Complexity Theory

Part Two

Recap from Last Time

The Complexity Class **P**

- The complexity class **P** (***polynomial time***) is defined as

$$\mathbf{P} = \{\ L \mid \text{There is a polynomial-time decider for } L\ \}$$

- Intuitively, **P** contains all decision problems that can be solved efficiently.
- This is like class **R**, except with "efficiently" tacked onto the end.

The Complexity Class **NP**

- The complexity class **NP** (***nondeterministic polynomial time***) contains all problems that can be verified in polynomial time.
- Formally:

  NP = { *L* | There is a polynomial-time verifier for *L* }

- Intuitively, **NP** is the set of problems where "yes" answers can be checked efficiently.
- This is like the class **RE**, but with "efficiently" tacked on to the definition.

The Biggest Unsolved Problem in Theoretical Computer Science:

$$\mathbf{P} \stackrel{?}{=} \mathbf{NP}$$

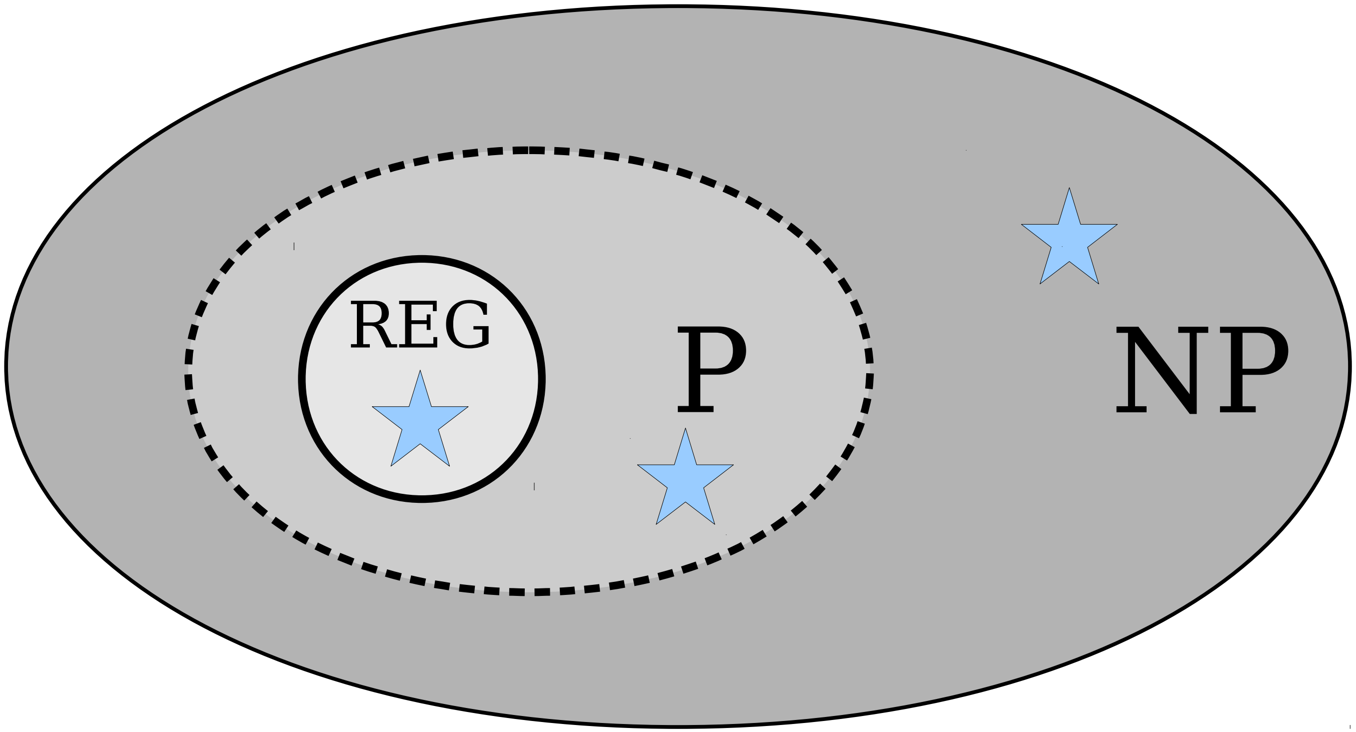
Theorem (Baker-Gill-Solovay): Any proof that purely relies on universality and self-reference cannot resolve $\mathbf{P} \stackrel{?}{=} \mathbf{NP}$.

Proof: Take CS154!

So how *are* we going to reason about **P** and **NP**?

New Stuff!

A Challenge



Problems in **NP** vary widely in their difficulty, even if **P** = **NP**.

How can we rank the relative difficulties of problems?

Reducibility

Maximum Matching

- Given an undirected graph *G*, a ***matching*** in *G* is a set of edges such that no two edges share an endpoint.
- A ***maximum matching*** is a matching with the largest number of edges.

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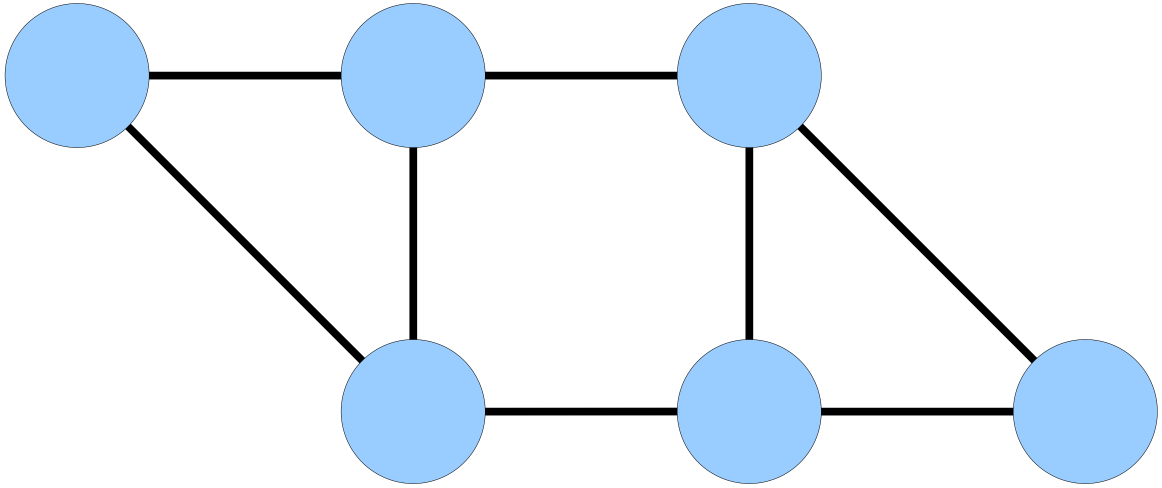

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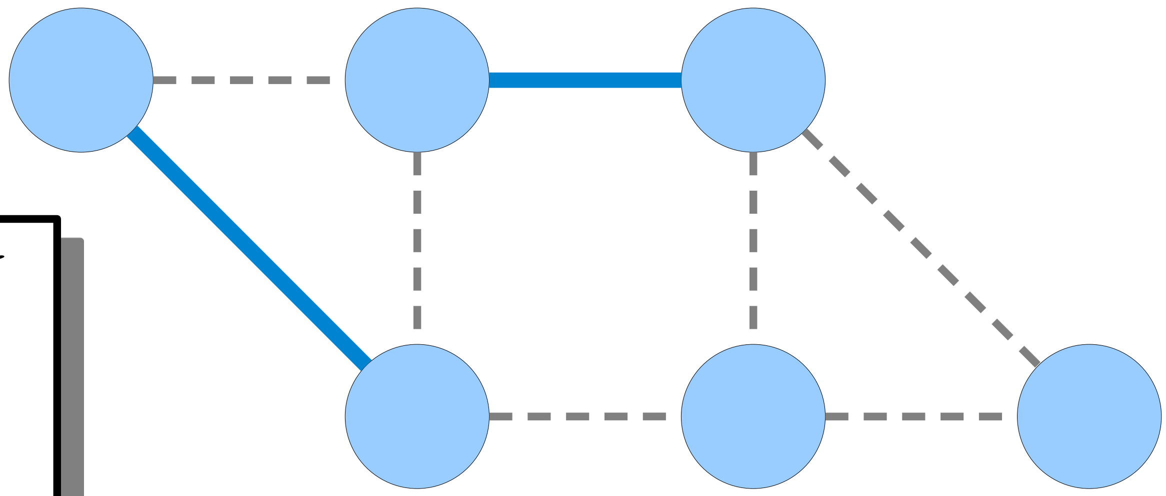

A matching, but not a maximum matching.

Maximum Matching

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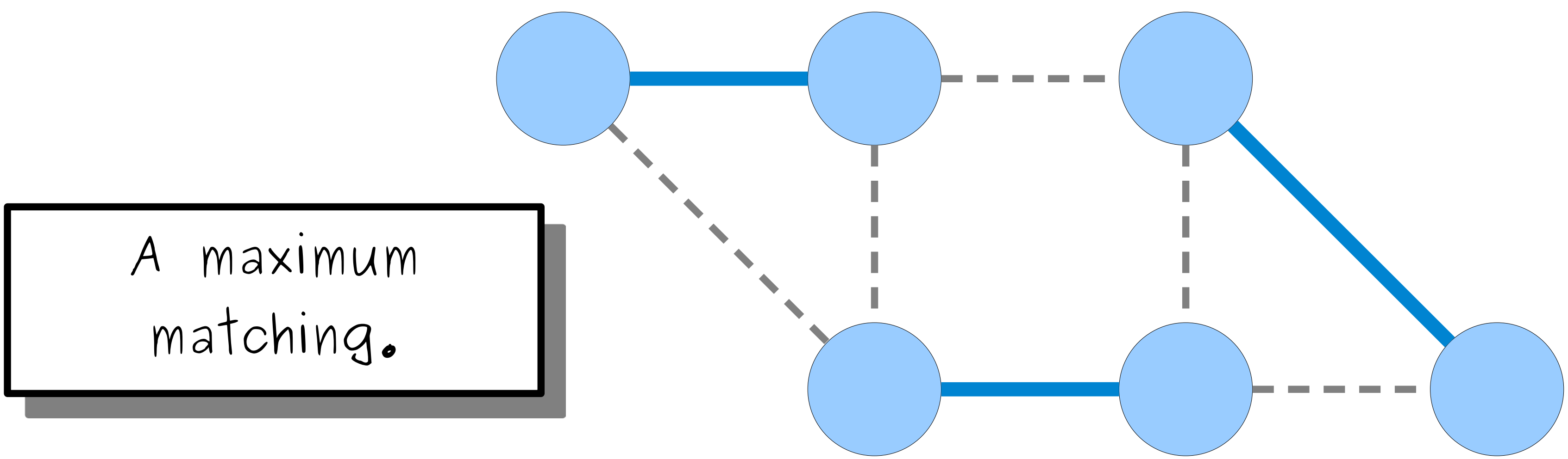

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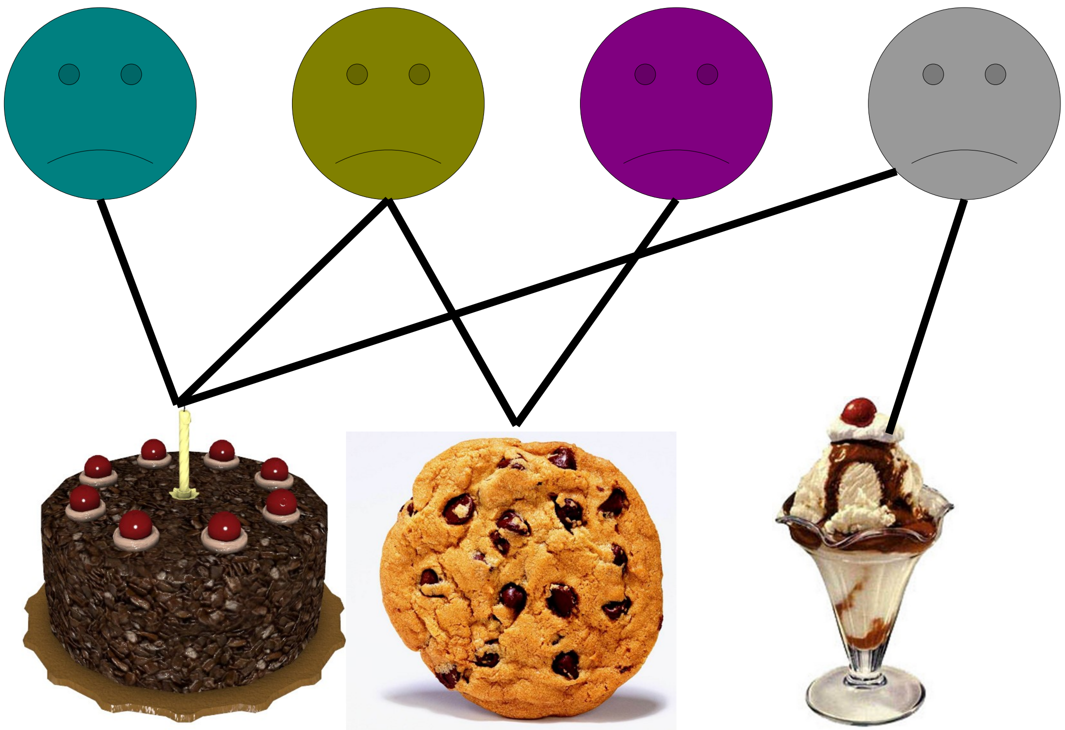

