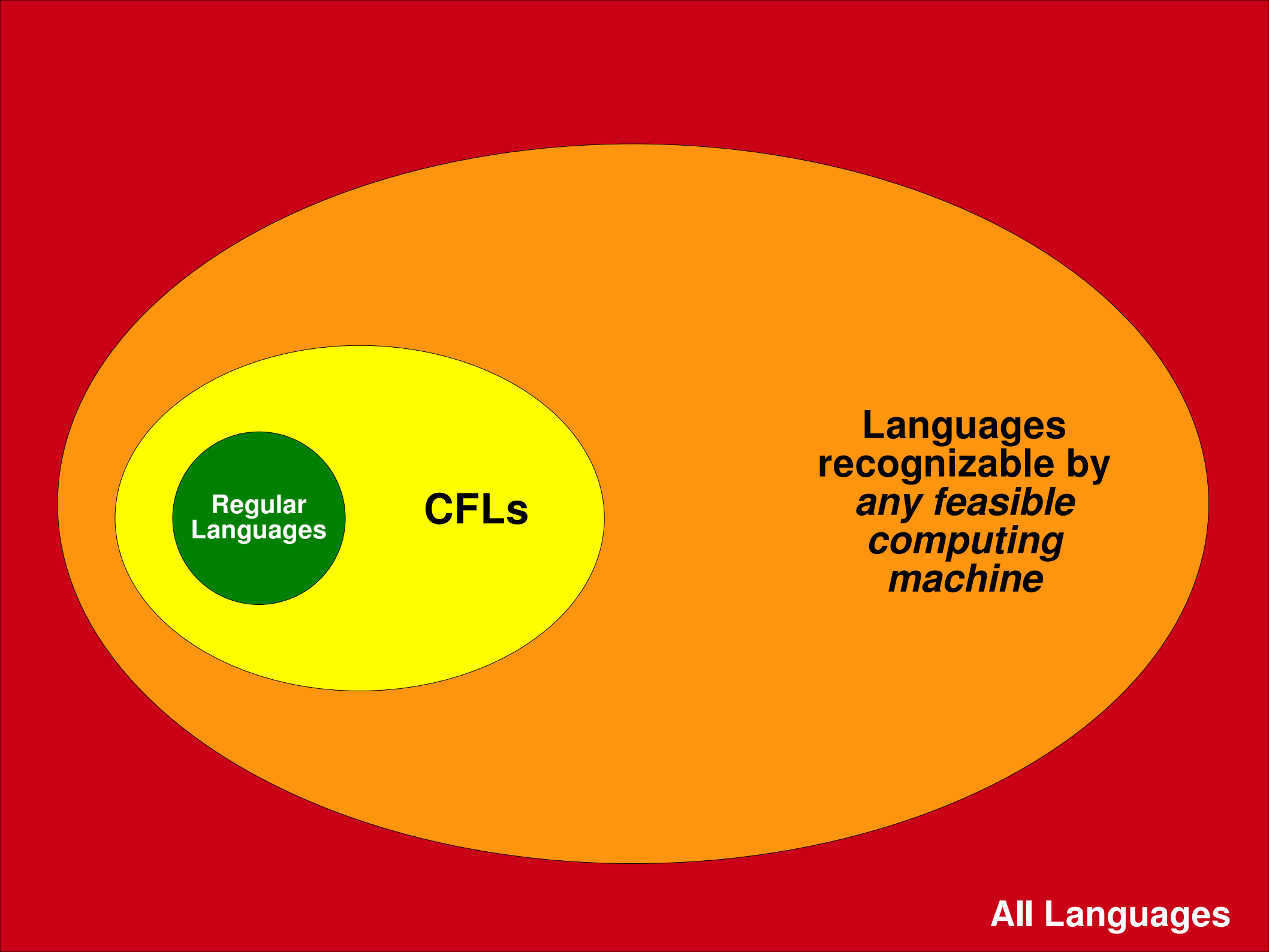
Turing Machines

Part One

What problems can we solve with a computer?


Regular Languages
CFLs
Languages recognizable by *any feasible computing machine*
All Languages

That same drawing, to scale.

