Unsolvable Problems

Part One

Outline for Today

- ***Self-Reference Revisited***
  - Programs that compute on themselves.
- ***Self-Defeating Objects***
  - Objects “too powerful” to exist.
- ***The Fortune Teller***
  - Can you escape the future?
- ***Why Do Programs Loop?***
  - … and can we eliminate loops?
- ***Undecidable Problems***
  - Something beyond the reach of algorithms.

Recap from Last Time

R and **RE**

- A language L is ***recognizable*** if there is a TM M with the following property:

$$\forall w \in \Sigma^*. \ (M \text{ accepts } w \leftrightarrow w \in L).$$

- That is, for any string w:
  - If $w \in L$, then M accepts w.
  - If $w \notin L$, them M does not accept w.
    - It might reject w, or it might loop on w.
- This is a "weak" notion of solving a problem.
- The class **RE** consists of all the recognizable languages.

R and **RE**

- A language L is ***decidable*** if there is a TM M with the following properties:

$$\forall w \in \Sigma^*. (M \text{ accepts } w \leftrightarrow w \in L).$$

***M* halts on all inputs.**

- That is, for any string w:
  - If $w \in L$, then M accepts w.
  - If $w \notin L$, then M rejects w.
- This is a "strong" notion of solving a problem.
- The class **R** consists of all the decidable languages.

The Universal TM

- The ***universal Turing machine***, denoted U_{TM}, is a TM with the following behavior: when run on a string $\langle M, w \rangle$, where M is a TM and w is a string, U_{TM} will

  … accept $\langle M, w \rangle$ if M accepts w,

  … reject $\langle M, w \rangle$ if M rejects w, and

  … loop on $\langle M, w \rangle$ if M loops on w.

- $\mathbf{A_{TM}}$ is the language recognized by the universal TM. This is the language

$$\mathbf{A_{TM} = \{ \langle \mathit{M}, \mathit{w} \rangle \mid \mathit{M} \text{ is a TM and } \mathit{M} \text{ accepts } \mathit{w} \}}$$

Self-Referential Programs

- Computing devices can compute on their own source code:

  Theorem: It is possible to construct TMs that perform arbitrary computations on their own source code.

- This allows us to write programs that work on their own source code.

```cpp
void cormorant() {
    string me = /* source code of
                 * cormorant
                 */;
    cout << me << endl;
}
```

```cpp
bool curlew(string input) {
    string me = /* source code of
                 * curlew
                 */;
    return input == me;
}
```

```cpp
int avocet() {
    string me = /* source code of
                 * avocet
                 */;
    int result = 0;
    for (char ch: me) {
        if (ch == 'a') result++;
    }
    return result;
}
```

What do each of these pieces of code do?

New Stuff!

Part One: Self-Defeating Objects

A ***self-defeating object*** is an object whose essential properties ensure it doesn't exist.

Question: Why is there no largest integer?

Answer: Because if n is the largest integer, what happens when we look at $n+1$?

Self-Defeating Objects

Theorem: There is no largest integer.

Proof sketch: Suppose for the sake of contradiction that there is a largest integer. Call that integer n.

Consider the integer $n+1$.

Notice that $n < n+1$.

But then n isn't the largest integer.

Contradiction! ■-*ish*

We're using n to construct something that undermines n, hence the term "self-defeating."

An Important Detail

Careful – we're assuming what we're trying to prove!

Claim: There is a largest integer.

Proof: Assume x is the largest integer.

Notice that $x > x - 1$.

So there's no contradiction. ■-*ish*

How do we know there's no contradiction? We just checked one case.

Self-Defeating Objects

- If you can show

$$\textit{\textbf{x exists}} \to \bot$$

  then you know that *x* doesn't exist. (This is a proof by contradiction.)

