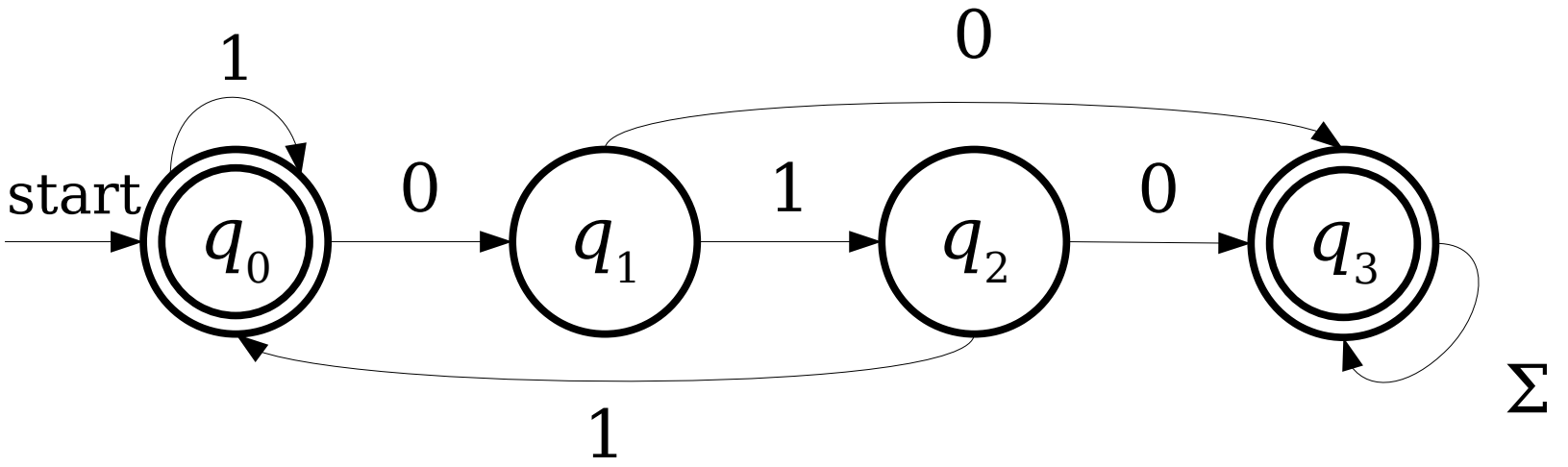
# Finite Automata

## Part Three

# Recap from Last Time

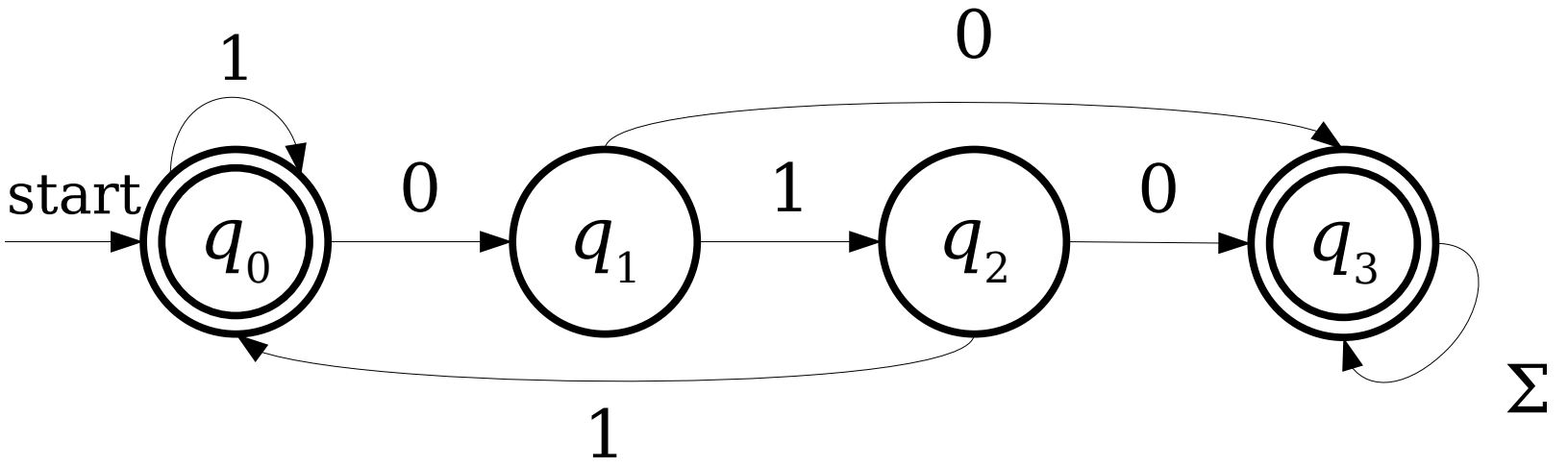
# Tabular DFAs




| | 0 | 1 |
|---|---|---|
| $*q_0$ | $q_1$ | $q_0$ |
| $q_1$ | $q_3$ | $q_2$ |
| $q_2$ | $q_3$ | $q_0$ |
| $*q_3$ | $q_3$ | $q_3$ |

These stars indicate accepting states.

# Tabular DFAs

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| $q_2$ | $q_3$ | $q_0$ |
| $*q_3$ | $q_3$ | $q_3$ |

Since this is the first row, it's the start state.

If $D$ is a DFA, the ***language of D***, denoted $\mathscr{L}(D)$, is $\{ w \in \Sigma^* \mid D \text{ accepts } w \}$.

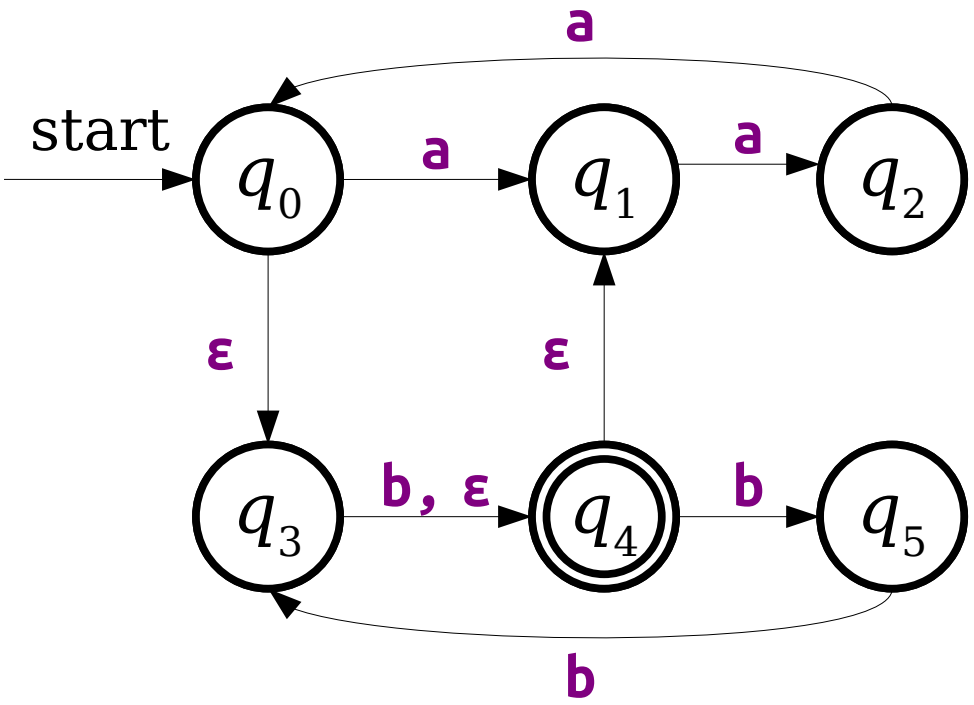
A language $L$ is called a ***regular language*** if there exists a DFA $D$ such that $\mathscr{L}(D) = L$.