All Languages

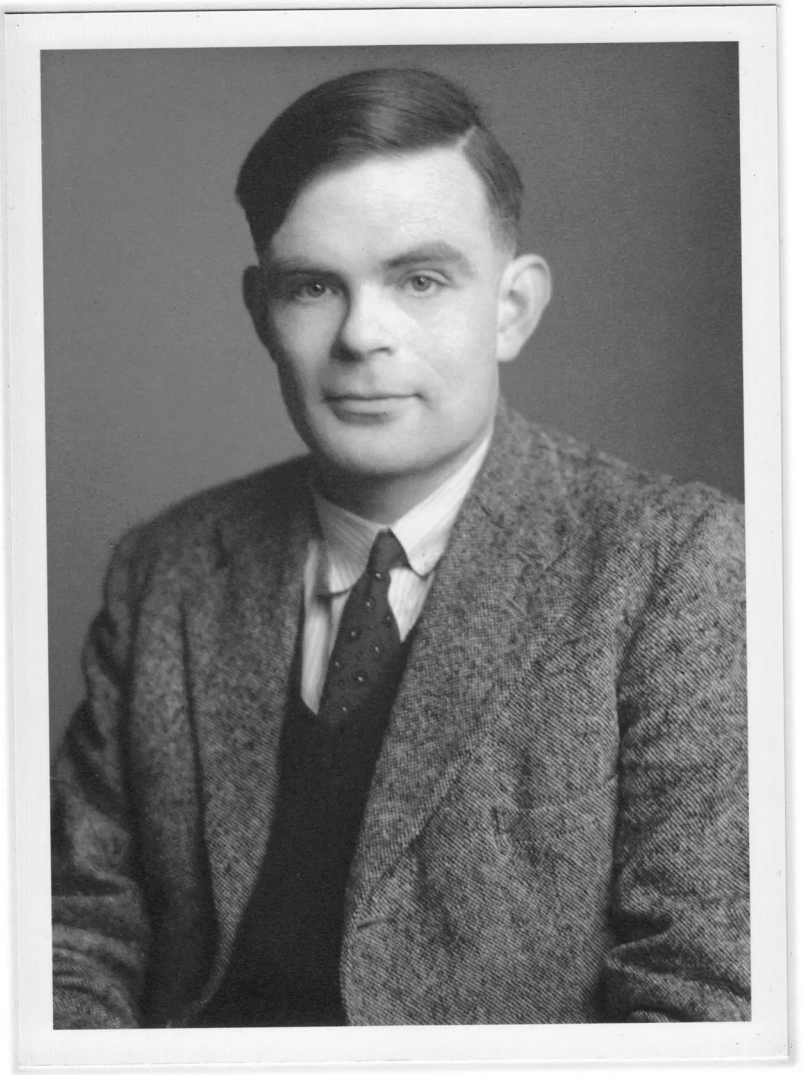
The Problem

- Finite automata accept precisely the regular languages.
- We may need unbounded memory to recognize context-free languages.
  - e.g. $\{ \mathtt{a}^n\mathtt{b}^n \mid n \in \mathbb{N} \}$ requires unbounded counting.
- How do we model a computing device that has unbounded memory?

A Brief History Lesson

Turing Machines

- In March 1936, Alan Turing (aged 23!) published a paper detailing the ***a-machine*** (for ***automatic machine***), an automaton for computing on real numbers.
- They're now more popularly referred to as ***Turing machines*** in his honor.
- He also later made contributions to computational biology, artificial intelligence, cryptography, etc. Seriously, Google this guy.



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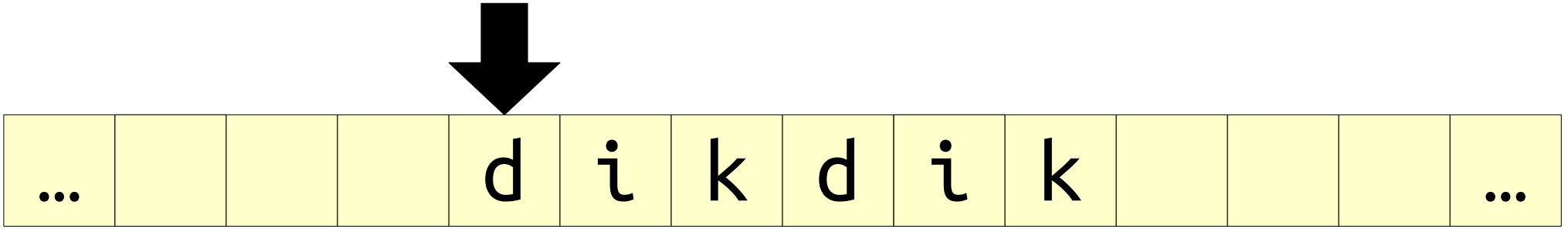
```
                  1     1     1  1  1  1
   2 7 1 8 2 8 1 8 2 8 4 5 9 0
+  3 1 4 1 5 9 2 6 5 3 5 8 9 7
-------------------------------
   5 8 5 9 8 7 4 4 8 2 0 4 8 7
```

Key Idea: Even if you need huge amounts of scratch space to perform a calculation, at each point in the calculation you only need access to a small amount of that scratch space.

Turing Machines

- To provide his machines extra memory, Turing gave his machines access to an ***infinite tape*** subdivided into a number of ***tape cells***.
- A Turing machine can only see one tape cell at a time, the one pointed at by the ***tape head***.
- The Turing machine can
  - read the cell under the tape head,
  - (possibly) change which symbol was written under the tape head, and
  - move its tape head to the left or to the right.



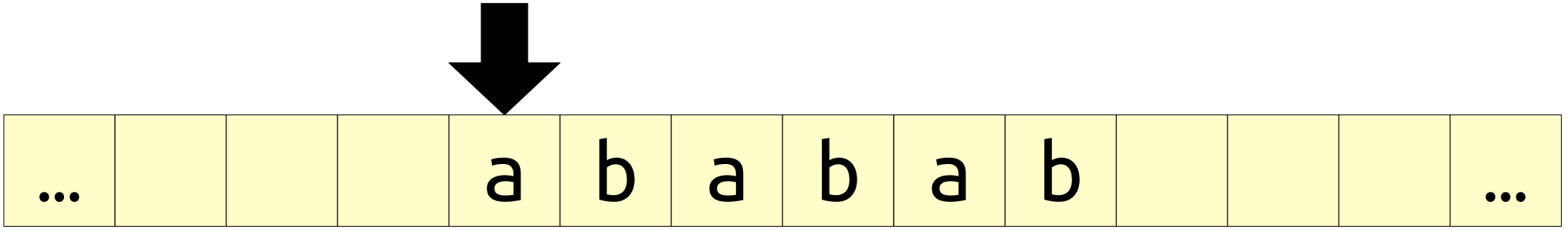
Turing Machines

- Over the years, there have been many simplifications and edits to Turing's original automata.
  - In practice, electronic computers are written in terms of individual instructions rather than states and transitions.
  - Turing's original paper deals with computing individual real numbers; we typically want to compute functions of inputs.
- What we're going to present as "Turing machines" in this class differ significantly from Turing's original description, while retaining the core essential ideas.
  - (Our model is closer to Emil Post's *Formulation 1* and Hao Wang's *Basic Machine B*, for those of you who are curious.)
- If you'd like to learn more about Turing's original version of the Turing machine, come chat with me after class!

