Mathematical Induction

Part One

Everybody – do the wave!

The Wave

- If done properly, everyone will eventually end up joining in.
- Why is that?
  - Someone (me!) started everyone off.
  - Once the person before you did the wave, you did the wave.

Let P be some predicate. The ***principle of mathematical induction*** states that if

If it starts true…

$P(0)$ is true

…and it stays true…

and

$$\forall k \in \mathbb{N}.\ (P(k) \rightarrow P(k+1))$$

then

$$\forall n \in \mathbb{N}.\ P(n)$$

…then it's always true.

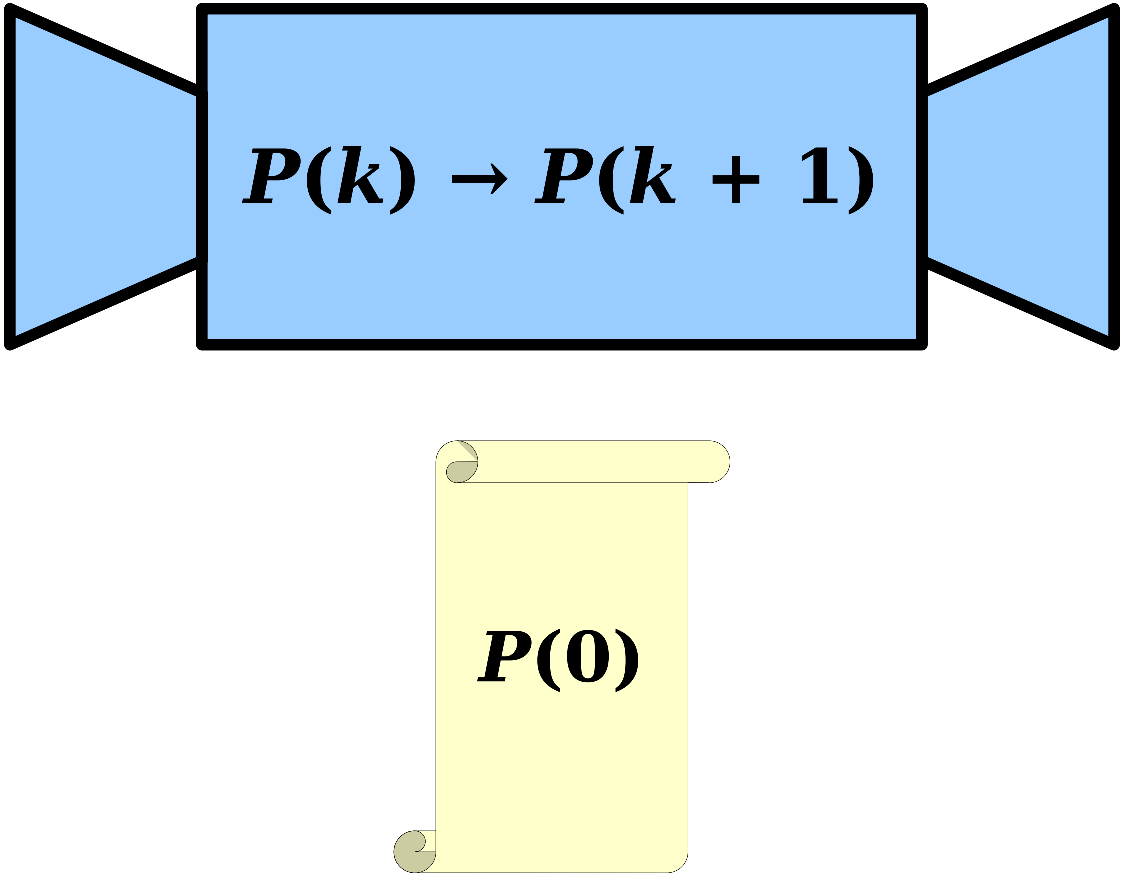
Induction, Intuitively

$$\boldsymbol{P(0)}$$

$$\forall k \in \mathbb{N}.\ (P(k) \rightarrow P(k+1))$$

- It's true for 0.
- Since it's true for 0, it's true for 1.
- Since it's true for 1, it's true for 2.
- Since it's true for 2, it's true for 3.
- Since it's true for 3, it's true for 4.
- Since it's true for 4, it's true for 5.
- Since it's true for 5, it's true for 6.
- …

Why Induction Works



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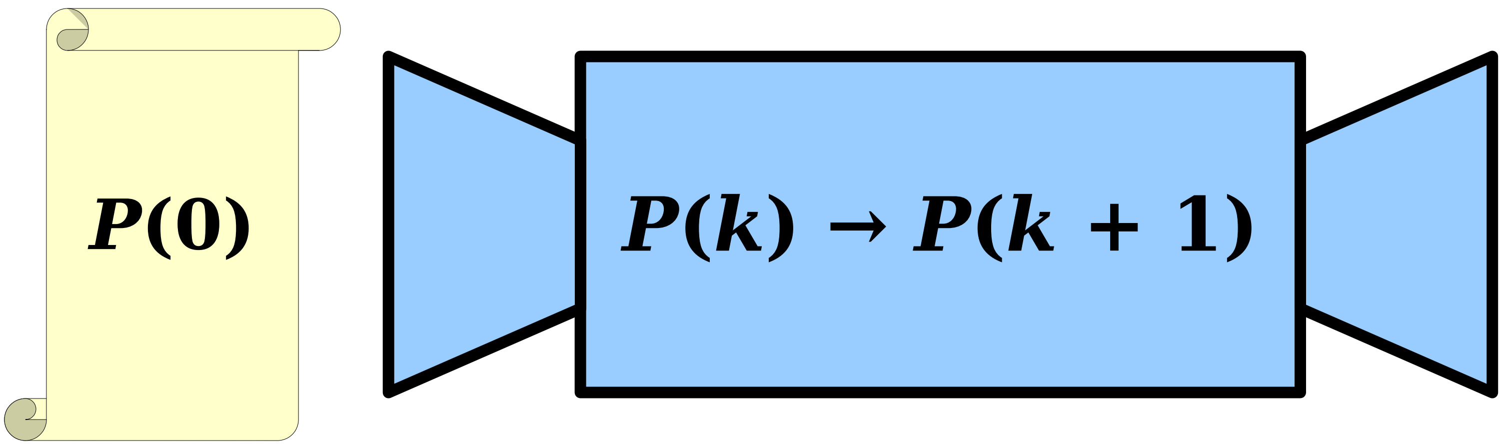

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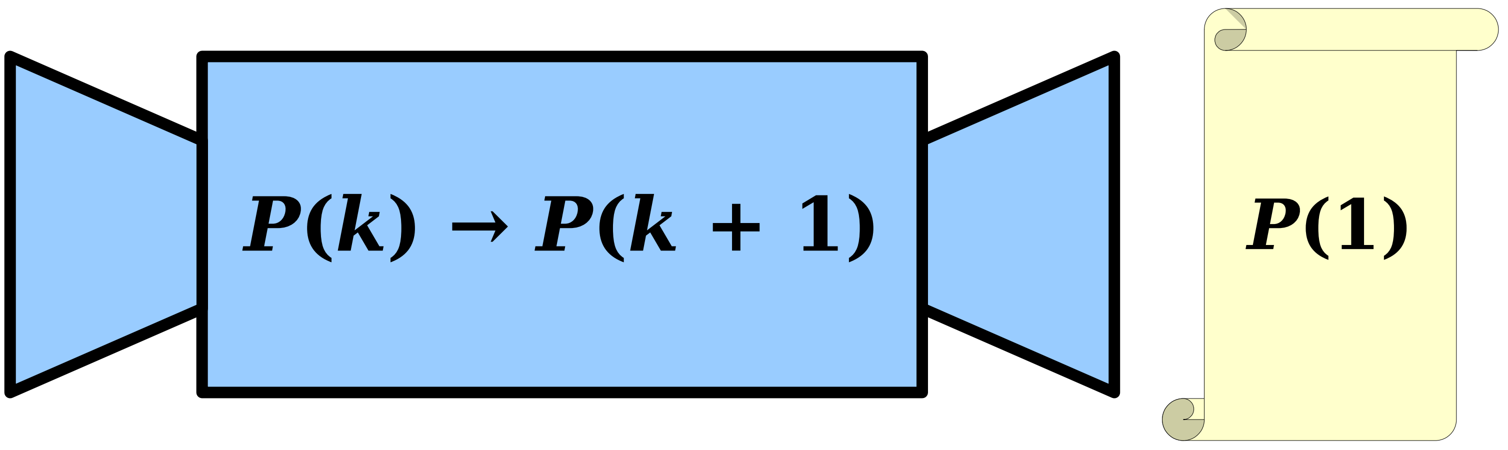

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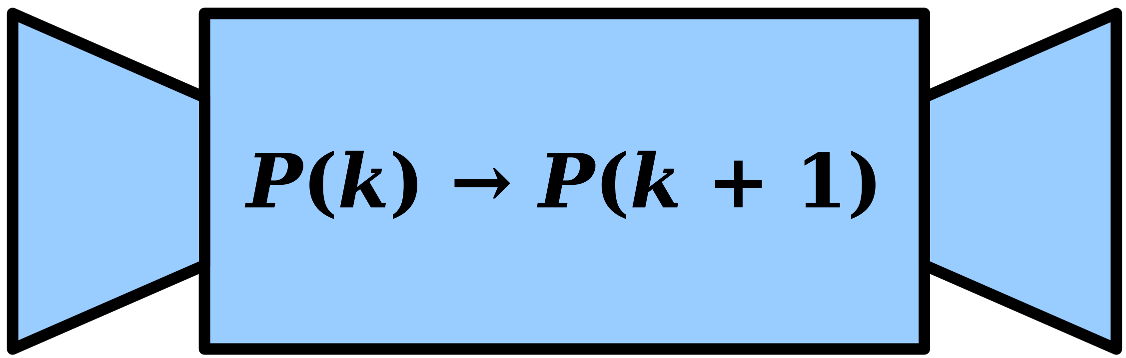

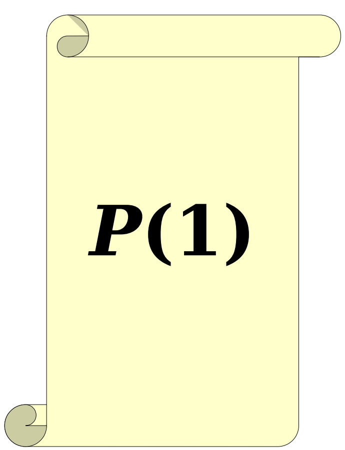

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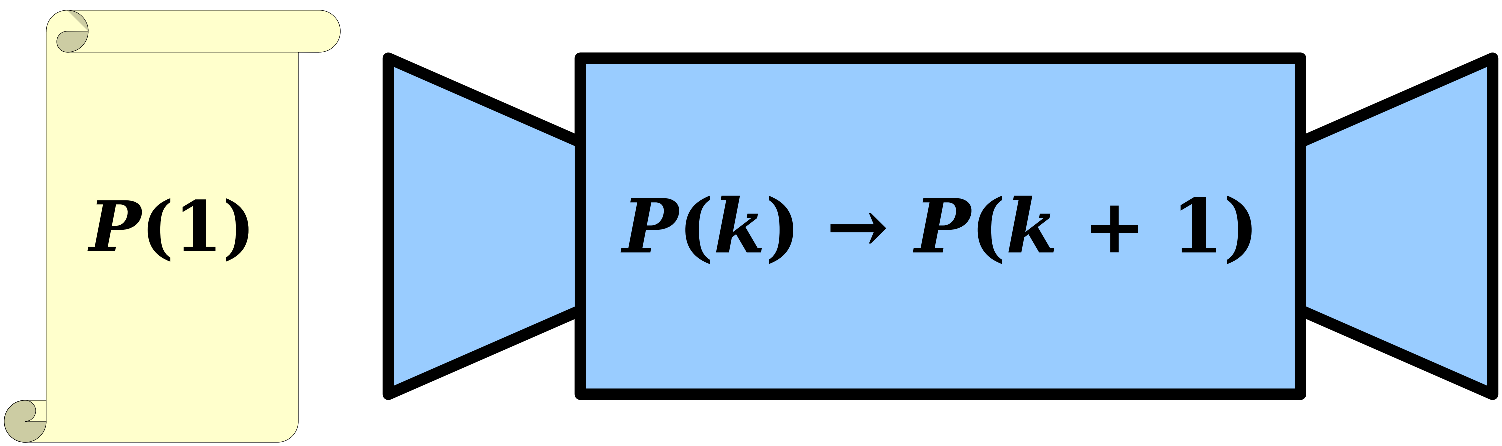

