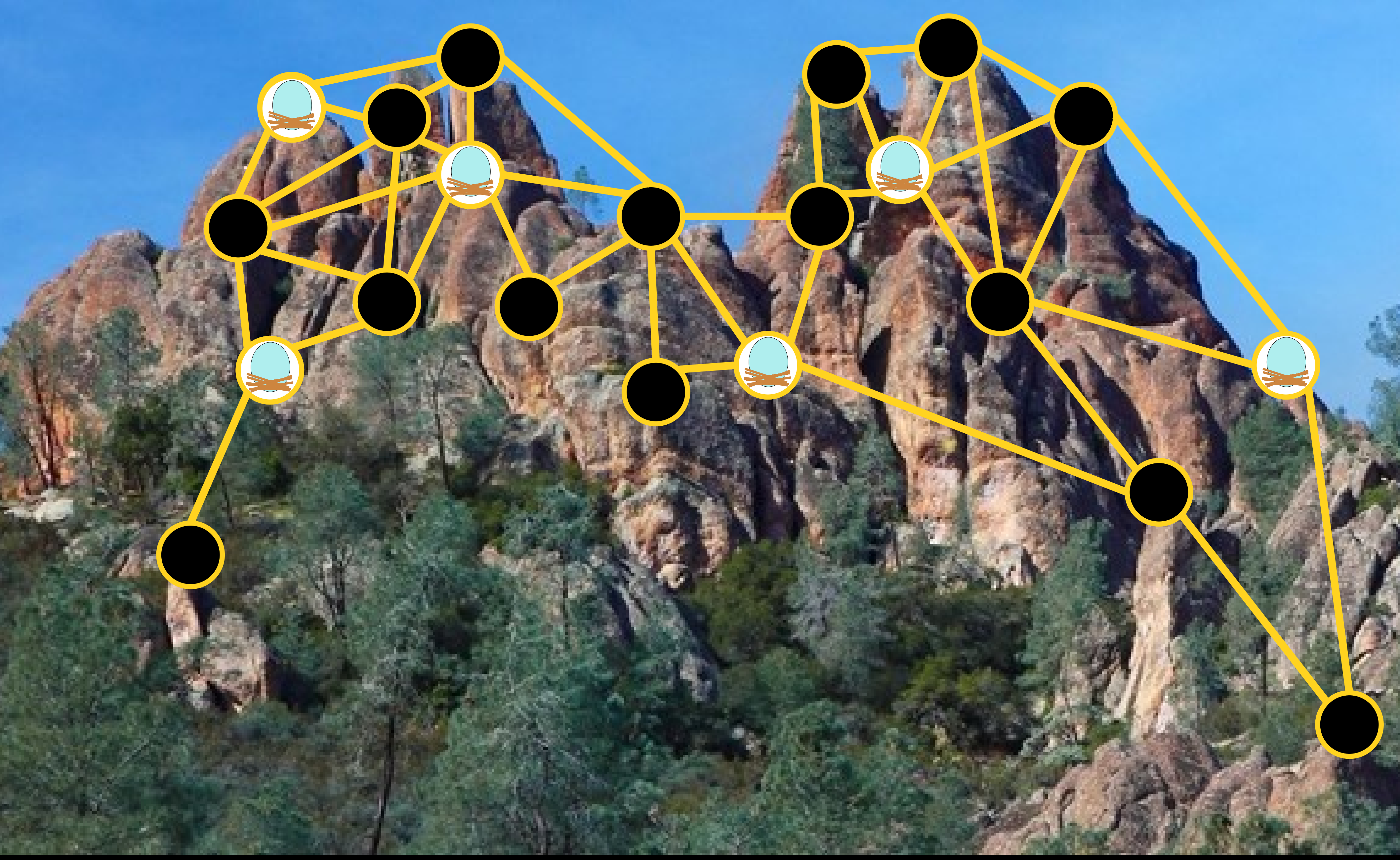
Set up nests for the California condor. Condors are territorial and won’t nest if they can see other condors.



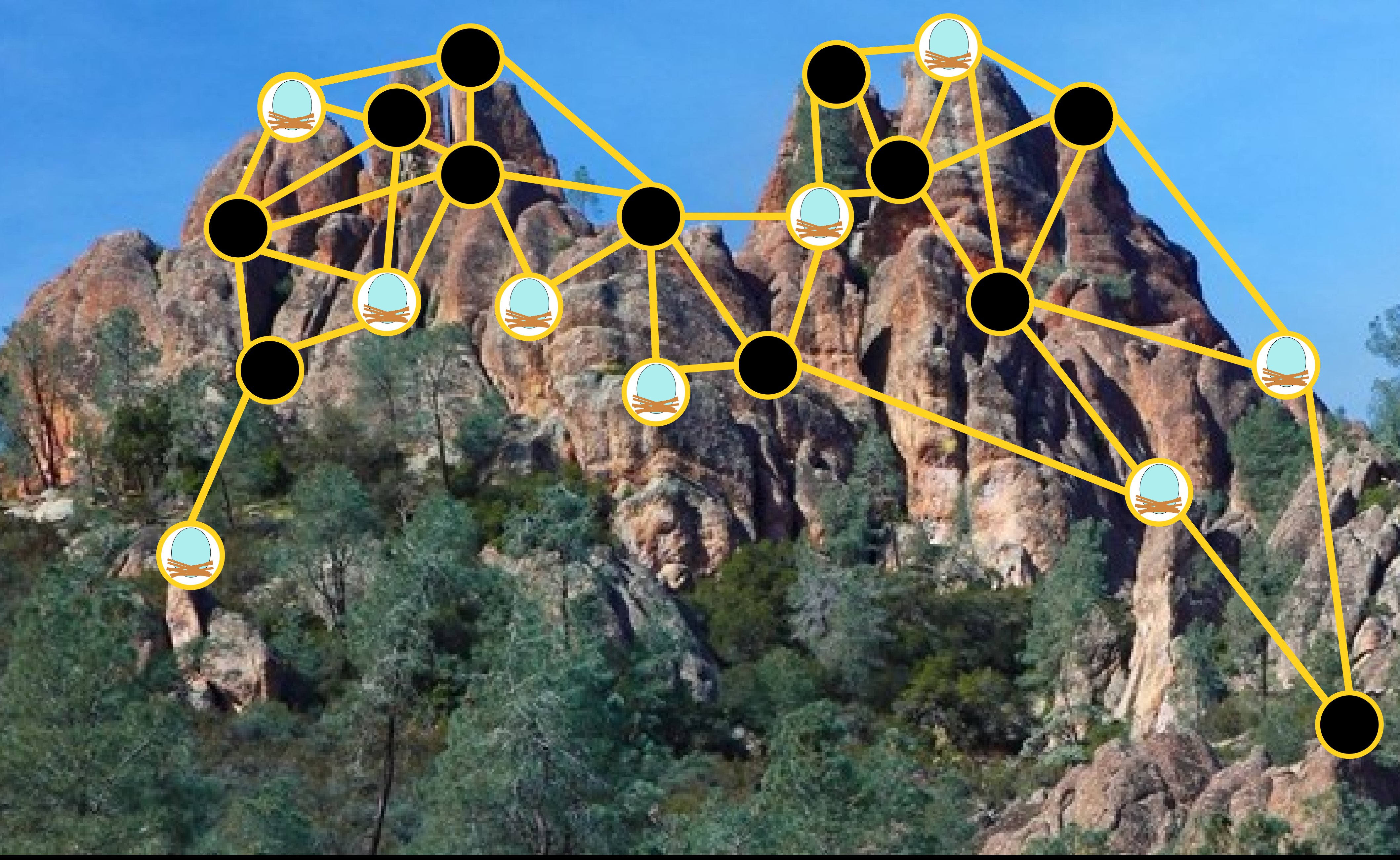
Set up nests for the California condor. Condors are territorial and won’t nest if they can see other condors.