- If you can show

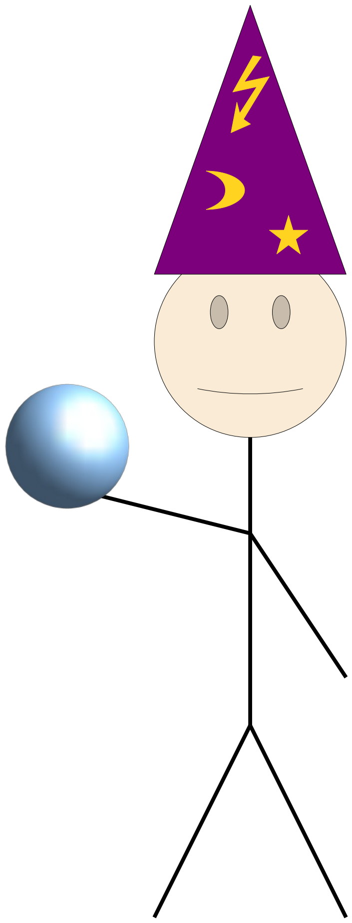
$$\textit{\textbf{x exists}} \to \top$$

  you cannot conclude that *x* exists. (This is not a valid proof technique.)

Part Two: The Fortune Teller

The Fortune Teller

- A fortune teller appears who claims they can see into anyone’s future.
- For a nominal fee, the fortune teller will tell you anything you want to know about the future.



The Fortune Teller

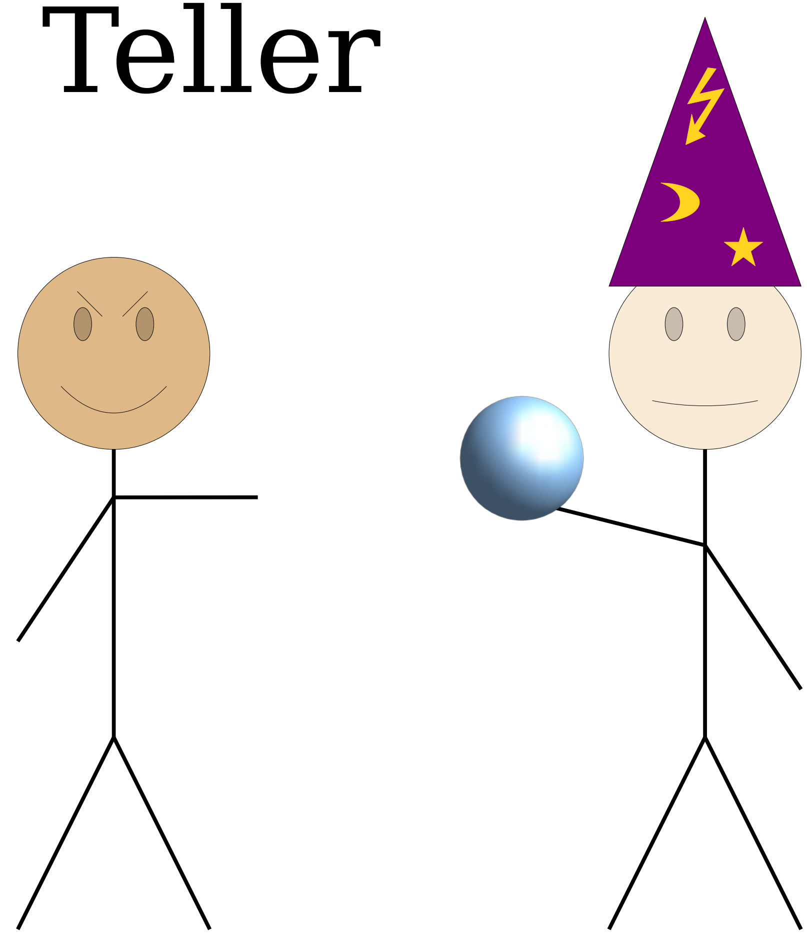
- One day, a trickster arrives. The trickster thinks the fortune teller is lying and can't really see the future.
- The trickster says the following:

"I have a yes/no question about the future. But before I ask my question, let's talk payment.

If you answer yes, then I'll pay you $137.

If you answer no, then I'll pay you $42.

- The fortune teller thinks for a moment, then agrees.



Trickster pays $137 if the fortune teller answers "yes."