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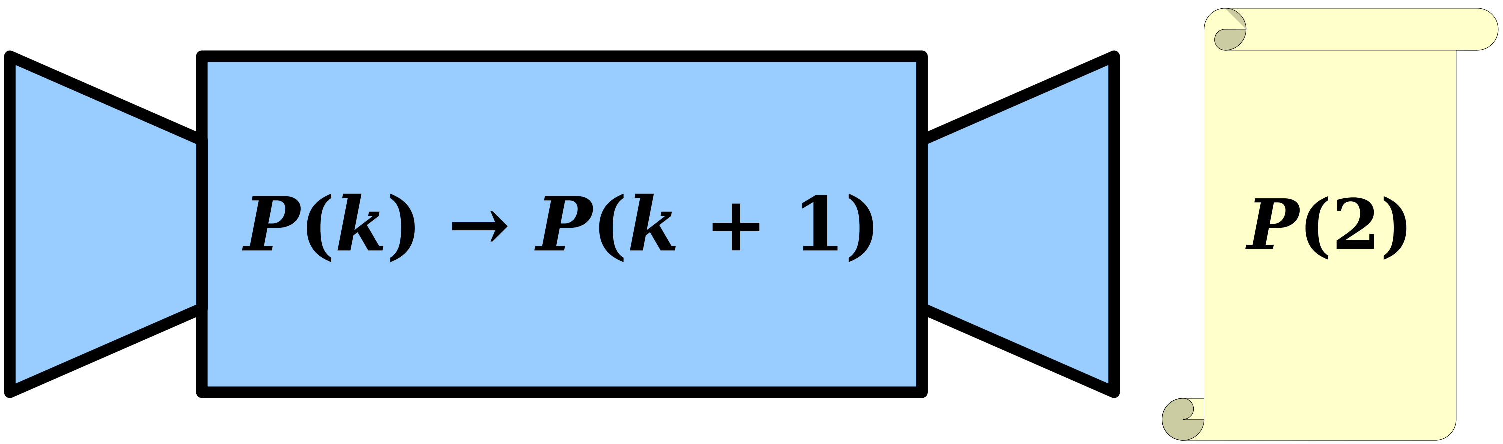

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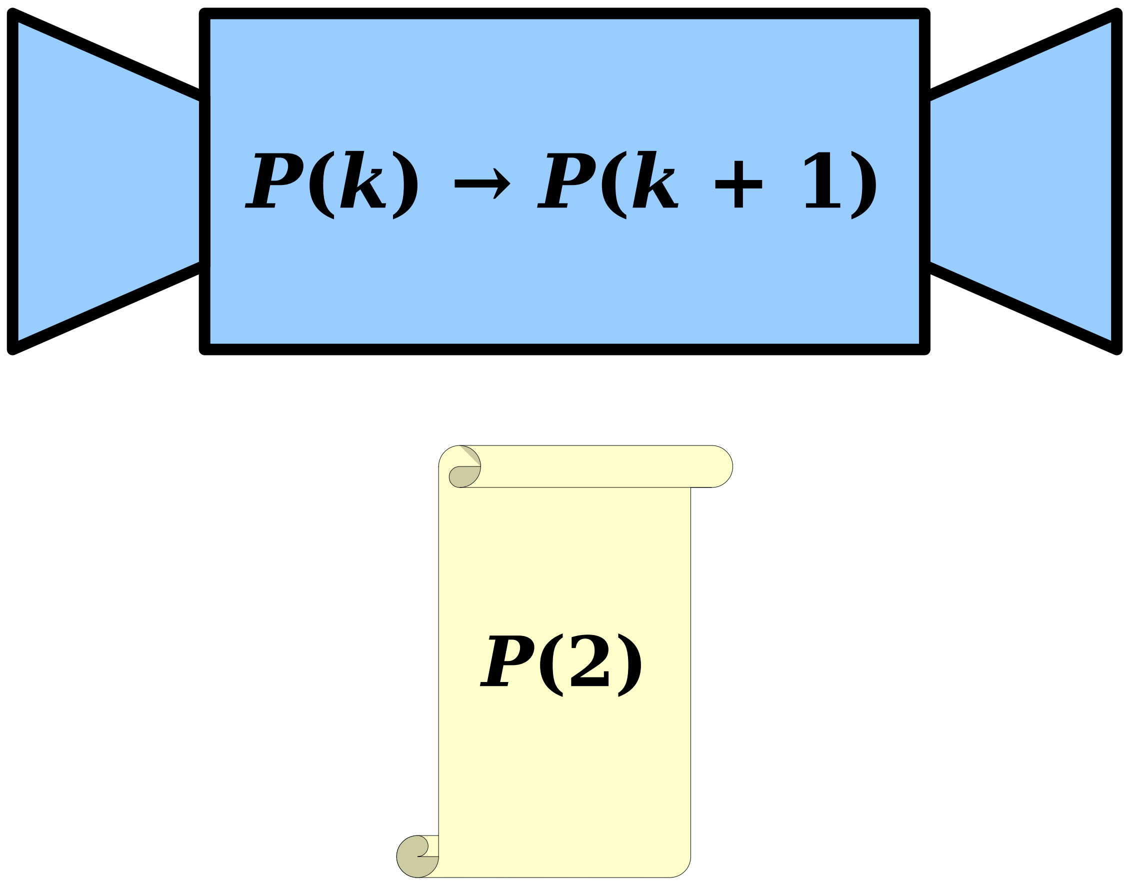

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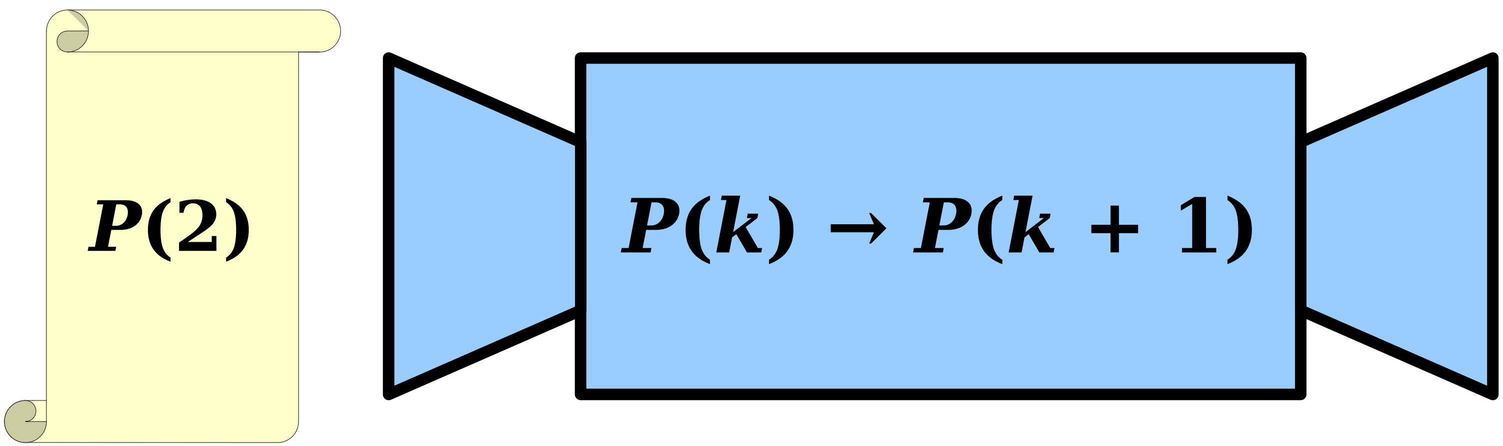

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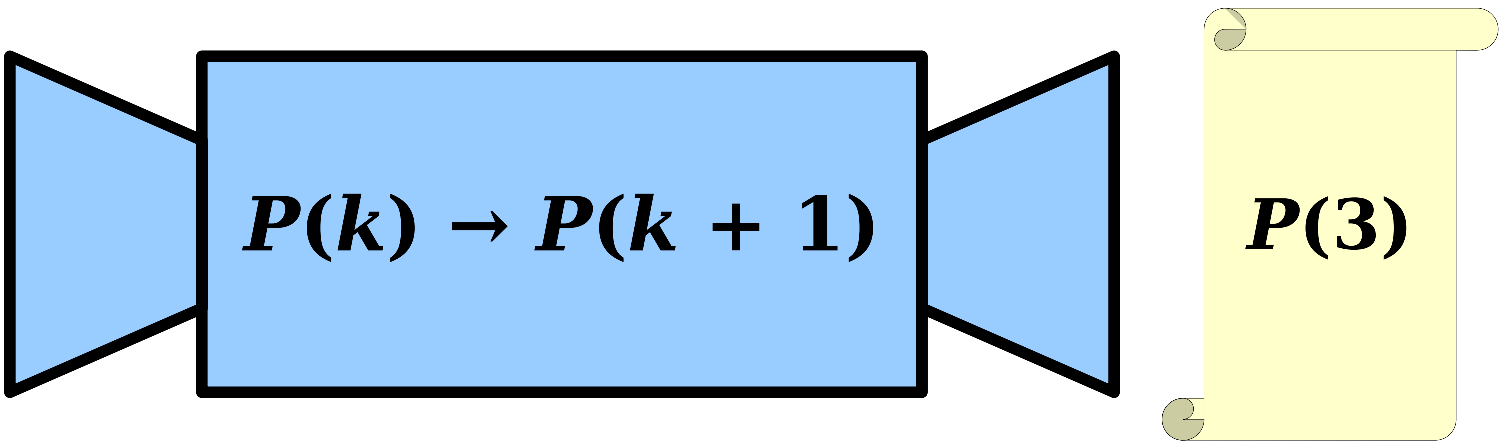

Proof by Induction

- A ***proof by induction*** is a way to use the principle of mathematical induction to show that some result is true for all natural numbers n.
- In a proof by induction, there are three steps:
  - Prove that $P(0)$ is true.
    - This is called the ***basis*** or the ***base case***.
  - Prove that if $P(k)$ is true, then $P(k+1)$ is true.
    - This is called the ***inductive step***.
    - The assumption that $P(k)$ is true is called the ***inductive hypothesis***.
  - Conclude, by induction, that $P(n)$ is true for all $n \in \mathbb{N}$.

Some Sums

$$2^0$$

$$2^0 + 2^1$$

$$2^0 + 2^1 + 2^2$$

$$2^0 + 2^1 + 2^2 + 2^3$$

$$2^0 + 2^1 + 2^2 + 2^3 + 2^4$$

$$\mathbf{2^0} = 1$$

$$\mathbf{2^0 + 2^1} = 1 + 2 = 3$$

$$\mathbf{2^0 + 2^1 + 2^2} = 1 + 2 + 4 = 7$$

$$\mathbf{2^0 + 2^1 + 2^2 + 2^3} = 1 + 2 + 4 + 8 = 15$$

$$\mathbf{2^0 + 2^1 + 2^2 + 2^3 + 2^4} = 1 + 2 + 4 + 8 + 16 = 31$$

$$2^0 = 1 = 2^1 - 1$$

$$2^0 + 2^1 = 1 + 2 = 3 = 2^2 - 1$$

$$2^0 + 2^1 + 2^2 = 1 + 2 + 4 = 7 = 2^3 - 1$$

$$2^0 + 2^1 + 2^2 + 2^3 = 1 + 2 + 4 + 8 = 15 = 2^4 - 1$$

$$2^0 + 2^1 + 2^2 + 2^3 + 2^4 = 1 + 2 + 4 + 8 + 16 = 31 = 2^5 - 1$$

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At the start of the proof, we tell the reader what predicate we're going to show is true for all natural numbers n, then tell them we're going to prove it by induction.

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Here, we state what P(0) actually says. Now, can go prove this using any proof techniques we'd like!

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The goal of this step is to prove

"If $P(k)$ is true, then $P(k+1)$ is true."

To do this, we'll choose an arbitrary k, assume that $P(k)$ is true, then try to prove $P(k+1)$.

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Here, we explicitly state P(k+1), which is what we want to prove. Now, we can use any proof technique we want to prove it.

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For ou… meaning that th… 1. Since the su… d $2^0 - 1$ is zero…

Here, we'll use our inductive hypothesis (the assumption that $P(k)$ is true) to simplify a complex expression. This is a common theme in inductive proofs.

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Therefore, $P(k + 1)$ is true, completing the induction.

Theorem: The sum of the first n powers of two is $2^n - 1$.

***Proof*:** Let $P(n)$ be the statement "the sum of the first n powers of two is $2^n - 1$." We will prove, by induction, that $P(n)$ is true for all $n \in \mathbb{N}$, from which the theorem follows.

For our base case, we need to show $P(0)$ is true, meaning that the sum of the first zero powers of two is $2^0 - 1$. Since the sum of the first zero powers of two is zero and $2^0 - 1$ is zero as well, we see that $P(0)$ is true.

For the inductive step, assume that for some arbitrary $k \in \mathbb{N}$ that $P(k)$ holds, meaning that

$$2^0 + 2^1 + \ldots + 2^{k-1} = 2^k - 1. \qquad (1)$$

We need to show that $P(k + 1)$ holds, meaning that the sum of the first $k + 1$ powers of two is $2^{k+1} - 1$. To see this, notice that

$$\begin{aligned} 2^0 + 2^1 + \ldots + 2^{k-1} + 2^k &= (2^0 + 2^1 + \ldots + 2^{k-1}) + 2^k \\ &= 2^k - 1 + 2^k \qquad \textit{(via (1))} \\ &= 2(2^k) - 1 \\ &= 2^{k+1} - 1. \end{aligned}$$

Therefore, $P(k + 1)$ is true, completing the induction. ■

A Quick Aside

- This result helps explain the range of numbers that can be stored in an `int`.
- If you have an unsigned 32-bit integer, the largest value you can store is given by $1 + 2 + 4 + 8 + \dots + 2^{31} = 2^{32} - 1$.
- This formula for sums of powers of two has many other uses as well. You'll see one on Friday.

Structuring a Proof by Induction

- Define some predicate P that you'll show, by induction, is true for all natural numbers.
- Prove the base case:
  - State that you're going to prove that $P(0)$ is true, then go prove it.
- Prove the inductive step:
  - Say that you're assuming $P(k)$ for some arbitrary natural number k, then write out exactly what that means.
  - Say that you're going to prove $P(k+1)$, then write out exactly what that means.
  - Prove that $P(k+1)$ using any proof technique you'd like!
- This is a rather verbose way of writing inductive proofs. As we get more experience with induction, we'll start leaving out some details from our proofs.