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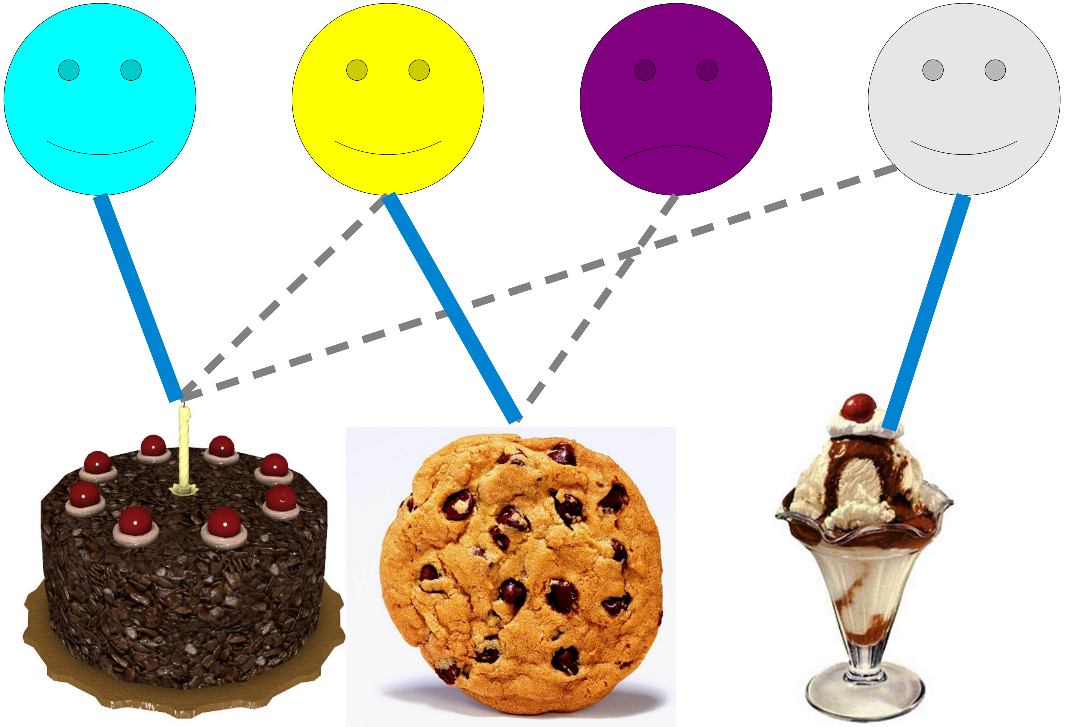
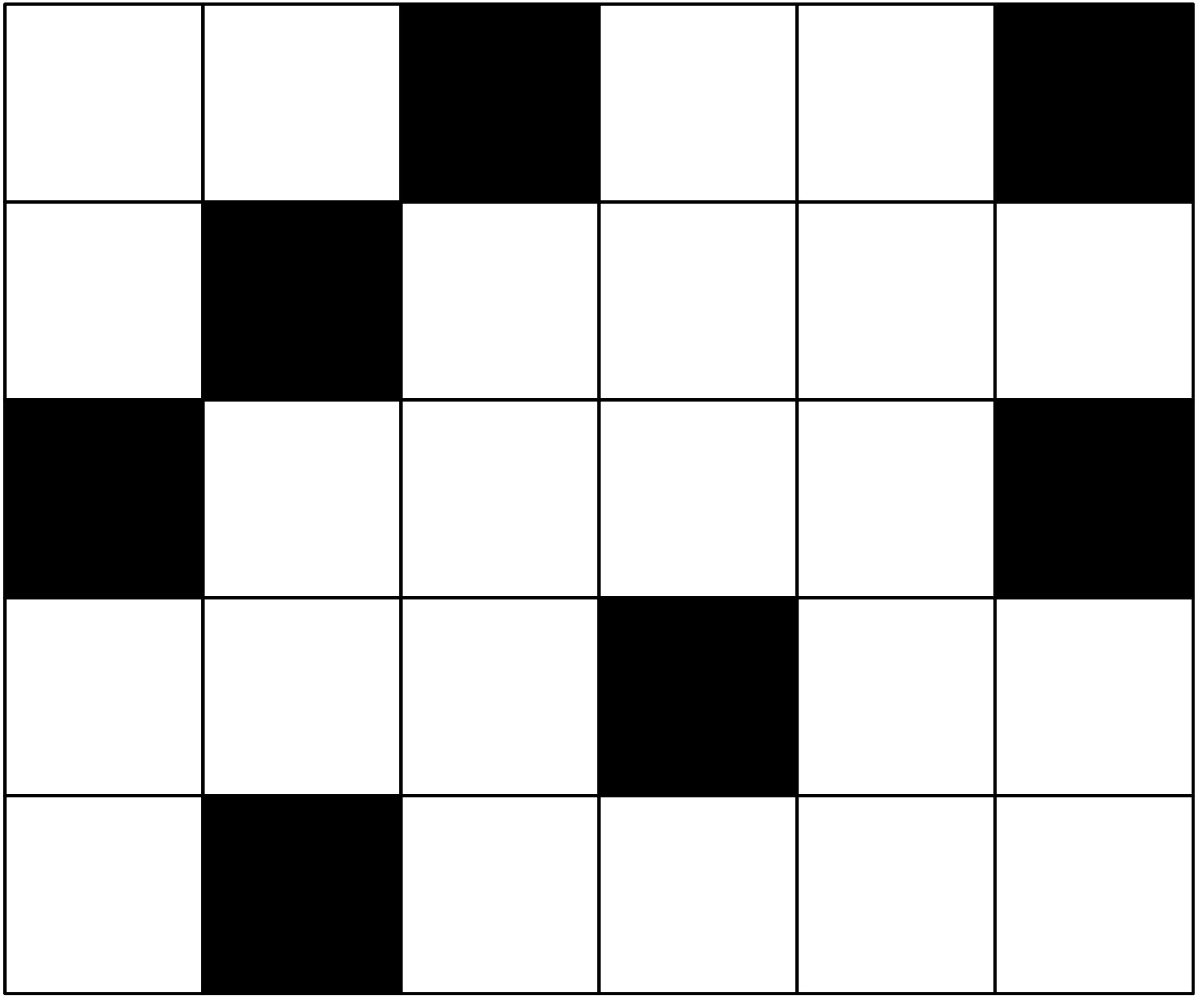


Maximum Matching

- Jack Edmonds' paper “Paths, Trees, and Flowers” gives a polynomial-time algorithm for finding maximum matchings.
  - He’s the guy from last time with the quote about “better than decidable.”
- Using this fact, what other problems can we solve?

Domino Tiling



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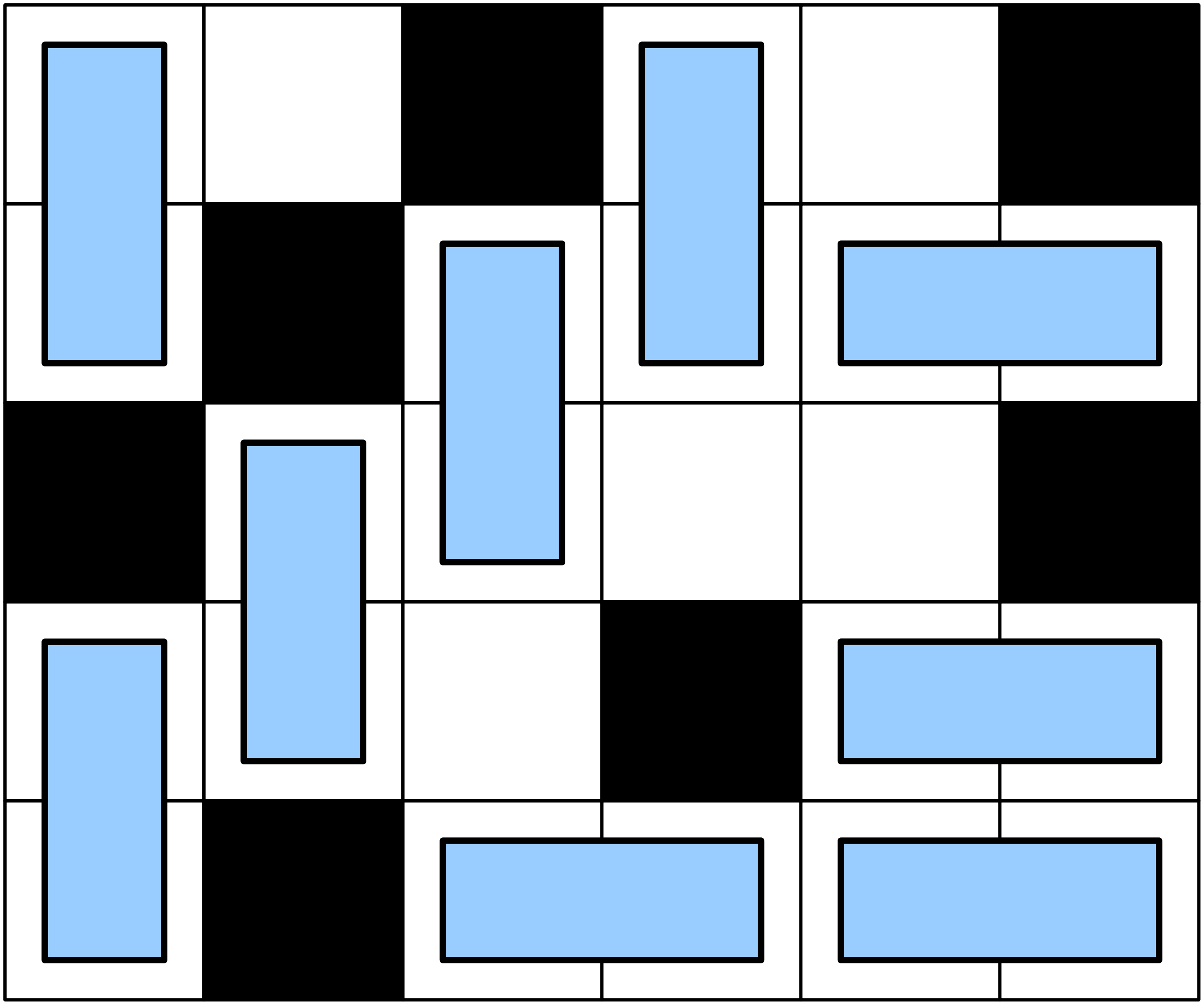

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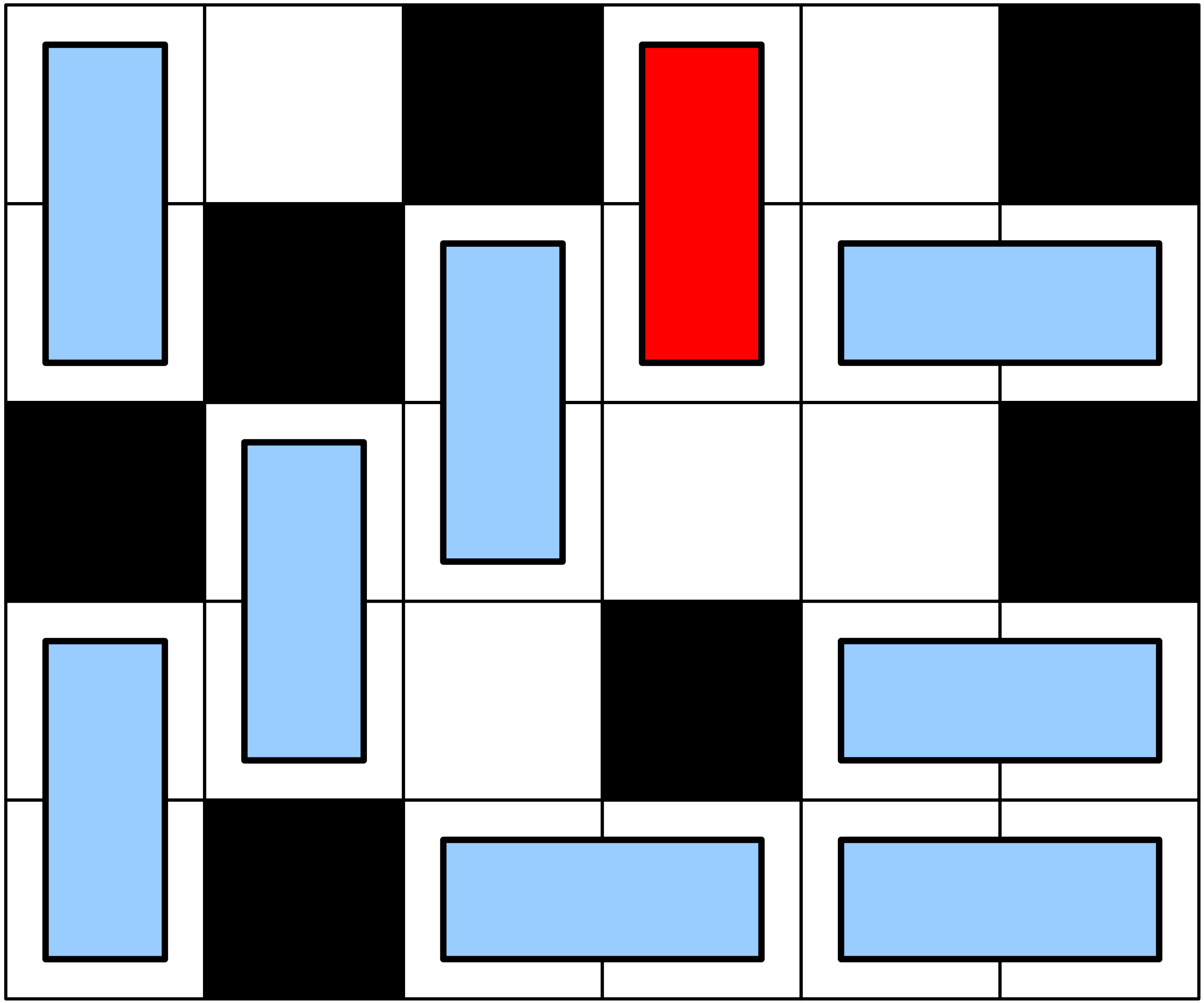

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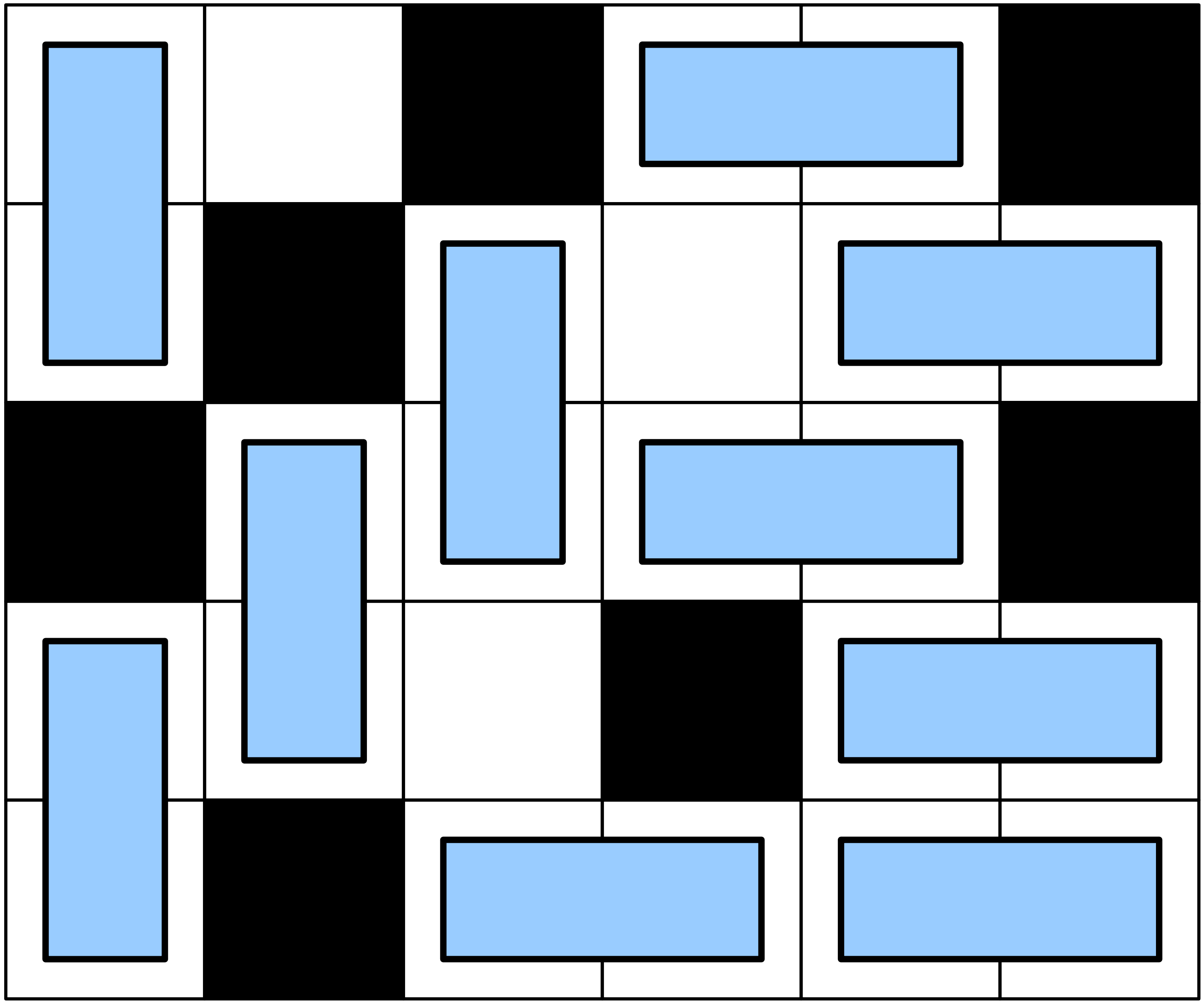