# NFAs

- An ***NFA*** is a
  - ***N***ondeterministic
  - ***F***inite
  - ***A***utomaton
- Can have missing transitions or multiple transitions defined on the same input symbol.
- Accepts if *any possible series of choices* leads to an accepting state.

# ε-Transitions

- NFAs have a special type of transition called the **ε-transition**.
- An NFA may follow any number of ε-transitions at any time without consuming any input.



# Massive Parallelism

- An NFA can be thought of as a DFA that can be in many states at once.
- At each point in time, when the NFA needs to follow a transition, it tries all the options at the same time.
- The NFA accepts if *any* of the states that are active at the end are accepting states. It rejects otherwise.

Just how powerful *are* NFAs?

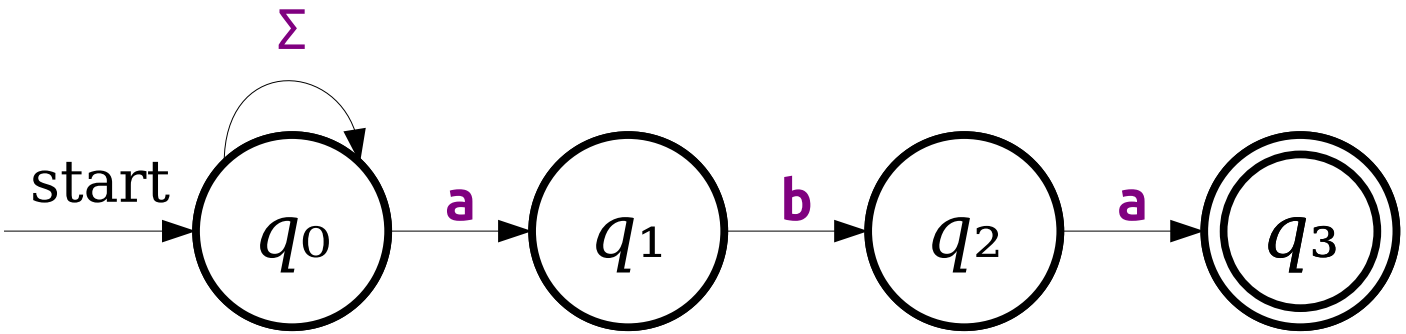
New Stuff!

# NFAs and DFAs

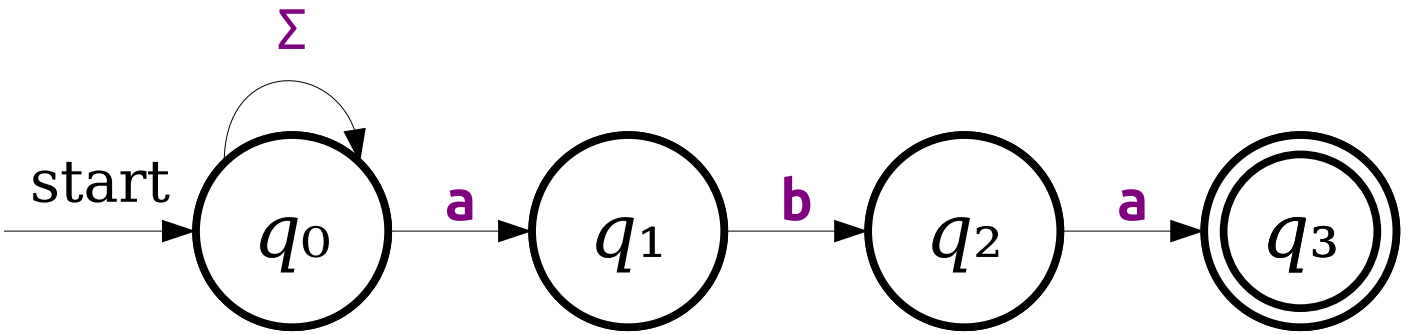
- Any language that can be accepted by a DFA can be accepted by an NFA.
- Why?
  - Every DFA essentially already *is* an NFA!
- ***Question:*** Can any language accepted by an NFA also be accepted by a DFA?
- Surprisingly, the answer is ***yes***!

***Thought Experiment:***

How would you simulate an NFA in software?




| | $a$ | $b$ |
|---|---|---|
| $\{q_0\}$ | $\{q_0, q_1\}$ | $\{q_0\}$ |
| $\{q_0, q_1\}$ | $\{q_0, q_1\}$ | $\{q_0, q_2\}$ |
| $\{q_0, q_2\}$ | $\{q_0, q_1, q_3\}$ | $\{q_0\}$ |
| $\{q_0, q_1, q_3\}$ | $\{q_0, q_1\}$ | $\{q_0, q_2\}$ |

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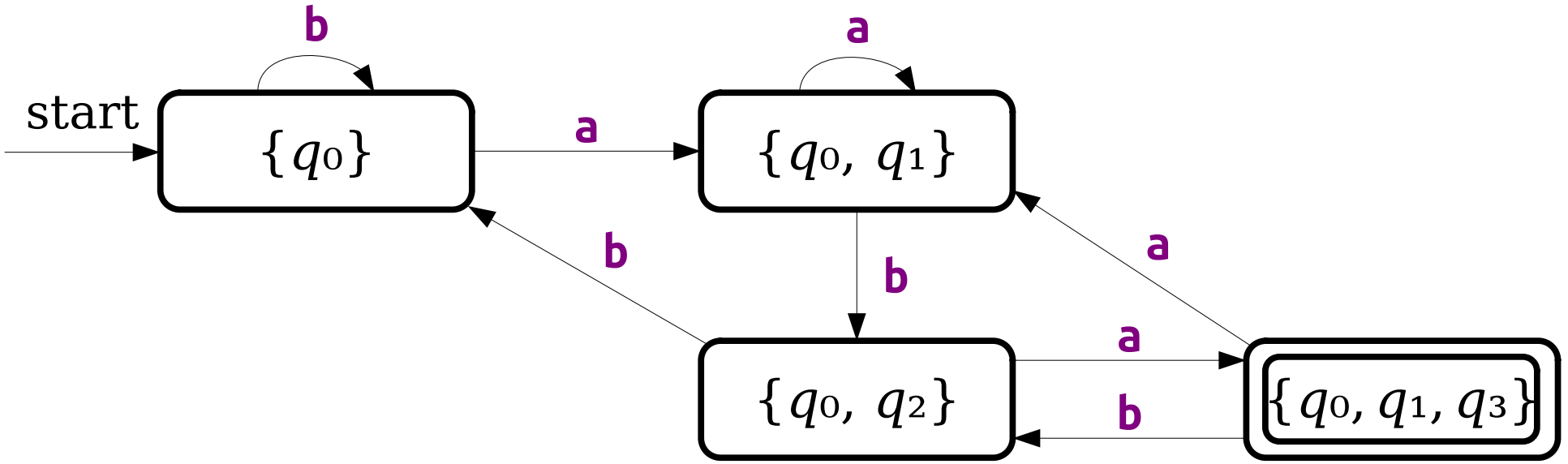
| $\{q_0, q_1\}$ | $\{q_0, q_1\}$ | $\{q_0, q_2\}$ |
| $\{q_0, q_2\}$ | $\{q_0, q_1, q_3\}$ | $\{q_0\}$ |
| $*\{q_0, q_1, q_3\}$ | $\{q_0, q_1\}$ | $\{q_0, q_2\}$ |



# The Subset Construction

- This procedure for turning an NFA for a language $L$ into a DFA for a language $L$ is called the ***subset construction***.
  - It's sometimes called the ***powerset construction***; it's different names for the same thing!
- Intuitively:
  - Each state in the DFA corresponds to a set of states from the NFA.
  - Each transition in the DFA corresponds to what transitions would be taken in the NFA when using the massive parallel intuition.
  - The accepting states in the DFA correspond to which sets of states would be considered accepting in the NFA when using the massive parallel intuition.
- There's an online ***Guide to the Subset Construction*** with a more elaborate example involving ε-transitions and cases where the NFA dies; check that for more details.