The Counterfeit Coin Problem

Problem Statement

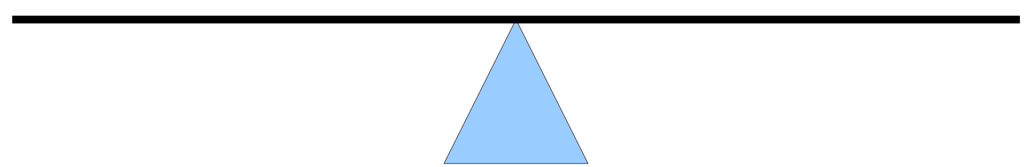
- You are given a set of three seemingly identical coins, two of which are real and one of which is counterfeit.
- The counterfeit coin weighs more than the rest of the coins.
- You are given a balance. Using only one weighing on the balance, find the counterfeit coin.

Finding the Counterfeit Coin

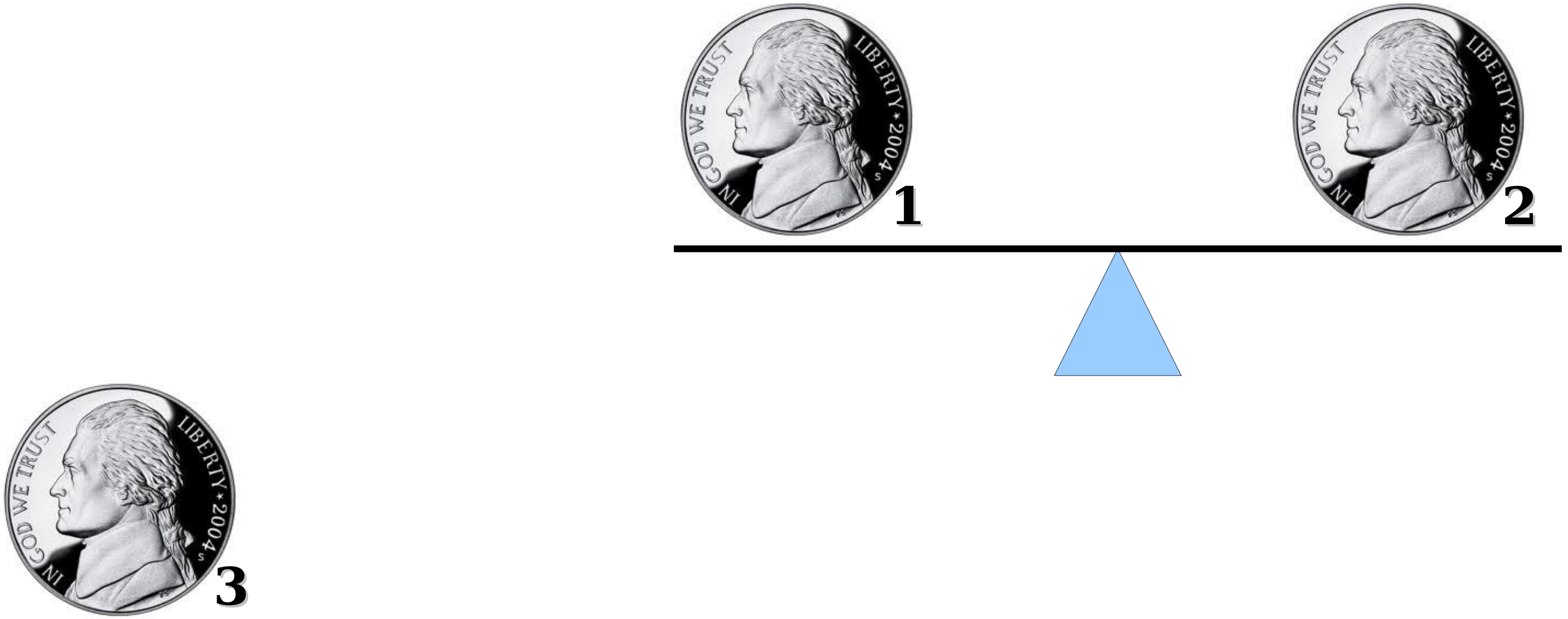
1

2

3



Finding the Counterfeit Coin



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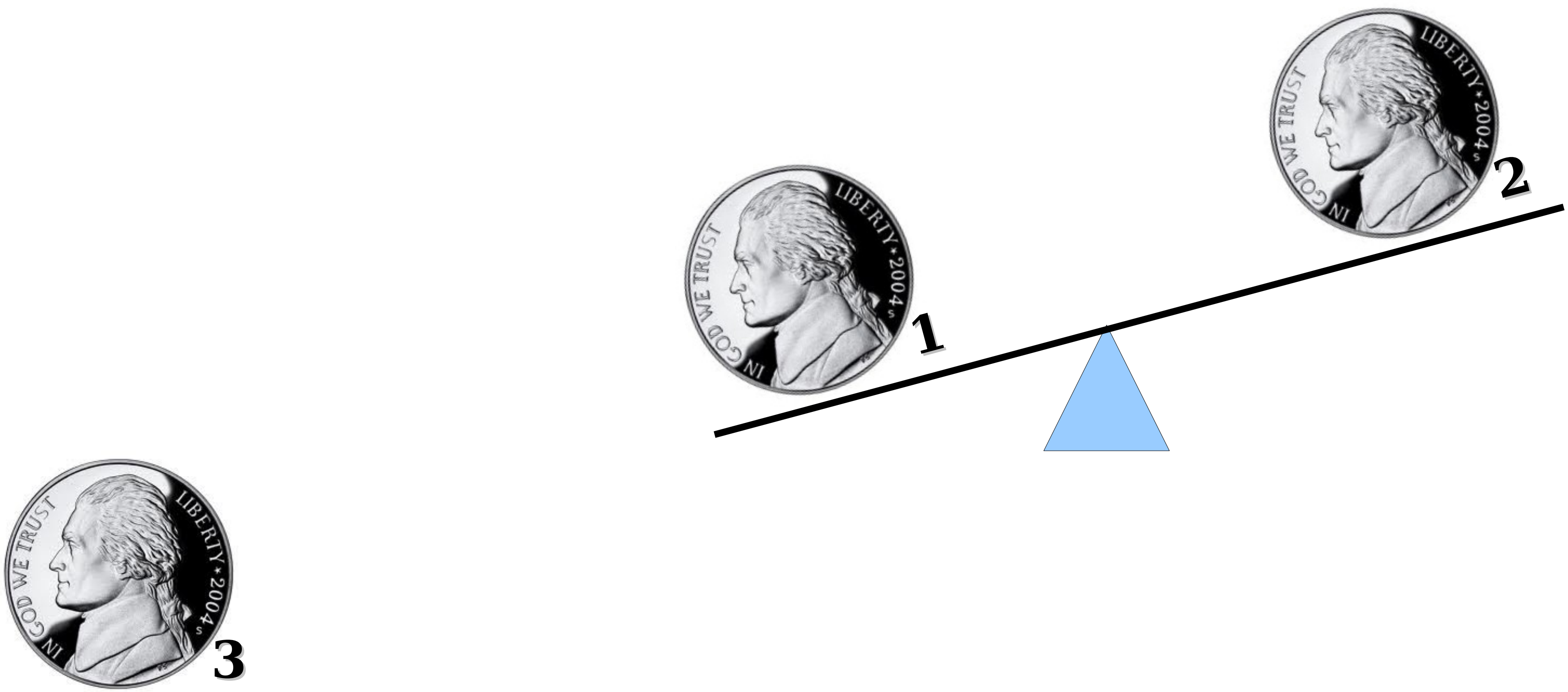

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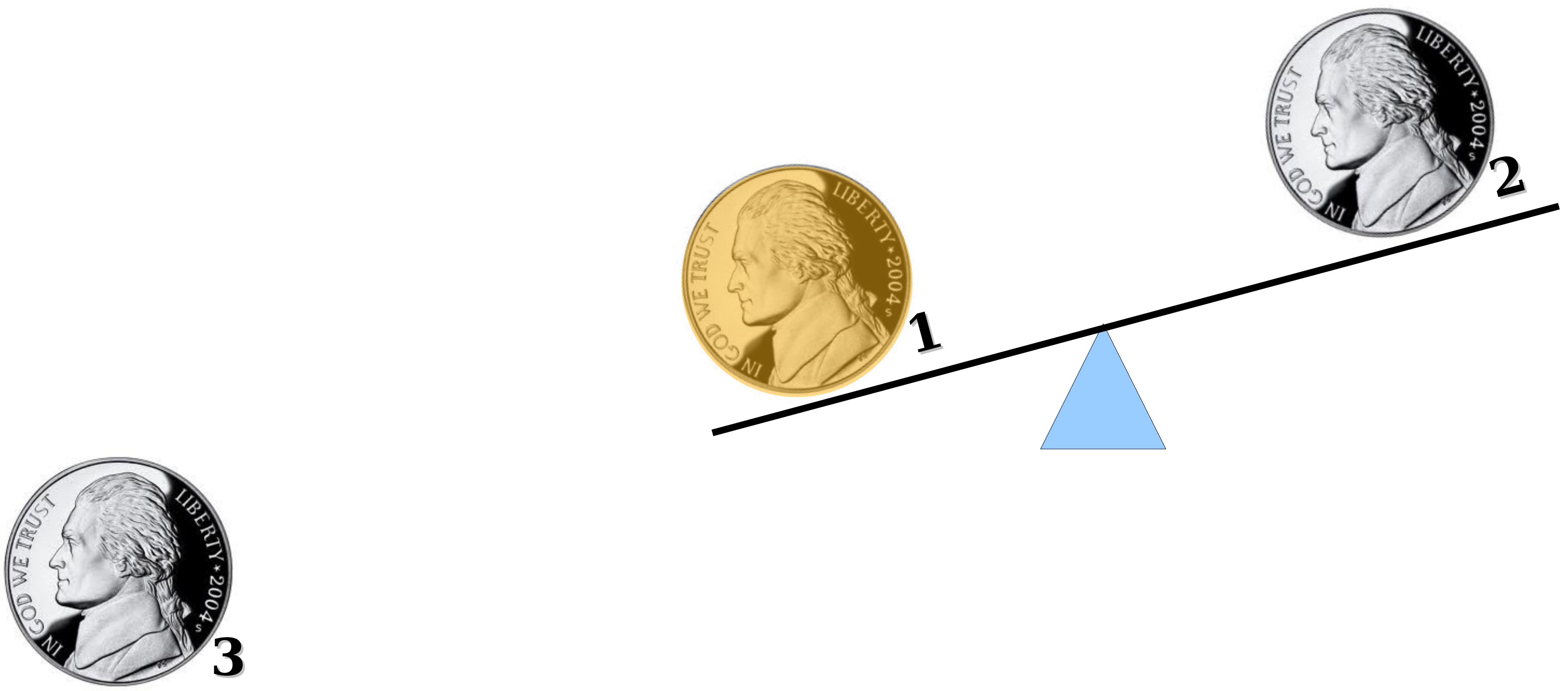

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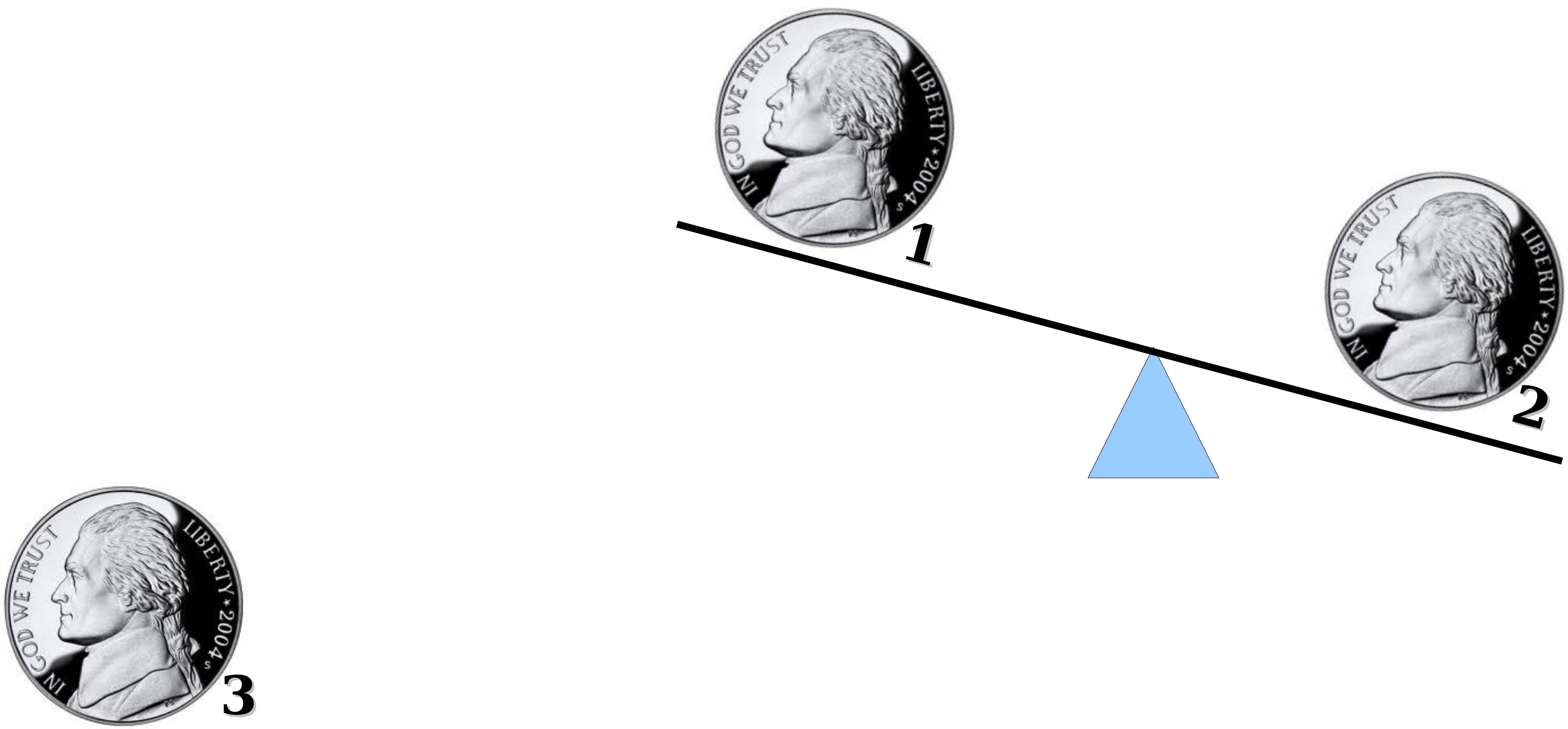