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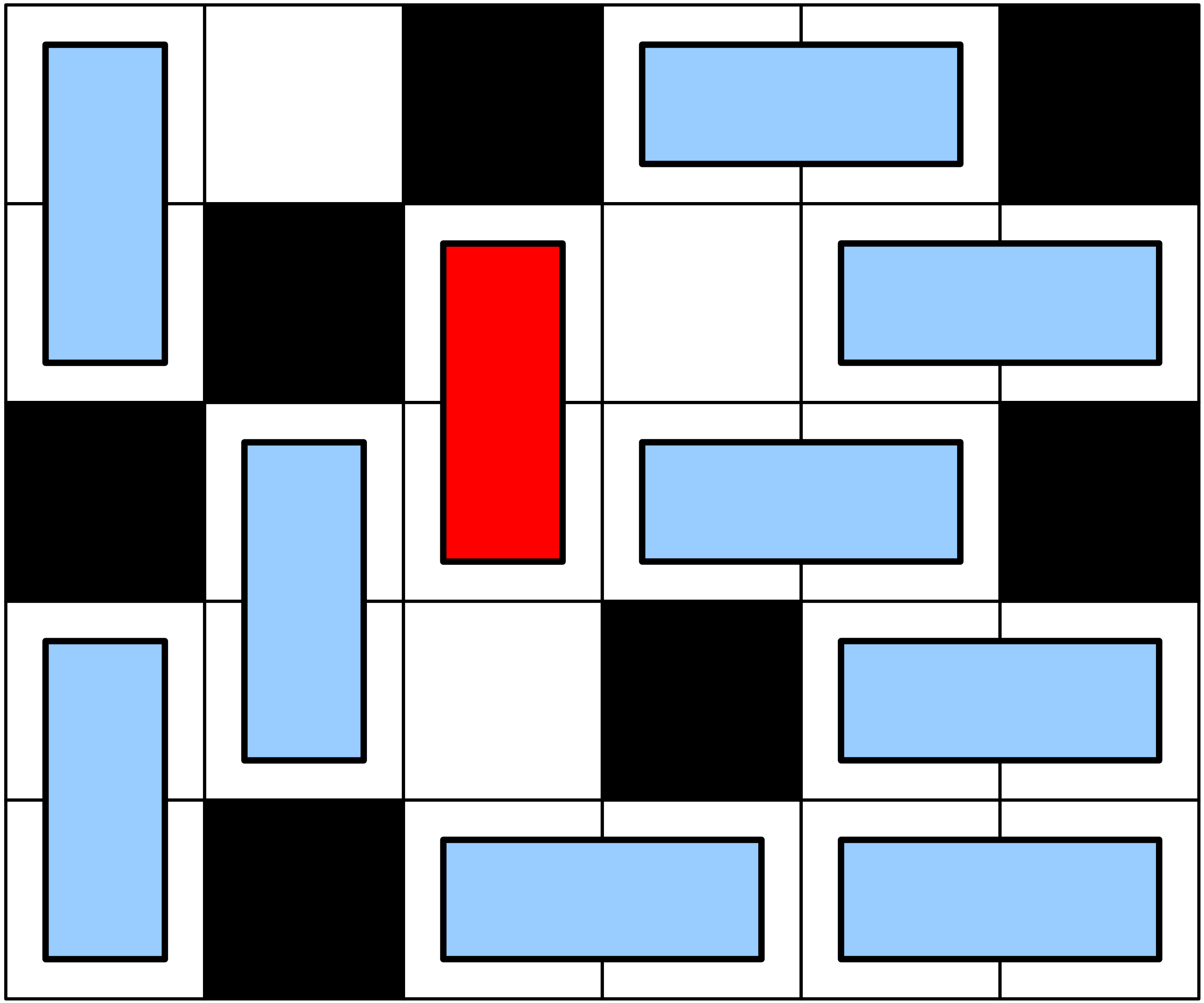

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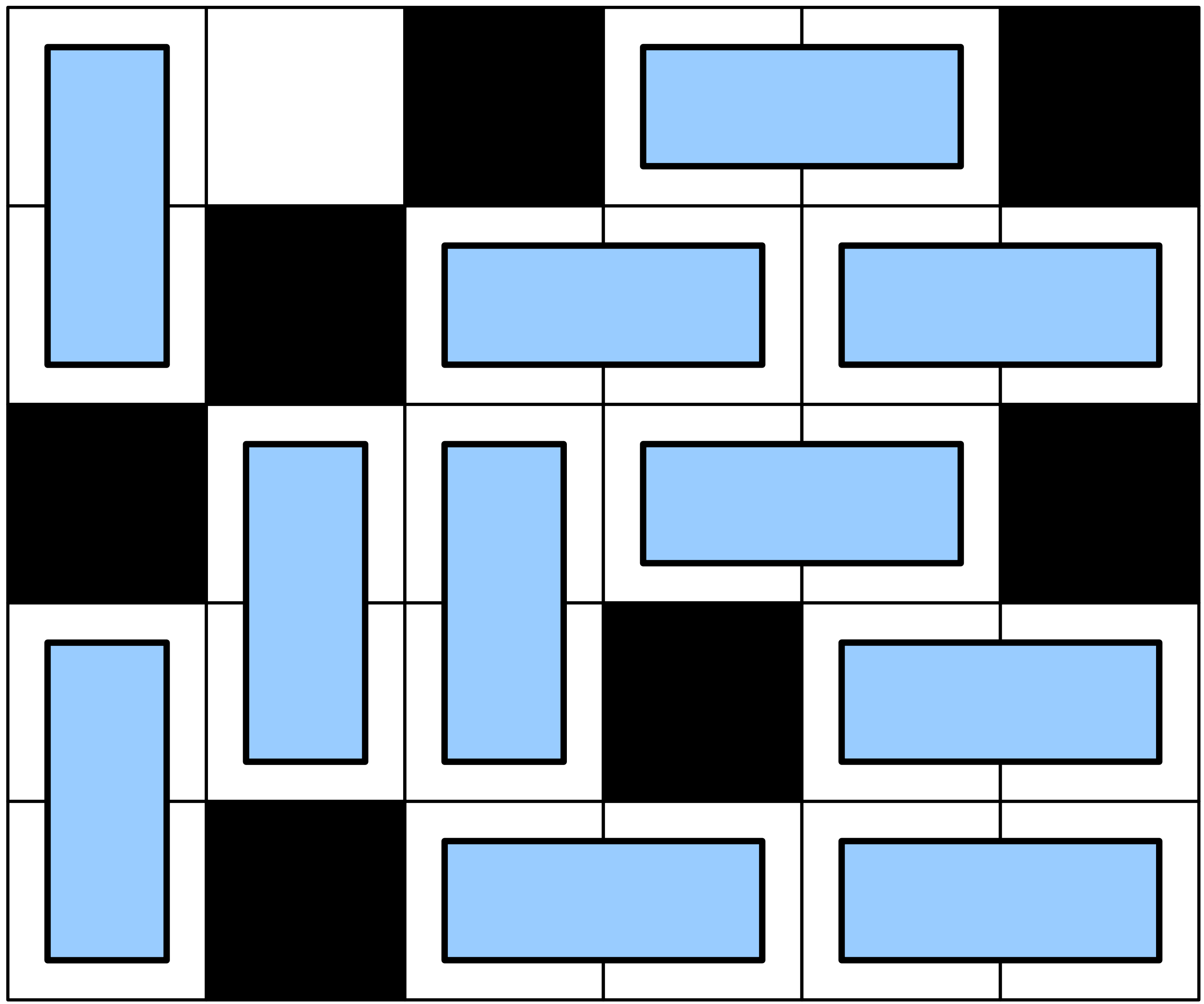
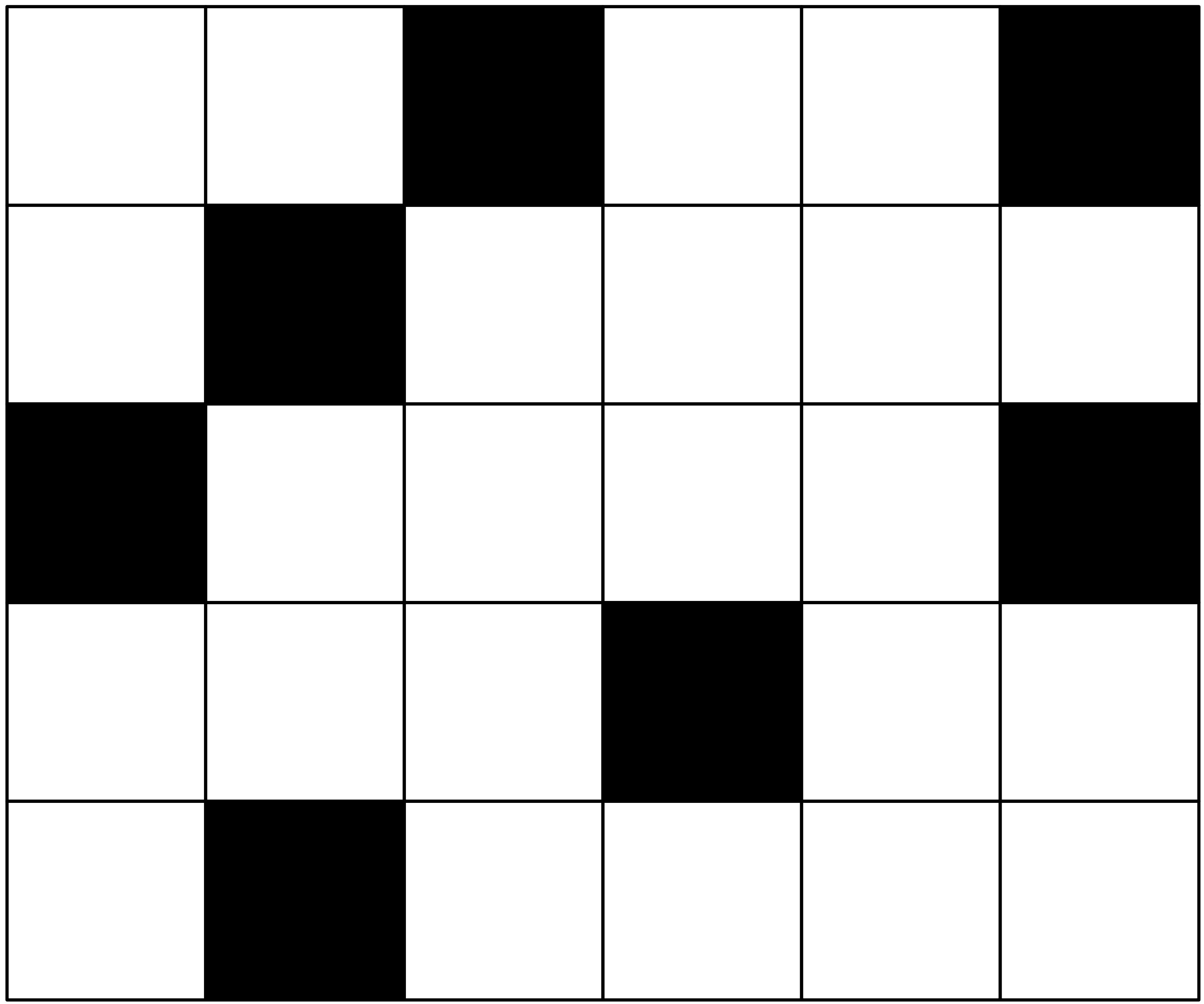


Solving Domino Tiling



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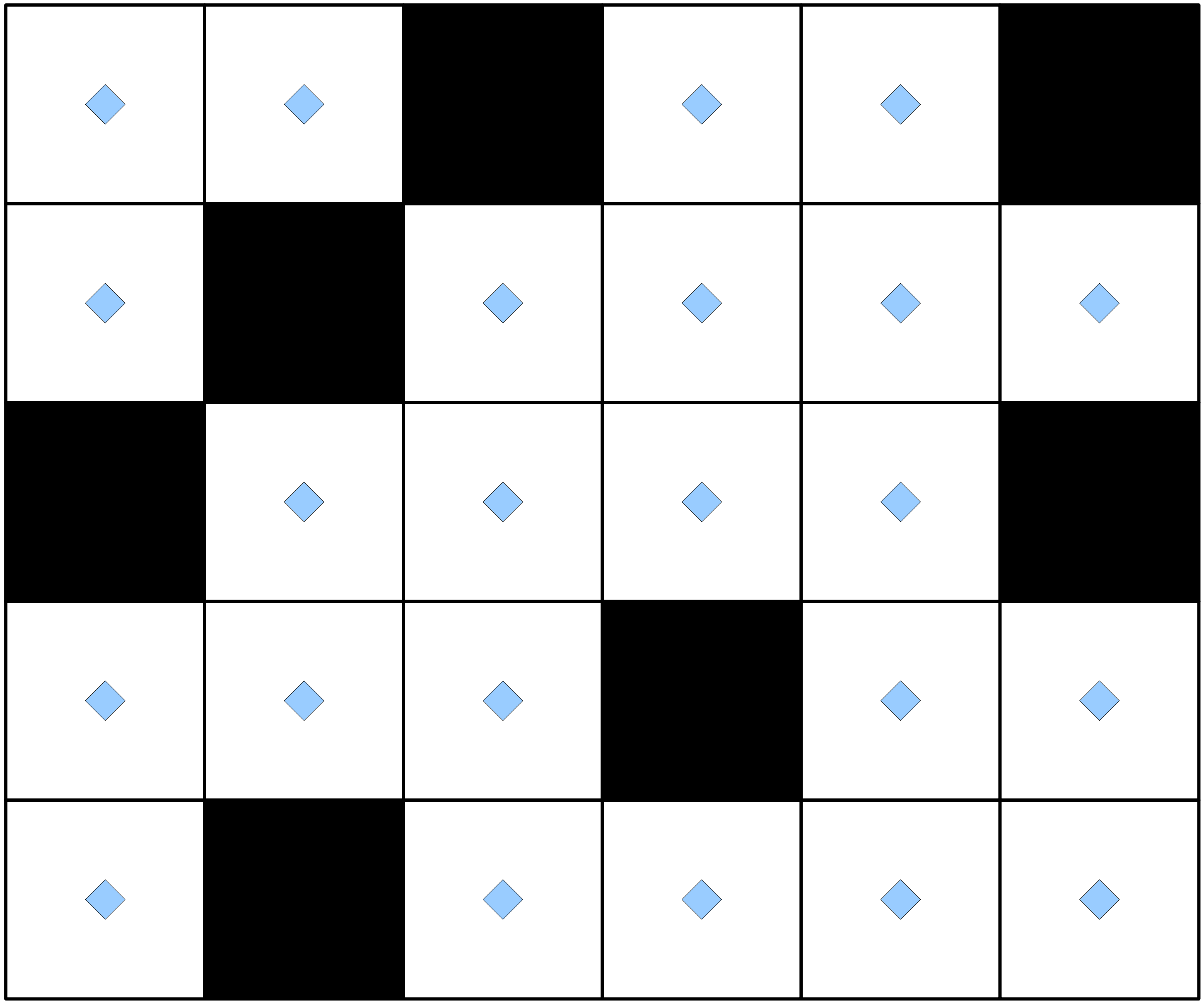

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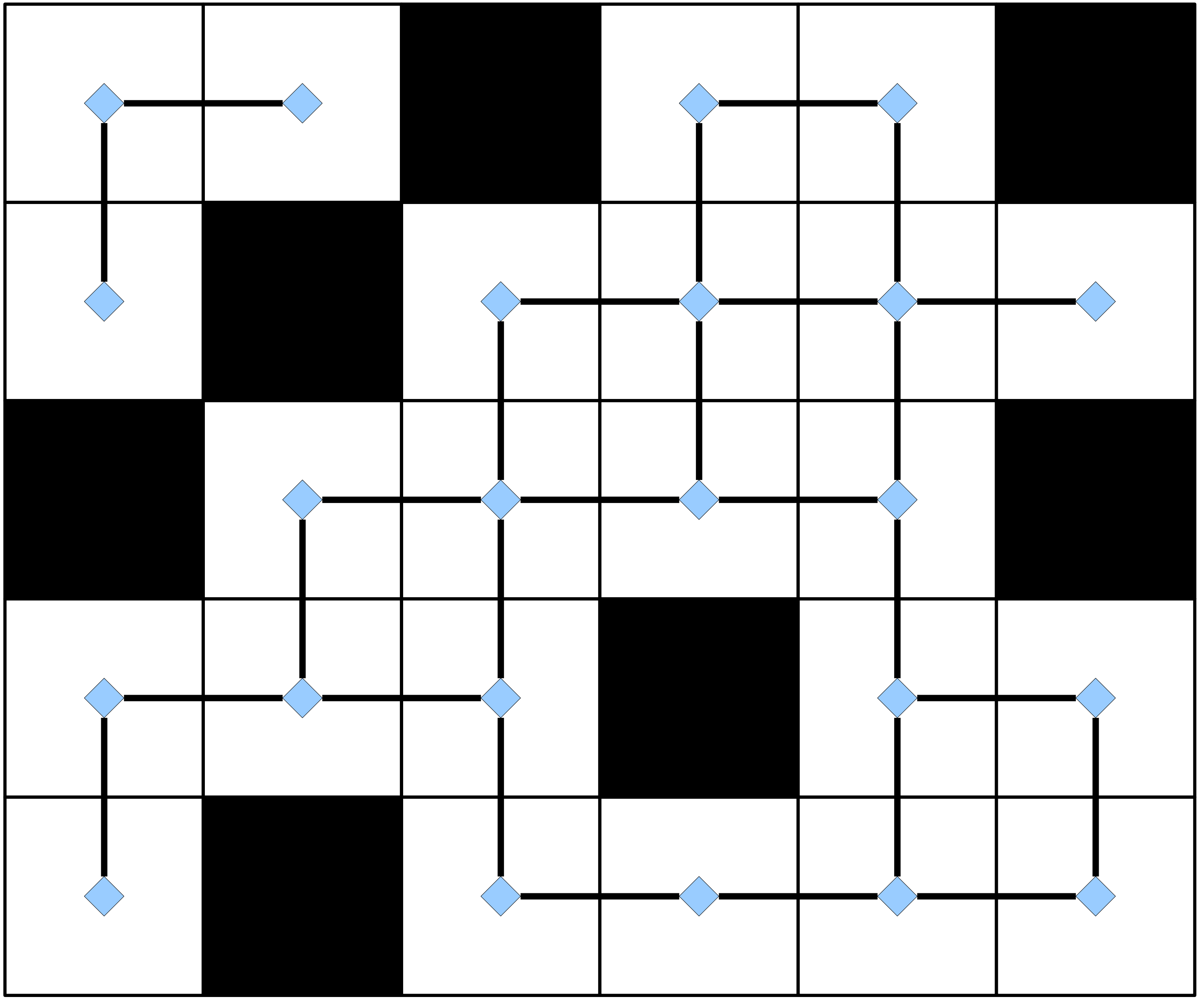