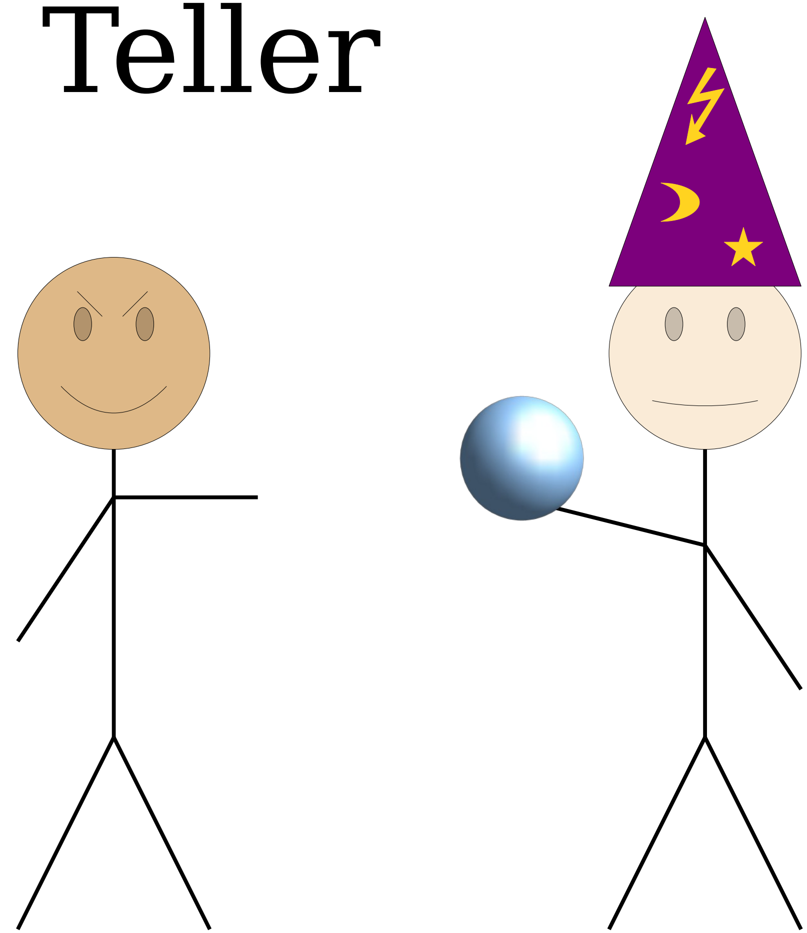
Trickster pays $42 if the fortune teller answers "no."

The Fortune Teller

- The trickster then asks this question:

  "Am I going to pay you $42?"

- The fortune teller is trapped!
- Talk to your neighbor – why?



Trickster pays $137 if the fortune teller answers "yes."

Trickster pays $42 if the fortune teller answers "no."

The Fortune Teller

- The payment scheme the fortune teller agreed to means

  Fortune Teller Says Yes ↔ ***Trickster Pays \$137***.

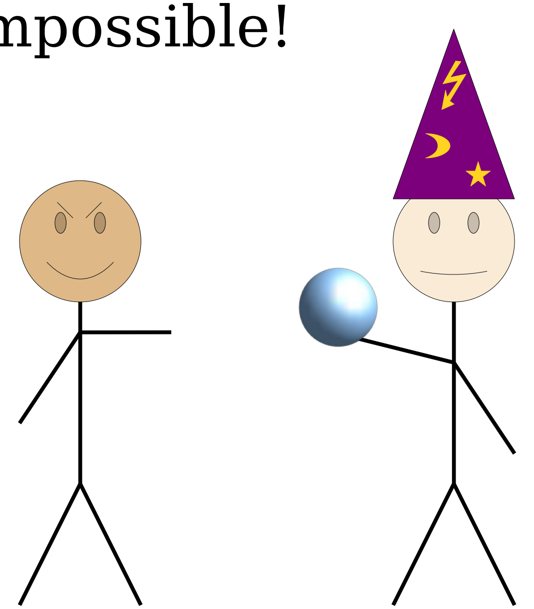
- The trickster's question to the fortune teller means

  Fortune Teller Says Yes ↔ ***Trickster Pays \$42***.

- Putting this together, we get

  Trickster Pays \$42 ↔ ***Trickster Pays \$137***.

- This is impossible!