Turing Machines

- A TM is a series of instructions that control a tape head as it moves across an infinite tape.
- The tape begins with the input string written somewhere, surrounded by infinitely many blank cells.
  - Rule: The input string cannot contain blank cells.
- The tape head begins above the first character of the input. (If the input is ε, the tape head points somewhere on a blank tape.)

```
Start:
  If Blank Return True
  If 'b' Return False
  Write 'x'
  Move Right
  If Not 'b' Return False
  Write 'x'
  Move Right
  Goto Start
```



Turing Machines

- We begin at the `Start` label.
- Labels indicate different sections of code. The name `Start` is special and means “begin here.”
- Labels have no effect when executed. We just move to the next line.

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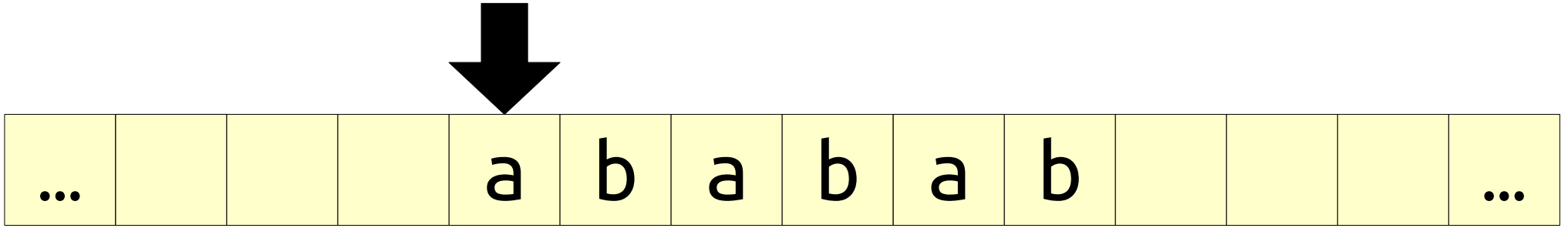

Turing Machines

- A statement of the form

  If *symbol command*

  checks if the character under the tape head is *symbol*.
- If so, it executes *command*.
- If not, nothing happens.

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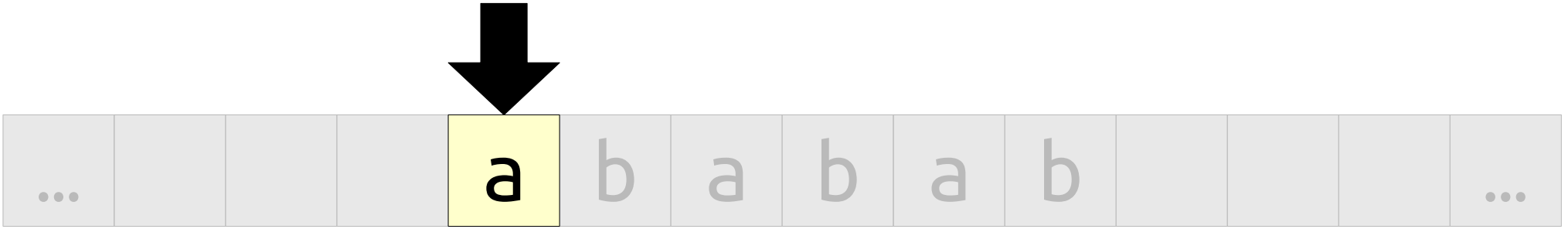

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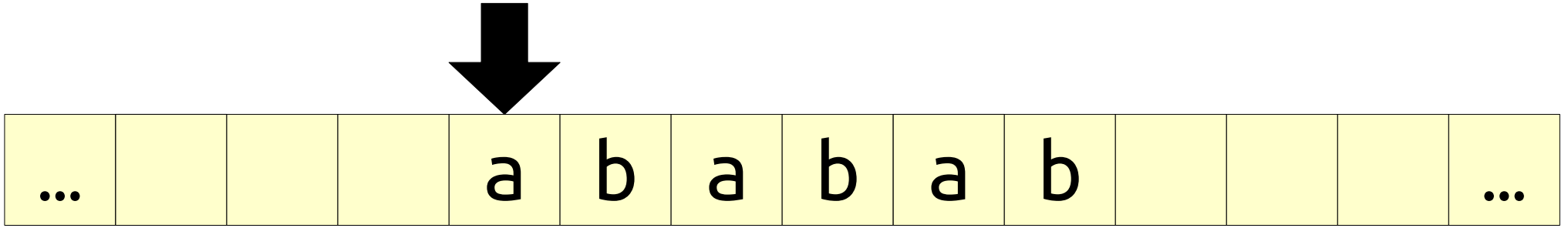

Turing Machines

- The statement

  `Write` *symbol*

  writes *symbol* to the cell under the tape head.
- The *symbol* can either be `Blank` or a character in quotes.

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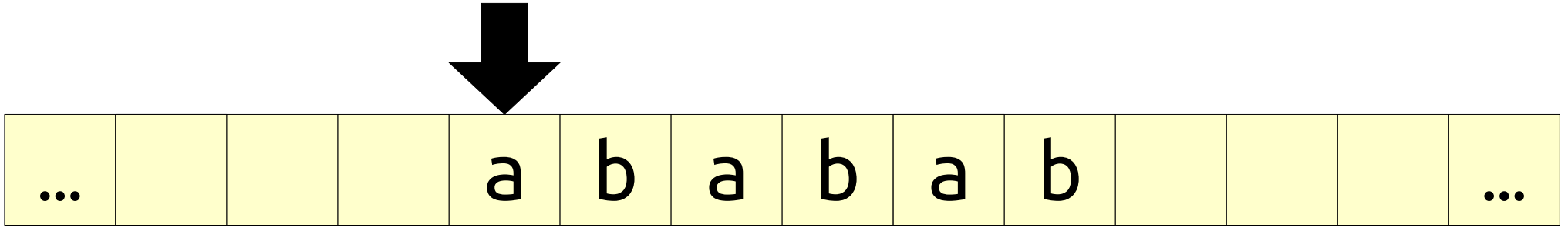