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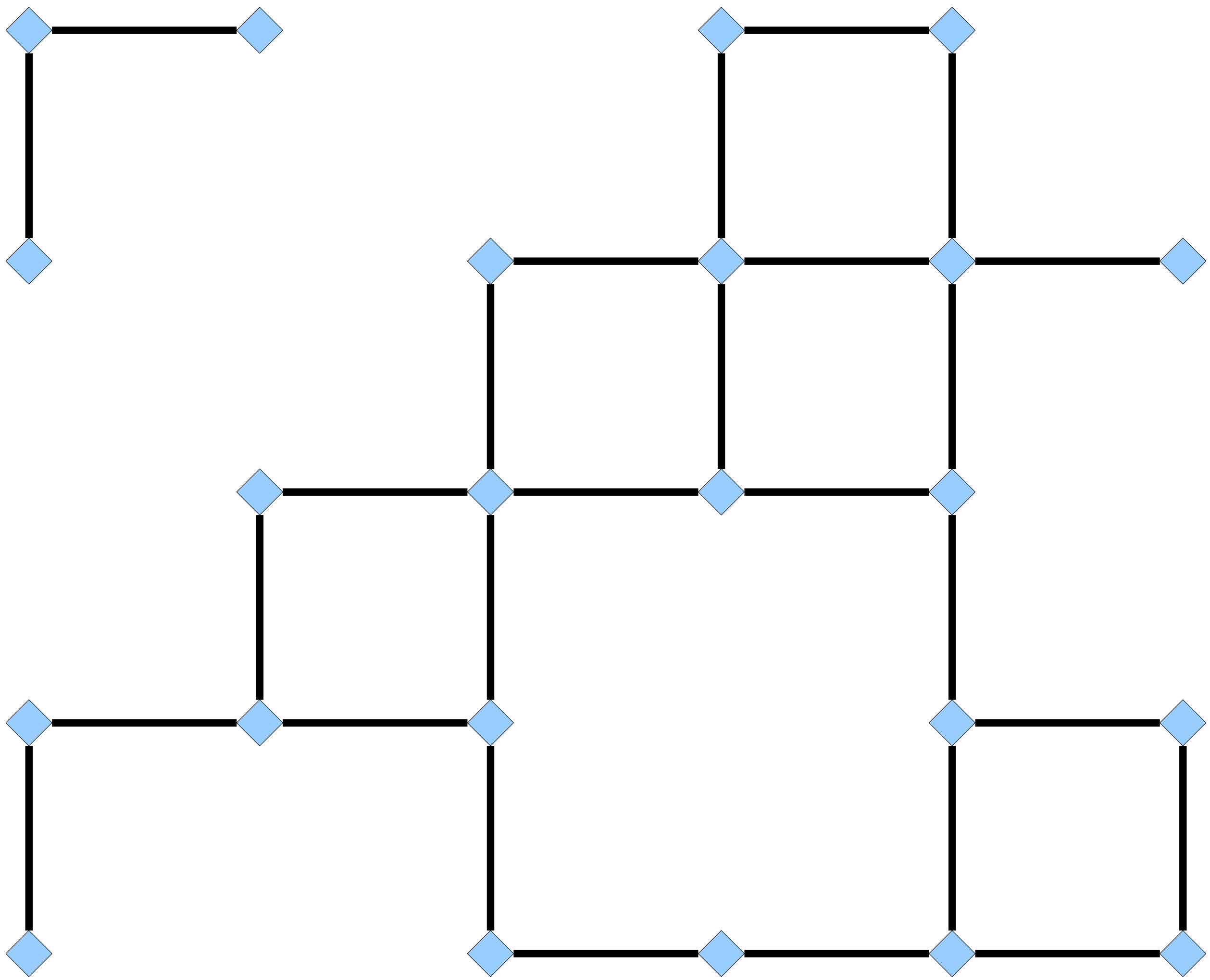

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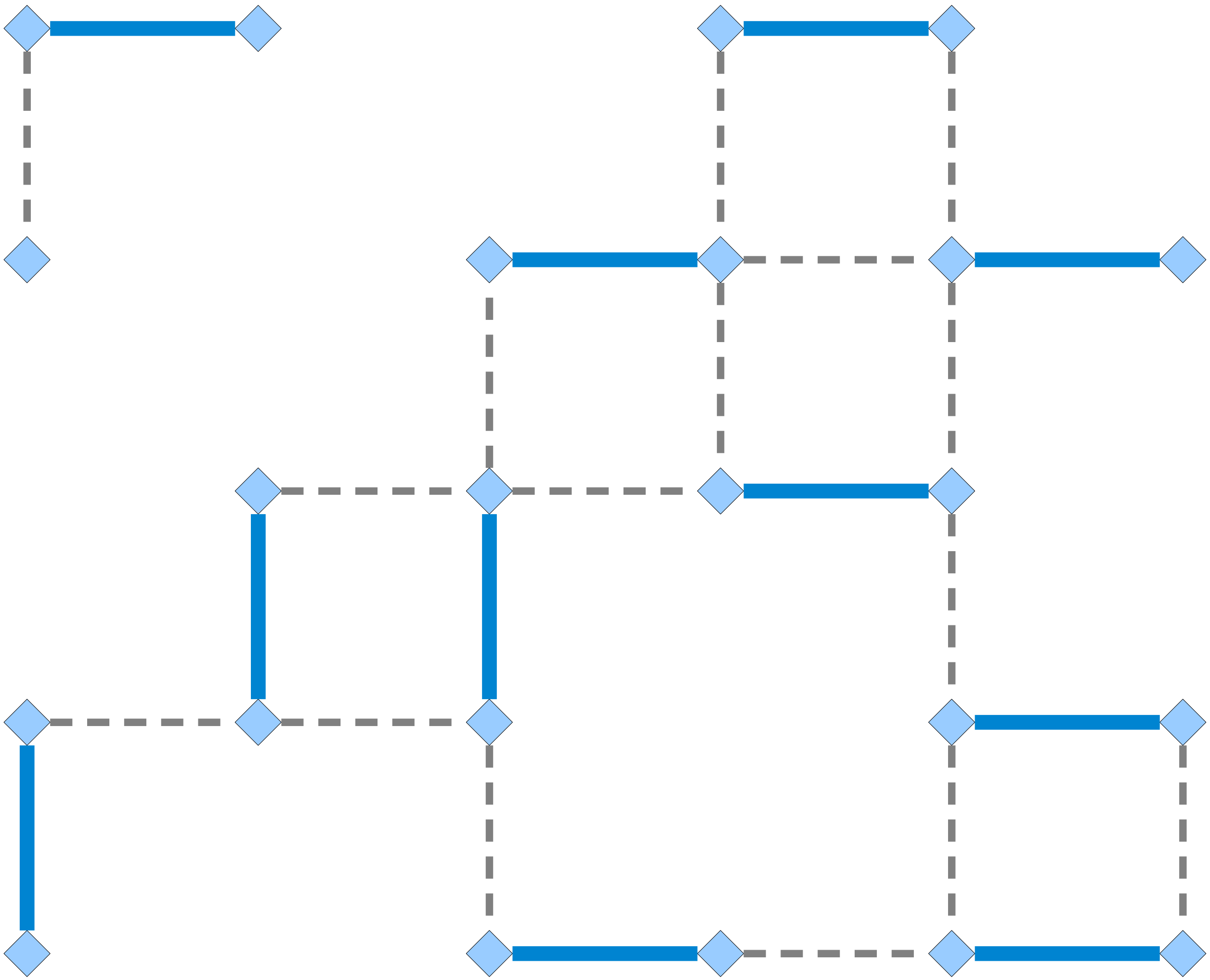

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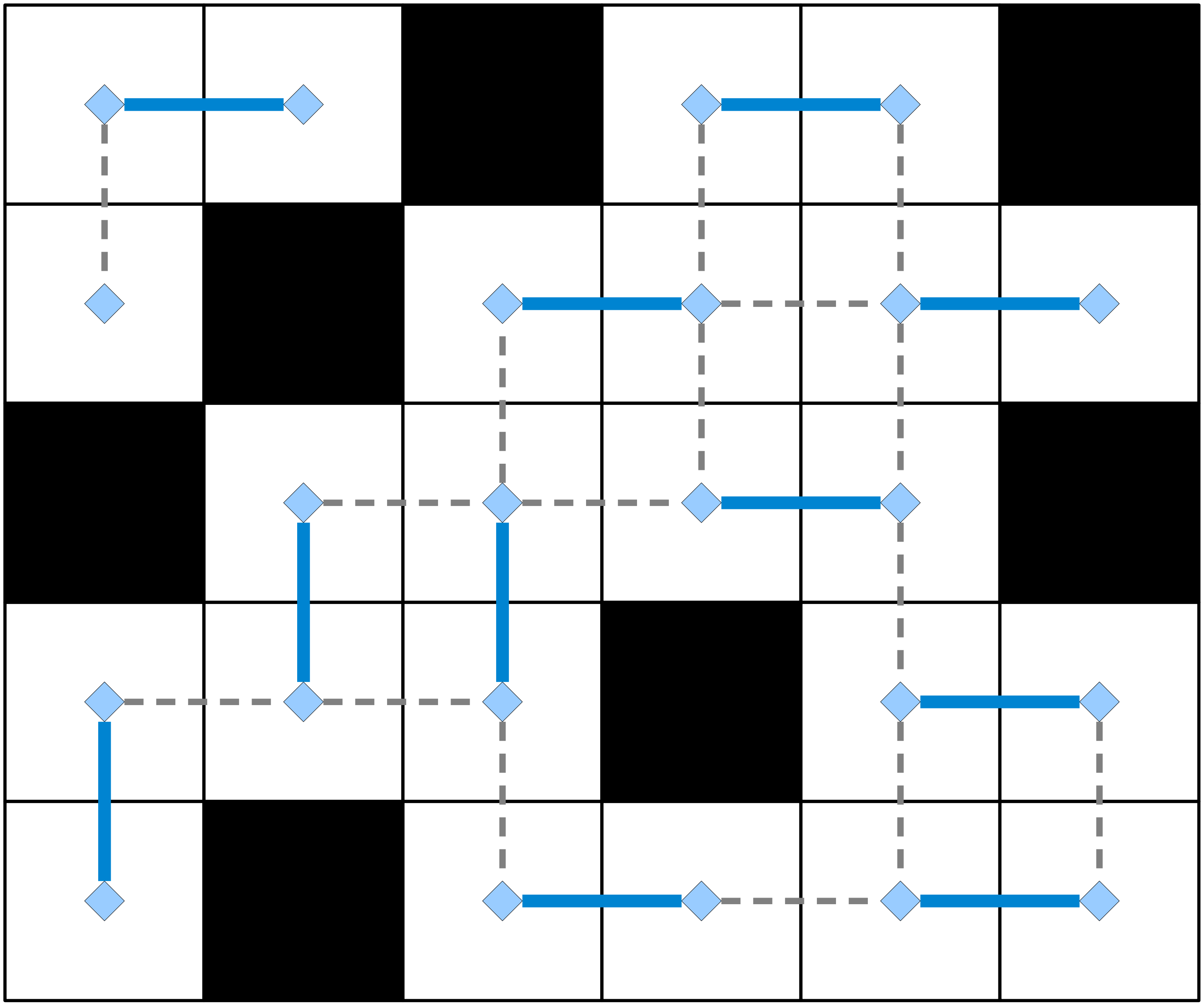

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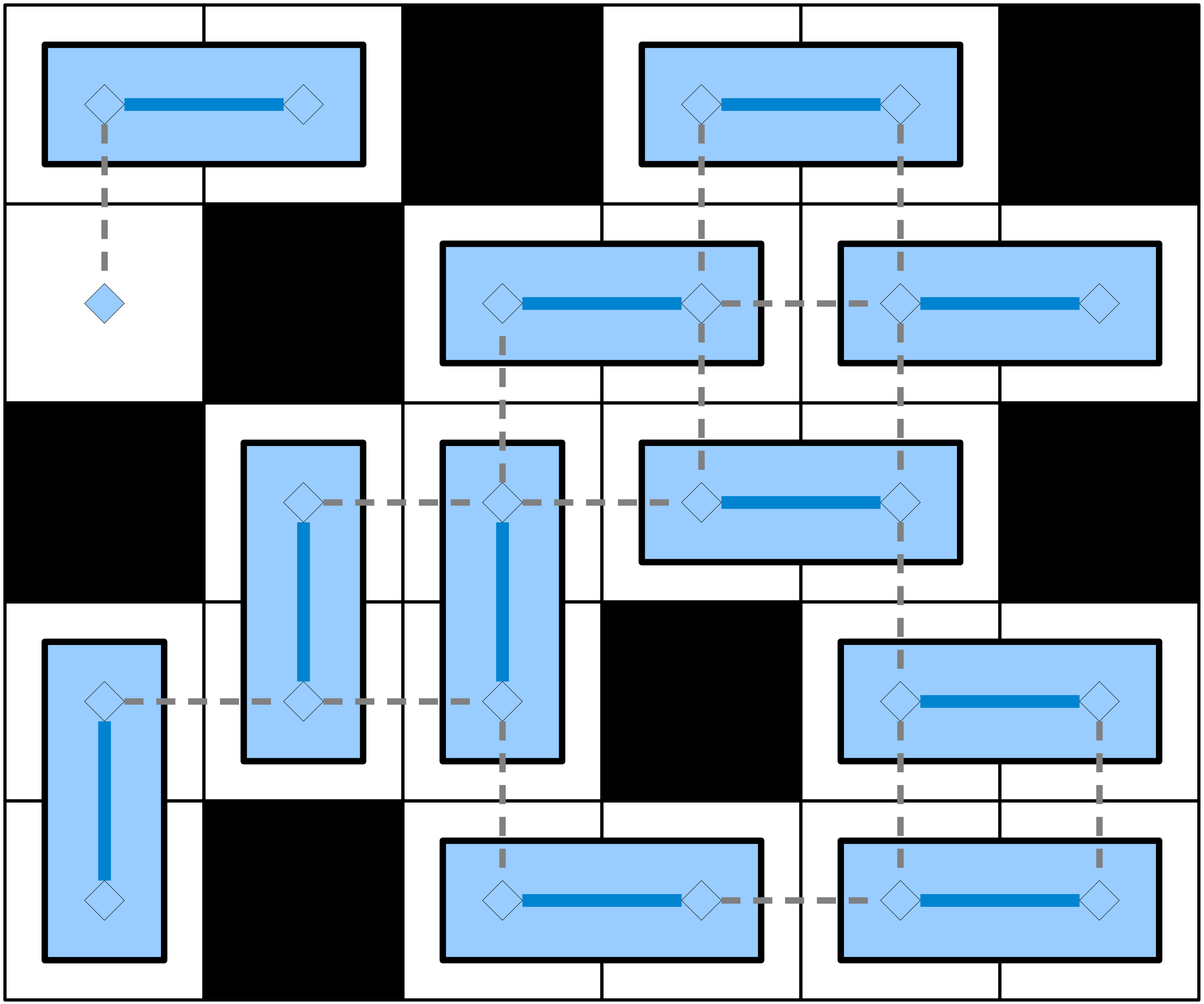