# The Subset Construction

- In converting an NFA to a DFA, the DFA's states correspond to sets of NFA states.
- ***Useful fact:*** $|\wp(S)| = 2^{|S|}$ for any finite set $S$.
- In the worst-case, the construction can result in a DFA that is *exponentially larger* than the original NFA.
- ***Question to ponder:*** Can you find a family of languages that have NFAs of size $n$, but no DFAs of size less than $2^n$?

A language $L$ is called a ***regular language*** if there exists a DFA $D$ such that $\mathscr{L}(D) = L$.

# An Important Result

***Theorem:*** A language $L$ is regular if and only if there is some NFA $N$ such that $\mathscr{L}(N) = L$.

***Proof Sketch:*** Pick a language $L$. First, assume $L$ is regular. That means there's a DFA $D$ where $\mathscr{L}(D) = L$. Every DFA is "basically" an NFA, so there's an NFA ($D$) whose language is $L$.

Next, assume there's an NFA $N$ such that $\mathscr{L}(N) = L$. Using the subset construction, we can build a DFA $D$ where $\mathscr{L}(N) = \mathscr{L}(D)$. Then we have that $\mathscr{L}(D) = L$, so $L$ is regular. ■-*ish*

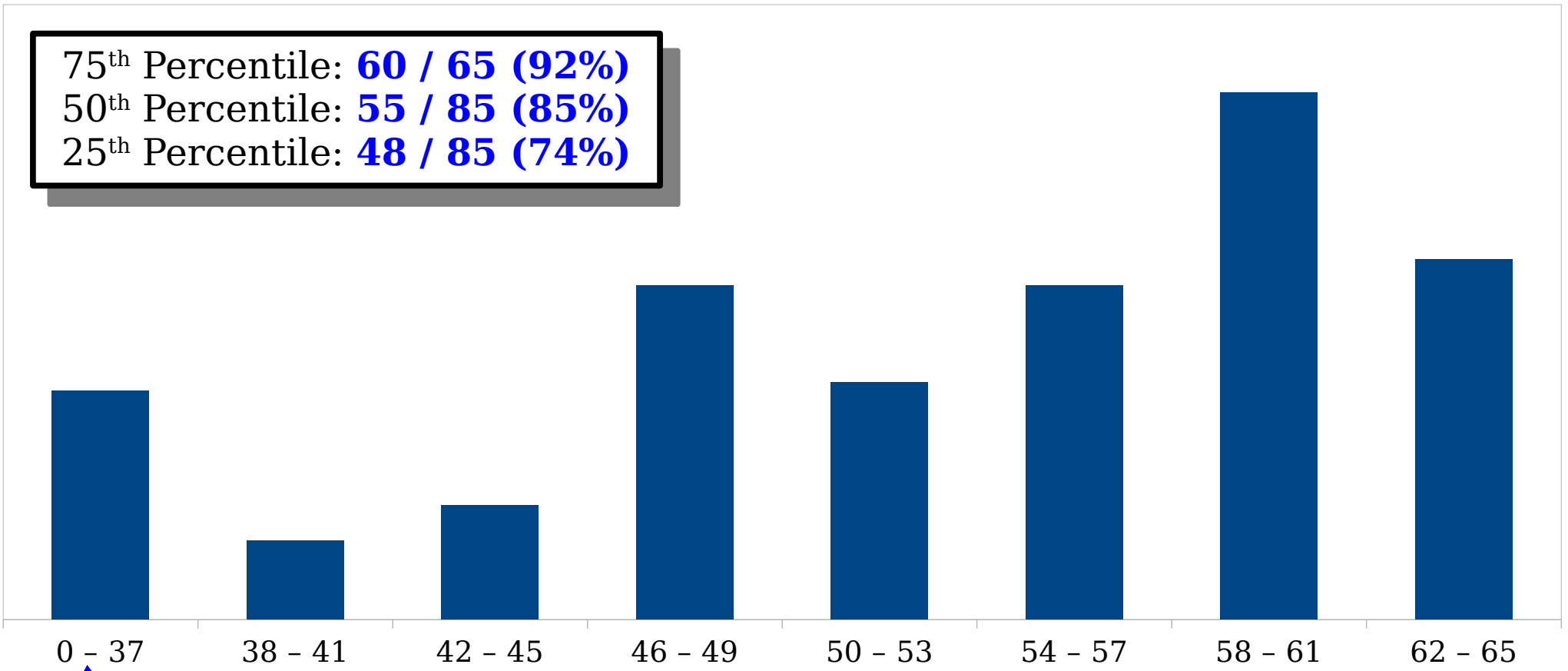
# Why This Matters

- We now have two perspectives on regular languages:
  - Regular languages are languages accepted by DFAs.
  - Regular languages are languages accepted by NFAs.
- We can now reason about the regular languages in two different ways.

Time-Out for Announcements!

# Problem Set Four Grades

75th Percentile: **60 / 65 (92%)**
50th Percentile: **55 / 85 (85%)**
25th Percentile: **48 / 85 (74%)**