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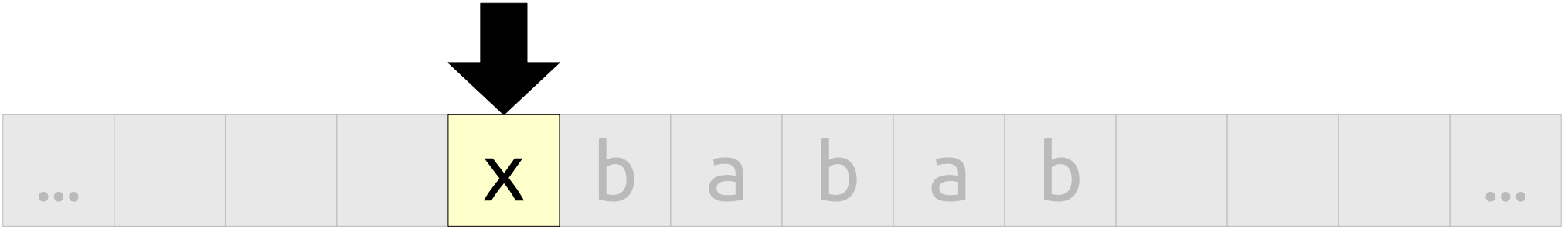

Turing Machines

- The command

  Move *direction*

  moves the tape head one step in the indicated direction (either `Left` or `Right`).

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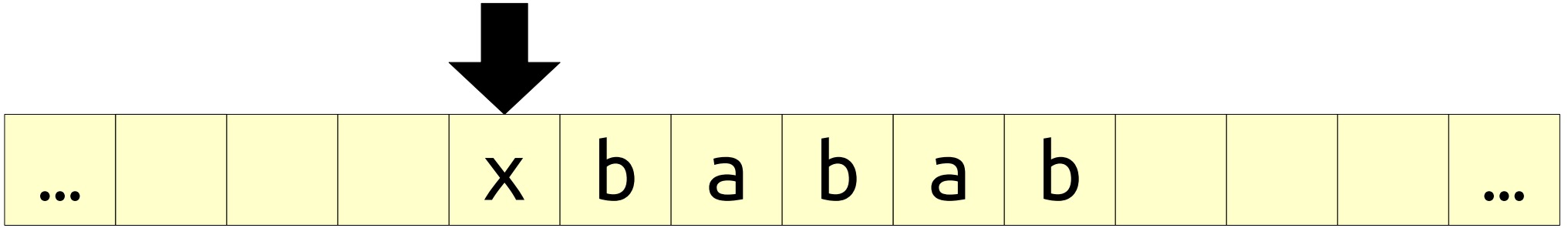

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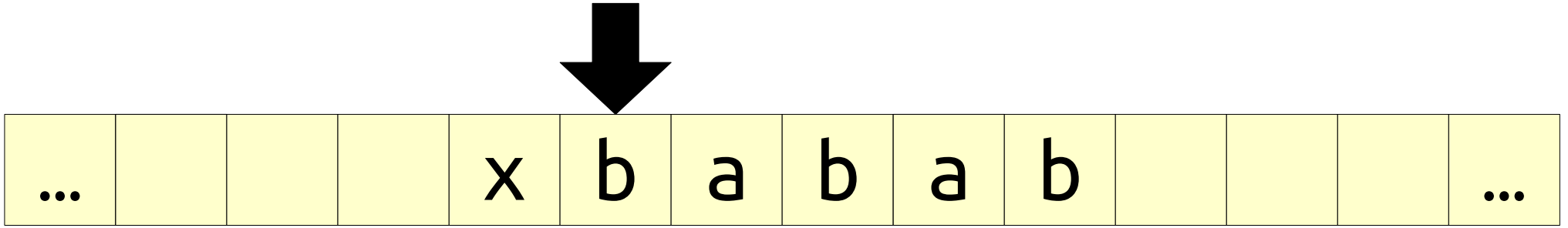

Turing Machines

- A statement of the form

  `If Not` *symbol command*

  sees if the cell under the tape head holds *symbol.*
- If so, nothing happens.
- If not, it executes *command*.

```
Start:
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   If 'b' Return False
   Write 'x'
   Move Right
   If Not 'b' Return False
   Write 'x'
   Move Right
   Goto Start
```