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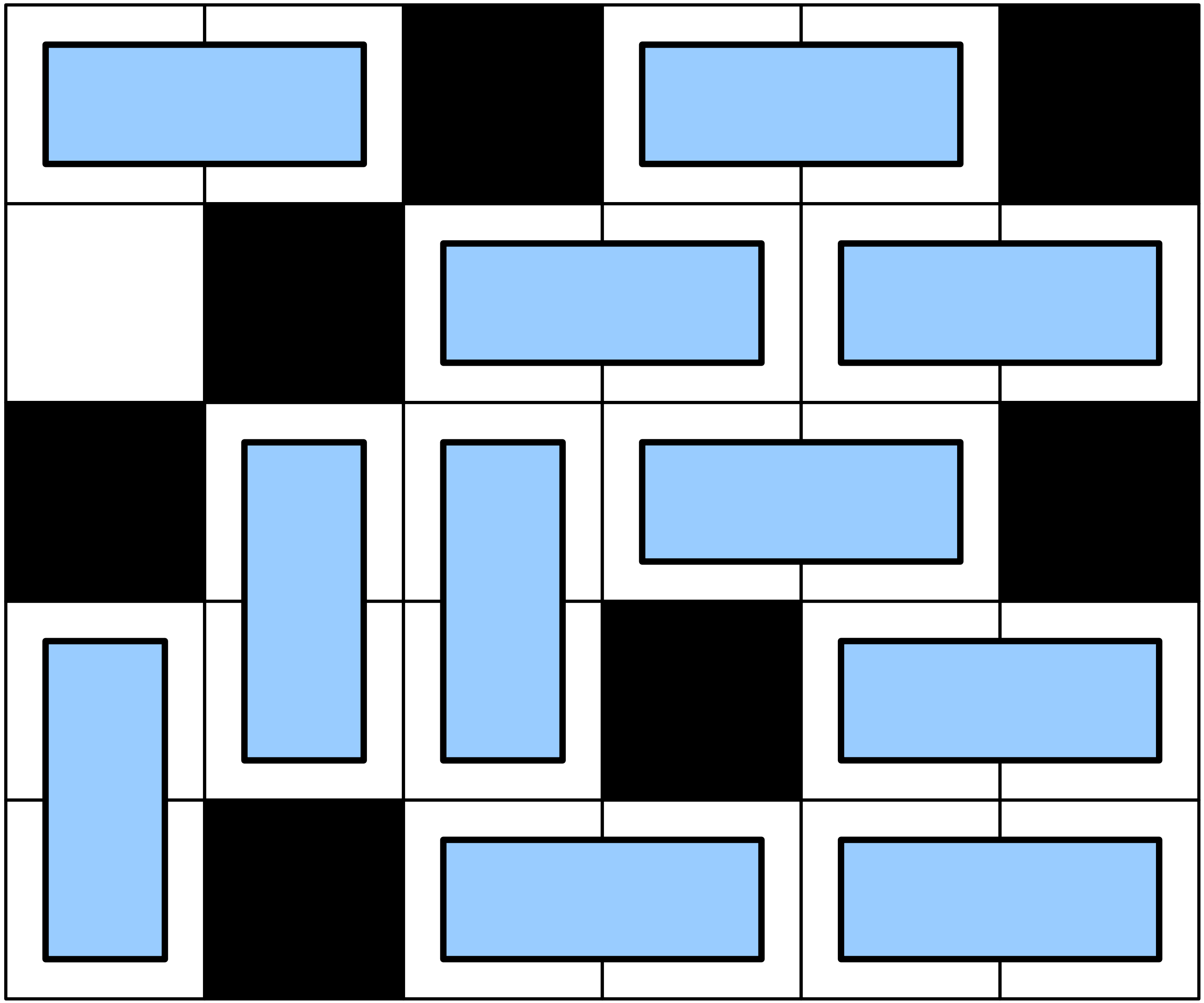

In Pseudocode

```
boolean canPlaceDominoes(Grid G, int k) {
    return hasMatching(gridToGraph(G), k);
}
```

Intuition:

Tiling a grid with dominoes can't be “harder” than solving maximum matching, because if we can solve maximum matching efficiently, we can solve domino tiling efficiently.

Another Example

Reachability

- Consider the following problem:

  Given an directed graph *G* and nodes *s* and *t* in *G*, is there a path from *s* to *t*?

- This problem can be solved in polynomial time (use DFS or BFS).

Converter Conundrums

- Suppose that you want to plug your laptop into a projector.
- Your laptop only has a VGA output, but the projector needs HDMI input.
- You have a box of connectors that convert various types of input into various types of output (for example, VGA to DVI, DVI to DisplayPort, etc.)
- ***Question:*** Can you plug your laptop into the projector?

Converter Conundrums

Connectors

RGB to USB
VGA to DisplayPort
DB13W3 to CATV
DisplayPort to RGB
DB13W3 to HDMI
DVI to DB13W3
S-Video to DVI
FireWire to SDI
VGA to RGB
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USB to S-Video
SDI to HDMI

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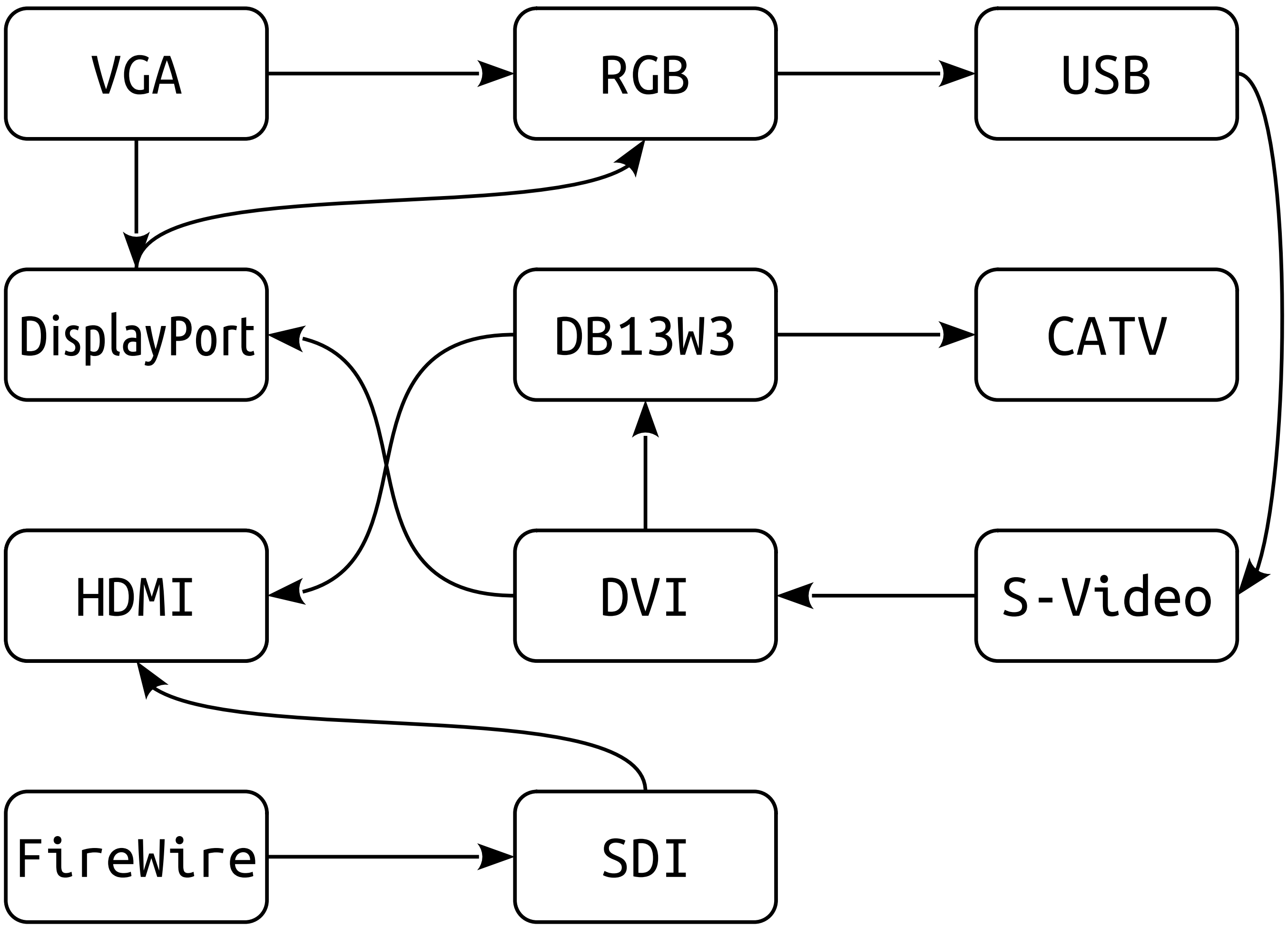

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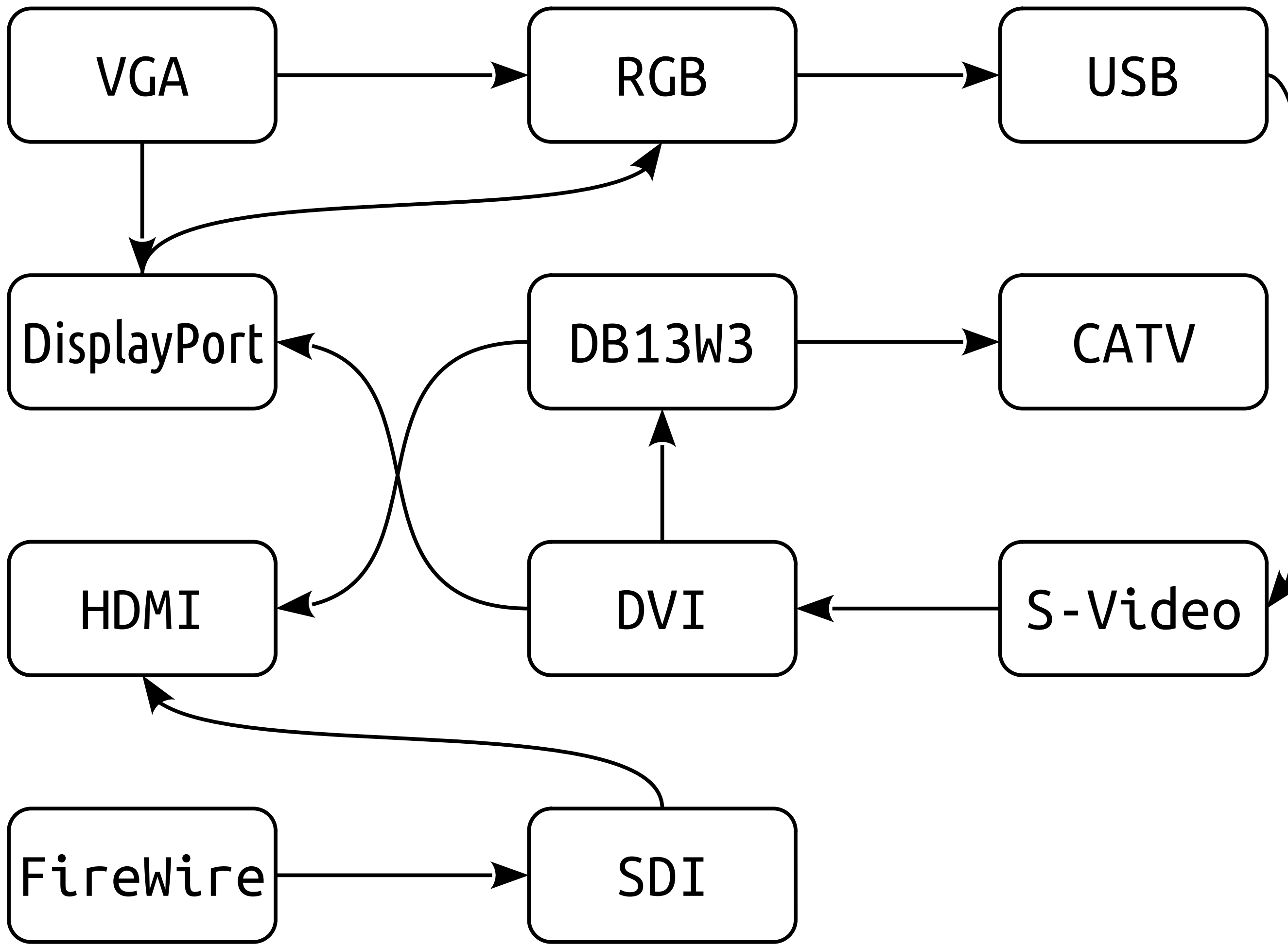

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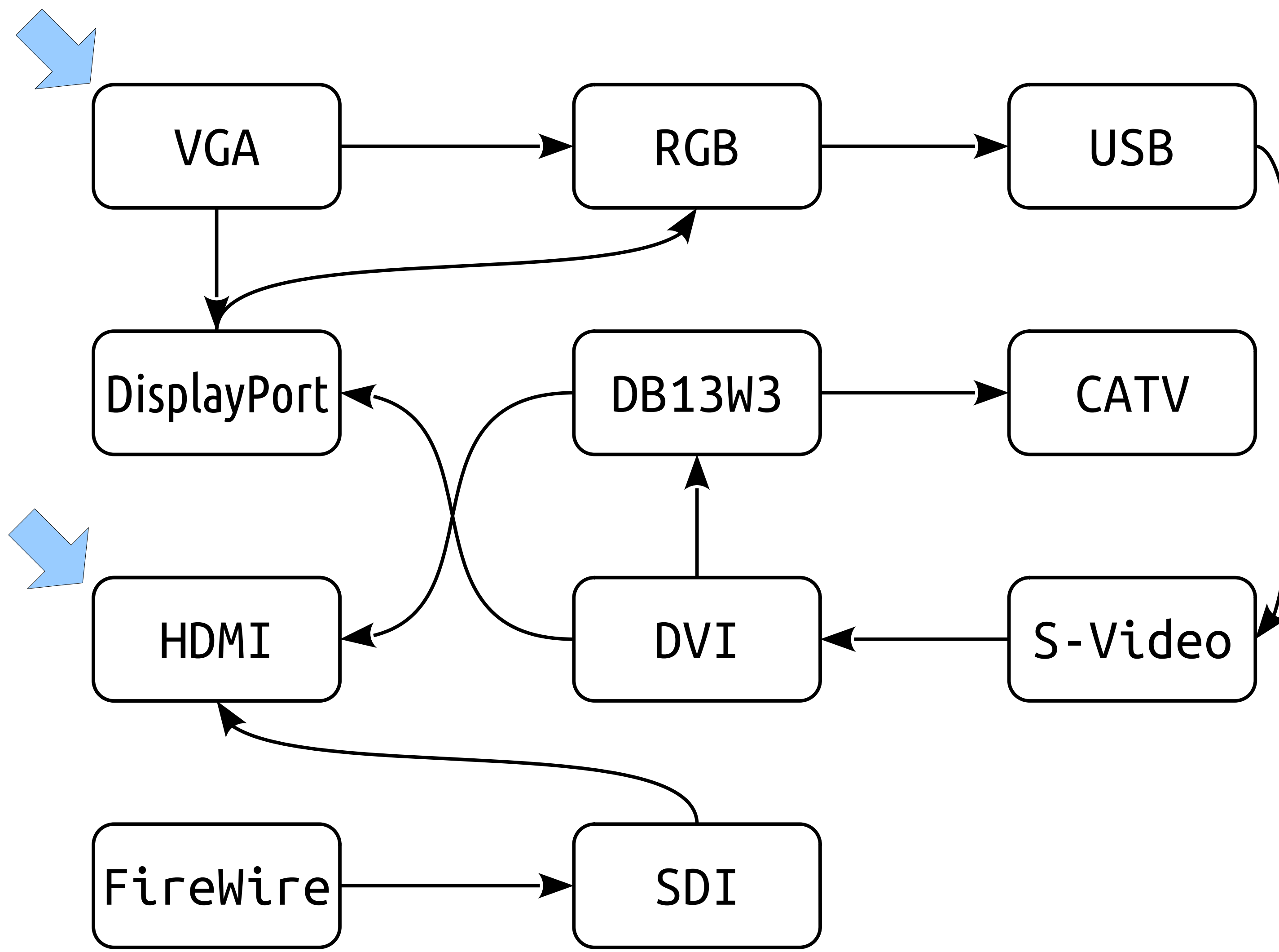