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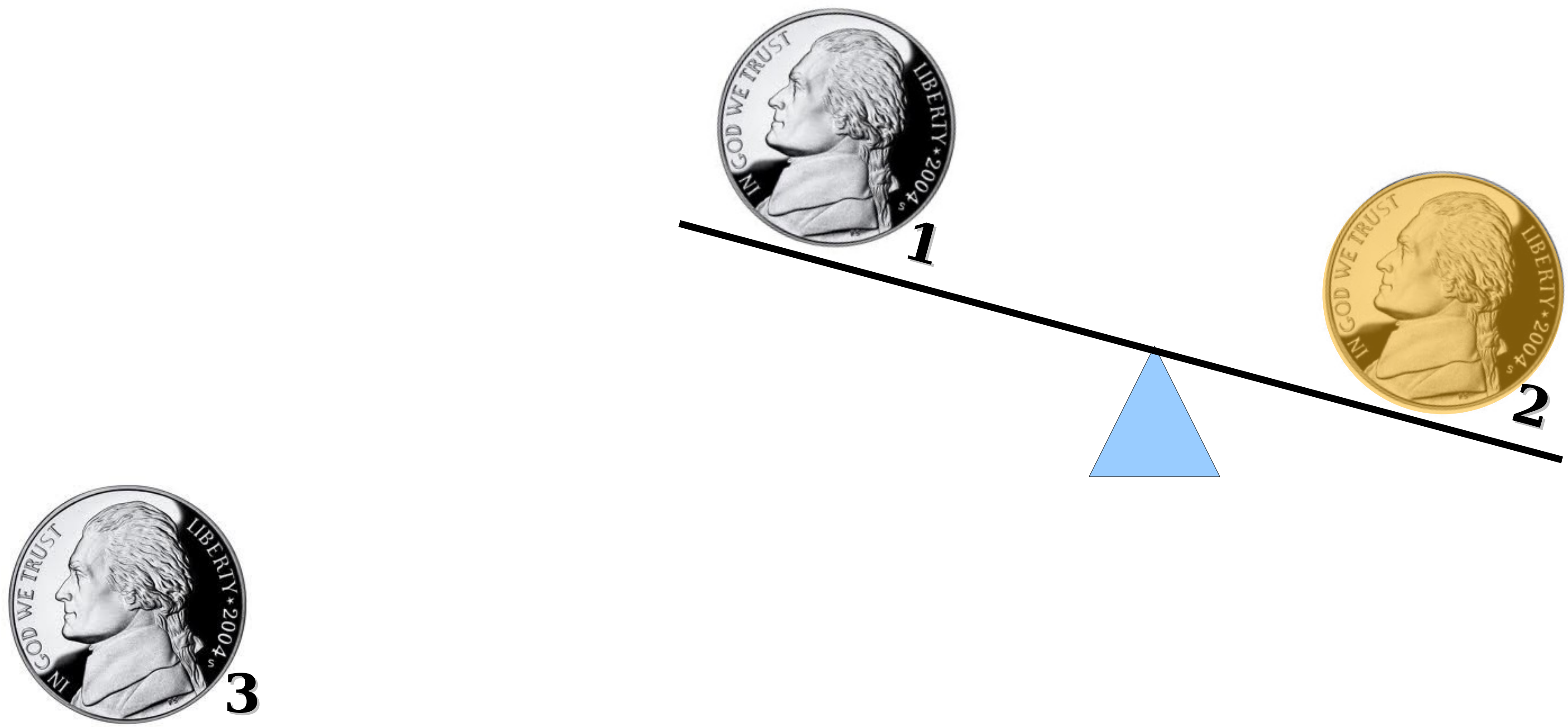

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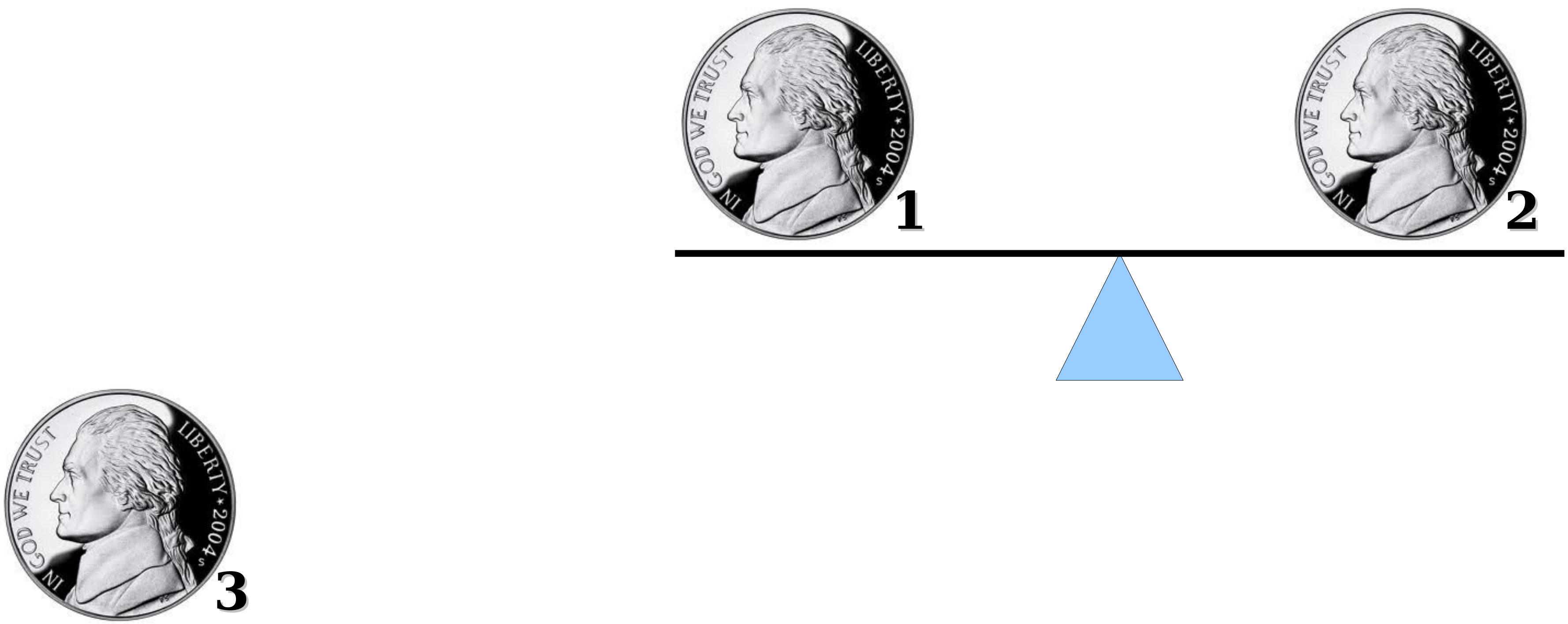

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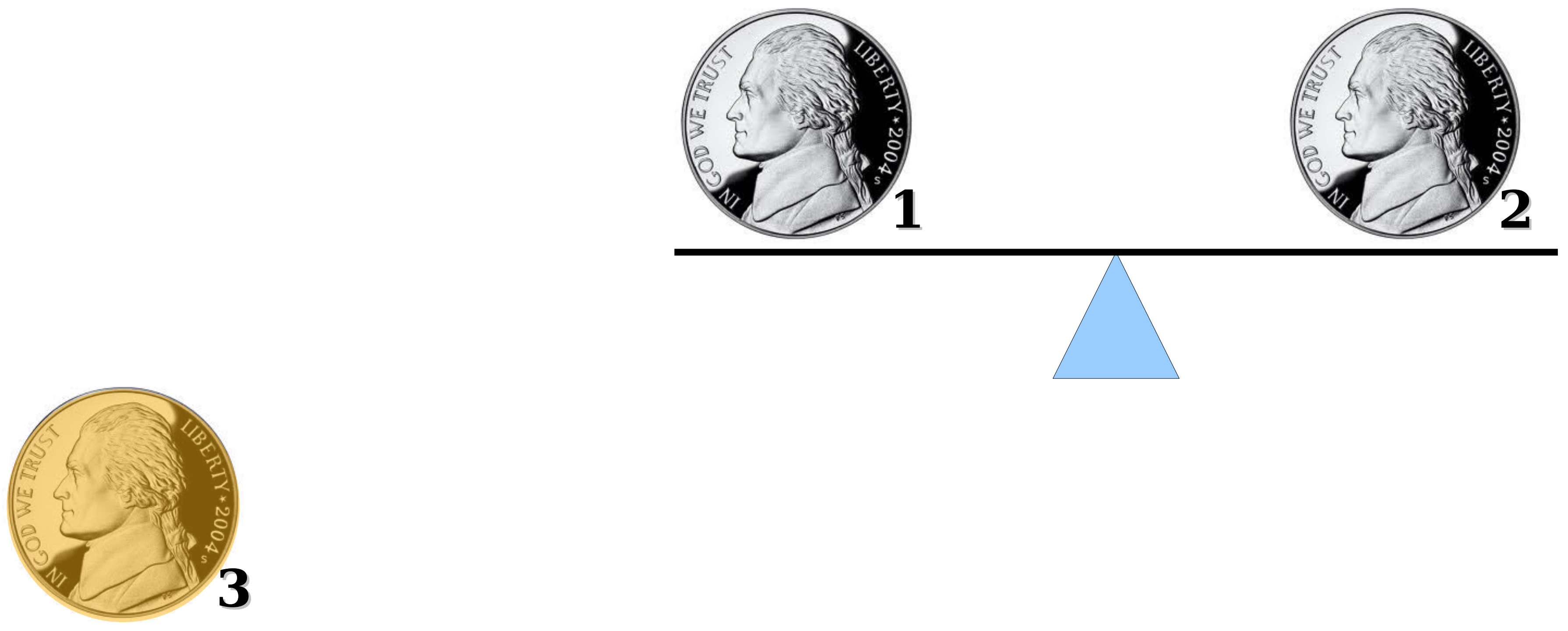
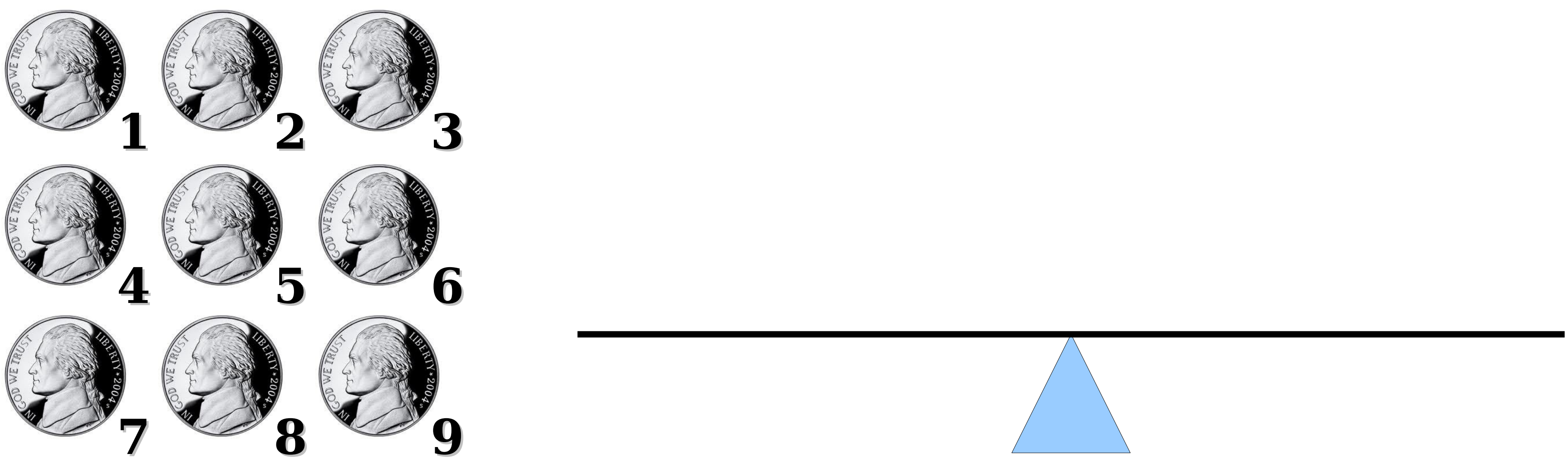


A Harder Problem

- You are given a set of ***nine*** seemingly identical coins, eight of which are real and one of which is counterfeit.
- The counterfeit coin weighs more than the rest of the coins.
- You are given a balance. Using only ***two*** weighings on the balance, find the counterfeit coin.