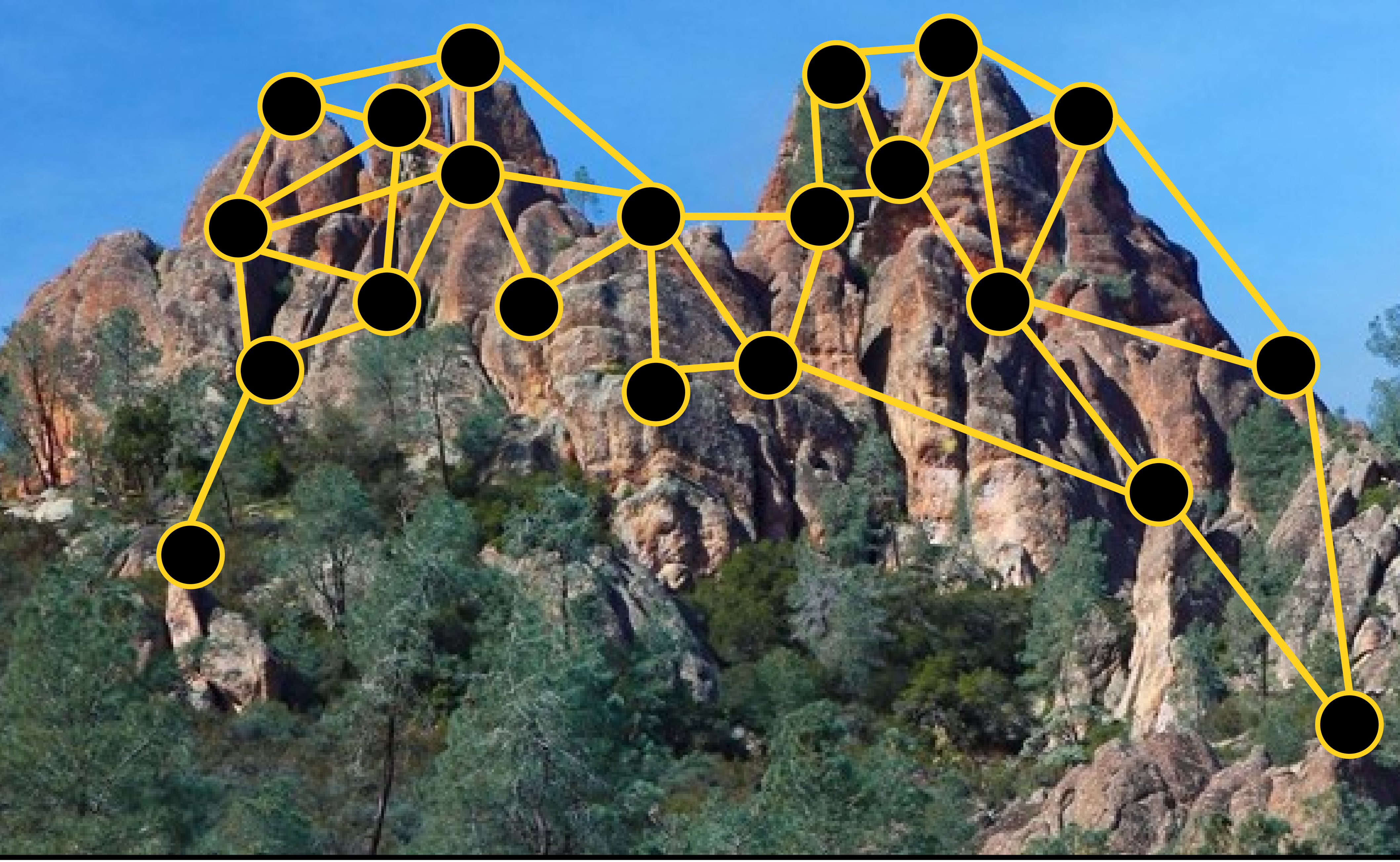
Set up nests for the California condor. Condors are territorial and won’t nest if they can see other condors.