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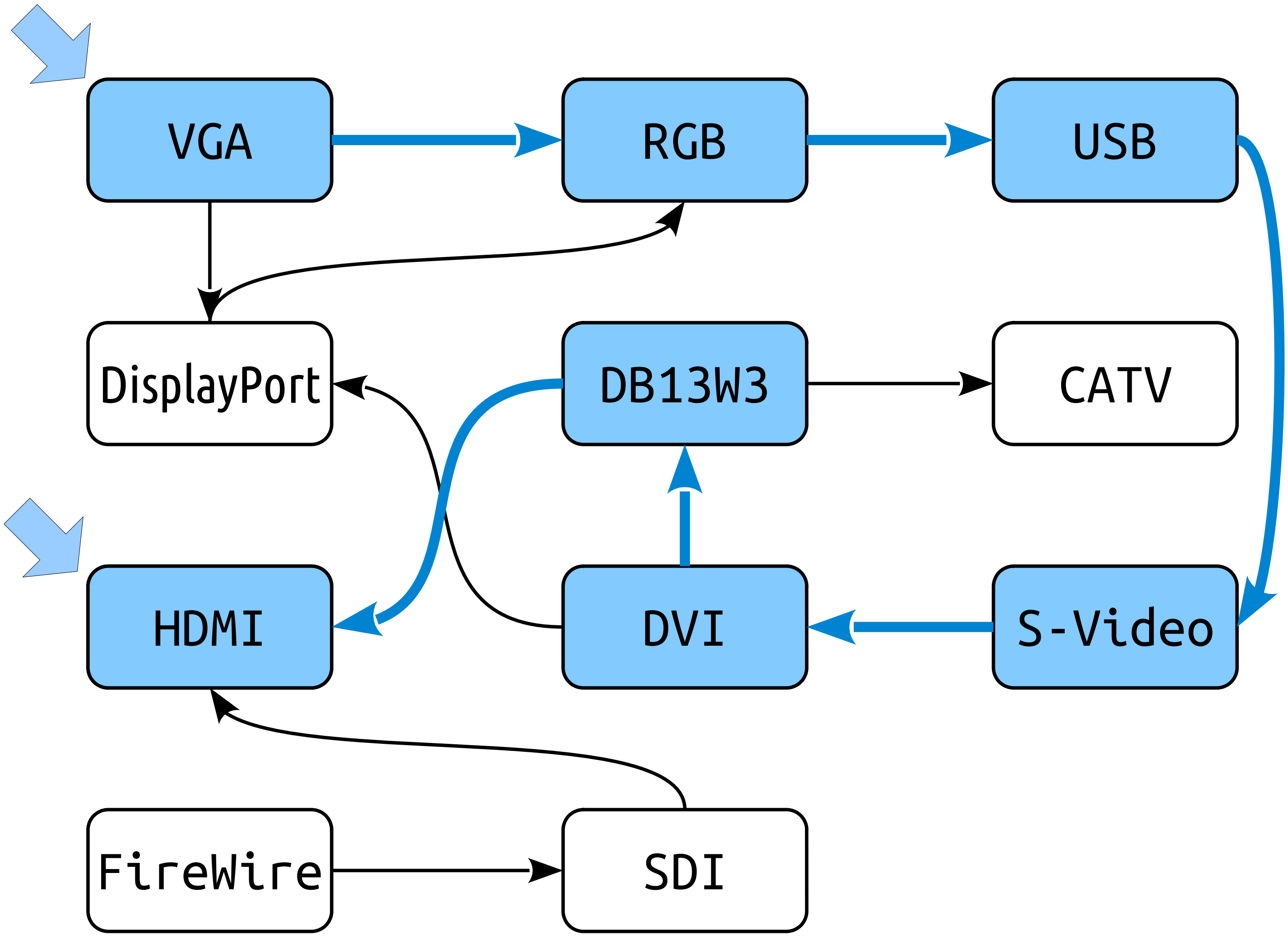

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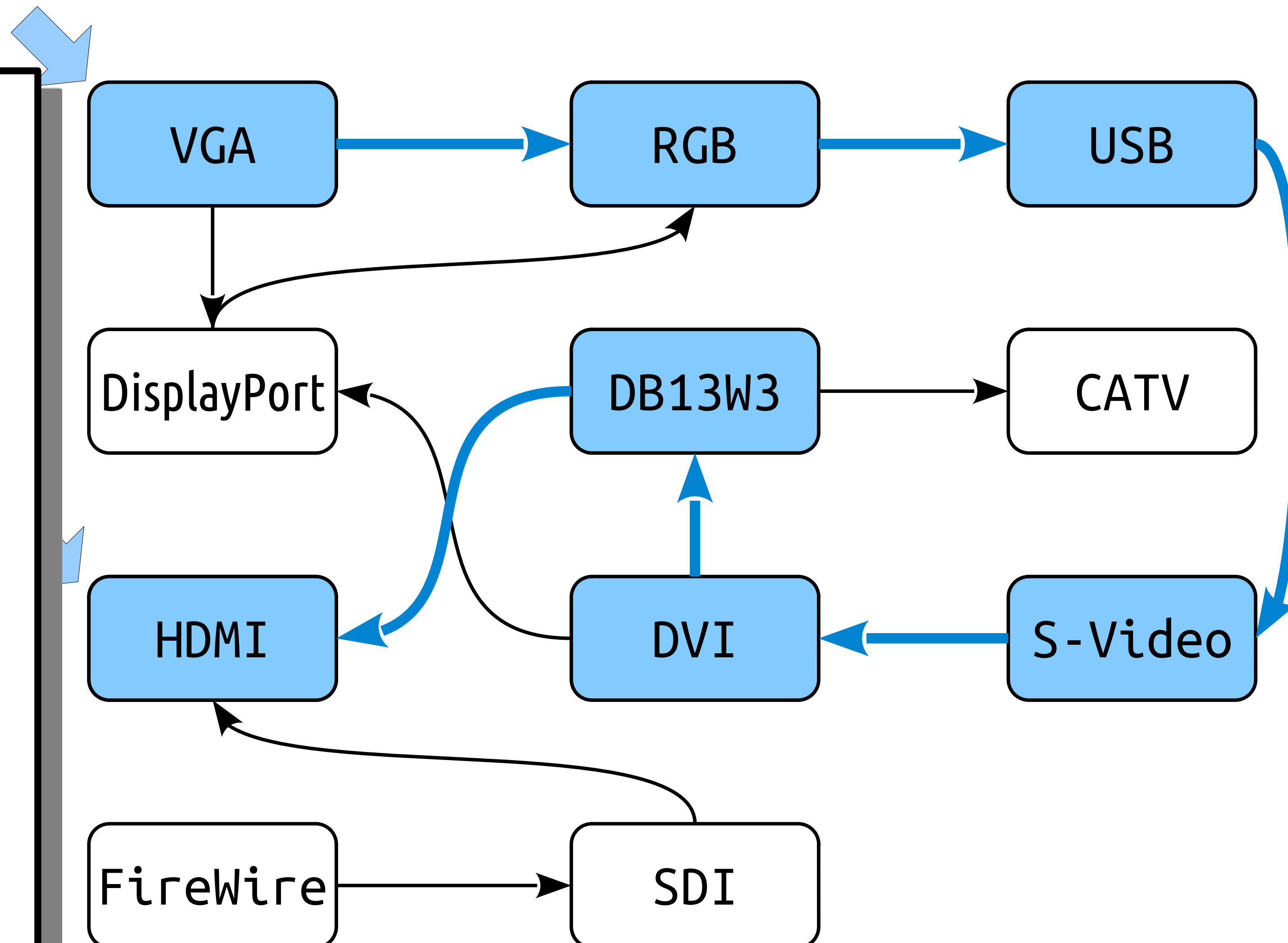


In Pseudocode

```
bool canPlugIn(vector<Plug> plugs) {
  return isReachable(plugsToGraph(plugs),
                     VGA, HDMI);
}
```

Intuition:

Finding a way to plug a computer into a projector can't be “harder” than determining reachability in a graph, since if we can determine reachability in a graph, we can find a way to plug a computer into a projector.

```
bool solveProblemA(string input) {
    return solveProblemB(translate(input));
}
```

Intuition:

Problem *A* can't be "harder" than problem *B*, because solving problem *B* lets us solve problem *A*.

```
bool solveProblemA(string input) {
    return solveProblemB(translate(input));
}
```

- If A and B are problems where it's possible to solve problem A using the strategy shown above*, we write

$$A \leq_p B.$$

- We say that ***A is polynomial-time reducible to B***.

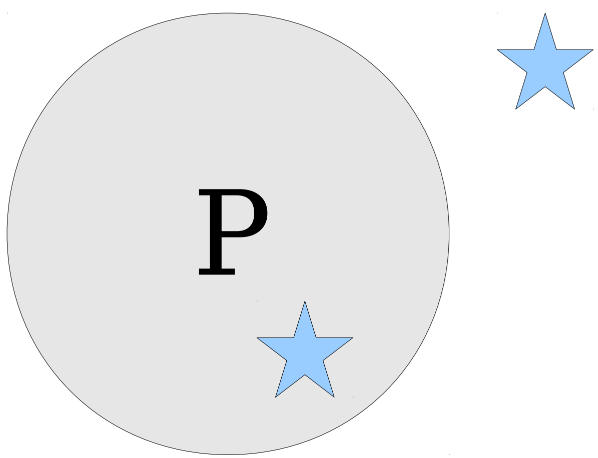
* Assuming that `translate` runs in polynomial time.

```
bool solveProblemA(string input) {
    return solveProblemB(translate(input));
}
```

- This is a powerful general problem-solving technique. You'll see it a lot in CS161.

Polynomial-Time Reductions

- If $A \leq_p B$ and $B \in \mathbf{P}$, then $A \in \mathbf{P}$.



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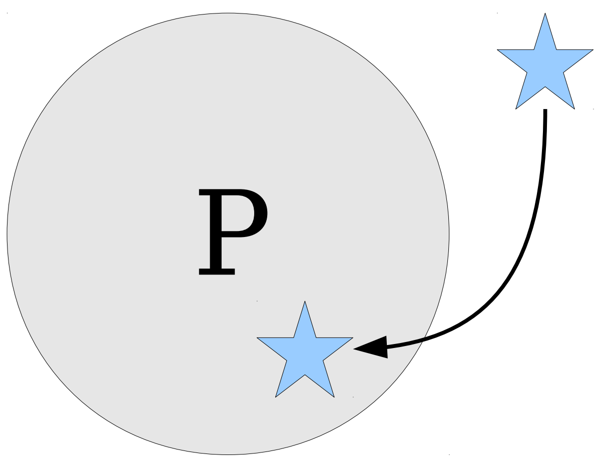

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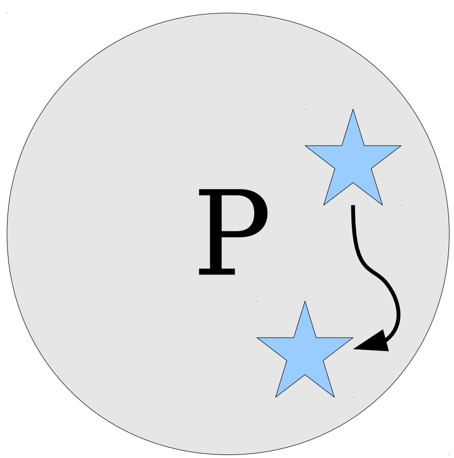
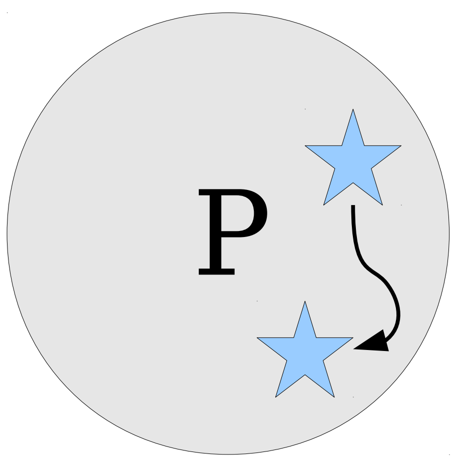


Polynomial-Time Reductions

- If $A \leq_p B$ and $B \in \mathbf{P}$, then $A \in \mathbf{P}$.
- If $A \leq_p B$ and $B \in \mathbf{NP}$, then $A \in \mathbf{NP}$.



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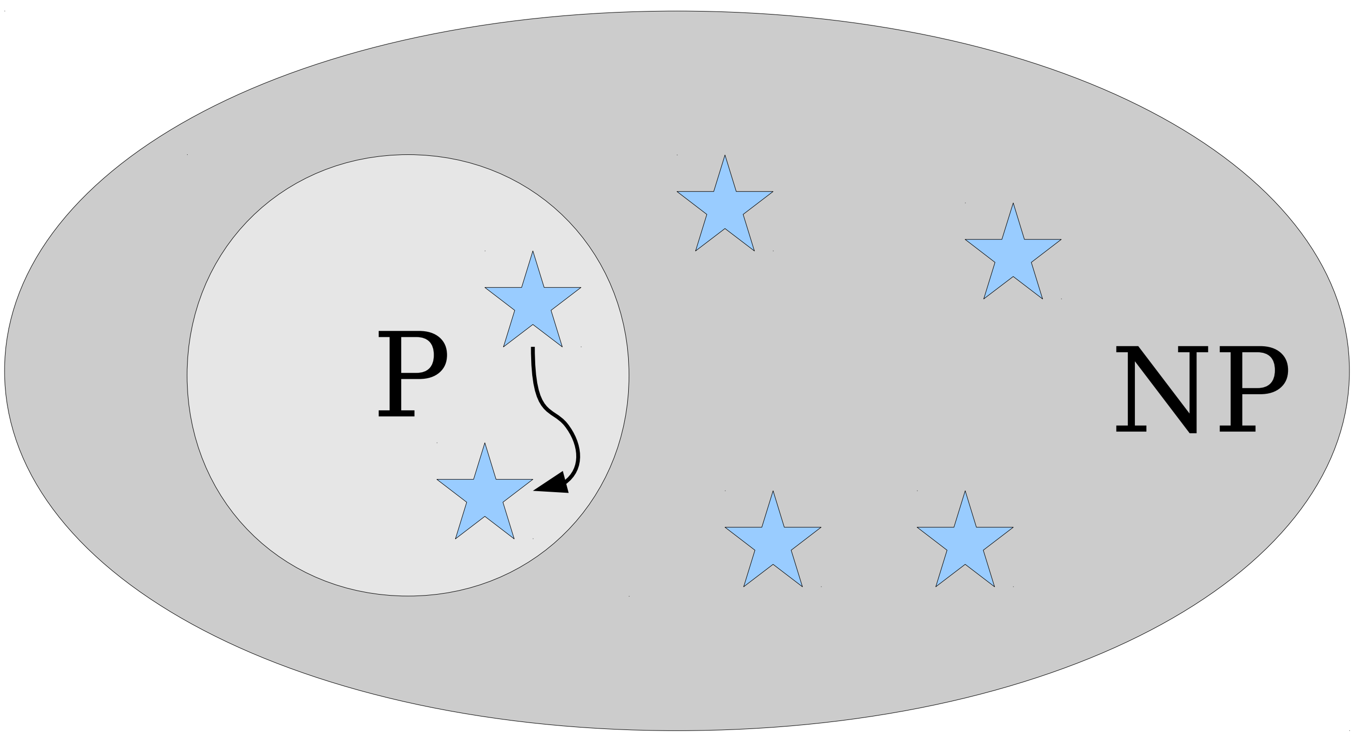