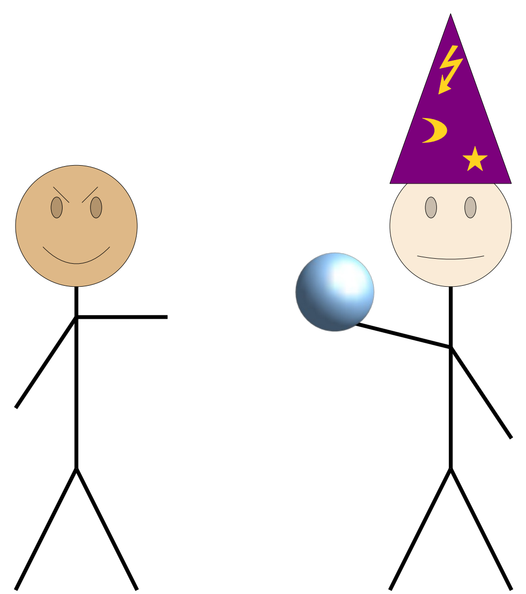


Trickster pays \$137 if the fortune teller answers "yes."

Trickster pays \$42 if the fortune teller answers "no."

The Fortune Teller

- The fortune teller is a self-defeating object.
- The trickster’s strategy is to couple the fortune teller’s behavior to what the future holds.
  - The trickster’s behavior is chosen in advance to make the fortune teller’s answer wrong.
- Therefore, the fortune teller can’t answer all questions about all people in the future.



Trickster pays $137 if the fortune teller answers “yes.”

Trickster pays $42 if the fortune teller answers “no.”

Part Three: Why Do Programs Loop?

Thoughts on Loops

- In practice, the programs we write sometimes go into infinite loops.
- In Theoryland, Turing machines are allowed to loop. This happens if they don't accept and don't reject.
- ***Question:*** Why are infinite loops possible?
- Or rather: are infinite loops an inherent part of computation, or are they some weird sort of "accident" in how we program computers?

Thoughts on Loops

- ***Theorem:*** The language A_{TM} is recognizable, but undecidable.
  - There's a *recognizer* for A_{TM} (specifically, the universal Turing machine U_{TM}).
  - It is impossible to build a *decider* for this language.
- Stated differently, there's a program we can write (a universal TM) that *has* to loop infinitely on some inputs.
- ***Goal:*** Prove this theorem, and explore its theoretical and philosophical implications.

A_{TM} Revisited

- As a refresher, the language A_{TM} is

  $$\mathbf{A_{TM} = \{ \langle \mathit{M}, \mathit{w} \rangle \mid \mathit{M} \textbf{ is a TM and } \mathit{M} \textbf{ accepts } \mathit{w} \}}.$$

- The universal TM U_{TM} has the following behavior when given as input a TM M and a string w:
  - If M accepts w, then U_{TM} accepts $\langle M, w \rangle$.
  - If M rejects w, then U_{TM} rejects $\langle M, w \rangle$.
  - If M loops on w, then U_{TM} loops on $\langle M, w \rangle$.
- U_{TM} is a recognizer for A_{TM}, but because of that last case it's not a decider for A_{TM}.

A_{TM} Revisited

- As a refresher, the language A_{TM} is

  $A_{TM} = \{ \langle M, w \rangle \mid M$ is a TM and M accepts $w \}$.

- Given a TM M and a string w, a decider D for A_{TM} would need to have this behavior:
  - If M accepts w, then D **?** $\langle M, w \rangle$.
  - If M rejects w, then D **?** $\langle M, w \rangle$.
  - If M loops on w, then D **?** $\langle M, w \rangle$.
- This is basically the same set of requirements as U_{TM}, except for what happens if M loops on w.
- Our goal is to prove that there is no way to build a program that meets these requirements.

A_{TM} Revisited

- We can envision a decider for A_{TM} as a function

```
bool willAccept(string fn, string input)
```

  that takes as input the source code of a function (`fn`) and a string representing an input to that function (`input`).
- It then does the following:
  - If `fn(input)` returns true, `willAccept(fn, input)` returns true.
  - If `fn(input)` returns false, `willAccept(fn, input)` returns false.
  - If `fn(input)` loops, then `willAccept(fn, input)` returns false.
- We're going to show it's impossible to write a function that actually does this. But for now, let's just explore what such a decider would do.

```
function = "bool f(string input) {
  if (input == "") return false;
  return input[0] == 'a';
}";

input = "abbababba";

willAccept(function, input) = ?
```

```
function = "bool g(string input) {
  while (true) {
    input += input;
  }
}";

input = "yay! ";

willAccept(function, input) = ?
```

```
function = "bool h(string input) {
  for (char c: input) {
    if (c != input[0]) return true;
  }
  return false;
}";

input = "aaaaaa";

willAccept(function, input) = ?
```

```
function = "bool j(string input) {
  int n = input.length();
  while (n > 1) {
    if (n % 2 == 0) n /= 2;
    else n = 3*n + 1;
  }
  return true;
}";

input = /* 10^137 a's */;

willAccept(function, input) = ?
```

For each of these instances, what does `willAccept(function, input)` return?