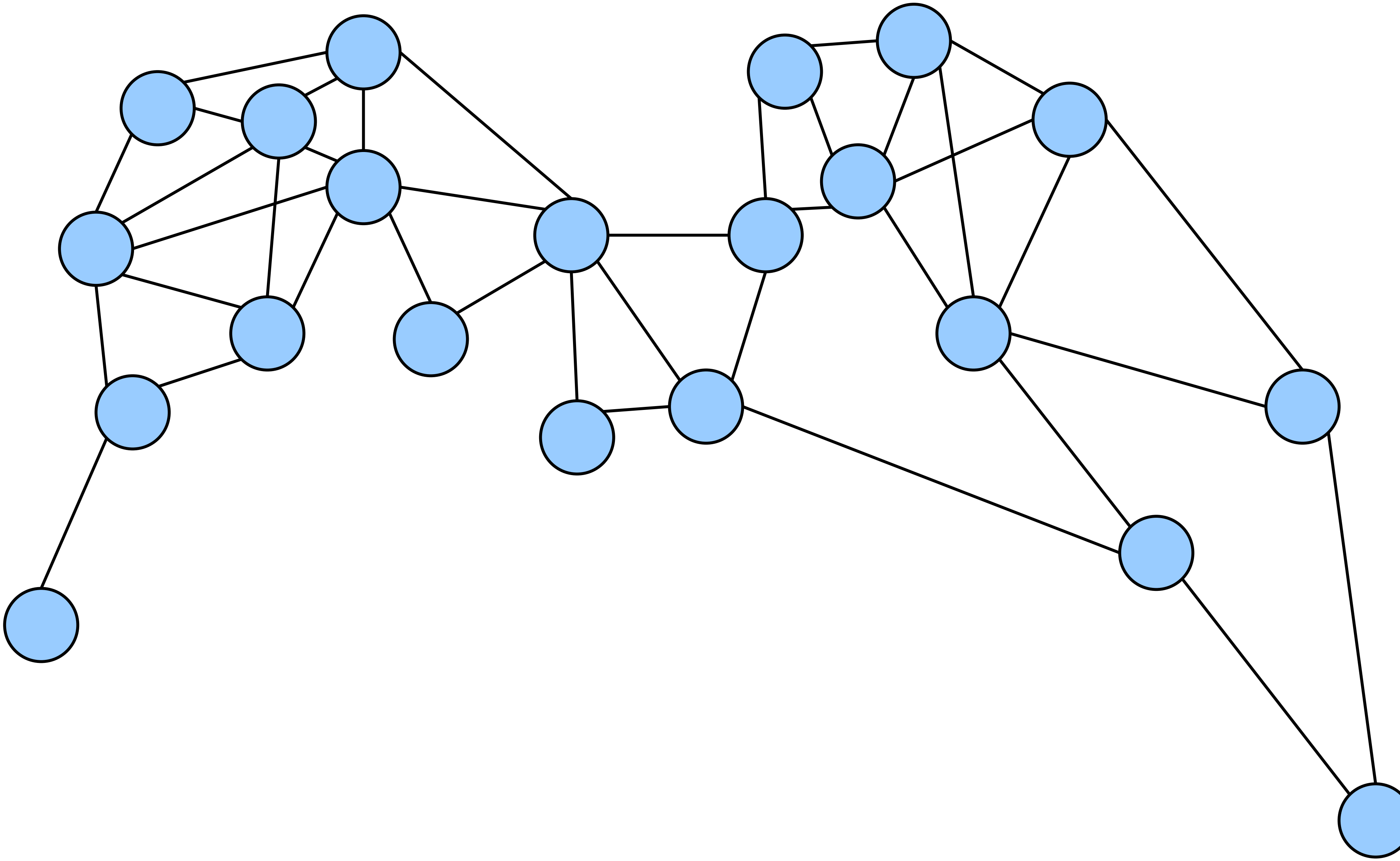
Set up nests for the California condor. Condors are territorial and won’t nest if they can see other condors.



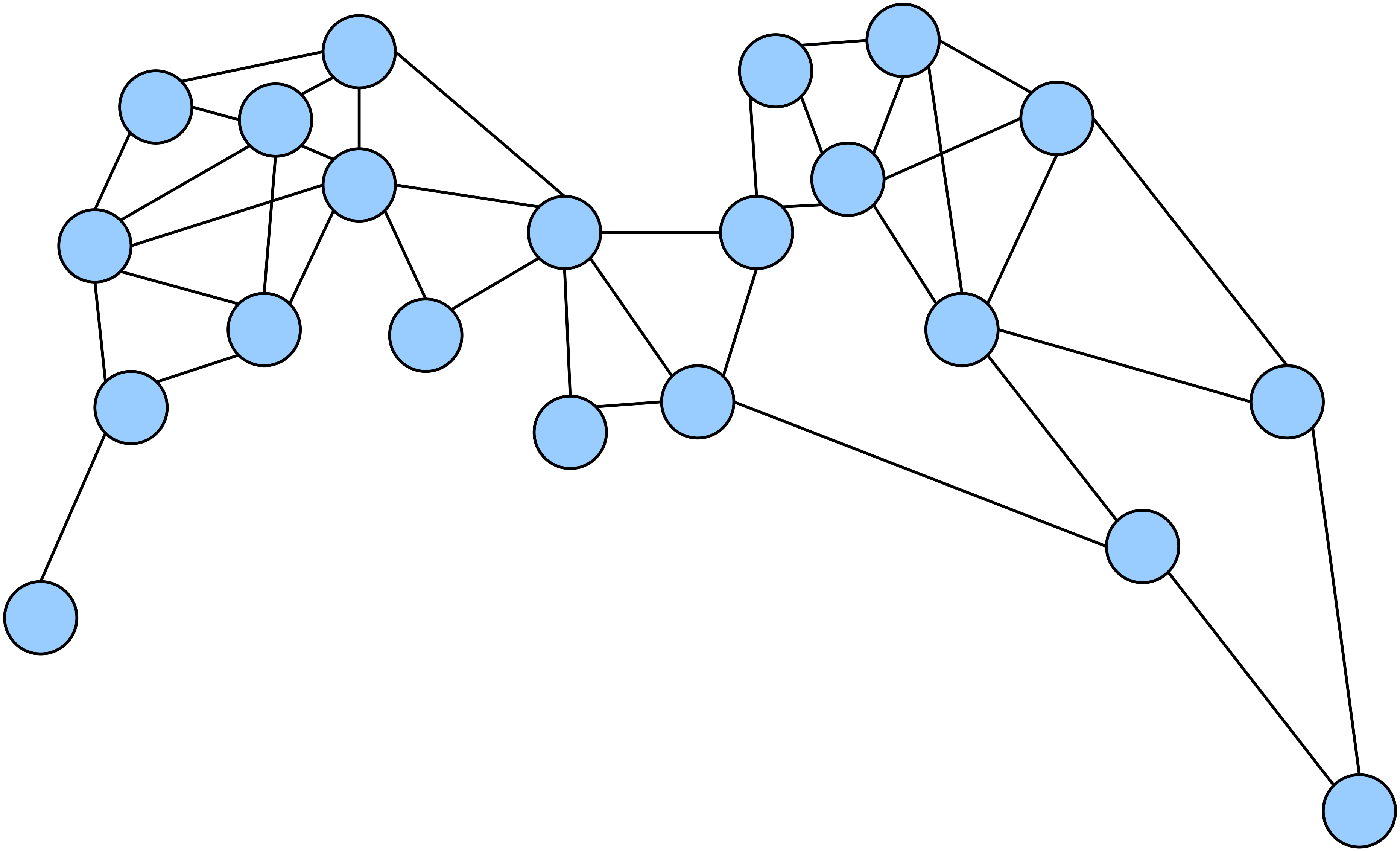
Set up nests for the California condor. Condors are territorial and won’t nest if they can see other condors.