Polynomial-Time Reductions

- If $A \leq_p B$ and $B \in \mathbf{P}$, then $A \in \mathbf{P}$.
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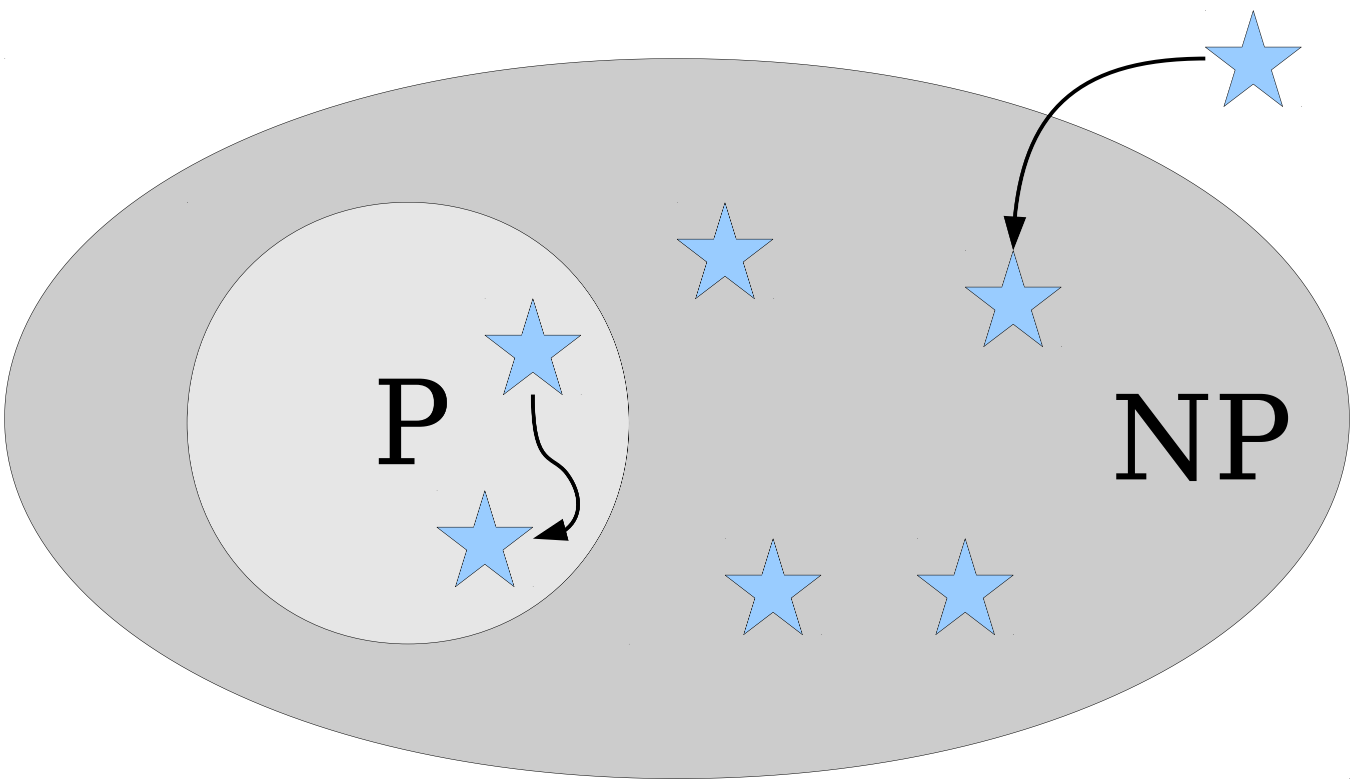

Polynomial-Time Reductions

- If $A \leq_p B$ and $B \in \mathbf{P}$, then $A \in \mathbf{P}$.
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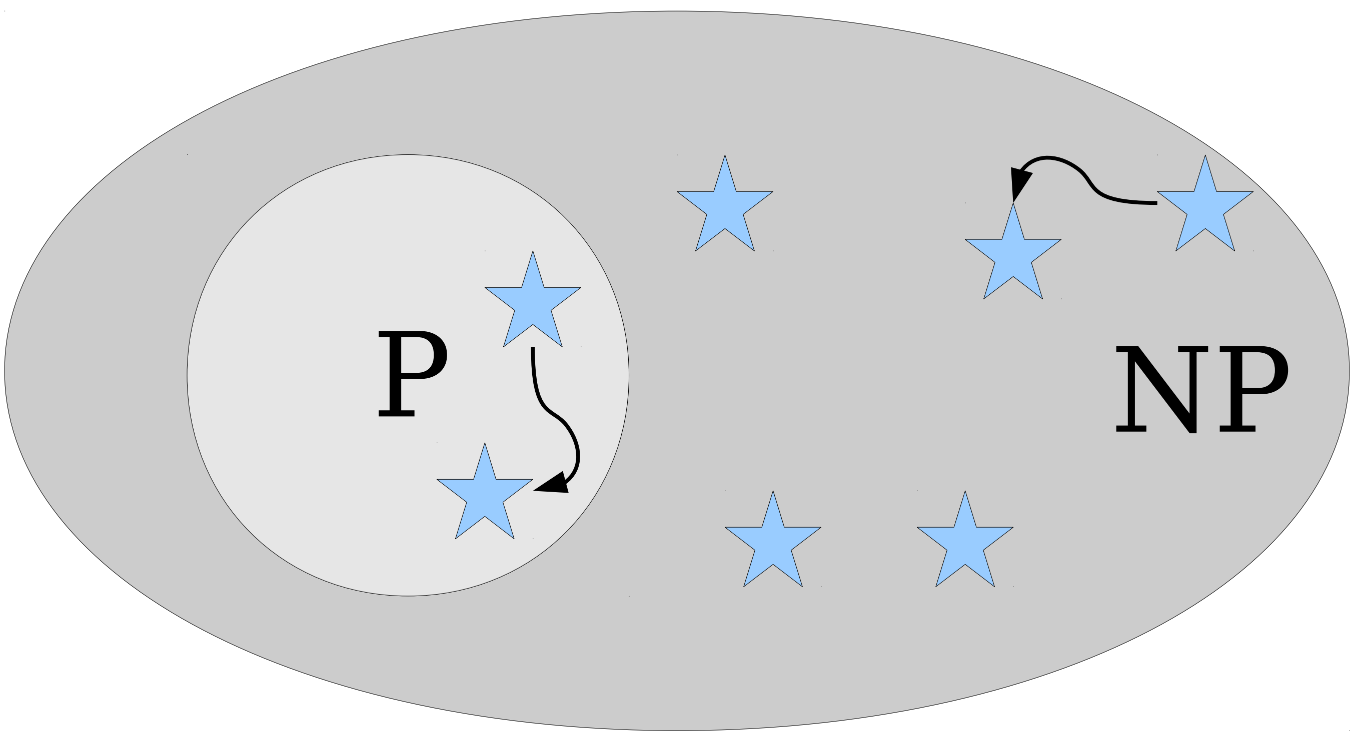


This $\leq_p$ relation lets us rank the relative difficulties of problems in **P** and **NP**.

What else can we do with it?

Time-Out for Announcements!

Please evaluate this course on Axess.

Your feedback makes a difference.

Final Exam Logistics

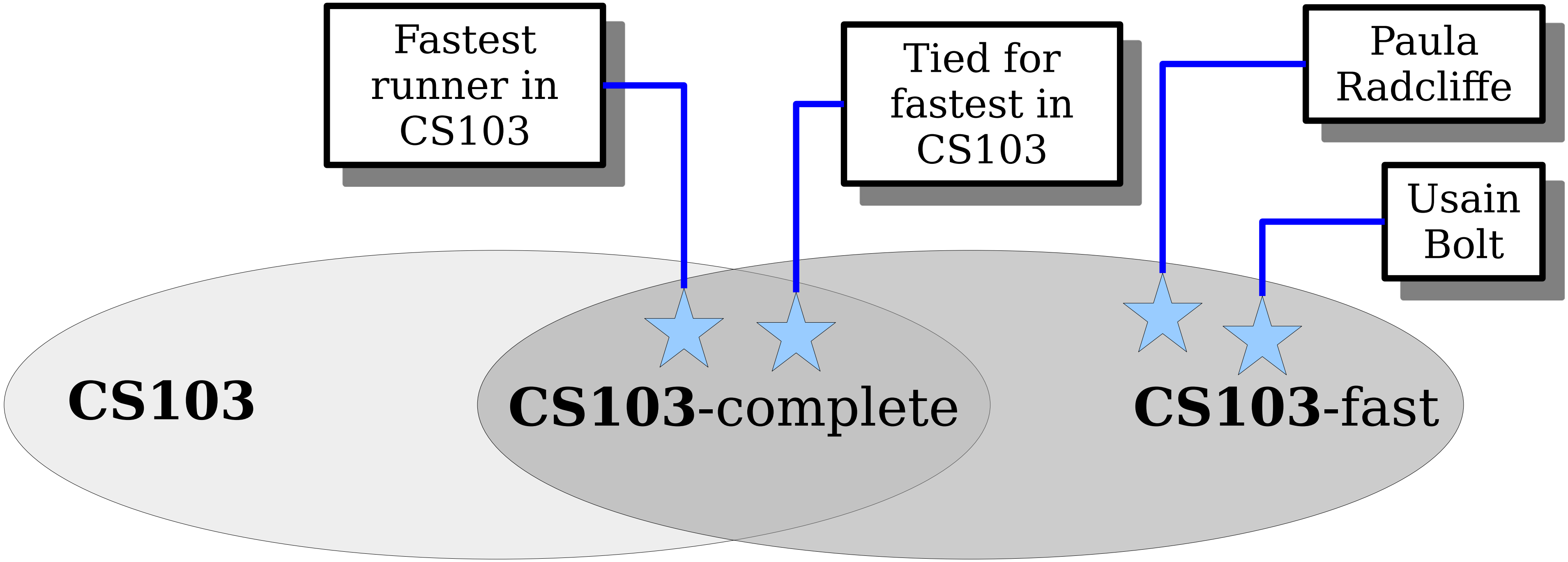
- Our final exam is a take-home exam that goes out this Friday at 2:30PM and comes due next Thursday (December 9th) at 3:30PM.
- Like the midterms, you can work on the exam for any amount of time in that period.
- Like the midterms, the exam is open-book and open-note, but you cannot communicate with other humans or solicit solutions.
- Unlike the midterms, this exam is designed to take about six hours to complete and covers all topics from the course (PS1 – PS9, plus L00 – L27).

Preparing for the Final

- We’ve posted a gigantic list of cumulative review problems on the course website that you can use to get more practice with whatever topics you’re interested in.
- Our recommendation: Look back over the exams and problem sets and redo any problems that you didn’t really get the first time around.
- Keep the TAs in the loop: stop by office hours to have them review your answers and offer feedback.

Back to CS103!

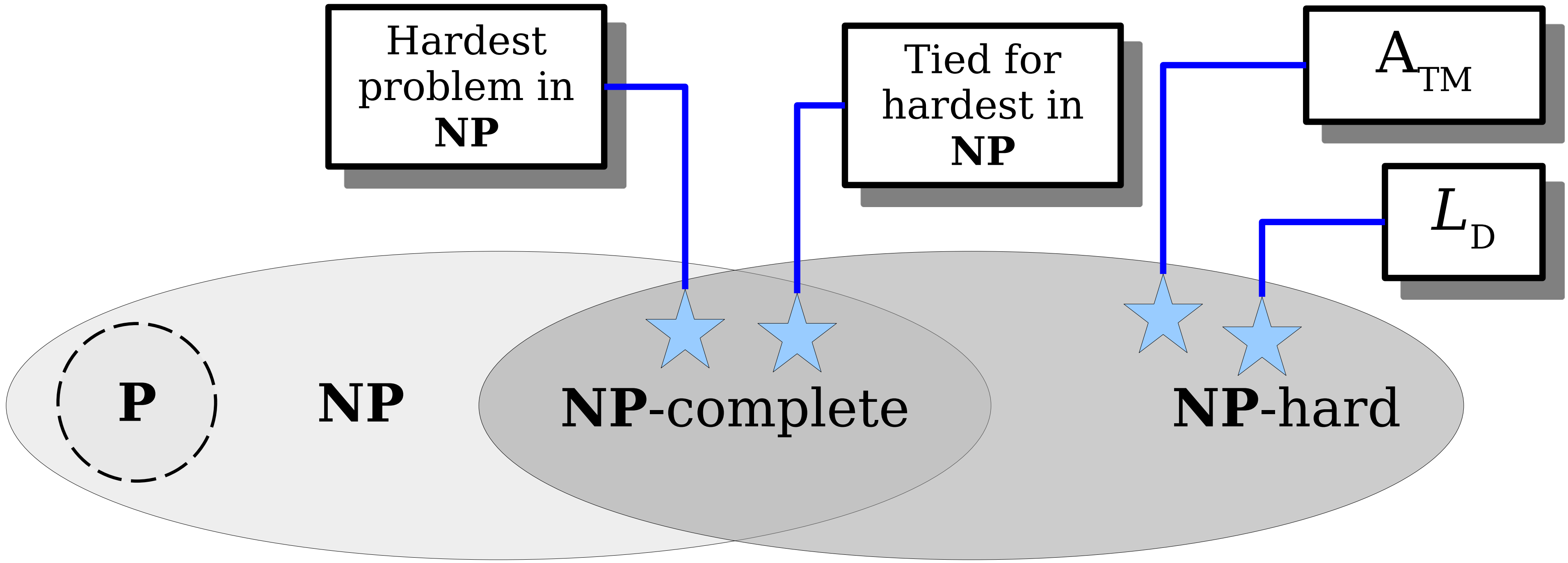
An Analogy: Running Really Fast



For people A and B, we say $\boldsymbol{A \leq_r B}$ if
A's top running speed is at most B's top speed.
(Intuitively: B can run at least as fast as A.)

We say that person P is ***CS103-fast*** if
$\forall A \in \textbf{CS103}.\ A \leq_r P.$
(How fast are you if you're CS103-fast?)

We say that person P is ***CS103-complete*** if
$P \in$ **CS103** and P is **CS103**-fast.
(How fast are you if you're CS103-complete?)



For languages A and B, we say $\boldsymbol{A \leq_p B}$ if A reduces to B in polynomial time.

(Intuitively: B is at least as hard as A.)

We say that a language L is ***NP-hard*** if

$\forall A \in \mathbf{NP}.\ A \leq_p L.$

(How hard is a problem that's NP-hard?)

We say that a language L is ***NP-complete*** if

$L \in \mathbf{NP}$ and L is $\mathbf{NP}$-hard.

(How hard is a problem that's NP-complete?)

Intuition: The **NP**-complete problems are the hardest problems in **NP**.

If we can determine how hard those problems are, it would tell us a lot about the $\mathbf{P} \stackrel{?}{=} \mathbf{NP}$ question.

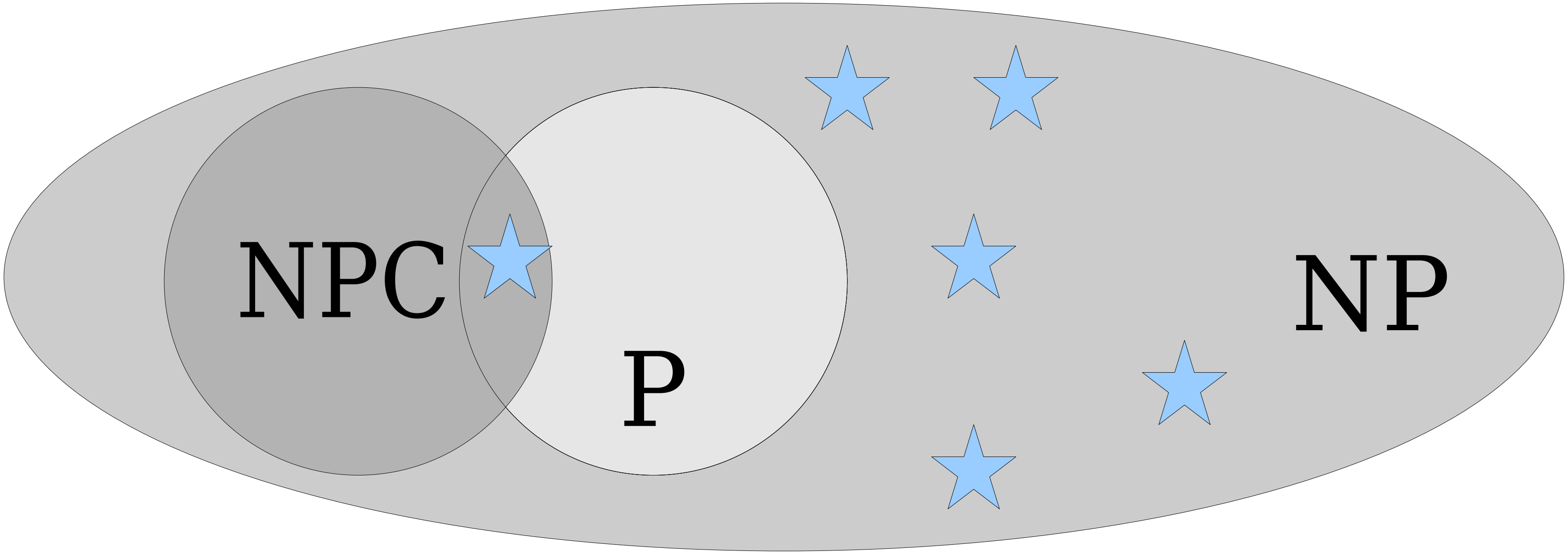
The Tantalizing Truth

Theorem: If *any* **NP**-complete language is in **P**, then **P** = **NP**.