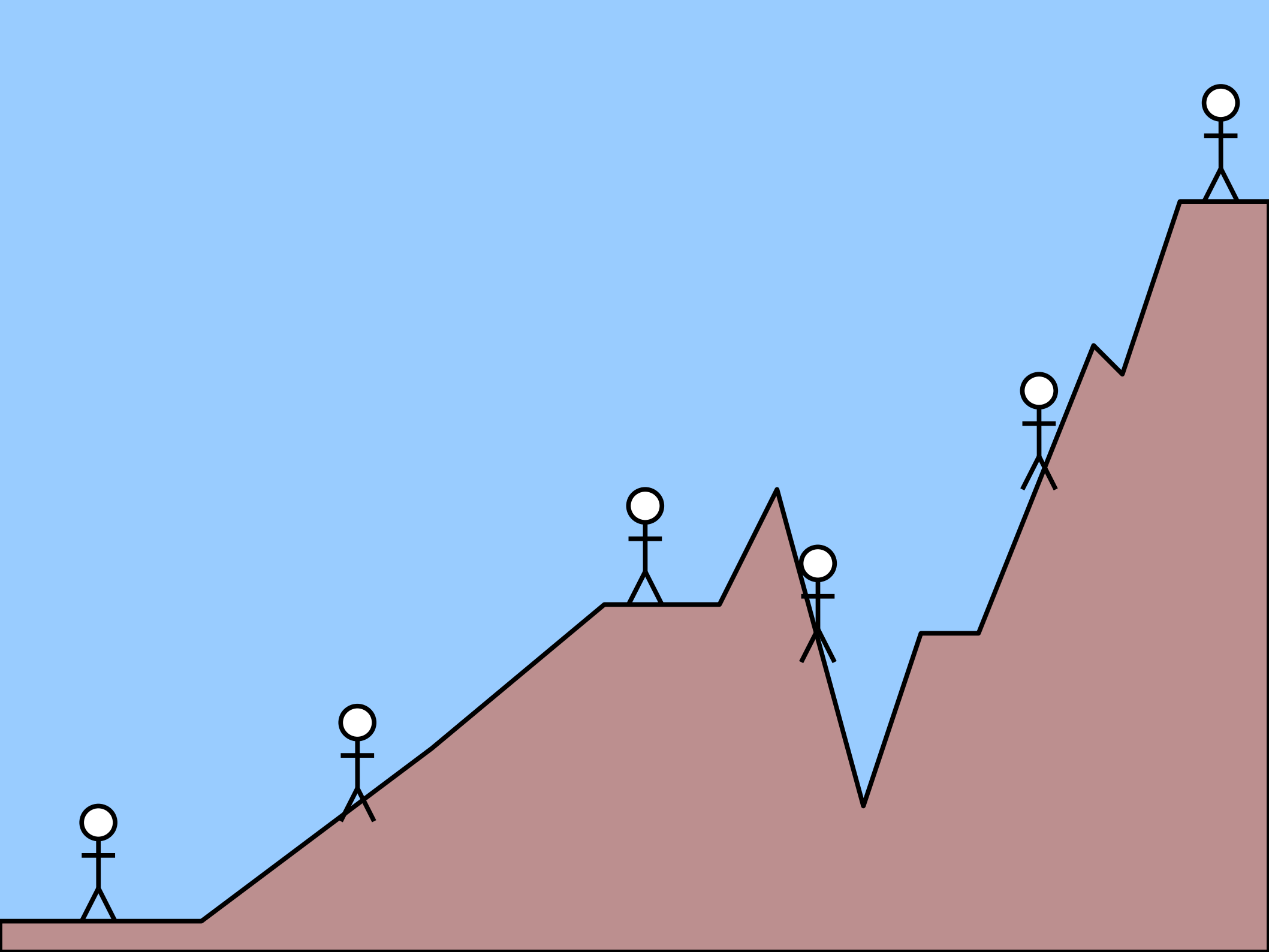

Many of these grades are because folks forgot to list partners – please check to make sure you're getting credit for the work you're doing, and let us know if your partner forgot to add you.

# Problem Set Six

- Problem Set Five was due at 2:30PM today.
- Problem Set Six goes out today. It's due next Friday at 2:30PM.
  - Design DFAs and NFAs for a range of problems!
  - Explore formal language theory!
  - See some clever applications!

# Second Midterm Logistics

- Our second midterm exam is a 49-hour take-home exam that goes out next Friday (November 5th) at 2:30PM and comes due next Sunday (November 7th) at 2:30PM Pacific time.
  - It's 49 hours long because of the switch to Daylight Saving Time.
- Topic coverage is PS3 – PS5 and lectures 07 – 13 (functions through induction). Later topics (automata, formal languages) won't be tested. Earlier topics are fair game for the exam, since the material in this class builds on itself.
- We've released Extra Practice Problems 2, a collection of 18 problems with solutions, to the course website to help you prepare.
- And always, keep the TAs in the loop! Let us know what we can do to help out.



# Three Questions

- What’s something you know now that, at the start of the quarter, you knew you didn’t know?
- What’s something you know now that, at the start of the quarter, you *didn’t* know you didn’t know?
- What’s something you *don’t* know now that, at the start of the quarter, you *didn’t* know you didn’t know?

~~Your Questions~~

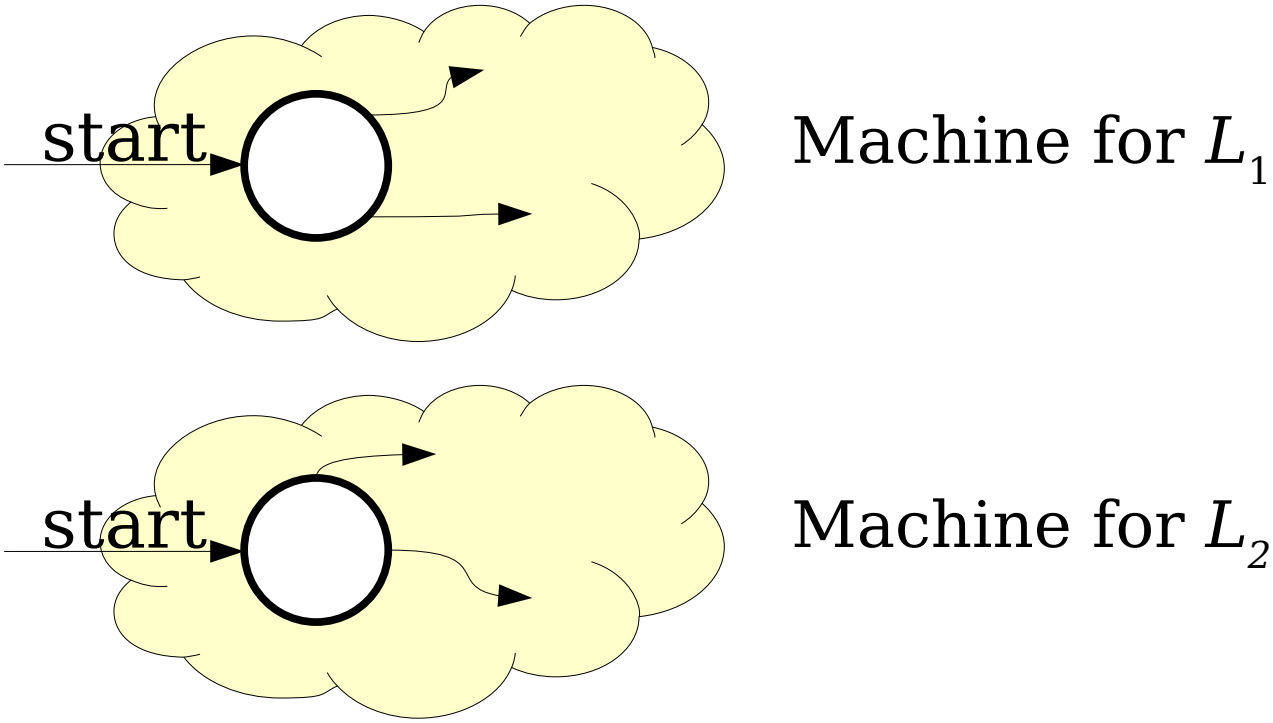
Next time, because I forgot to set that up today. Oops.

Back to CS103!