Finding the Counterfeit Coin



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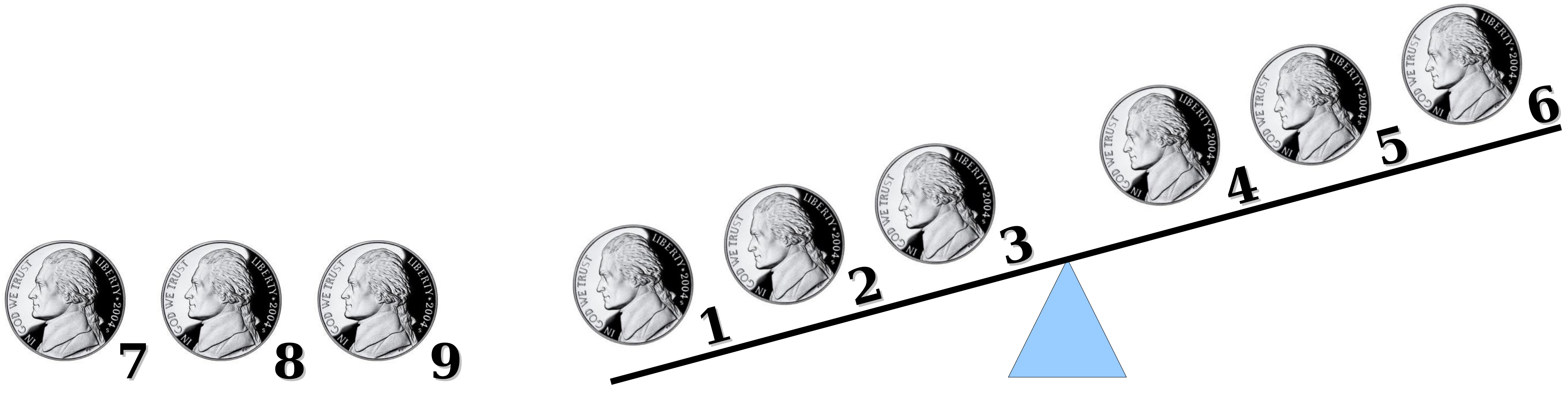

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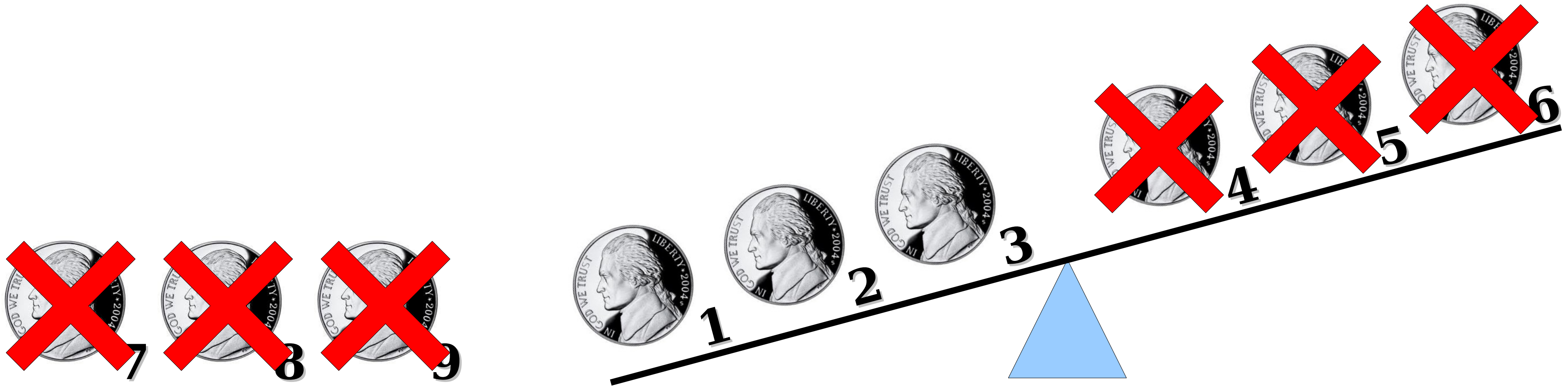
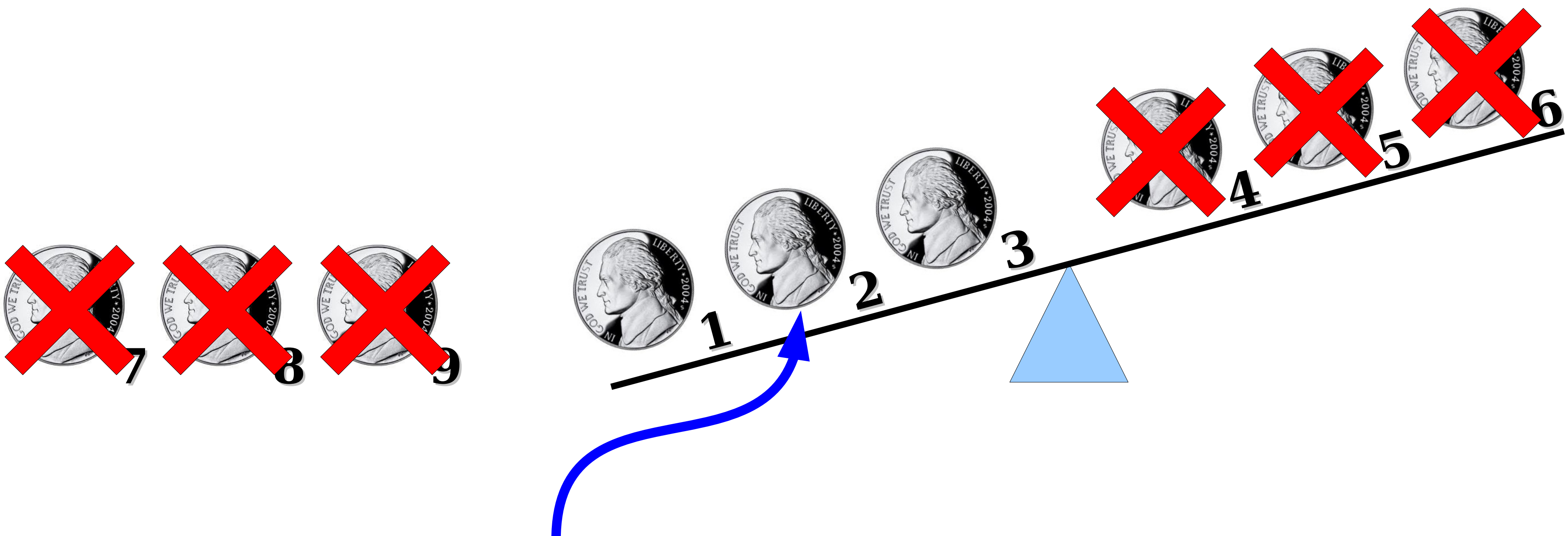
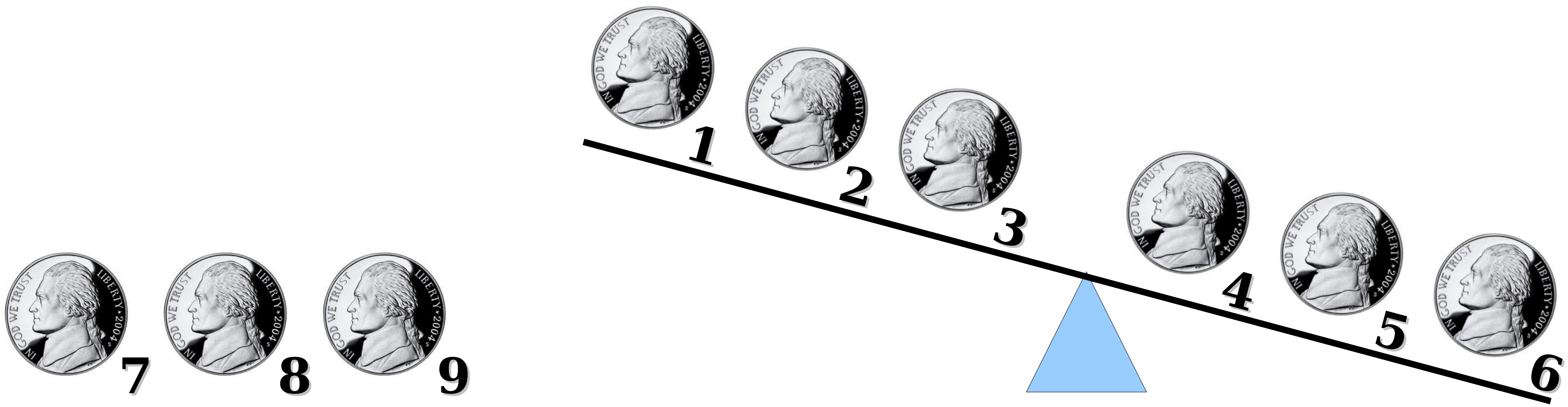


Finding the Counterfeit Coin



Now we have one weighing to find the counterfeit out of these three coins.

Finding the Counterfeit Coin



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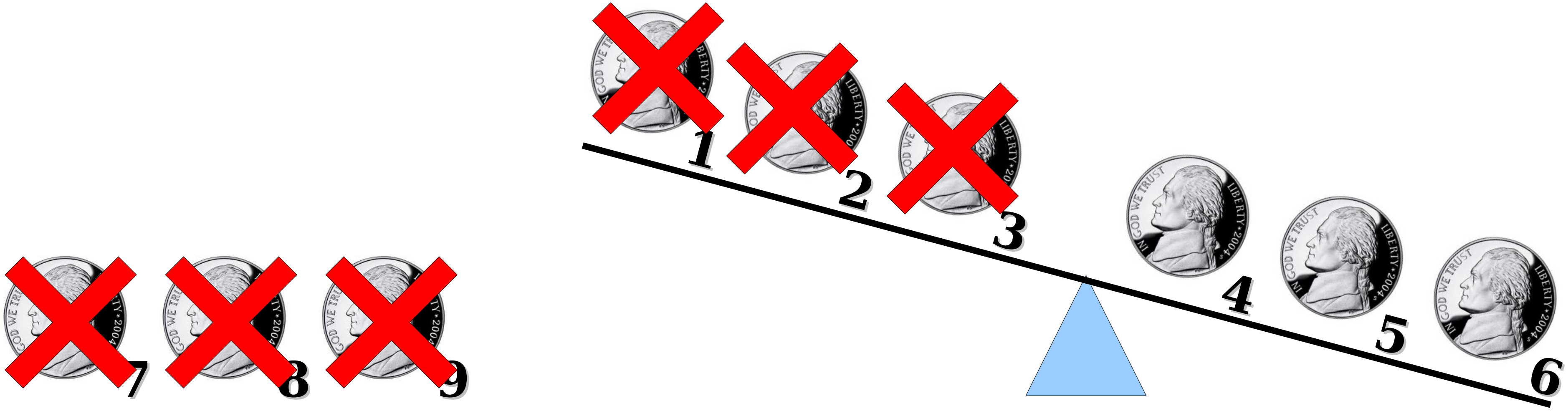

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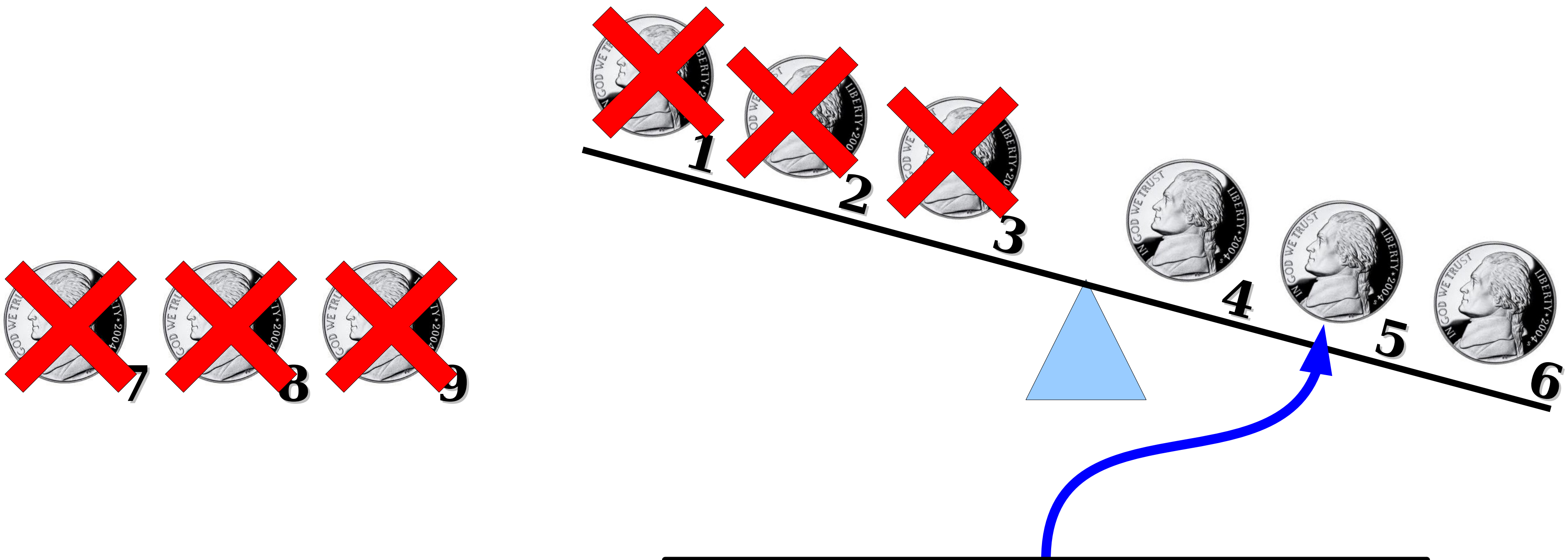