Deciding A_{TM}

- Earlier this quarter you explored sums of four squares. Now, let's talk about sums of three cubes.
- Are there integers x, y, and z where…
  - $x^3 + y^3 + z^3 = 10$?
  - $x^3 + y^3 + z^3 = 11$?
  - $x^3 + y^3 + z^3 = 12$?
  - $x^3 + y^3 + z^3 = 13$?

Deciding A_{TM}

- Surprising fact: until 2019, no one knew whether there were integers *x*, *y*, and *z* where

$$x^3 + y^3 + z^3 = 33.$$

- A heavily optimized computer search found this answer:

$$x = 8{,}866{,}128{,}975{,}287{,}528$$
$$y = -8{,}778{,}405{,}442{,}862{,}239$$
$$z = -2{,}736{,}111{,}468{,}807{,}040$$

- As of November 2021, no one knows whether there are integers *x*, *y*, and *z* where

$$x^3 + y^3 + z^3 = 114.$$

Deciding A_{TM}

- Consider the language

$$L = \{ \mathtt{a}^n \mid \exists x \in \mathbb{Z}.\ \exists y \in \mathbb{Z}.\ \exists z \in \mathbb{Z}.\ x^3 + y^3 + z^3 = n \}$$

- Here's code for a recognizer to see whether such a triple exists:

```
bool hasTriple(int n) {
  for (int max = 0; ; max++)
    for (int x = -max; x <= max; x++)
      for (int y = -max; y <= max; y++)
        for (int z = -max; z <= max; z++)
          if (x*x*x + y*y*y + z*z*z == n)
            return true;
}
```

- Imagine calling `willAccept(/* hasTriple code */, 114)`.
  - If such a triple exists, `willAccept` returns true.
  - If no such triple exists, `willAccept` returns false.
- ***Key Intuition:*** However `willAccept` is implemented, it has to be clever enough to resolve open problems in mathematics!

Why is A_{TM} Hard?

- ***Intuition:*** A decider for A_{TM} would be able to…
  - … determine whether the hailstone sequence terminates for any input. (Write a recognizer that runs the hailstone sequence, then feed it into the decider for A_{TM}.)
  - … see if any number is the sum of three cubes. (Write a recognizer that tries all infinitely many triples of integers, then feed it into the decider for A_{TM}.)
  - … and much, much more.
- In other words, this seemingly simple problem of "is this program going to terminate?" accidentally scoops up a bunch of other seemingly harder problems.

Time-Out for Announcements!

Stanford Engineering Research Introductions

Are you curious about STEM research? Do you come from or identify with an underrepresented background or community? Are you a college underclassman?

If so, SERI (Stanford Engineering Research Introductions) can introduce you to graduate research, including what to expect and how to best prepare yourself to enter the graduate research program of your choice

WHAT
Presentations, panels, lab tours, and more during a two-day program

WHERE
In person at Stanford University, COVID19-restrictions permitting

WHEN
Late January or Early February 2022

HOW
Apply online by December 17th, 2021
tinyurl.com/SERIApplication

Info session tomorrow (Thursday) at 2:30PM. RSVP using ***this link***.

App. QR Code

Questions?
https://seri.sites.stanford.edu/
seri-program@lists.stanford.edu

Stanford | ENGINEERING

On Rigor and Formalism in Math

- Terry Tao, considered by many to be the greatest living mathematician, has an essay about learning mathematics.
- It explains why formal proofs and rigorous arguments are an important part of learning math – and why it can be a bit tricky at times.
- You can read it ***online here***.

Your Questions

“What should my roommate and I name our plants?”

I have two suggestions!

Back to CS103!