Set up nests for the California condor. Condors are territorial and won’t nest if they can see other condors.



Set up nests for the California condor. Condors are territorial and won’t nest if they can see other condors.



Choose a set of nodes, no two of which are adjacent.

Independent Sets

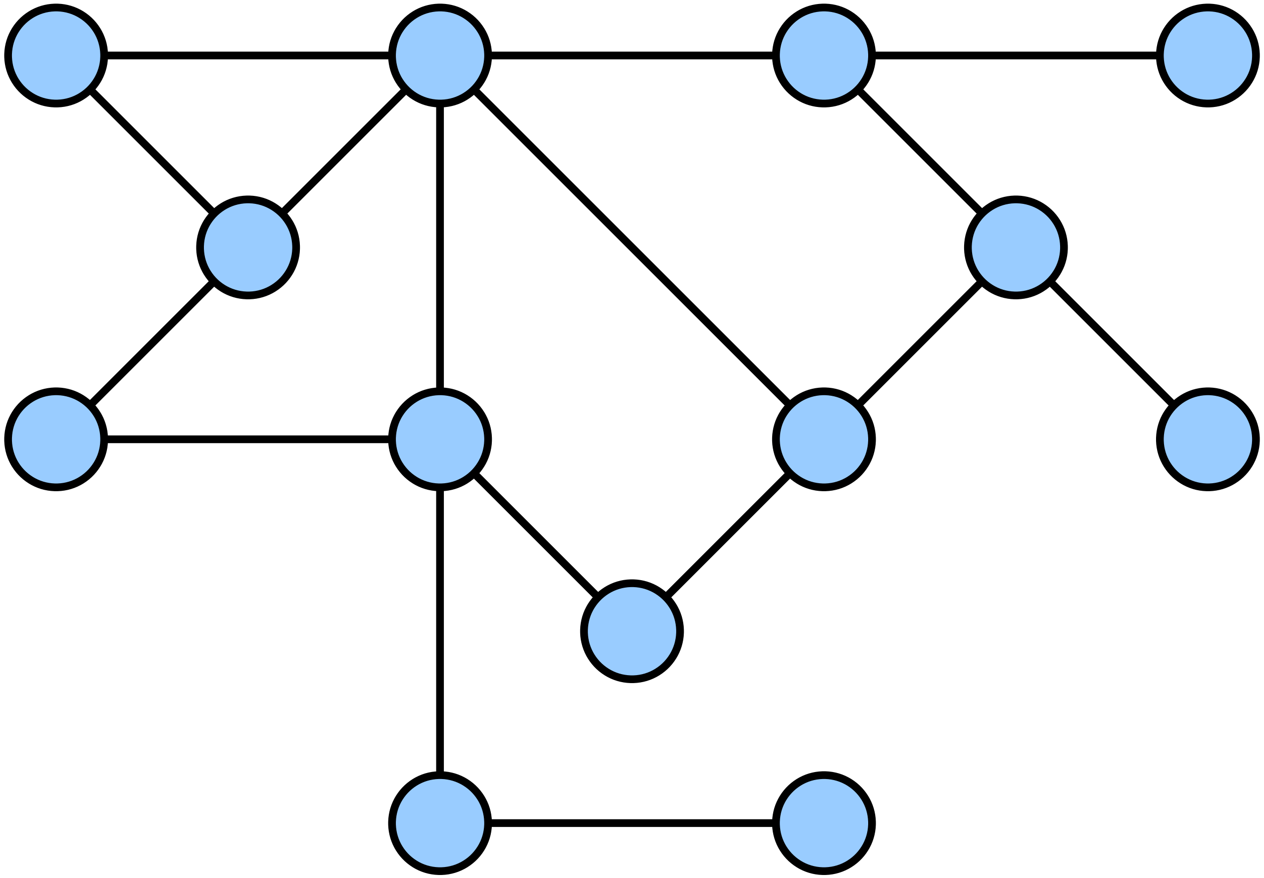
- If $G = (V, E)$ is an (undirected) graph, then an ***independent set*** in G is a set $I \subseteq V$ such that