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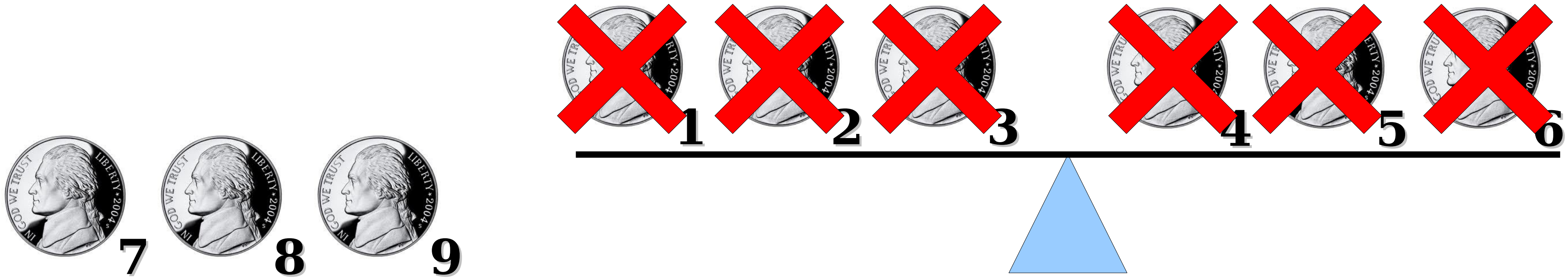

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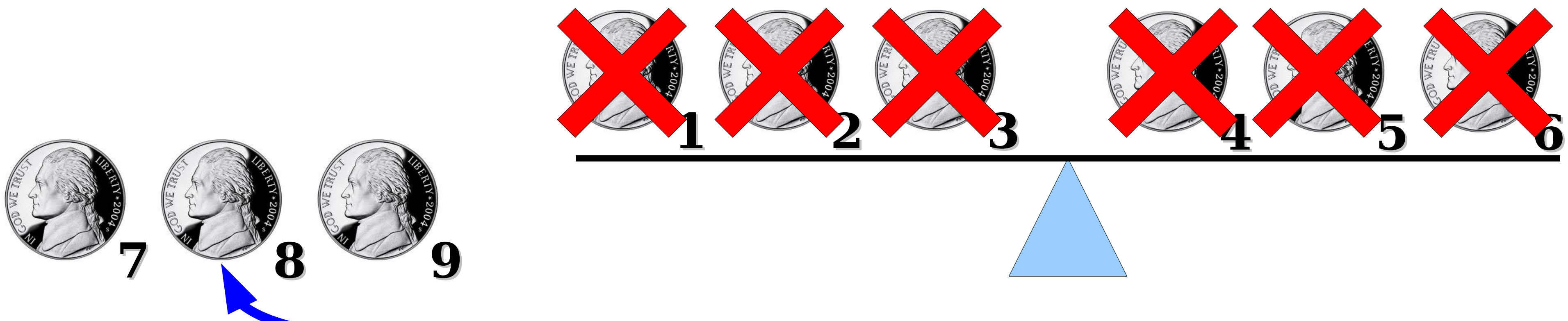

Now we have one weighing to find the counterfeit out of these three coins.

Can we generalize this?

A Pattern

- Assume out of the coins that are given, exactly one is counterfeit and weighs more than the other coins.
- If we have no weighings, how many coins can we have while still being able to find the counterfeit?
  - ***One*** coin, since that coin has to be the counterfeit!
- If we have one weighing, we can find the counterfeit out of ***three*** coins.
- If we have two weighings, we can find the counterfeit out of ***nine*** coins.

So far, we have

1, 3, 9 = $3^0, 3^1, 3^2$

Does this pattern continue?

Theorem: If exactly one coin in a group of 3^n coins is heavier than the rest, that coin can be found using only n weighings on a balance.

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We'll use induction to prove that $P(n)$ holds for every $n \in \mathbb{N}$, from which the theorem follows.

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At the start of the proof, we tell the reader what predicate we're going to show is true for all natural numbers n, then tell them we're going to prove it by induction.

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In a proof by induction, we need to prove that

$P(0)$ is true

If $P(k)$ is true, then $P(k+1)$ is true.

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As our base case, we'll prove that $P(0)$ is true, meaning that if we have a set of $3^0=1$ coins with one coin heavier than the rest, we can find that coin with zero weighings.

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Here, we state what P(0) actually says. Now, can go prove this using any proof techniques we'd like!

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For the inductive step, suppose $P(k)$ is true for some arbitrary $k \in \mathbb{N}$, so we can find the heavier of 3^k coins in k weighings.

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The goal of this step is to prove

"If $P(k)$ is true, then $P(k+1)$ is true."

To do this, we'll choose an arbitrary k, assume that $P(k)$ is true, then try to prove $P(k+1)$.

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Here, we explicitly state $P(k+1)$, which is what we want to prove. Now, we can use any proof technique we want to try to prove it.

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Suppose we have 3^{k+1} coins with one heavier than the others. Split the coins into three groups of 3^k coins each.

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Suppose we have 3^{k+1} coins with one heavier than the others. Split the coins into three groups of 3^k coins each. Weigh two of the groups against one another.

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Suppose we have 3^{k+1} coins with one heavier than the others. Split the coins into three groups of 3^k coins each. Weigh two of the groups against one another. If one group is heavier than the other, the coins in that group must contain the heavier coin.

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If exact … rest, that … ce.

We'll use … from which the theore…

Here, we use our inductive hypothesis (the assumption that $P(k)$ is true) to solve this simpler version of the overall problem.

As our ba… if we have a set of 3… an find that coin with zero weighings. This is true because if we have just one coin, it's vacuously heavier than all the others, and no weighings are needed.

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For the … ℕ, so we can … $+1)$: that we …

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Some Fun Problems

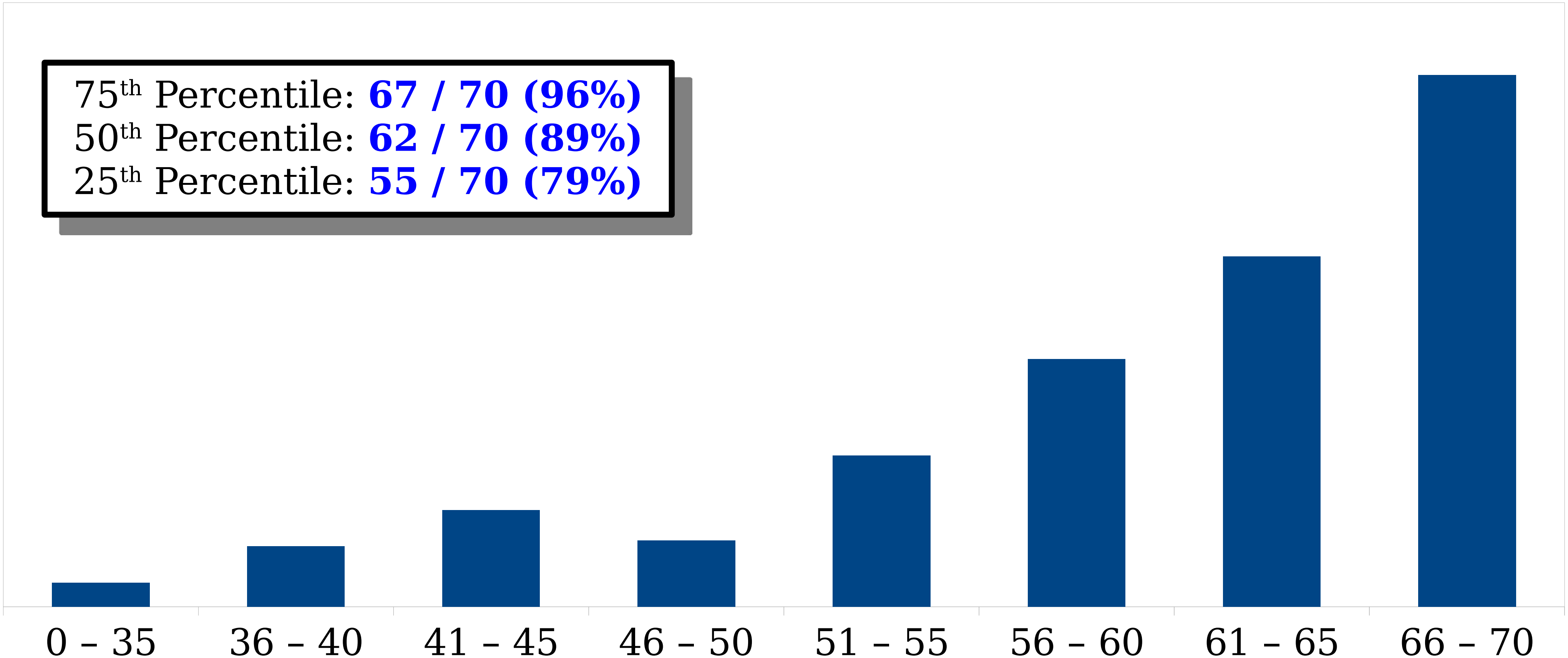
- Here's some nifty variants of this problem that you can work through:
  - Suppose that you have a group of coins where there's either exactly one heavier coin, or all coins weigh the same amount. If you only get k weighings, what's the largest number of coins where you can find the counterfeit or determine none exists?
  - What happens if the counterfeit can be either heavier or lighter than the other coins? What's the maximum number of coins where you can find the counterfeit if you have k weighings?
  - Can you find the counterfeit out of a group of more than 3^k coins with k weighings?
  - Can you find the counterfeit out of any group of at most 3^k coins with k weighings?

Time-Out for Announcements!

First Midterm Exam

- You’re done with the first midterm! Woohoo!
- The TAs will be grading exams this weekend. We’ll release solutions and stats once they finish.
- You’re welcome to come chat with us about the questions on the exam if you’d like. We can’t discuss how we’ll grade things, though, since the criteria are still under development.