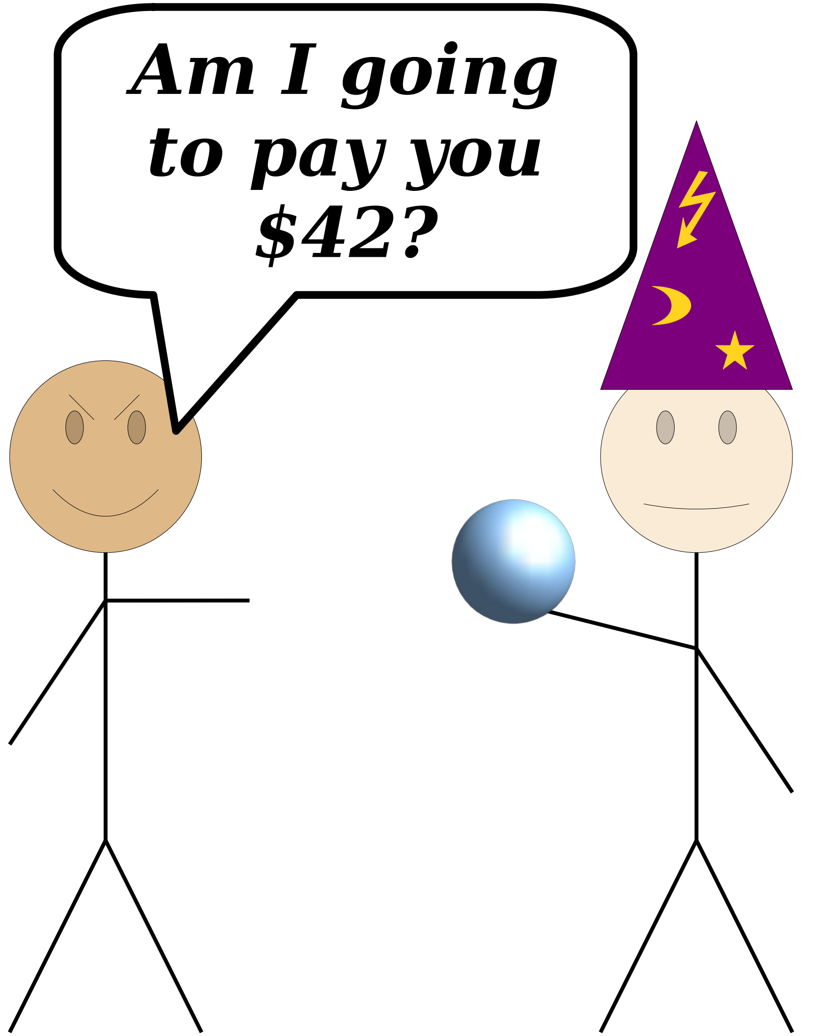
Part Four: Putting It All Together

To Recap

- We’re assuming that, somehow, someone wrote a function

```
bool willAccept(string function, string input);
```

  that takes the code of a function and an input to that function, then
  - returns true if `function(input)` returns true, and
  - returns false if `function(input)` doesn’t return true.
- ***Goal:*** Show that this decider is “self-defeating;” its power is so great that it undermines itself.
- ***Idea:*** Convert the fortune teller story into a program.

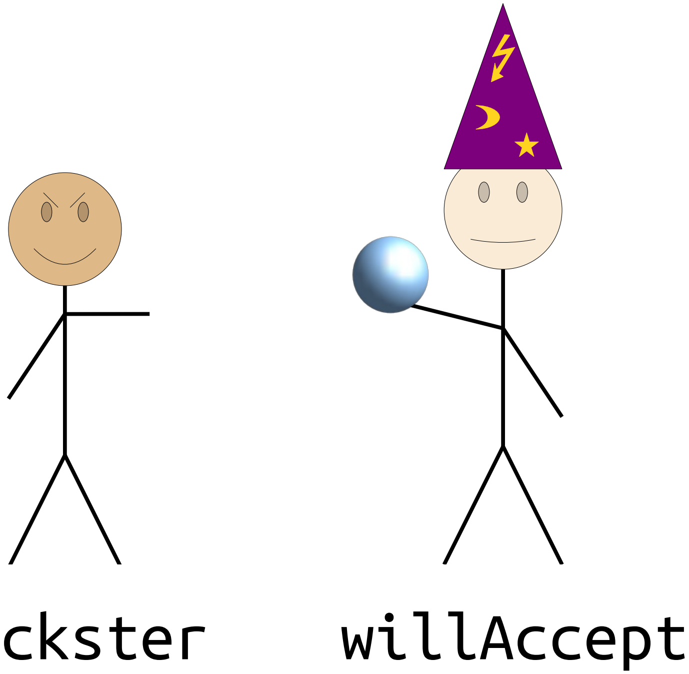


Trickster pays $137 if the fortune teller answers “yes.”