$$\forall u \in I.\ \forall v \in I.\ \{u, v\} \notin E.$$

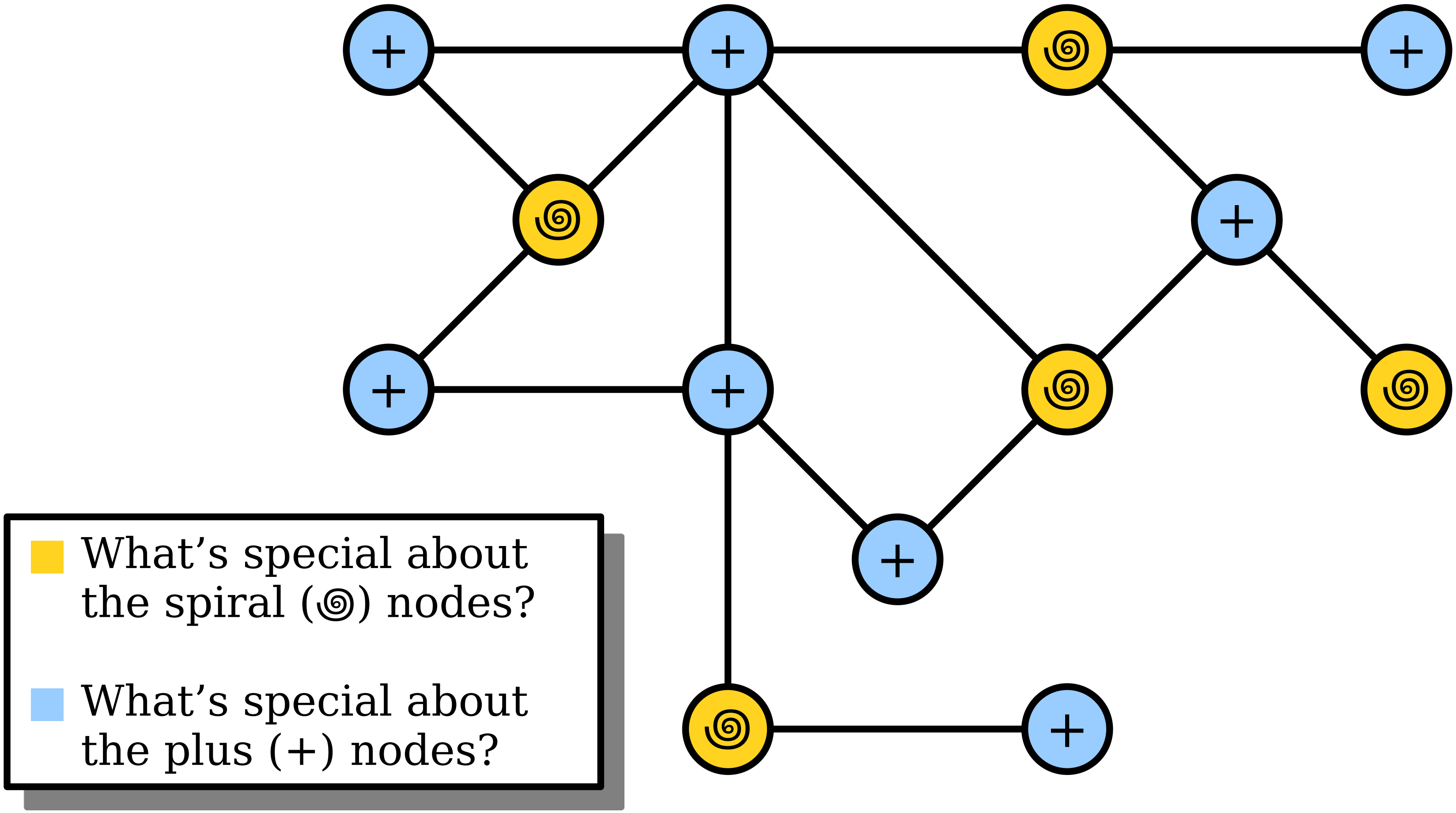
("No two nodes in I are adjacent.")

- Independent sets have applications to resource optimization, conflict minimization, error-correcting codes, cryptography, and more.