# Properties of Regular Languages

# The Union of Two Languages

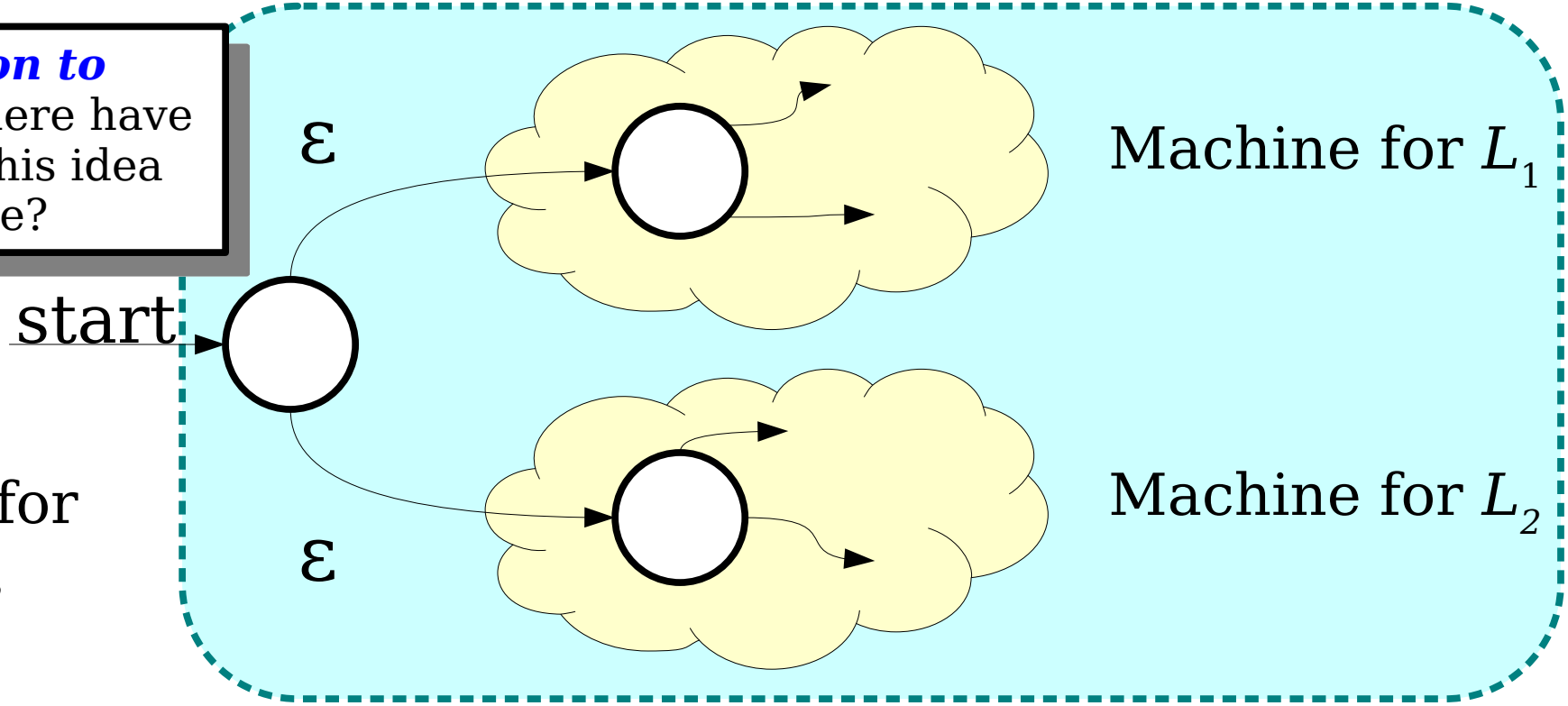
- If $L_1$ and $L_2$ are languages over the alphabet $\Sigma$, the language $L_1 \cup L_2$ is the language of all strings in at least one of the two languages.
- If $L_1$ and $L_2$ are regular languages, is $L_1 \cup L_2$?



# The Union of Two Languages

- If $L_1$ and $L_2$ are languages over the alphabet Σ, the language $L_1 \cup L_2$ is the language of all strings in at least one of the two languages.
- If $L_1$ and $L_2$ are regular languages, is $L_1 \cup L_2$?

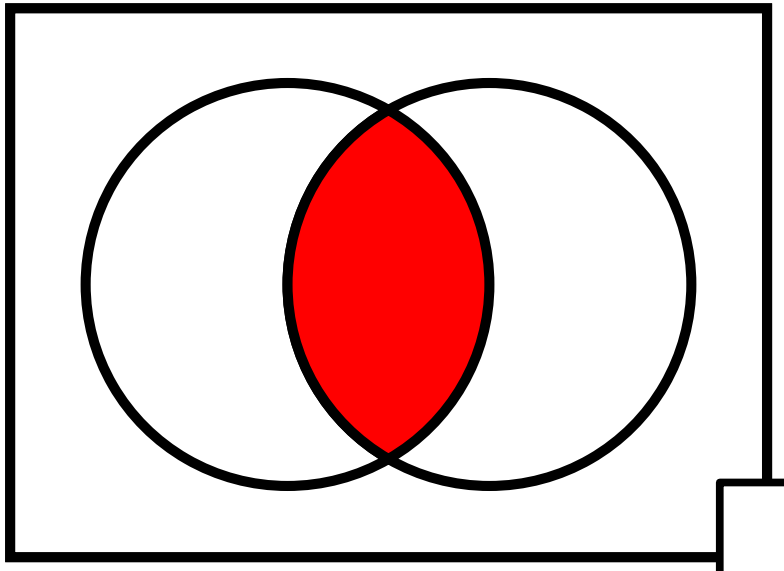
***Question to ponder:*** where have you seen this idea before?




Machine for $L_1 \cup L_2$

# The Intersection of Two Languages

- If $L_1$ and $L_2$ are languages over $\Sigma$, then $L_1 \cap L_2$ is the language of strings in both $L_1$ and $L_2$.
- Question: If $L_1$ and $L_2$ are regular, is $L_1 \cap L_2$ regular as well?



$$\overline{\overline{L_1} \cup \overline{L_2}}$$

Hey, it's De Morgan's laws!

# Concatenation

# String Concatenation

- If $w \in \Sigma^*$ and $x \in \Sigma^*$, the ***concatenation*** of $w$ and $x$, denoted ***wx***, is the string formed by tacking all the characters of $x$ onto the end of $w$.
- Example: if $w$ = `quo` and $x$ = `kka`, the concatenation $wx$ = `quokka`.
- This is analogous to the + operator for strings in many programming languages.
- Some facts about concatenation:
  - The empty string ε is the ***identity element*** for concatenation:

$$w\varepsilon = \varepsilon w = w$$

  - Concatenation is ***associative***:

$$wxy = w(xy) = (wx)y$$

# Concatenation

- The ***concatenation*** of two languages $L_1$ and $L_2$ over the alphabet $\Sigma$ is the language

$$L_1L_2 = \{ wx \in \Sigma^* \mid w \in L_1 \wedge x \in L_2 \}$$

# Concatenation Example

- Let Σ = { `a`, `b`, …, `z`, `A`, `B`, …, `Z` } and consider these languages over Σ:
  - ***Noun*** = { `Puppy`, `Rainbow`, `Whale`, … }
  - ***Verb*** = { `Hugs`, `Juggles`, `Loves`, … }
  - ***The*** = { `The` }
- The language ***TheNounVerbTheNoun*** is
  - { `ThePuppyHugsTheWhale`, `TheWhaleLovesTheRainbow`, `TheRainbowJugglesTheRainbow`, … }

# Concatenation

- The ***concatenation*** of two languages $L_1$ and $L_2$ over the alphabet $\Sigma$ is the language

$$L_1L_2 = \{ wx \in \Sigma^* \mid w \in L_1 \wedge x \in L_2 \}$$

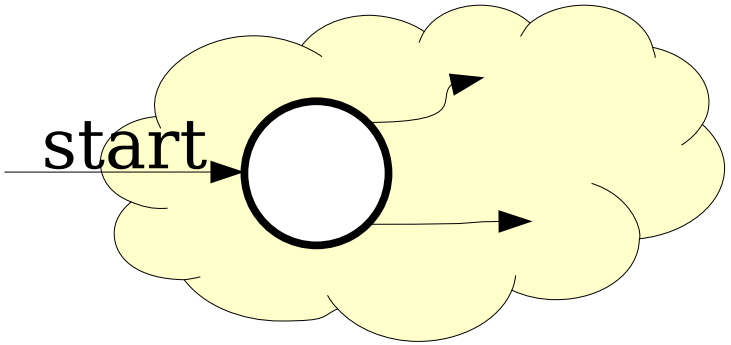
- Two views of $L_1L_2$:
  - The set of all strings that can be made by concatenating a string in $L_1$ with a string in $L_2$.
  - The set of strings that can be split into two pieces: a piece from $L_1$ and a piece from $L_2$.

This is closely related to, but different than, the Cartesian product.