Problem Set Three Grades



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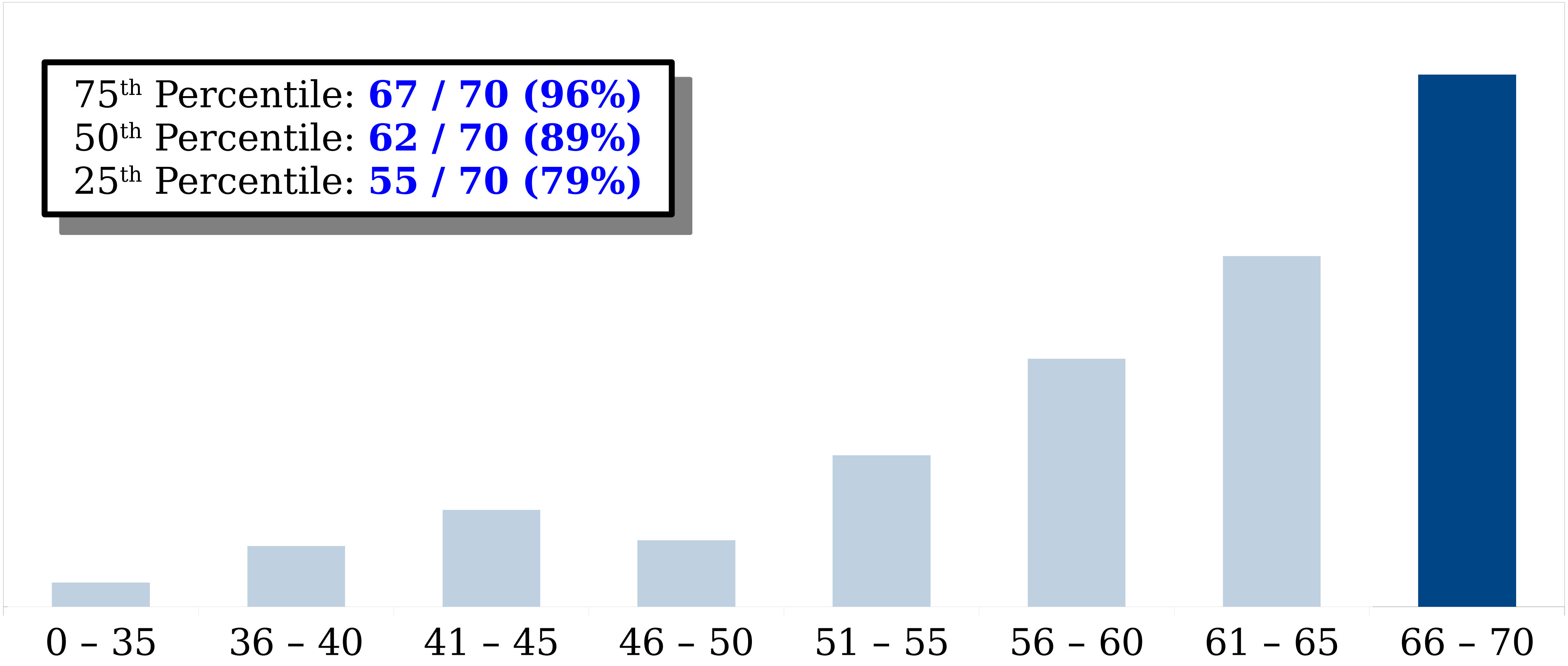
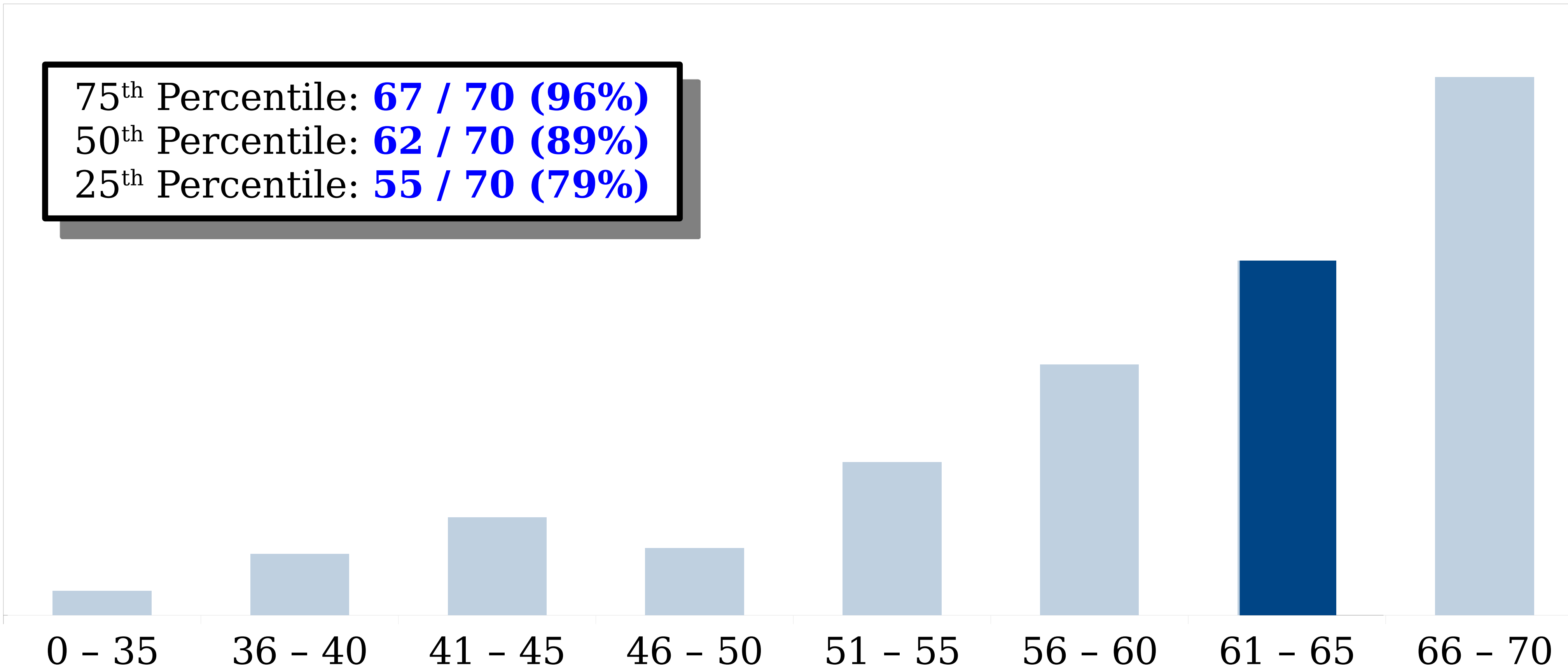


"Awesome job! Take a look at the feedback to see where you need to make final tweaks and adjustments."

Problem Set Three Grades

75th Percentile: **67 / 70 (96%)**
50th Percentile: **62 / 70 (89%)**
25th Percentile: **55 / 70 (79%)**



"Well done! There are few spots you may need to brush up on, so take a look to see what they are and keep them in mind going forward."

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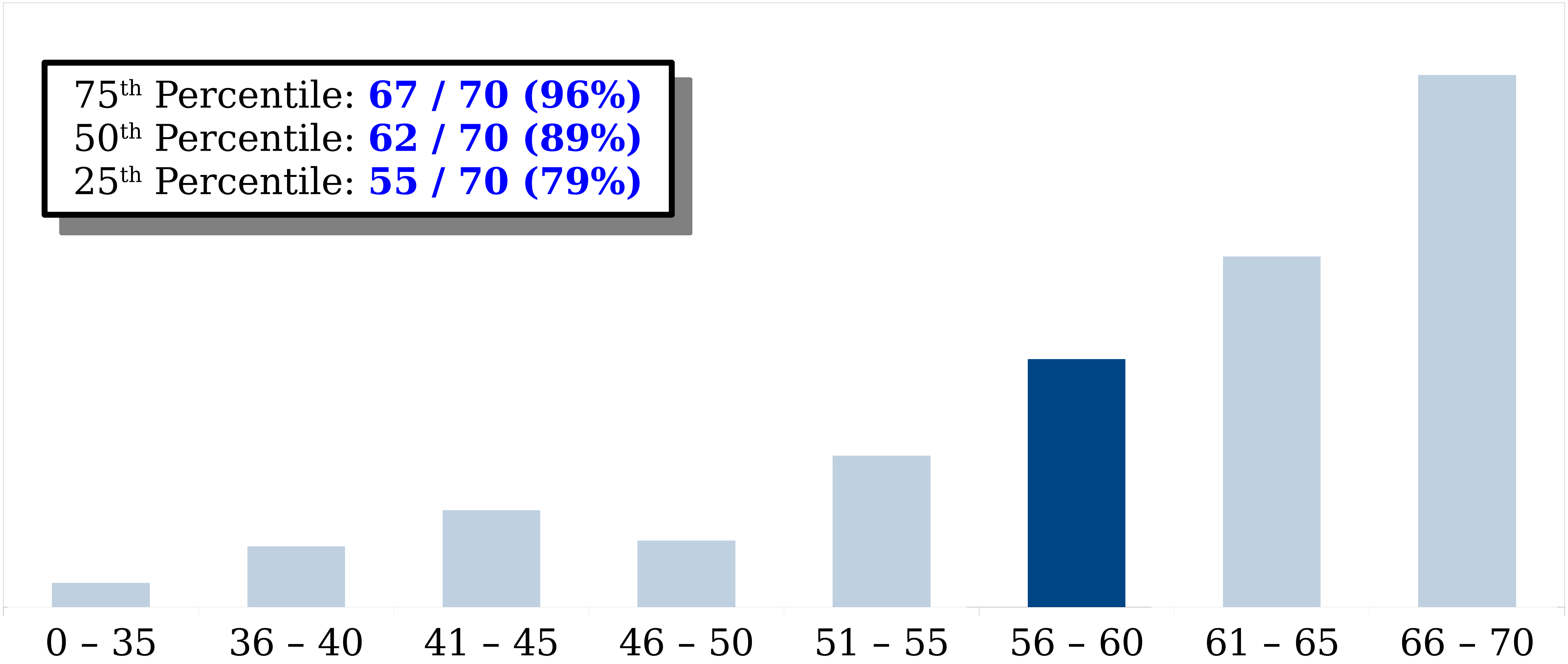

"Nice work. Take a look at the feedback to see what concrete areas you need to focus on – chances are there's some skill you may need some more practice with – and keep up the good work!"

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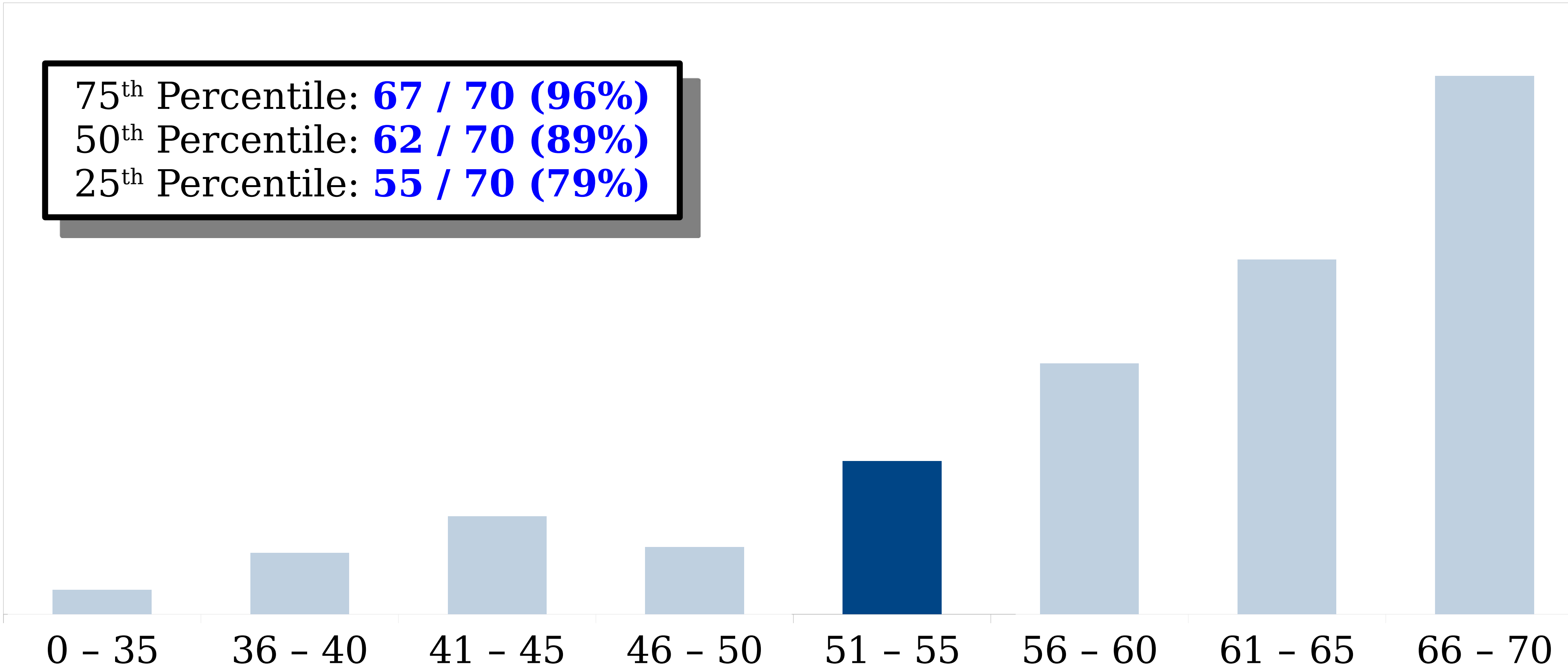

"You're on the right track, and chances are there's some key skill you may need to get some more practice with. Take a look at your feedback, figure out where to focus your energy going forward, and let us know how we can help."

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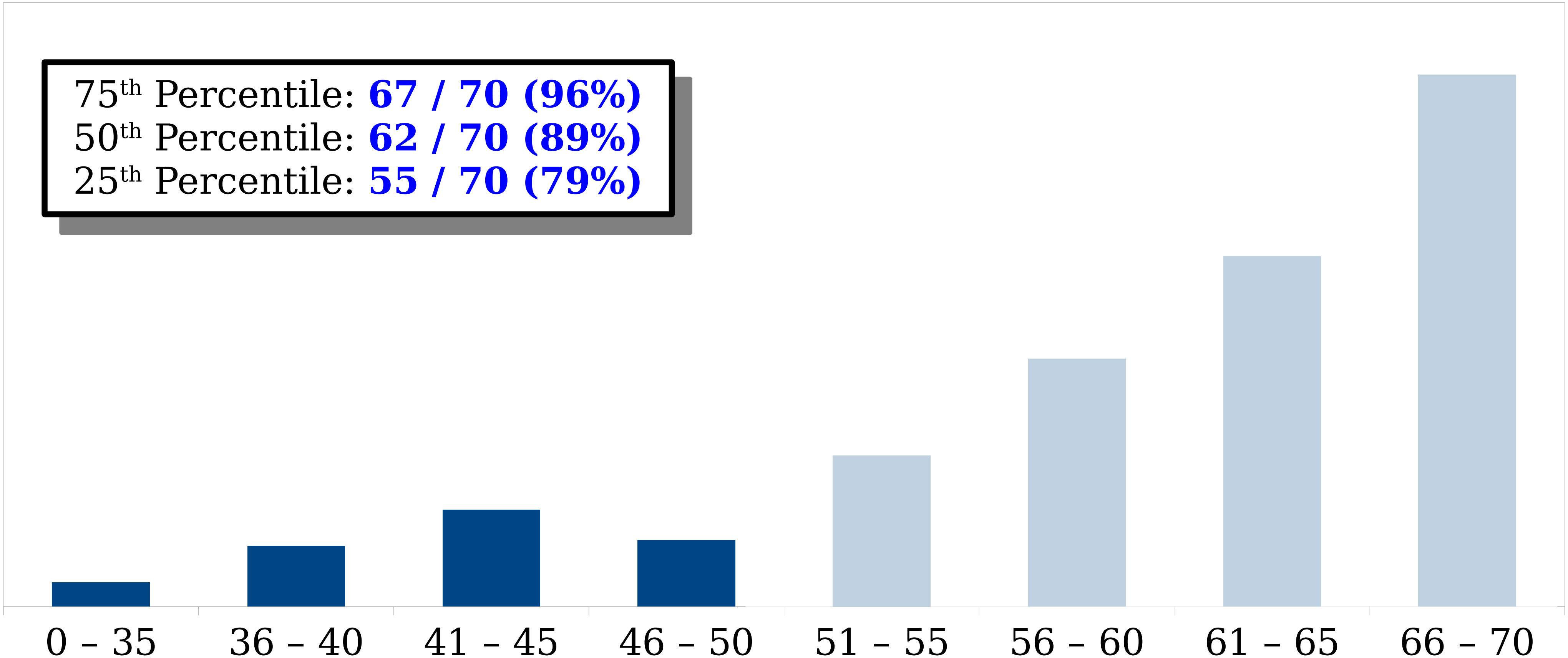

"Seems like there's some skill or technique that hasn't clicked yet. Feel free to chat with us in office hours or over Piazza about your feedback, and let us know how we can help you practice and improve."