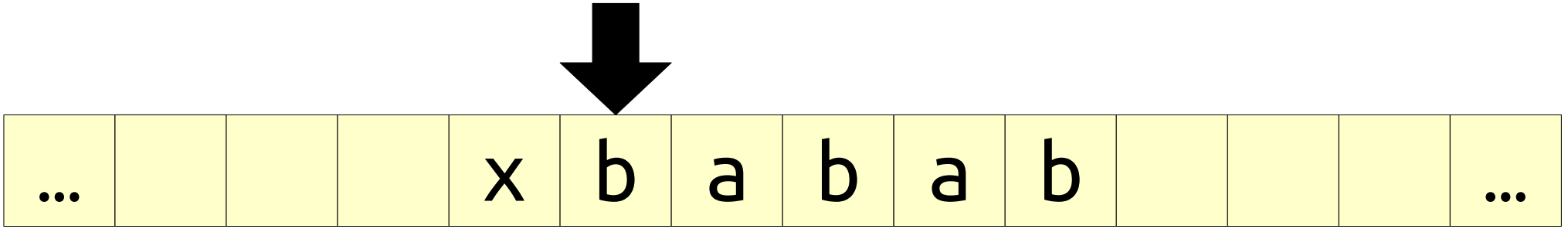

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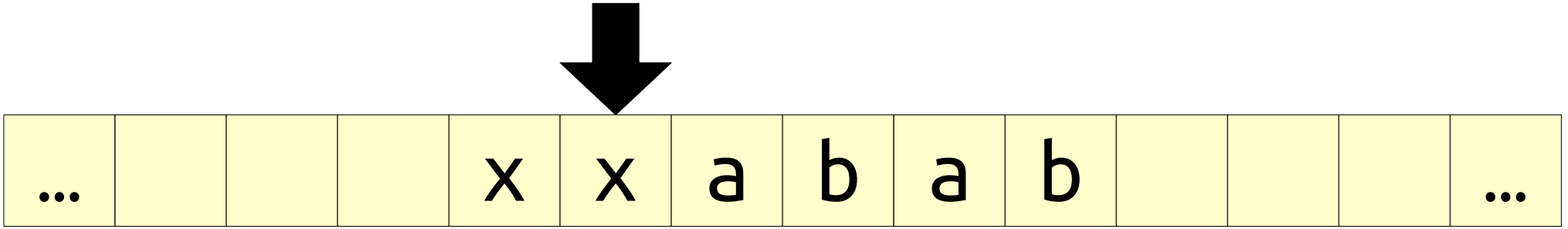

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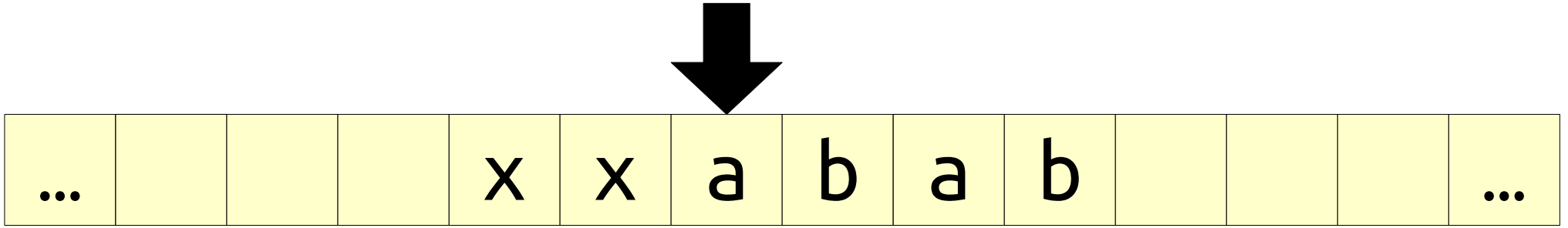

Turing Machines

- The command

  `Goto` *label*

  jumps to the indicated label.
- This program just has a `Start` label, but most interesting programs have other labels beyond this.

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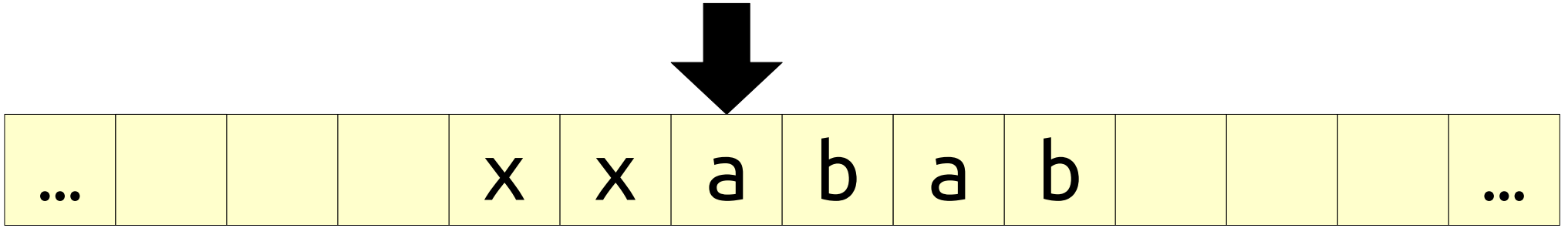

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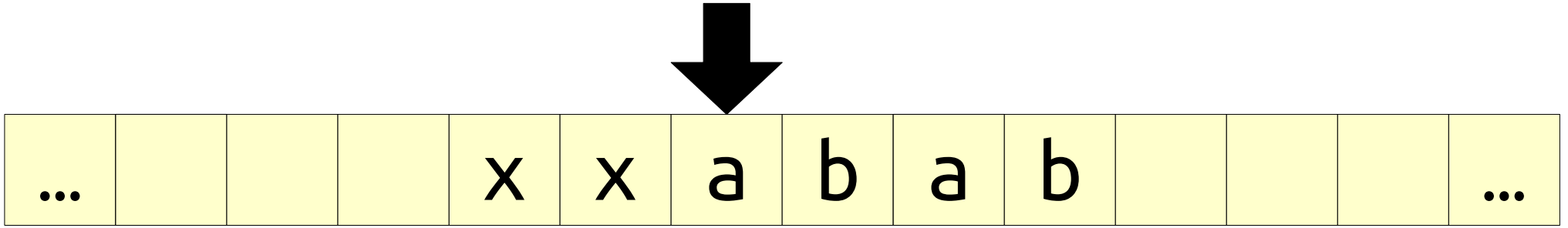