Trickster pays $42 if the fortune teller answers “no.”

```
bool willAccept(string function, string input) {
   // Returns true if function(input) returns
   // true. Returns false otherwise.
}

bool trickster(string input) {
   string me = /* source code of trickster */;
   return !willAccept(me, input);
}
```



If `willAccept` says `trickster` will return true, then `trickster` returns false.

If `willAccept` says `trickster` will not return true, then `trickster` returns true.

```
bool willAccept(string function, string input) {
   // Returns true if function(input) returns
   // true. Returns false otherwise.
}

bool trickster(string input) {
   string me = /* source code of trickster */;
   return !willAccept(me, input);
}
```

A self-defeating object.

Using that object against itself.

```
bool willAccept(string function, string input) {
    // Returns true if function(input) returns
    // true. Returns false otherwise.
}

bool trickster(string input) {
    string me = /* source code of trickster */;
    return !willAccept(me, input);
}
```

"The largest integer *n*."

"The integer *n* + 1."

Theorem: There is no largest integer.

Proof sketch: Suppose for the sake of contradiction that there is a largest integer. Call that integer n.

Consider the integer $n+1$.

Notice that $n < n+1$.

But then n isn't the largest integer.

Contradiction! ■-*ish*

Theorem: $A_{TM} \notin \mathbf{R}$.

Proof: By contradiction; assume that $A_{TM} \in \mathbf{R}$. Then there is a decider D for A_{TM}. We can represent D as a function

```
bool willAccept(string function, string w);
```

that takes in the source code of a function `function` and a string `w`, then returns true if `function(w)` returns true and returns false otherwise.

Given this, consider this function `trickster`:

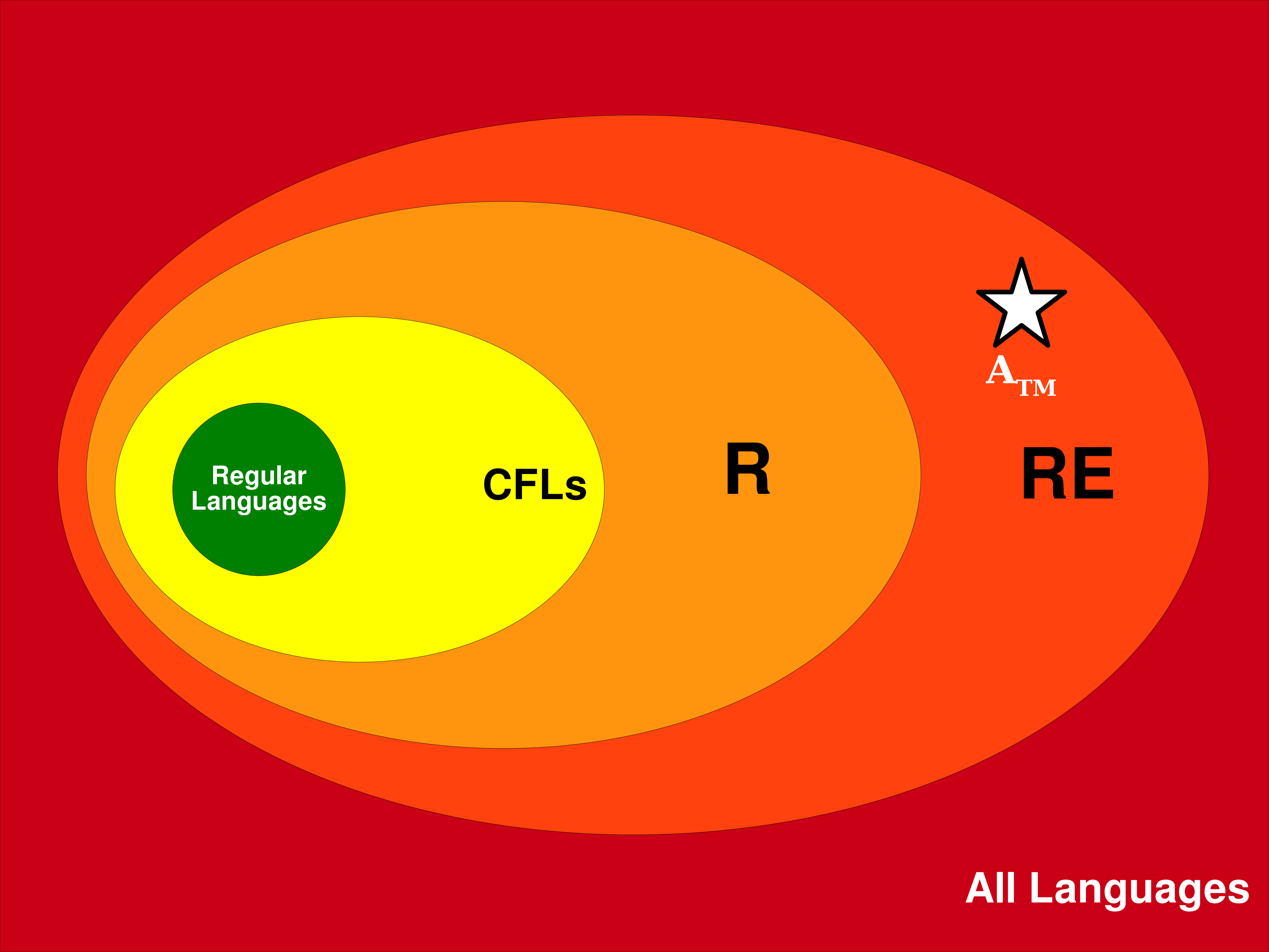
```
bool trickster(string input) {
    string me = /* source code of trickster */;
    return !willAccept(me, input);
}
```