Intuition: This means the hardest problems in **NP** aren't actually that hard. We can solve them in polynomial time. So that means we can solve all problems in **NP** in polynomial time.

The Tantalizing Truth

Theorem: If *any* **NP**-complete language is in **P**, then **P = NP**.



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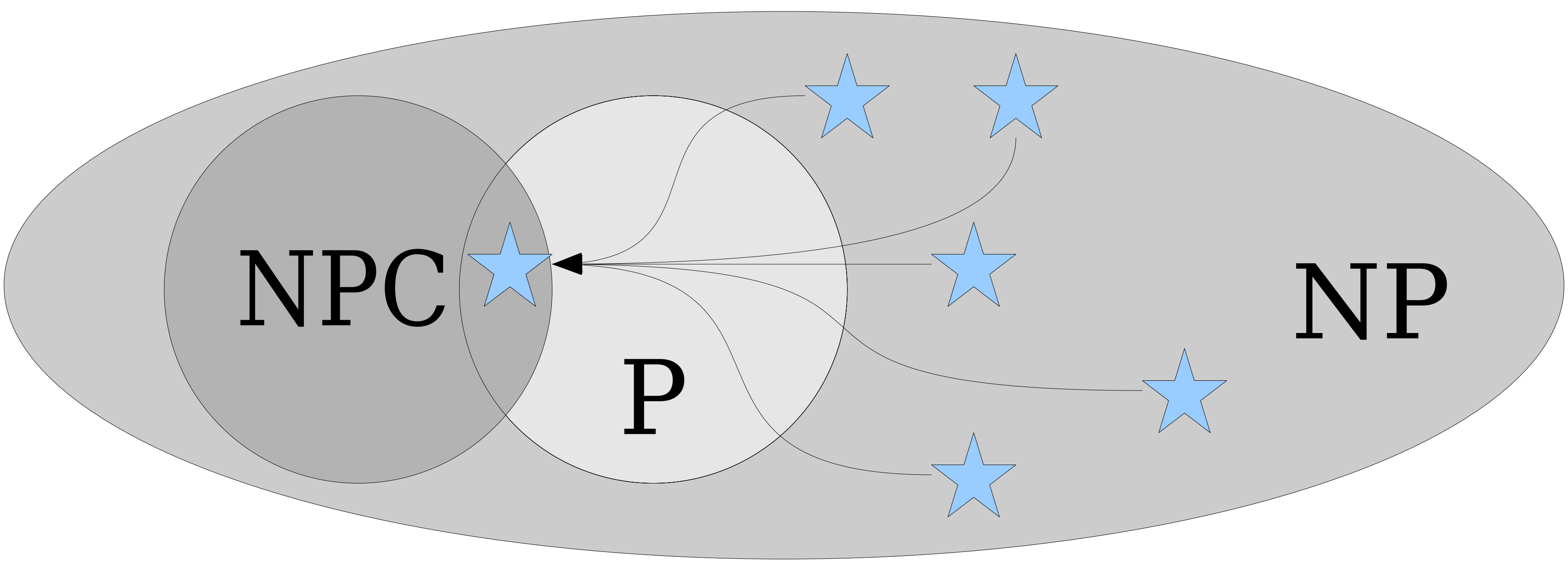

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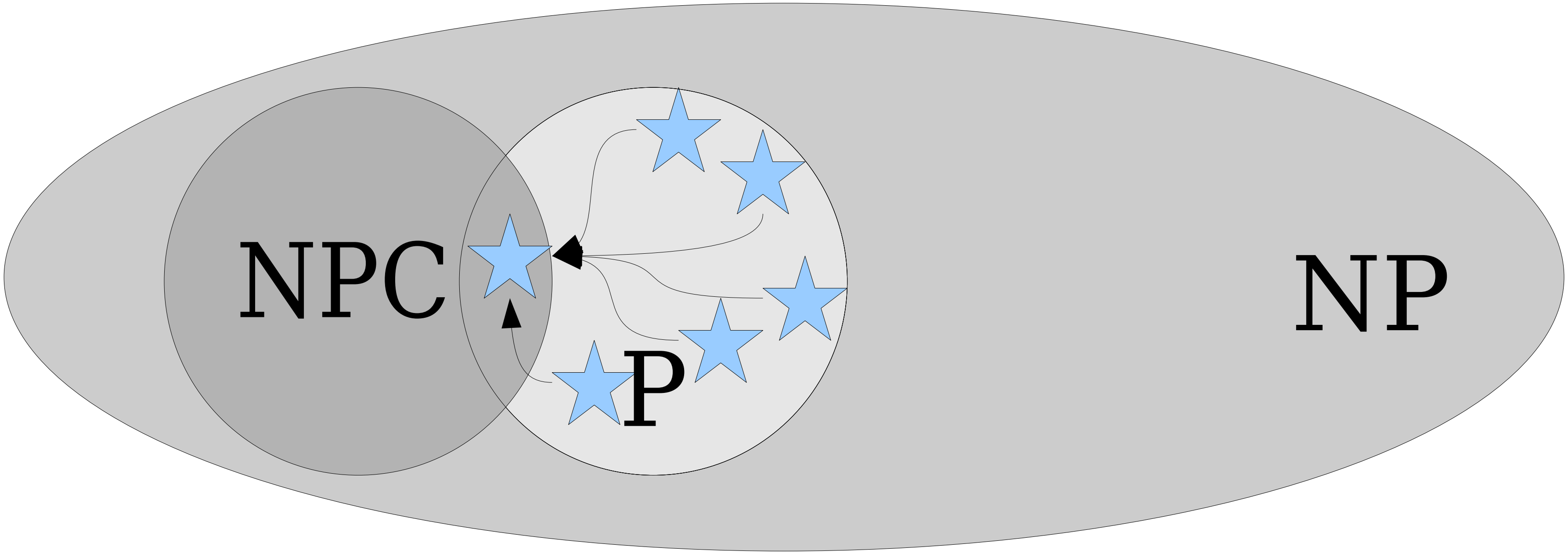

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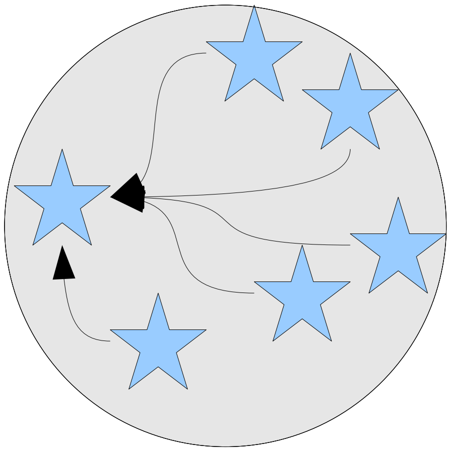
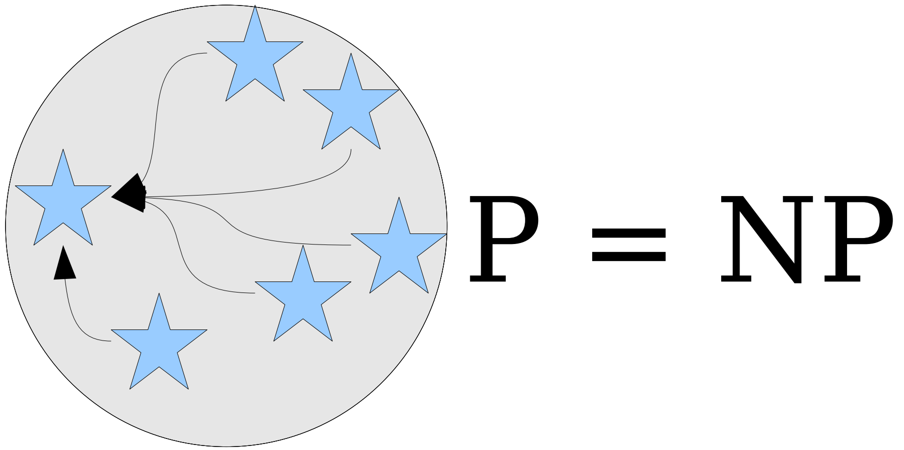


P = NP

The Tantalizing Truth

Theorem: If *any* **NP**-complete language is in **P**, then **P** = **NP**.

Proof: Suppose that L is **NP**-complete and $L \in$ **P**. Now consider any arbitrary **NP** problem A. Since L is **NP**-complete, we know that $A \leq_p L$. Since $L \in$ **P** and $A \leq_p L$, we see that $A \in$ **P**. Since our choice of A was arbitrary, this means that **NP** ⊆ **P**, so **P** = **NP**. ■



The Tantalizing Truth

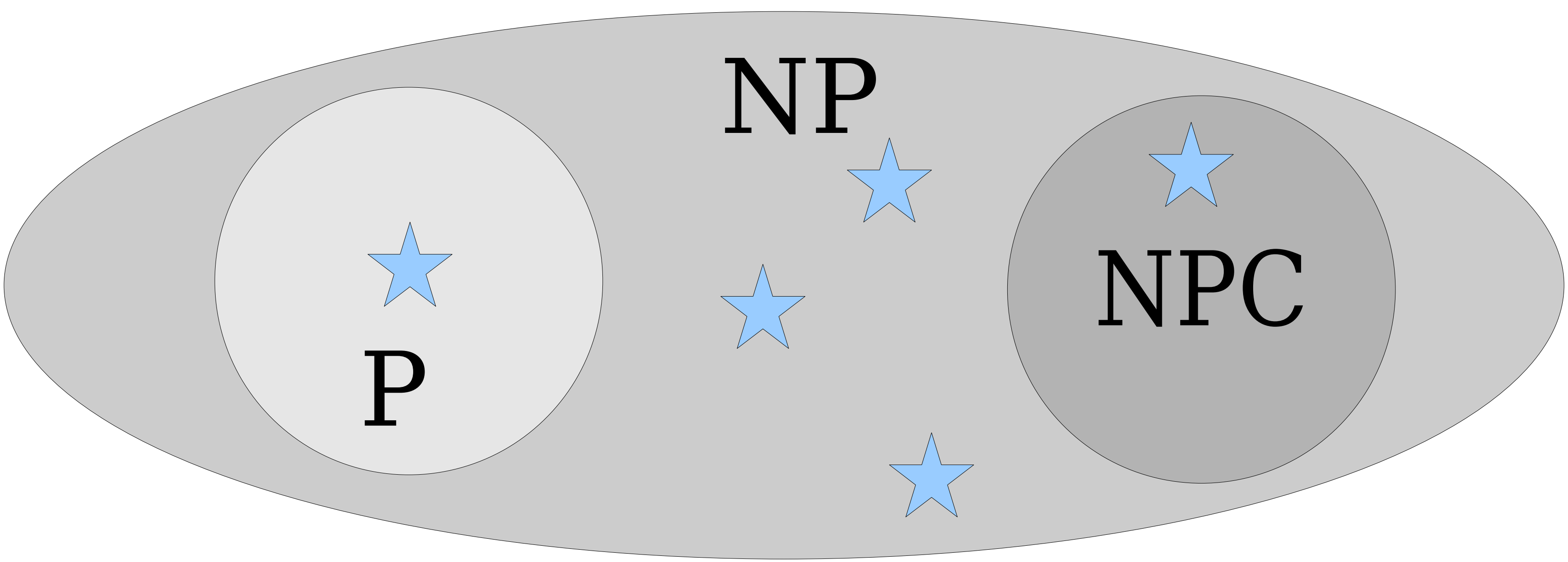
Theorem: If *any* **NP**-complete language is not in **P**, then **P** ≠ **NP**.

Intuition: This means the hardest problems in **NP** are so hard that they can't be solved in polynomial time. So the hardest problems in **NP** aren't in **P**, meaning **P** ≠ **NP**.

The Tantalizing Truth

Theorem: If *any* **NP**-complete language is not in **P**, then **P** ≠ **NP**.

Proof: Suppose that L is an **NP**-complete language not in **P**. Since L is **NP**-complete, we know that $L \in$ **NP**. Therefore, we know that $L \in$ **NP** and $L \notin$ **P**, so **P** ≠ **NP**. ■



How do we even know NP-complete problems exist in the first place?

Satisfiability

- A propositional logic formula φ is called ***satisfiable*** if there is some assignment to its variables that makes it evaluate to true.
  - $p \land q$ is satisfiable.
  - $p \land \neg p$ is unsatisfiable.
  - $p \to (q \land \neg q)$ is satisfiable.
- An assignment of true and false to the variables of φ that makes it evaluate to true is called a ***satisfying assignment***.

SAT

- The ***boolean satisfiability problem*** (***SAT***) is the following:

Given a propositional logic formula φ, is φ satisfiable?

- Formally:

$$\textbf{\textit{SAT}} = \{ \langle\varphi\rangle \mid \varphi \text{ is a satisfiable PL formula} \}$$

Theorem (Cook-Levin): SAT is **NP**-complete.

Proof Idea: To see that **SAT** ∈ **NP**, show how to make a polynomial-time verifier for it. Key idea: have the certificate be a satisfying assignment.

To show that **SAT** is **NP**-hard, given a polymomial-time verifier V for an arbitrary **NP** language L, for any string w you can construct a polynomially-sized formula $\varphi(w)$ that says "there is a certificate c where V accepts $\langle w, c\rangle$." This formula is satisfiable if and only if $w \in L$, so deciding whether the formula is satisfiable decides whether w is in L. ■-*ish*

Proof: Take CS154!

Why All This Matters

- Resolving $\mathbf{P} \stackrel{?}{=} \mathbf{NP}$ is equivalent to just figuring out how hard SAT is.

$$\text{SAT} \in \mathbf{P} \quad \leftrightarrow \quad \mathbf{P} = \mathbf{NP}$$

- We've turned a huge, abstract, theoretical problem about solving problems versus checking solutions into the concrete task of seeing how hard one problem is.
- You can get a sense for how little we know about algorithms and computation given that we can't yet answer this question!

Why All This Matters

- You will almost certainly encounter **NP**-hard problems in practice – they're everywhere!
- If a problem is **NP**-hard, then there is no known algorithm for that problem that
  - is efficient on all inputs,
  - always gives back the right answer, and
  - runs deterministically.
- ***Useful intuition:*** If you need to solve an **NP**-hard problem, you will either need to settle for an approximate answer, an answer that's likely but not necessarily right, or have to work on really small inputs.

Sample **NP**-Hard Problems

- ***Computational biology:*** Given a set of genomes, what is the most probable evolutionary tree that would give rise to those genomes? *(Maximum parsimony problem)*
- ***Game theory:*** Given an arbitrary perfect-information, finite, two-player game, who wins? *(Generalized geography problem)*
- ***Operations research:*** Given a set of jobs and workers who can perform those tasks in parallel, can you complete all the jobs within some time bound? *(Job scheduling problem)*
- ***Machine learning:*** Given a set of data, find the simplest way of modeling the statistical patterns in that data *(Bayesian network inference problem)*
- ***Medicine:*** Given a group of people who need kidneys and a group of kidney donors, find the maximum number of people who can receive transplants. *(Cycle cover problem)*
- ***Systems:*** Given a set of processes and a number of processors, find the optimal way to assign those tasks so that they complete as soon as possible *(Processor scheduling problem)*

Coda: What if $\mathbf{P} \stackrel{?}{=} \mathbf{NP}$ is resolved?

Next Time

- ***Why All This Matters***
- ***Where to Go from Here***
- ***A Final "Your Questions"***
- ***Parting Words!***