Turing Machines

- A TM stops when executing the

  `Return` *result*

  command.
- Here, *result* can be either `True` or `False`.
- (If we “fall off” the bottom of the program, the TM acts as though it executes the `Return False` command.)

```
Start:
   If Blank Return True
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   Move Right
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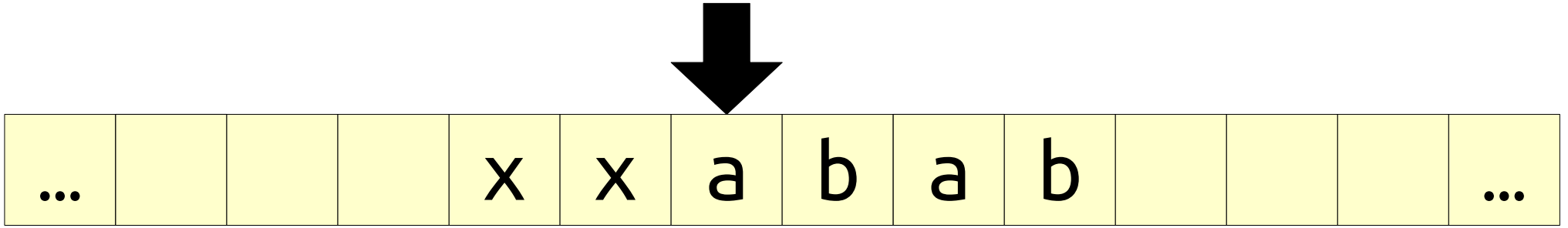

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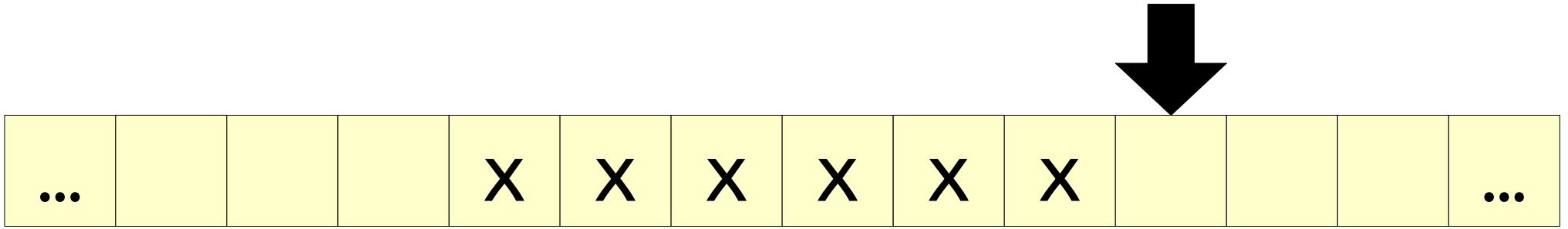

Turing Machines

- This TM initially started up with the string ababab on its tape, so this means that TM returns true on the input ababab, not xxxxxx.
- An intuition for this: we gave this program an input. It therefore returned true with respect to that input, not whatever internal data it generated in making its decision.

```
Start:
   If Blank Return True
   If 'b' Return False
   Write 'x'
   Move Right
   If Not 'b' Return False
   Write 'x'
   Move Right
   Goto Start
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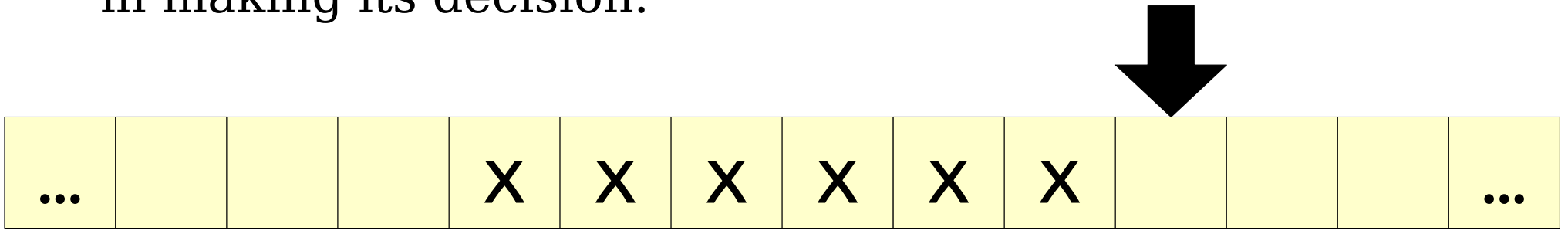