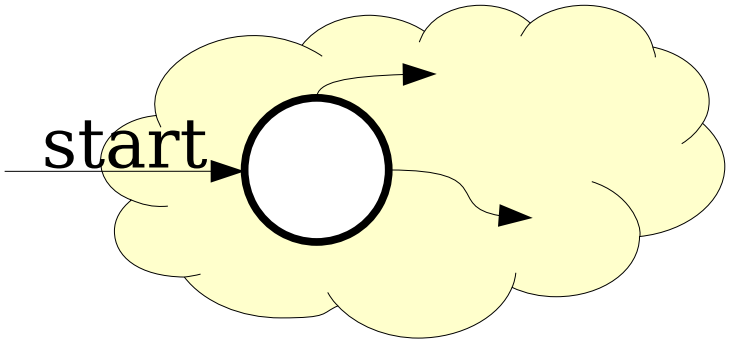
***Question to ponder:*** In what ways are concatenations similar to Cartesian products? In what ways are they different?

# Concatenating Regular Languages

- If $L_1$ and $L_2$ are regular languages, is $L_1L_2$?
- Intuition – can we split a string $w$ into two strings $xy$ such that $x \in L_1$ and $y \in L_2$?




Machine for $L_1$

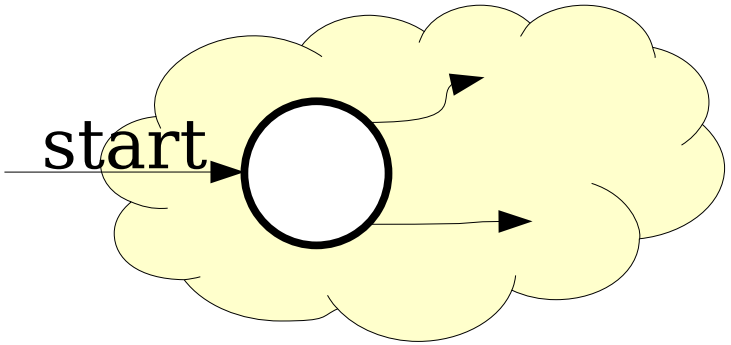



Machine for $L_2$

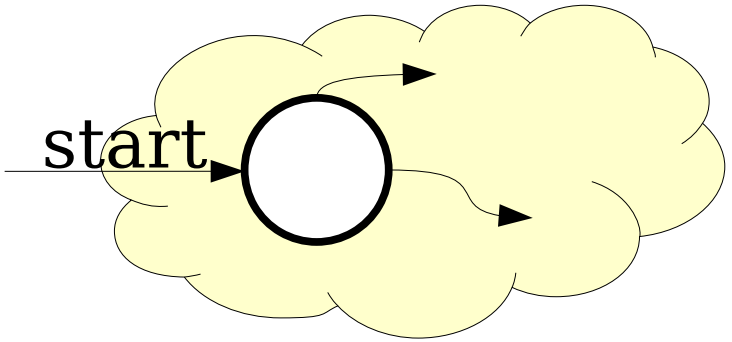
| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| **b** | **o** | **o** | **k** | **k** | **e** | **e** | **p** | **e** | **r** |

# Concatenating Regular Languages

- If $L_1$ and $L_2$ are regular languages, is $L_1L_2$?
- Intuition – can we split a string $w$ into two strings $xy$ such that $x \in L_1$ and $y \in L_2$?




Machine for $L_1$




Machine for $L_2$

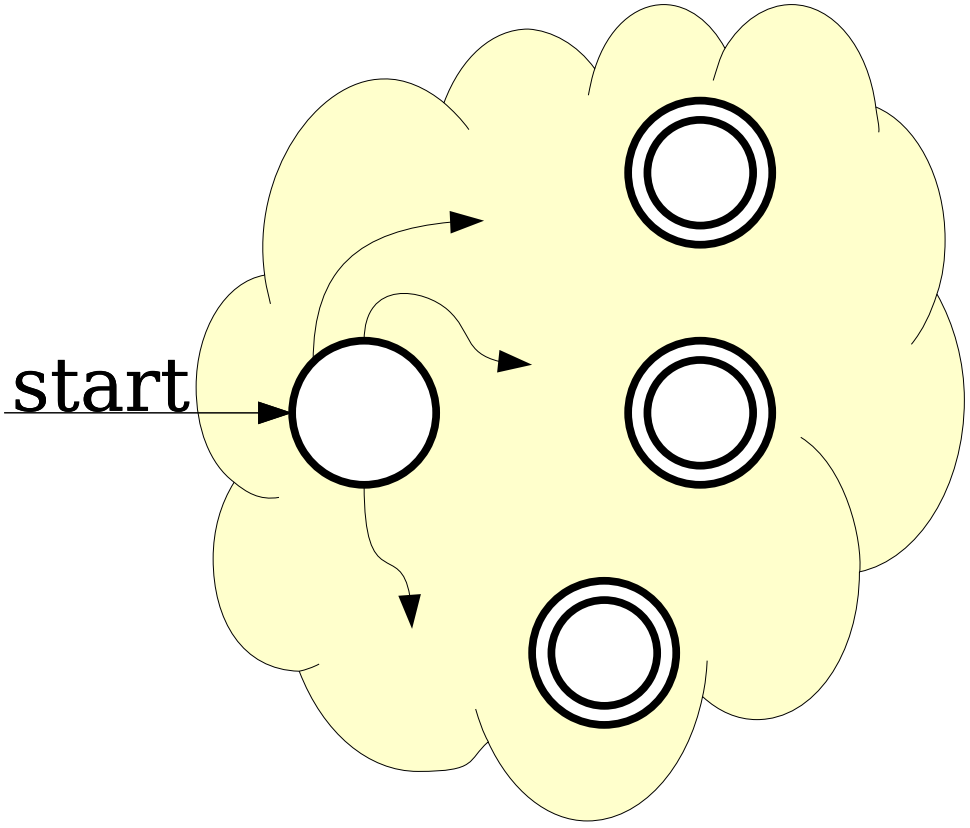
| b | o | o | k |
|---|---|---|---|

| k | e | e | p | e | r |
|---|---|---|---|---|---|

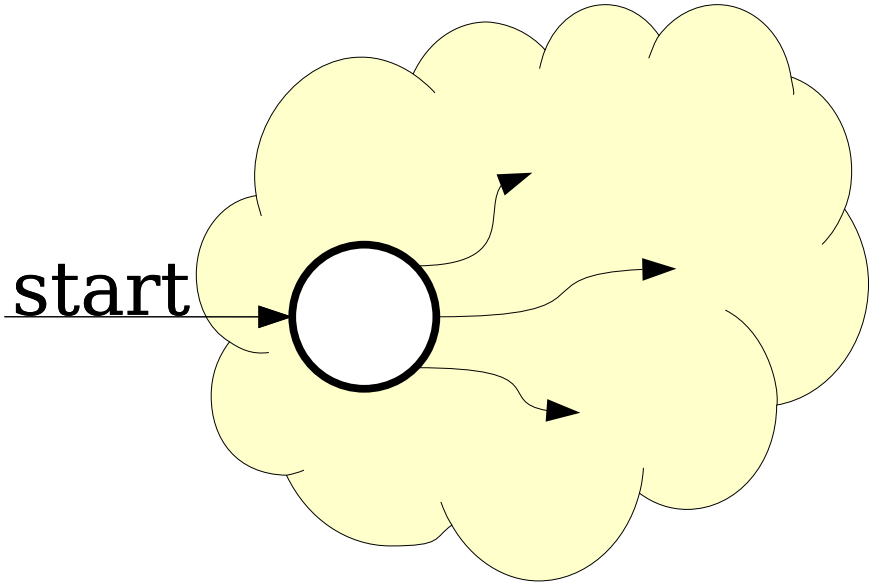
# Concatenating Regular Languages

- If $L_1$ and $L_2$ are regular languages, is $L_1L_2$?
- Intuition – can we split a string $w$ into two strings $xy$ such that $x \in L_1$ and $y \in L_2$?
- ***Idea***:
  - Run a DFA/NFA for $L_1$ on $w$.
  - Whenever it reaches an accepting state, optionally hand the rest of $w$ to a DFA/NFA for $L_2$.
  - If the automaton for $L_2$ accepts the rest, $w \in L_1L_2$.
  - If the automaton for $L_2$ rejects the remainder, the split was incorrect.