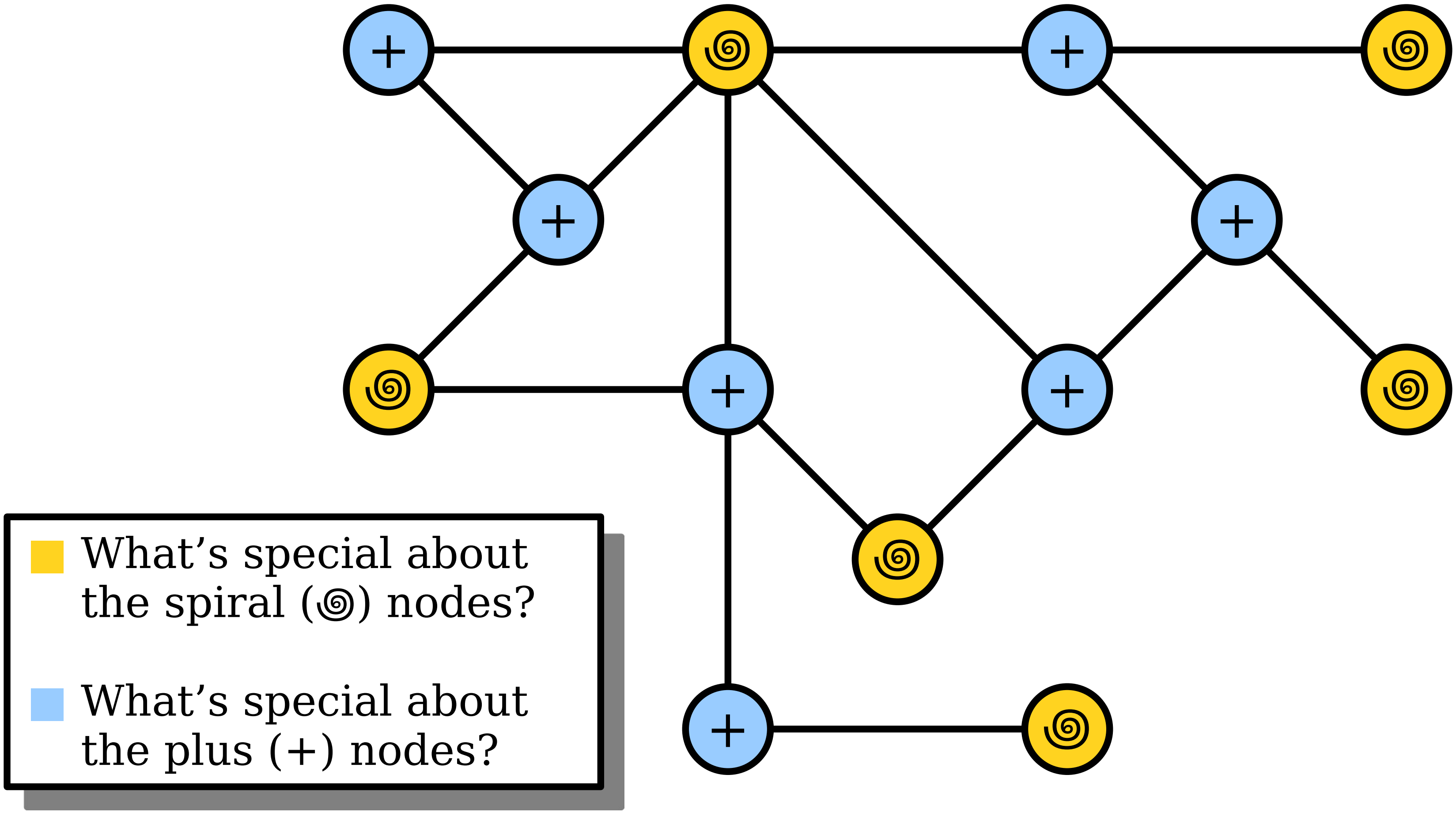
A Connection



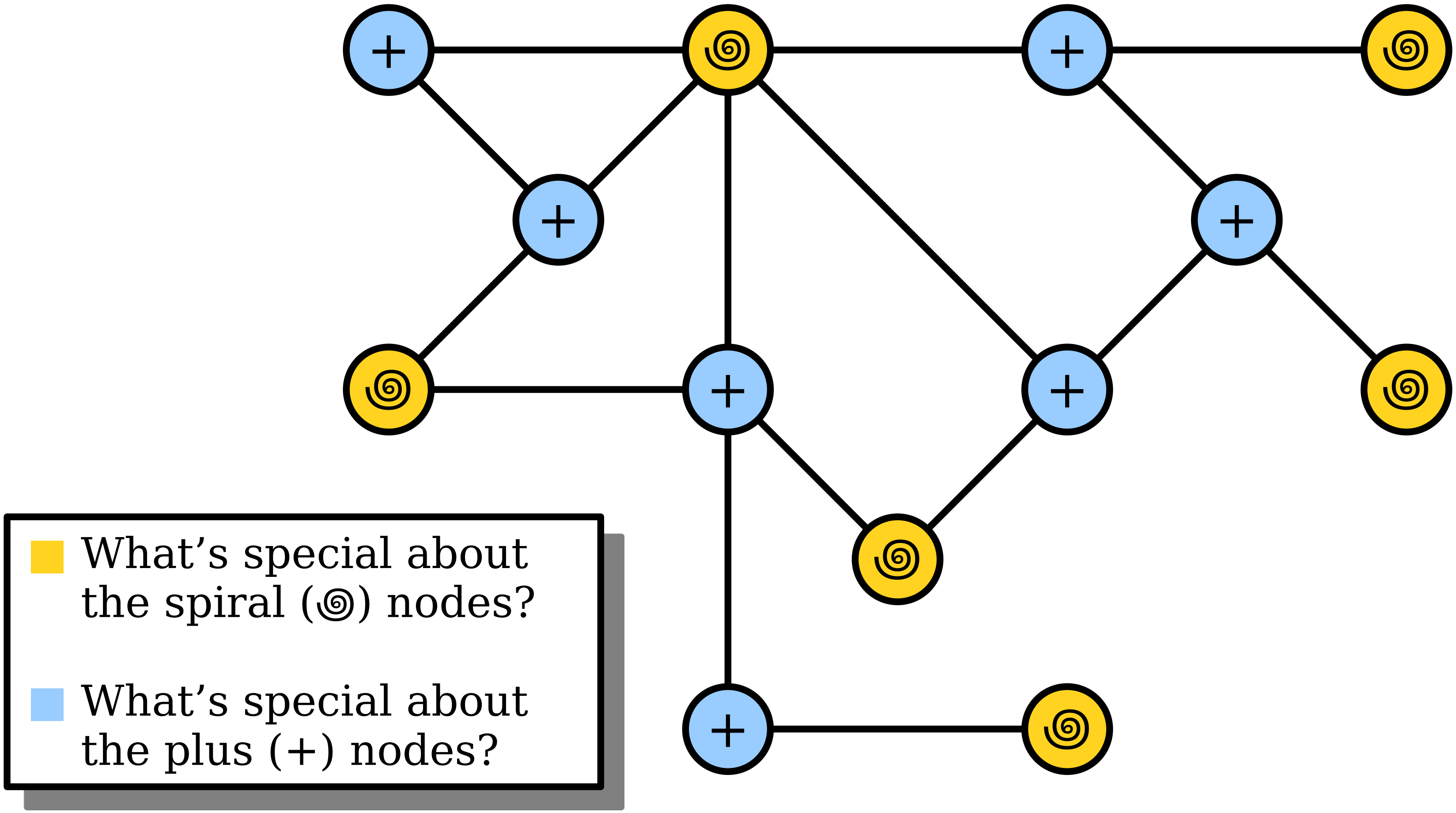
Independent sets and vertex covers are related.



Independent sets and vertex covers are related.



Independent sets and vertex covers are related.



Theorem: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. Then C is a vertex cover of G if and only if $V - C$ is an independent set in G.

Lemma 1: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If C is a vertex cover of G, then $V - C$ is an independent set in G.

What We're Assuming	What We Need To Show
G is a graph.	$V - C$ is an independent set in G.
C is a vertex cover of G.	$\forall x \in V - C.$
$\forall u \in V. \forall v \in V. (\{u, v\} \in E \to$	$\forall y \in V - C.$
$u \in C \lor v \in C$	$\{x, y\} \notin E.$
$)$	

Lemma 1: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If C is a vertex cover of G, then $V - C$ is an independent set in G.

What We're Assuming

G is a graph.

C is a vertex cover of G.

$$\forall u \in V.\ \forall v \in V.\ (\{u, v\} \in E \to u \in C \lor v \in C)$$

We're assuming a universally-quantified statement. That means we *don't do anything right now* and instead wait for an edge to present itself.

What We Need To Show

$V - C$ is an independent set in G.

$$\forall x \in V - C.\ \forall y \in V - C.\ \{x, y\} \notin E.$$

We need to prove a universally-quantified statement. We'll ask the reader to pick arbitrary choices of x and y for us to work with.

Lemma 1: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If C is a vertex cover of G, then $V - C$ is an independent set in G.

What We're Assuming	What We Need To Show
G is a graph.	$V - C$ is an independent set in G.
C is a vertex cover of G.	$\forall x \in V - C.$ $\forall y \in V - C.$ $\{x, y\} \notin E.$
$\forall u \in V. \forall v \in V. (\{u, v\} \in E \to u \in C \lor v \in C)$	

We need to prove a universally-quantified statement. We'll ask the reader to pick arbitrary choices of x and y for us to work with.

Lemma 1: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If C is a vertex cover of G, then $V - C$ is an independent set in G.

What We're Assuming

G is a graph.

C is a vertex cover of G.