Turing Machines

- To summarize, we only have six commands:
  - `Move` *direction*
  - `Write` *symbol*
  - `Goto` *label*
  - `Return` *result*
  - `If` *symbol command*
  - `If Not` *symbol command*
- Despite their simplicitly, TMs are *surprisingly* powerful. The rest of this lecture explores why.

```
Start:
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  If 'b' Return False
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  Move Right
  If Not 'b' Return False
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```
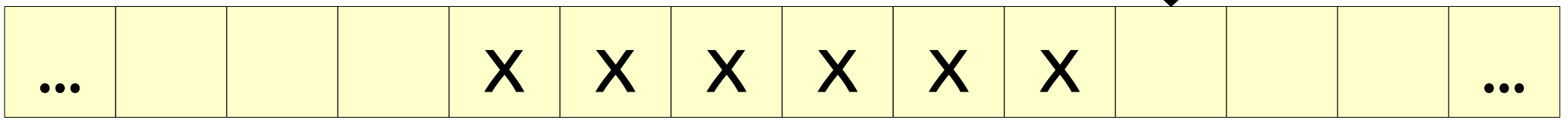



Programming Turing Machines

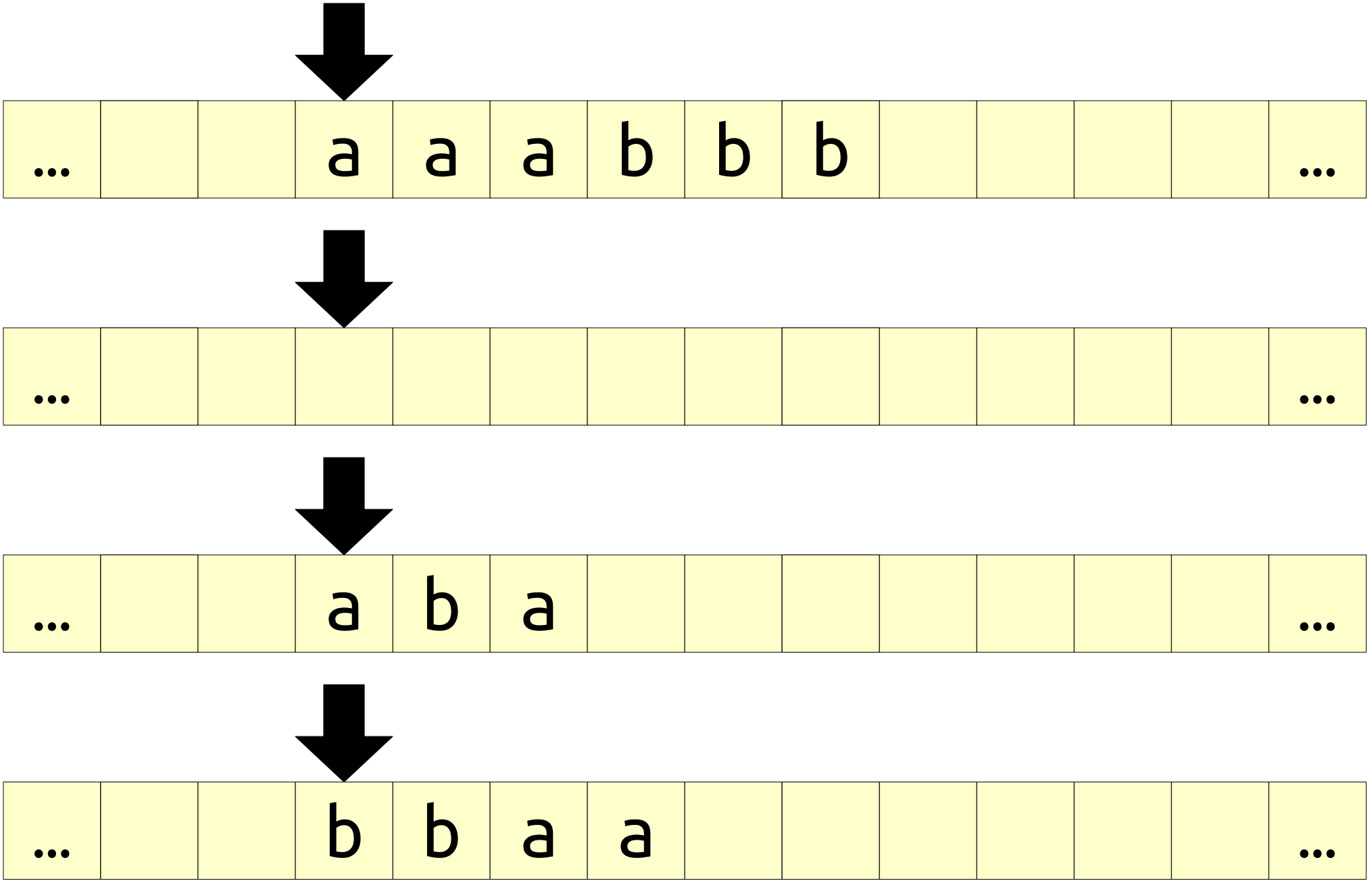
Our First Challenge

- The language

$$\{ \mathtt{a}^n\mathtt{b}^n \mid n \in \mathbb{N} \}$$

  is a canonical example of a nonregular language. It's not possible to check if a string is in this language given only finite memory.
- Turing machines, however, are powerful enough to do this. Let's see how.

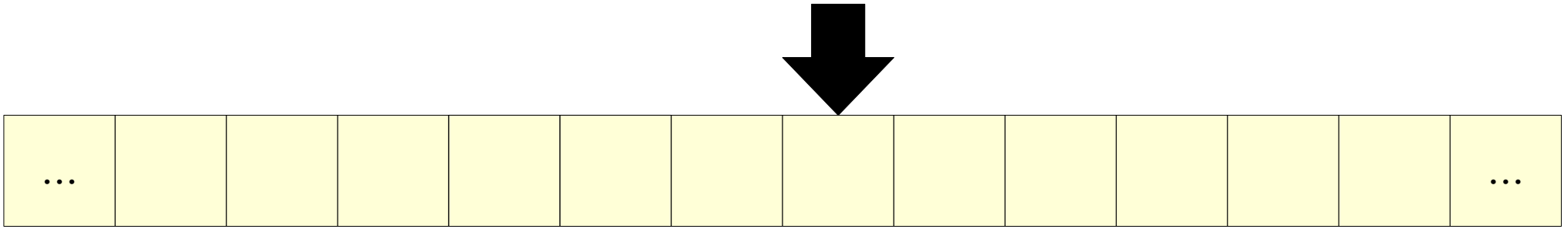
$$L = \{ \mathtt{a}^n \mathtt{b}^n \mid n \in \mathbb{N} \}$$



A Recursive Approach

- We can process our string using this recursive approach:
  - The string ε is in L.
  - The string **a**w**b** is in L if and only if w is in L.
  - Any string starting with **b** is not in L.
  - Any string ending with **a** is not in L.
- All that's left to do now is write a TM that implements this.

```
Start:
  If Blank Return True
  If 'b' Return False
  Write Blank

ZipRight:
  Move Right
  If Not Blank Goto ZipRight
  Move Left
  If Not 'b' Return False
  Write Blank