# Concatenating Regular Languages

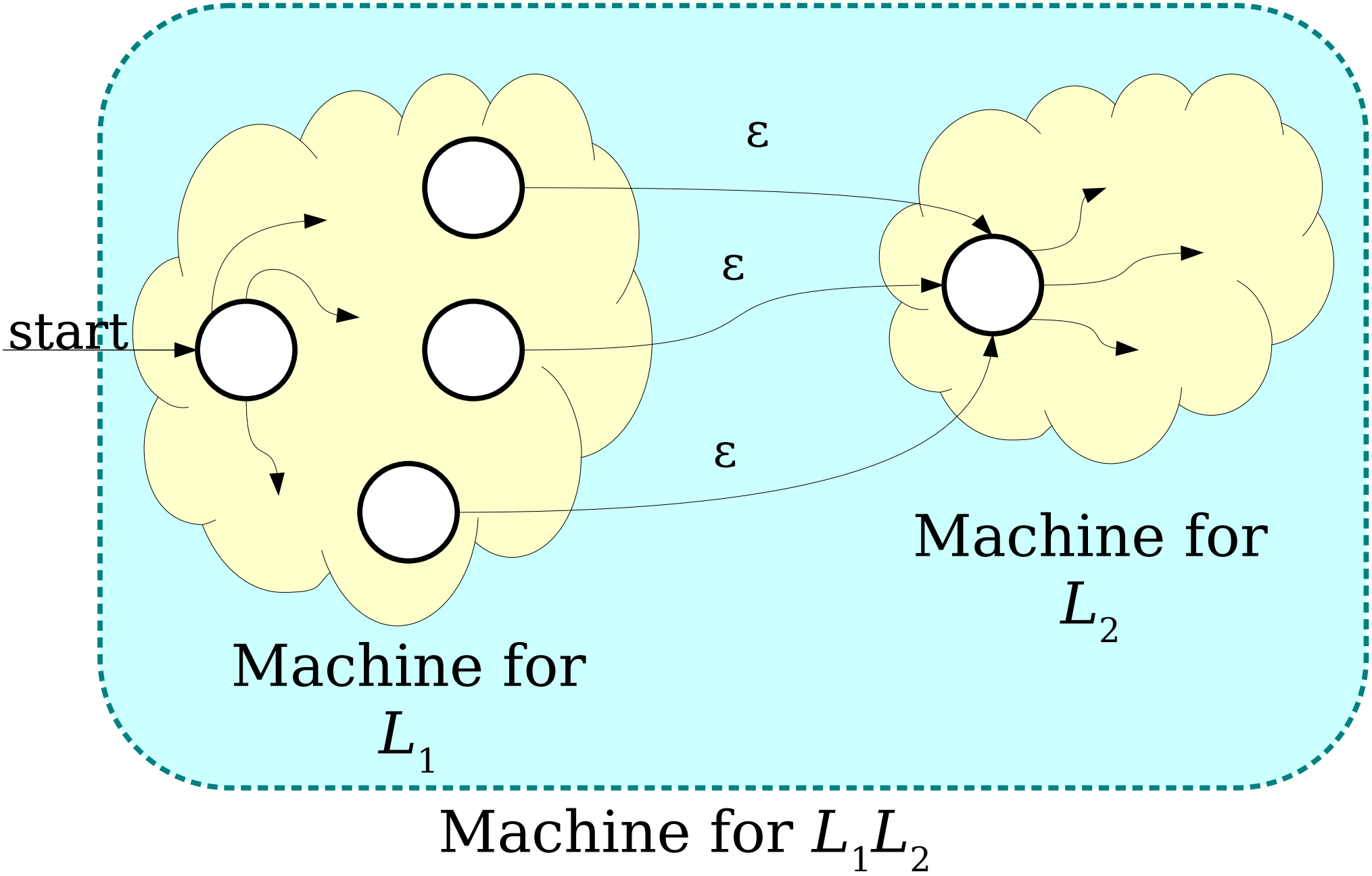



Machine for $L_1$




Machine for $L_2$

# Concatenating Regular Languages



# Lots and Lots of Concatenation

- Consider the language $L$ = { **aa**, **b** }
- $LL$ is the set of strings formed by concatenating pairs of strings in $L$.

{ **aaaa**, **aab**, **baa**, **bb** }

- $LLL$ is the set of strings formed by concatenating triples of strings in $L$.

{ **aaaaaa**, **aaaab**, **aabaa**, **aabb**, **baaaa**, **baab**, **bbaa**, **bbb**}

- $LLLL$ is the set of strings formed by concatenating quadruples of strings in $L$.

{ **aaaaaaaa**, **aaaaaab**, **aaaabaa**, **aaaabb**, **aabaaaa**,
**aabaab**, **aabbaa**, **aabbb**, **baaaaaa**, **baaaab**, **baabaa**,
**baabb**, **bbaaaa**, **bbaab**, **bbbaa**, **bbbb**}

# Language Exponentiation

- We can define what it means to “exponentiate” a language as follows:
- $L^0 = \{\varepsilon\}$
  - Intuition: The only string you can form by gluing no strings together is the empty string.
  - Notice that $\{\varepsilon\} \neq \varnothing$. Can you explain why?
- $L^{n+1} = LL^n$
  - Idea: Concatenating ($n$+1) strings together works by concatenating $n$ strings, then concatenating one more.
- ***Question to ponder:*** Why define $L^0 = \{\varepsilon\}$?
- ***Question to ponder:*** What is $\varnothing^0$?

# The Kleene Star

# The Kleene Closure

- An important operation on languages is the ***Kleene Closure***, which is defined as

$$L^* = \{ w \in \Sigma^* \mid \exists n \in \mathbb{N}.\ w \in L^n \}$$

- Mathematically:

$$w \in L^* \quad \leftrightarrow \quad \exists n \in \mathbb{N}.\ w \in L^n$$

- Intuitively, $L^*$ is the language all possible ways of concatenating zero or more strings in $L$ together, possibly with repetition.
- ***Question to ponder:*** What is $\emptyset^*$?

# The Kleene Closure

If $L$ = { **a**, **bb** }, then $L^*$ = {

ε,

**a**, **bb**,

**aa**, **abb**, **bba**, **bbbb**,

**aaa**, **aabb**, **abba**, **abbbb**, **bbaa**, **bbabb**, **bbbba**, **bbbbbb**,

...

}

Think of L* as the set of strings you can make if you have a collection of stamps – one for each string in L – and you form every possible string that can be made from those stamps.