$\forall u \in V. \forall v \in V. (\{u, v\} \in E \to$
$u \in C \lor v \in C$
$)$

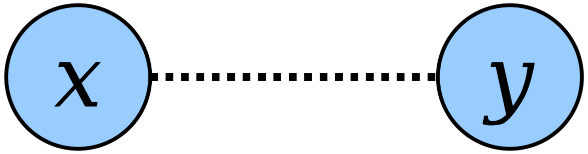
$x \in V - C.$

$y \in V - C.$

What We Need To Show

$V - C$ is an independent set in G.

$\forall x \in V - C.$
$\forall y \in V - C.$
$\{x, y\} \notin E.$



Lemma 1: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If C is a vertex cover of G, then $V - C$ is an independent set of G.

Lemma 1: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If C is a vertex cover of G, then $V - C$ is an independent set of G.

Proof:

Lemma 1: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If C is a vertex cover of G, then $V - C$ is an independent set of G.

Proof: Assume C is a vertex cover of G.

There's no need to introduce G or C here. That's done in the statement of the lemma itself.

Lemma 1: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If C is a vertex cover of G, then $V - C$ is an independent set of G.

Proof: Assume C is a vertex cover of G. We need to show that $V - C$ is an independent set of G.

Lemma 1: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If C is a vertex cover of G, then $V - C$ is an independent set of G.

Proof: Assume C is a vertex cover of G. We need to show that $V - C$ is an independent set of G. To do so, pick any nodes $x, y \in V - C$; we will show that $\{x, y\} \notin E$.

Lemma 1: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If C is a vertex cover of G, then $V - C$ is an independent set of G.

Proof: Assume C is a vertex cover of G. We need to show that $V - C$ is an independent set of G. To do so, pick any nodes $x, y \in V - C$; we will show that $\{x, y\} \notin E$.

Suppose for the sake of contradiction that $\{x, y\} \in E$.

Lemma 1: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If C is a vertex cover of G, then $V - C$ is an independent set of G.

Proof: Assume C is a vertex cover of G. We need to show that $V - C$ is an independent set of G. To do so, pick any nodes $x, y \in V - C$; we will show that $\{x, y\} \notin E$.

Suppose for the sake of contradiction that $\{x, y\} \in E$. Because $x, y \in V - C$, we know that $x \notin C$ and $y \notin C$.

Lemma 1: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If C is a vertex cover of G, then $V - C$ is an independent set of G.

Proof: Assume C is a vertex cover of G. We need to show that $V - C$ is an independent set of G. To do so, pick any nodes $x, y \in V - C$; we will show that $\{x, y\} \notin E$.

Suppose for the sake of contradiction that $\{x, y\} \in E$. Because $x, y \in V - C$, we know that $x \notin C$ and $y \notin C$. However, since C is a vertex cover of G and $\{x, y\} \in E$, we also see that $x \in C$ or $y \in C$, contradicting our previous statement.

Lemma 1: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If C is a vertex cover of G, then $V - C$ is an independent set of G.

Proof: Assume C is a vertex cover of G. We need to show that $V - C$ is an independent set of G. To do so, pick any nodes $x, y \in V - C$; we will show that $\{x, y\} \notin E$.

Suppose for the sake of contradiction that $\{x, y\} \in E$. Because $x, y \in V - C$, we know that $x \notin C$ and $y \notin C$. However, since C is a vertex cover of G and $\{x, y\} \in E$, we also see that $x \in C$ or $y \in C$, contradicting our previous statement.

We've reached a contradiction, so our assumption was wrong.

Lemma 1: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If C is a vertex cover of G, then $V - C$ is an independent set of G.

Proof: Assume C is a vertex cover of G. We need to show that $V - C$ is an independent set of G. To do so, pick any nodes $x, y \in V - C$; we will show that $\{x, y\} \notin E$.

Suppose for the sake of contradiction that $\{x, y\} \in E$. Because $x, y \in V - C$, we know that $x \notin C$ and $y \notin C$. However, since C is a vertex cover of G and $\{x, y\} \in E$, we also see that $x \in C$ or $y \in C$, contradicting our previous statement.

We've reached a contradiction, so our assumption was wrong. Therefore, we have $\{x, y\} \notin E$, as required.

Lemma 1: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If C is a vertex cover of G, then $V - C$ is an independent set of G.

Proof: Assume C is a vertex cover of G. We need to show that $V - C$ is an independent set of G. To do so, pick any nodes $x, y \in V - C$; we will show that $\{x, y\} \notin E$.

Suppose for the sake of contradiction that $\{x, y\} \in E$. Because $x, y \in V - C$, we know that $x \notin C$ and $y \notin C$. However, since C is a vertex cover of G and $\{x, y\} \in E$, we also see that $x \in C$ or $y \in C$, contradicting our previous statement.

We've reached a contradiction, so our assumption was wrong. Therefore, we have $\{x, y\} \notin E$, as required. ■

Taking Negations

- What is the negation of this statement, which says "C is a vertex cover?"

$$
\begin{aligned}
&\forall u \in C.\ \forall v \in C.\ (\{u, v\} \in C \to \\
&\quad u \in C \quad \lor \quad v \in C \\
&)
\end{aligned}
$$

Taking Negations

- What is the negation of this statement, which says "*C* is a vertex cover?"

$$\neg \forall u \in C.\ \forall v \in C.\ (\{u, v\} \in C \rightarrow$$
$$u \in C \quad \lor \quad v \in C$$
$$)$$

Taking Negations

- What is the negation of this statement, which says “C is a vertex cover?”

$$
\begin{aligned}
&\exists u \in C.\ \neg\forall v \in C.\ (\{u, v\} \in C \to \\
&\quad u \in C \quad \lor \quad v \in C \\
&)
\end{aligned}
$$

Taking Negations

- What is the negation of this statement, which says "C is a vertex cover?"

$$\exists u \in C.\ \exists v \in C.\ \neg(\{u, v\} \in C \rightarrow$$
$$u \in C \quad \lor \quad v \in C$$
$$)$$

Taking Negations

- What is the negation of this statement, which says "*C* is a vertex cover?"

$$\exists u \in C.\ \exists v \in C.\ (\{u, v\} \in C\ \land$$
$$\neg(u \in C \quad \lor \quad v \in C)$$
$$)$$

Taking Negations

- What is the negation of this statement, which says "*C* is a vertex cover?"

$$\exists u \in C.\ \exists v \in C.\ (\{u, v\} \in C\ \land$$
$$u \notin C \quad \land \quad v \notin C$$
$$)$$

Taking Negations

- What is the negation of this statement, which says "*C* is a vertex cover?"

$$\exists u \in C.\ \exists v \in C.\ (\{u, v\} \in C \land$$
$$u \notin C \quad \land \quad v \notin C$$
$$)$$

- This says "there is an edge where both endpoints aren't in *C*."