ZipLeft:
  Move Left
  If Not Blank Goto ZipLeft
  Move Right
  Goto Start
```



Time-Out for Announcements!

Second Midterm Complete

- You’re done with the second midterm exam – congratulations!
- We’re going to do our best to get the second midterm graded before Friday. However, as of now we can’t promise “yes, it will definitely be graded by then.”
- Have any questions about the exam? Post them on EdStem or come chat with us in person.

Your Questions

"What class at Stanford were you most surprised by (i.e. you didn't think you would like it/had to take it and ended up loving it)?"

Without a doubt, ENGR 50 (Intro to Materials Science and Engineering). I signed up for the course because it was required and was floored by how interesting and useful it was. I still remember some of the demos the professor and the TAs did in lecture and am surprised by how helpful it's been to know the basic concepts from that class.

“whats your favorite cuisine / food / comfort food”

It's really hard for me to just pick one, so I won't. I'm a huge fan of just about everything spanning from the Eastern Mediterranean through the Indian subcontinent. But in terms of straight-up comfort food, probably a bowl of French lentils cooked with aromatics and bok choy.

“What software do you use for your slides?”

It's LibreOffice, the default presentation software that comes with Ubuntu Linux. It's free, and I get what I pay for.

Back to CS103!

Our Next Challenge

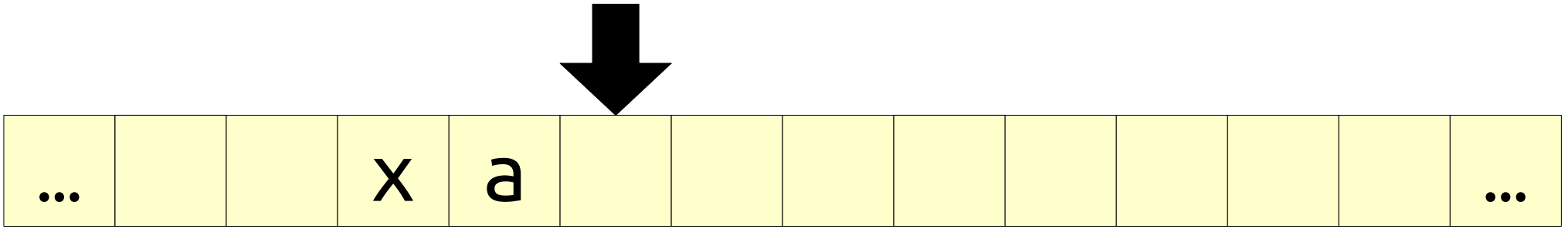
- Let's now take aim at this more general language:

$$\{ w \in \{\texttt{a}, \texttt{b}\}^* \mid w \text{ has an equal number of } \texttt{a}\text{'s and } \texttt{b}\text{'s } \}$$

- This language is not regular (do you see why?)
- It is context-free, but it's a bit tricky to write a CFG for it. (This is a great exercise!)
- Let's see how to design a TM for it.

```
Start:
  If 'a' Goto FoundA
  If 'b' Goto FoundB
  If Blank Return True
  Move Right
  Goto Start

FoundA:
  Write 'x'
LoopA:
  Move Right
  If 'a' Goto LoopA
  If 'x' Goto LoopA
  If Blank Return False
  Write 'x'
  Goto GoHome
```

```
GoHome:
  Move Left
  If Not Blank Goto GoHome
  Move Right
  Goto Start


FoundB:
  Write 'x'
LoopB:
  Move Right
  If 'b' Goto LoopB
  If 'x' Goto LoopB
  If Blank Return False
  Write 'x'
  Goto GoHome
```



Another Idea

- We just built a TM for the language

  $$\{ w \in \{\mathtt{a}, \mathtt{b}\}^* \mid w \text{ has the same number of } \mathtt{a}\text{'s and } \mathtt{b}\text{'s } \}.$$

- An observation: this would be a *lot* easier to test for if all the **a**'s came before all the **b**'s.
  - In fact, that would turn this into checking if the string has the form $\mathtt{a}^n\mathtt{b}^n$, which we already know how to do!
- ***Idea:*** Could we sort the characters of our input string?

Exploring This Idea

Cool TM Tricks 1: ***Fibonacci Numbers***

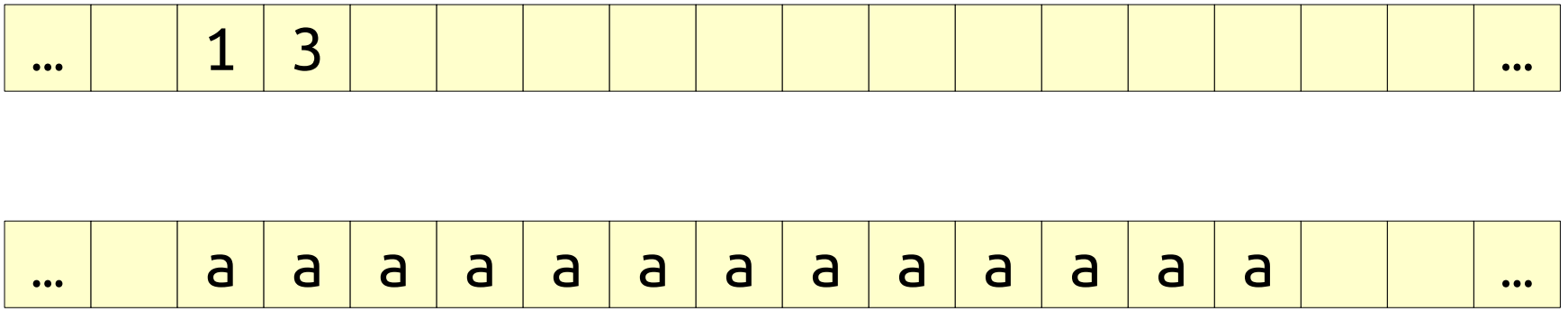
Fibonacci Numbers

$\{ \mathbf{a}^n \mid n \text{ is a Fibonacci number} \}$

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, …

Cool TM Tricks 2: ***Decimal Fibonacci***

Decimal Fibonacci



{ $w \in \{$**0**, **1**, **2**, … **9**$\}^*$ | w, interpreted as a base-10 number, is a Fibonacci number. }

Summary for Today

- Turing machines are abstract computers that issue commands to an infinite tape subdivided into cells.
- Each step of the TM can move the tape head, change what's on the tape, or jump to a different part of the program.
- TMs can be composed together to build larger TMs out of smaller ones.

Next Time

- ***The Church-Turing Thesis***
  - How powerful are Turing machines?
- ***Decidability and Recognizability***
  - Two notions of “solving a problem.”