Problem Set Four

- Problem Set Four is due this Friday at 2:30PM.
- ***Recommendation:*** As soon as you can, review all the feedback you got on PS3 and ask yourself these questions:
  - Based on the proofwriting and style feedback you received, do you know what specific changes you'd make to your answers?
  - If you made any logic errors, do you understand what those errors are to the point that you could explain them to someone else?
- Feel free to stop by office hours or to visit EdStem if you have questions. We're happy to help out! You can do this!

Your Questions

"I have noticed that with Computer Science, I know the right questions to ask so I can learn on my own if I am struggling. However, I have not developed that intuition with formal logic or math in general. For example, if I do not know how to do something I want to do in C, I know the types of questions to ask to get me to where I want to be. However, when I am thinking about applications of graph theory in my everyday life, I do not even know where to begin when I have a question. What are some tips you have for developing this intuition for use in this class and in life in general?"

A lot of this is a function of experience. I assume you probably have more experience writing code than you do proofs about graphs. Take stock of the time difference. How long have you been coding? How many coding classes have you taken? Compare that to what we're doing here. How long have you been working with graphs? How many classes have you taken in it? As you get more experience working with it your intuitions will be better.

Also – this stuff can be pretty tricky! Many "obvious" results about graphs are hard to prove, and many conjectures have turned out false, and many accepted proofs found to have flaws. There's new ground broken all the time! The more exposure you have and the more practice you get, the better you'll get at this.

“How should female, gendered marginalized, FLI (First-gen Low Income), ESL (English as Second Language) or folks from minority community go into CS where it's mostly dominated mostly by well-off white male?”

This is a large question and I can't address it in full. Here's a partial answer:

1. Fnd a community of people you feel welcome and supported in. Feeling a sense of belonging is huge. CS folks are more heterogeneous than you might initially expect.

2. Distinguish between what's under your control and what isn't. You can't change your past circumstances. You can control who you associate with, how you build your skills, etc.

3. Find an environment that works well for you. Different companies, research labs, nonprofits, etc. have different cultural values and norms. Seek out places that treat you as a human being and value you for your perspective, personality, skills, and abilities.

4. Get good mentorship. Find a mentor who is invested in your growth and development, advocates for you, and can offer advice when you need it.

Happy to chat about this more in person if you'd like!

Back to CS103!

How Not To Induct

Something's Wrong...

Something's Wrong...

Theorem: The sum of the first n powers of two is 2^n.

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$$2^0 + 2^1 + \ldots + 2^{k-1} = 2^k. \qquad (1)$$

We need to show that $P(k + 1)$ holds, meaning that the sum of the first $k + 1$ powers of two is 2^{k+1}. To see this, notice that

$$\begin{aligned} 2^0 + 2^1 + \ldots + 2^{k-1} + 2^k &= (2^0 + 2^1 + \ldots \\ &= 2^k + 2^k \\ &= 2(2^k) \\ &= 2^{k+1}. \end{aligned}$$

Where did we prove the base case?

Therefore, $P(k + 1)$ is true, completing the induction. ■

When writing a proof by induction,

make sure to prove the base case!

Otherwise, your proof is incomplete!

Why did this work?

Theorem: The sum of the first n powers of two is 2^n.

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$$2^0 + 2^1 + \ldots + 2^{k-1} = 2^k. \qquad (1)$$

We need to show that $P(k + 1)$ holds, meaning that the sum of the first $k + 1$ powers

that

$$2^0 + 2^1 + \ldots + 2^{k-1} + 2$$

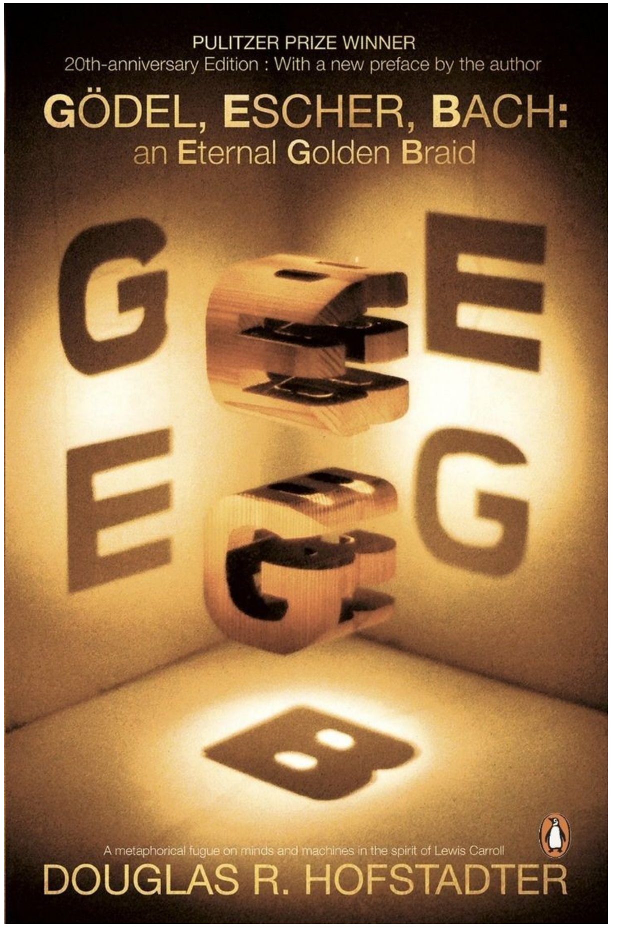
You can prove *anything* from a faulty assumption. This is called the **principle of explosion**.

Therefore, $P(k + 1)$ is true, completing the induction. ■

The MU Puzzle

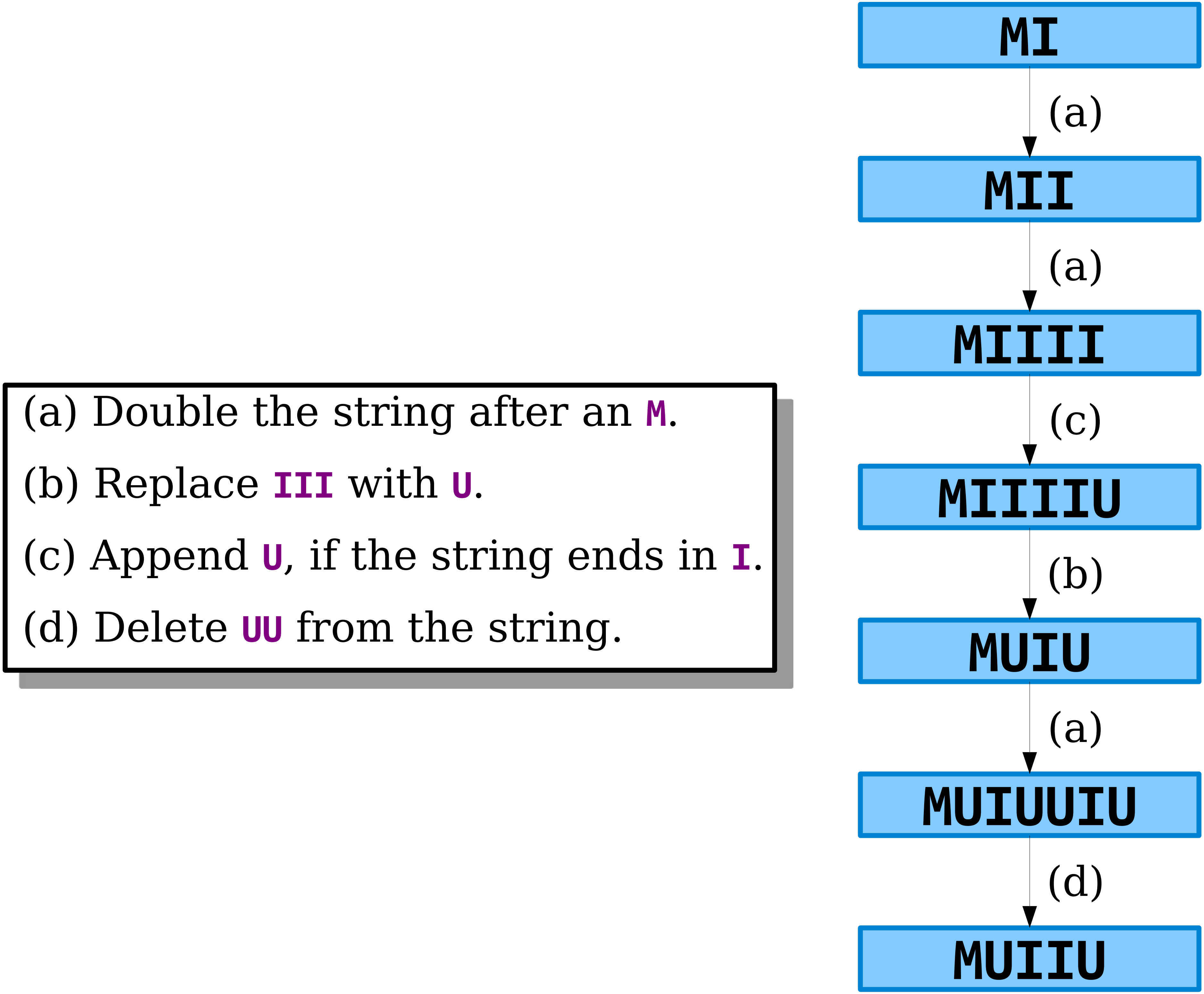
Gödel, Escher Bach: An Eternal Golden Braid

- Douglas Hofstadter, cognitive scientist at the University of Indiana, wrote this Pulitzer-Prize-winning mind trip of a book.
- It's a great read after you've finished CS103 – you'll see so many of the ideas we'll cover presented in a totally different way!



The **MU** Puzzle

- Begin with the string **MI**.
- Repeatedly apply one of the following operations:
    - Double the contents of the string after the **M**: for example, **MIIU** becomes **MIIUIIU**, or **MI** becomes **MII**.
    - Replace **III** with **U**: **MIIII** becomes **MUI** or **MIU**.
    - Append **U** to the string if it ends in **I**: **MI** becomes **MIU**.
    - Remove any **UU**: **MUUU** becomes **MU**.
- ***Question***: How do you transform **MI** to **MU**?



(a) Double the string after an M.
(b) Replace III with U.
(c) Append U, if the string ends in I.
(d) Delete UU from the string.
MI
(a)
MII
(a)
MIIII
(c)
MIIIIU
(b)
MUIU
(a)
MUIUUIU
(d)
MUIIU

Try It!

Starting with **MI**, apply these operations to make **MU**:

(a) Double the string after an **M**.

(b) Replace **III** with **U**.

(c) Append **U**, if the string ends in **I**.

(d) Delete **UU** from the string.

Not a single person in this room
was able to solve this puzzle.

Are we even sure that there is a solution?

Counting **I**'s


MI
1
MII
2
MIIII
4
MIIIIU
4
MIIIIUIIIIU
8
MIIIIUUIU
5
MIIIIUUIUIIIIUUIU
10
MUIUUIUIIIIUUIU
7

The Key Insight

- Initially, the number of **`I`**'s is *not* a multiple of three.
- To make **`MU`**, the number of **`I`**'s must end up as a multiple of three.
- Can we *ever* make the number of **`I`**'s a multiple of three?

Lemma 1: If n is an integer that is not a multiple of three, then $n - 3$ is not a multiple of three.

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Proof: By contrapositive; we'll prove that if $n - 3$ is a multiple of three, then n is also a multiple of three. Because $n - 3$ is a multiple of three, we can write $n - 3 = 3k$ for some integer k. Then $n = 3(k+1)$, so n is also a multiple of three, as required. ■

Lemma 2: If n is an integer that is not a multiple of three, then $2n$ is not a multiple of three.

Proof: Let n be a number that isn't a multiple of three. If n is congruent to one modulo three, then $n = 3k + 1$ for some integer k. This means $2n = 2(3k+1) = 6k + 2 = 3(3k) + 2$, so $2n$ is not a multiple of three. Otherwise, n must be congruent to two modulo three, so $n = 3k + 2$ for some integer k. Then $2n = 2(3k+2) = 6k+4 = 3(2k+1) + 1$, and so $2n$ is not a multiple of three. ■

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Theorem: The `MU` puzzle has no solution.

Proof: Assume for the sake of contradiction that the `MU` puzzle has a solution and that we can convert `MI` to `MU`. This would mean that at the very end, the number of `I`'s in the string must be zero, which is a multiple of three. However, we've just proven that the number of `I`'s in the string can never be a multiple of three.

We have reached a contradiction, so our assumption must have been wrong. Thus the `MU` puzzle has no solution. ■

Algorithms and Loop Invariants

- The proof we just made had the form
  - "If P is true before we perform an action, it is true after we perform an action."
- We could therefore conclude that after any series of actions of any length, if P was true beforehand, it is true now.
- In algorithmic analysis, this is called a ***loop invariant***.
- Proofs on algorithms often use loop invariants to reason about the behavior of algorithms.
  - Take CS161 for more details!

Next Time

- ***Variations on Induction***
  - Starting induction later.
  - Taking larger steps.
  - Complete induction.