# Reasoning about Infinity

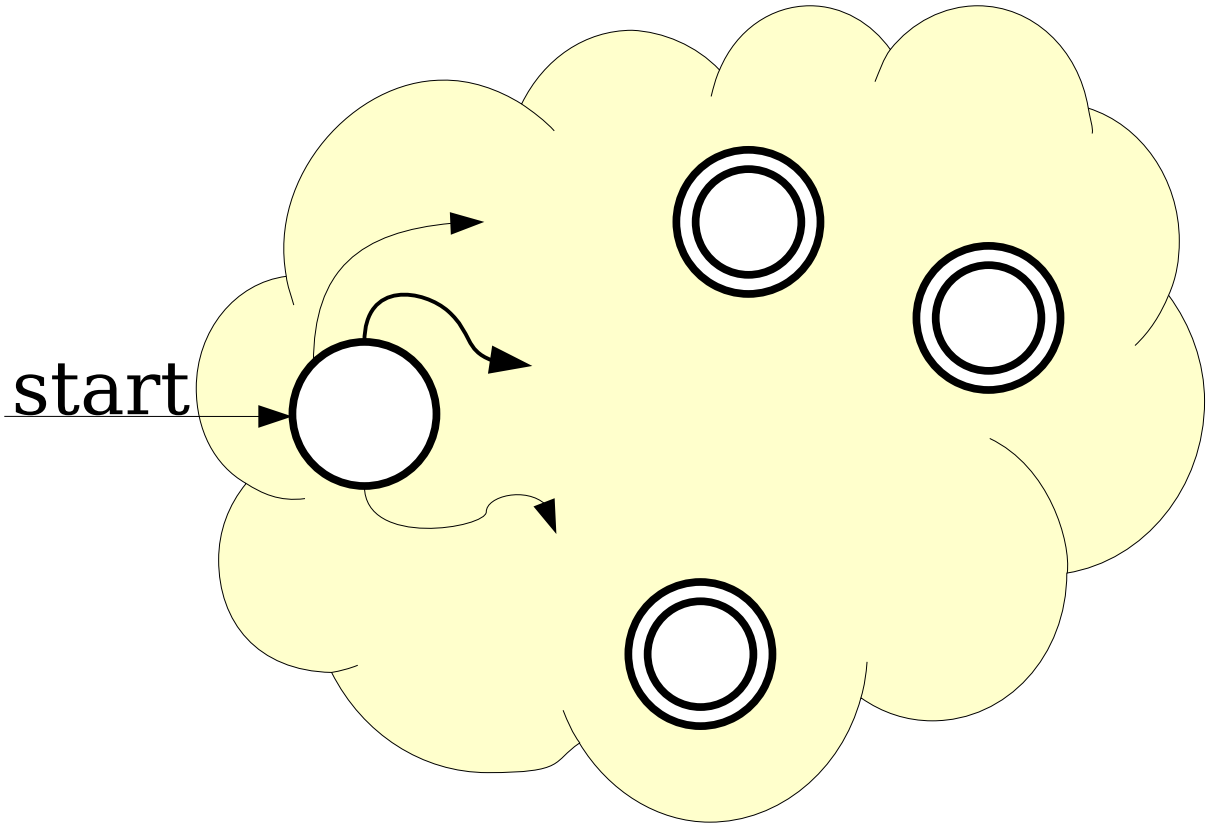
- If $L$ is regular, is $L^*$ necessarily regular?
- **A Bad Line of Reasoning:**
  - $L^0 = \{ \varepsilon \}$ is regular.
  - $L^1 = L$ is regular.
  - $L^2 = LL$ is regular
  - $L^3 = L(LL)$ is regular
  - …
  - Regular languages are closed under union.
  - So the union of all these languages is regular.

# Reasoning About the Infinite

- If a series of finite objects all have some property, the “limit” of that process *does not* necessarily have that property.
- In general, it is not safe to conclude that some property that always holds in the finite case must hold in the infinite case.
  - (This is why calculus is interesting).
- So our earlier argument ($L^* = L^0 \cup L^1 \cup \ldots$) isn’t going to work.
- We need a different line of reasoning.

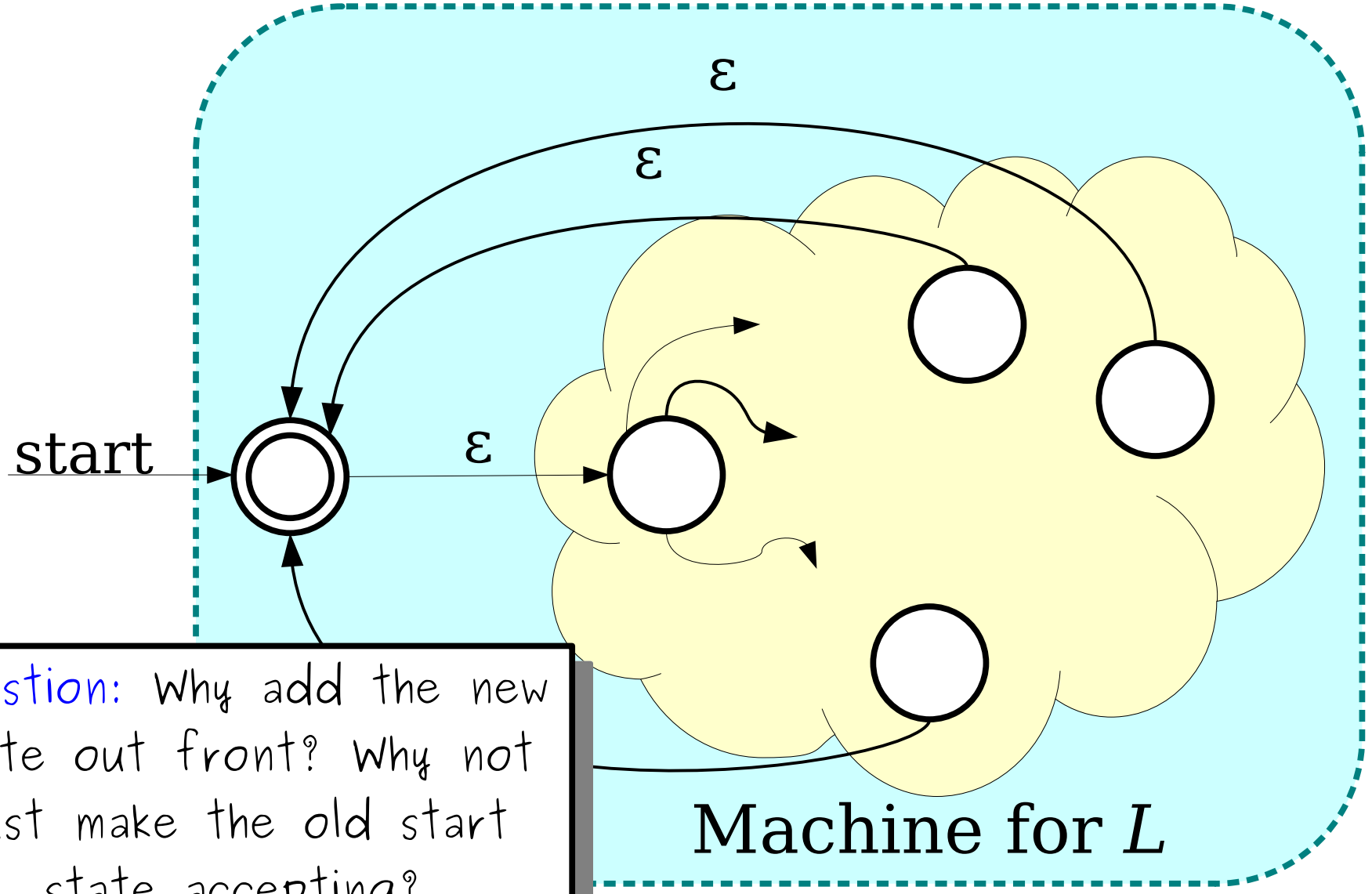
***Idea:*** Can we directly convert an NFA for language $L$ to an NFA for language $L$*?

# The Kleene Star




Machine for $L$

# The Kleene Star



# Closure Properties

- ***Theorem:*** If $L_1$ and $L_2$ are regular languages over an alphabet $\Sigma$, then so are the following languages:
  - $\overline{L}_1$
  - $L_1 \cup L_2$
  - $L_1 \cap L_2$
  - $L_1 L_2$
  - $L_1^*$
- These properties are called ***closure properties of the regular languages***.

# Next Time

- ***Regular Expressions***
  - Building languages from the ground up!
- ***Thompson's Algorithm***
  - A UNIX Programmer in Theoryland.
- ***Kleene's Theorem***
  - From machines to programs!