Lemma 2: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If C is not a vertex cover of G, then $V - C$ is not an independent set in G.

What We're Assuming	**What We Need To Show**
G is a graph.	$V - C$ is not an ind. set in G.
C is a not a vertex cover of G.	$\exists x \in V - C.$
$\exists u \in V.\ \exists v \in V.\ (\{u, v\} \in E \land$	$\exists y \in V - C.$
$u \notin C \quad \land \quad v \notin C$	$\{x, y\} \in E.$
$)$	

Lemma 2: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If C is not a vertex cover of G, then $V - C$ is not an independent set in G.

What We're Assuming	***What We Need To Show***
G is a graph.	$V - C$ is not an ind. set in G.
C is a not a vertex cover of G.	$\exists x \in V - C.$
$\exists u \in V. \exists v \in V. (\{u, v\} \in E \land u \notin C \land v \notin C)$	$\exists y \in V - C.$
	$\{x, y\} \in E.$

We're assuming an existentially-quantified statement, so we'll *immediately* introduce variables u and v.

We're proving an existentially-quantified statement, so we *don't* introduce variables x and y. We're on a scavenger hunt!

Lemma 2: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If C is not a vertex cover of G, then $V - C$ is not an independent set in G.

What We're Assuming	***What We Need To Show***
G is a graph.	$V - C$ is not an ind. set in G.
C is a not a vertex cover of G.	$\exists x \in V - C$.
$u \in V - C$.	$\exists y \in V - C$.
$v \in V - C$.	$\{x, y\} \in E$.
$\{u, v\} \in E$.	

We're assuming an existentially-quantified statement, so we'll *immediately* introduce variables u and v.

Lemma 2: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If C is not a vertex cover of G, then $V - C$ is not an independent set in G.

What We're Assuming	What We Need To Show
G is a graph.	$V - C$ is not an ind. set in G.
C is a not a vertex cover of G.	$\exists x \in V - C$.
$u \in V - C$.	$\exists y \in V - C$.
$v \in V - C$.	$\{x, y\} \in E$.
$\{u, v\} \in E$.	

Any ideas about what we should pick x and y to be?

Lemma 2: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If C is not a vertex cover of G, then $V - C$ is not an independent set of G.

Lemma 2: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If C is not a vertex cover of G, then $V - C$ is not an independent set of G.

Proof:

Lemma 2: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If C is not a vertex cover of G, then $V - C$ is not an independent set of G.

Proof: Assume C is not a vertex cover of G.

Lemma 2: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If C is not a vertex cover of G, then $V - C$ is not an independent set of G.

Proof: Assume C is not a vertex cover of G. We need to show that $V - C$ is not an independent set of G.

Lemma 2: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If C is not a vertex cover of G, then $V - C$ is not an independent set of G.

Proof: Assume C is not a vertex cover of G. We need to show that $V - C$ is not an independent set of G.

Since C is not a vertex cover of G, we know that there exists nodes $x, y \in V$ where $\{x, y\} \in E$, where $x \notin C$, and where $y \notin C$.

Lemma 2: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If C is not a vertex cover of G, then $V - C$ is not an independent set of G.

Proof: Assume C is not a vertex cover of G. We need to show that $V - C$ is not an independent set of G.

Since C is not a vertex cover of G, we know that there exists nodes $x, y \in V$ where $\{x, y\} \in E$, where $x \notin C$, and where $y \notin C$. Because $x \in V$ and $x \notin C$, we know that $x \in V - C$.

Lemma 2: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If C is not a vertex cover of G, then $V - C$ is not an independent set of G.

Proof: Assume C is not a vertex cover of G. We need to show that $V - C$ is not an independent set of G.

Since C is not a vertex cover of G, we know that there exists nodes $x, y \in V$ where $\{x, y\} \in E$, where $x \notin C$, and where $y \notin C$. Because $x \in V$ and $x \notin C$, we know that $x \in V - C$. Similarly, we see that $y \in V - C$.

Lemma 2: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If C is not a vertex cover of G, then $V - C$ is not an independent set of G.

Proof: Assume C is not a vertex cover of G. We need to show that $V - C$ is not an independent set of G.

Since C is not a vertex cover of G, we know that there exists nodes $x, y \in V$ where $\{x, y\} \in E$, where $x \notin C$, and where $y \notin C$. Because $x \in V$ and $x \notin C$, we know that $x \in V - C$. Similarly, we see that $y \in V - C$.

This means that $\{x, y\} \in E$, that $x \in V - C$, and that $y \in V - C$, and therefore that $V - C$ is not an independent set of G, as required.

Lemma 2: Let $G = (V, E)$ be a graph and let $C \subseteq V$ be a set. If C is not a vertex cover of G, then $V - C$ is not an independent set of G.

Proof: Assume C is not a vertex cover of G. We need to show that $V - C$ is not an independent set of G.

Since C is not a vertex cover of G, we know that there exists nodes $x, y \in V$ where $\{x, y\} \in E$, where $x \notin C$, and where $y \notin C$. Because $x \in V$ and $x \notin C$, we know that $x \in V - C$. Similarly, we see that $y \in V - C$.

This means that $\{x, y\} \in E$, that $x \in V - C$, and that $y \in V - C$, and therefore that $V - C$ is not an independent set of G, as required. ■

Recap for Today

- A ***graph*** is a structure for representing items that may be linked together. ***Digraphs*** represent that same idea, but with a directionality on the links.
- Graphs can’t have ***self-loops***; digraphs can.
- ***Vertex covers*** and ***independent sets*** are useful tools for modeling problems with graphs.
- The complement of a vertex cover is an independent set, and vice-versa.

Next Time

- ***Paths and Trails***
  - Walking from one point to another.
- ***Indegrees and Outdegrees***
  - Counting how many neighbors you have, in the directed case.
- ***Teleporting a Train***
  - Can you get stuck in a loop?
- ***The Cantor-Bernstein-Schroeder Theorem***
  - A proof on set cardinality that's really a proof about graphs.