Choose a string w. We consider two cases:

Case 1: `willAccept(me, input)` returns true. Since `willAccept` decides A_{TM}, this means `trickster(w)` returns true. However, given how `trickster` is written, in this case `trickster(w)` returns false.

Case 2: `willAccept(me, input)` returns false. Since `willAccept` decides A_{TM}, this means `trickster(w)` doesn't return true. However, given how `trickster` is written, in this case `trickster(w)` returns true.

In both cases we reach a contradiction, so our assumption must have been wrong. Therefore, $A_{TM} \notin \mathbf{R}$. ■


Regular Languages
CFLs
R
RE
A_{TM}
All Languages

What Does This Mean?

- In one fell swoop, we've proven that
  - A_{TM} is ***undecidable***; there is no general algorithm that can determine whether a TM will accept a string.
  - $\mathbf{R} \neq \mathbf{RE}$, because $A_{TM} \notin \mathbf{R}$ but $A_{TM} \in \mathbf{RE}$.
- What do these three statements really mean? As in, why should you care?

$A_{TM} \notin \mathbf{R}$

- What exactly does it mean for A_{TM} to be undecidable?

  Intuition: The only general way to find out what a program will do is to run it.

- As you'll see, this means that it's provably impossible for computers to be able to answer most questions about what a program will do.

$A_{TM} \notin \mathbf{R}$

- At a more fundamental level, the existence of undecidable problems tells us the following:

There is a difference between what is true and what we can discover is true.

- Given a TM M and a string w, one of these two statements is true:

M accepts w **M does not accept w**

But since A_{TM} is undecidable, there is no algorithm that can always determine which of these statements is true!

R ≠ RE

- Because **R ≠ RE**, there is a difference between decidability and recognizability:

  In some sense, it is fundamentally harder to solve a problem than it is to check an answer.

- There are problems where, when the answer is "yes," you can confirm it (run a recognizer), but where if you don't have the answer, you can't come up with it in a mechanical way (build a decider).

Next Time

- ***Why All This Matters***
  - Important, practical, undecidable problems.
- ***Intuiting RE***
  - What exactly is the class **RE** all about?
- ***Verifiers***
  - A totally different perspective on problem solving.
- ***Beyond RE***
  - Finding an impossible problem using very familiar techniques.