Complexity Theory
Part Two
Recap from Last Time
The Complexity Class $\mathbf{P}$

- The complexity class $\mathbf{P}$ (polynomial time) is defined as
  \[ \mathbf{P} = \{ L \mid \text{There is a polynomial-time decider for } L \} \]

- Intuitively, $\mathbf{P}$ contains all decision problems that can be solved efficiently.

- This is like class $\mathbf{R}$, except with “efficiently” tacked onto the end.
The Complexity Class $\textbf{NP}$

- The complexity class $\textbf{NP}$ (nondeterministic polynomial time) contains all problems that can be verified in polynomial time.

- Formally:
  \[
  \textbf{NP} = \{ L \mid \text{There is a polynomial-time verifier for } L \}
  \]

- Intuitively, $\textbf{NP}$ is the set of problems where “yes” answers can be checked efficiently.

- This is like the class $\textbf{RE}$, but with “efficiently” tacked on to the definition.
The Biggest Unsolved Problem in Theoretical Computer Science:

\[ P \overset{?}{=} NP \]
**Theorem (Baker-Gill-Solovay):** Any proof that purely relies on universality and self-reference cannot resolve $P \neq NP$.

**Proof:** Take CS154!
So how *are* we going to reason about $\mathbf{P}$ and $\mathbf{NP}$?
New Stuff!
A Challenge
Problems in **NP** vary widely in their difficulty, even if **P = NP**.

How can we rank the relative difficulties of problems?
Reducibility
Maximum Matching

- Given an undirected graph $G$, a **matching** in $G$ is a set of edges such that no two edges share an endpoint.

- A **maximum matching** is a matching with the largest number of edges.
Maximum Matching

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Maximum Matching

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A matching, but not a maximum matching.
Maximum Matching

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Maximum Matching

- Jack Edmonds' paper “Paths, Trees, and Flowers” gives a polynomial-time algorithm for finding maximum matchings.
  - He’s the guy from last time with the quote about “better than decidable.”
- Using this fact, what other problems can we solve?
Domino Tiling
Domino Tiling
Domino Tiling
Domino Tiling
Solving Domino Tiling
Solving Domino Tiling
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Solving Domino Tiling
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In Pseudocode

```java
boolean canPlaceDominoes(Grid G, int k) {
    return hasMatching(gridToGraph(G), k);
}
```
**Intuition:**

Tiling a grid with dominoes can't be "harder" than solving maximum matching, because if we can solve maximum matching efficiently, we can solve domino tiling efficiently.
Another Example
Reachability

• Consider the following problem:

  Given an directed graph $G$ and nodes $s$ and $t$ in $G$, is there a path from $s$ to $t$?

• This problem can be solved in polynomial time (use DFS or BFS).
Converter Conundrums

- Suppose that you want to plug your laptop into a projector.
- Your laptop only has a VGA output, but the projector needs HDMI input.
- You have a box of connectors that convert various types of input into various types of output (for example, VGA to DVI, DVI to DisplayPort, etc.)
- **Question:** Can you plug your laptop into the projector?
Converter Conundrums

Connectors
- RGB to USB
- VGA to DisplayPort
- DB13W3 to CATV
- DisplayPort to RGB
- DB13W3 to HDMI
- DVI to DB13W3
- S-Video to DVI
- FireWire to SDI
- VGA to RGB
- DVI to DisplayPort
- USB to S-Video
- SDI to HDMI
Converter Conundrums

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Converter Conundrums

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USB to S-Video
SDI to HDMI
In Pseudocode

```cpp
bool canPlugIn(vector<Plug> plugs) {
    return isReachable(plugsToGraph(plugs), VGA, HDMI);
}
```
Intuition:

Finding a way to plug a computer into a projector can't be “harder” than determining reachability in a graph, since if we can determine reachability in a graph, we can find a way to plug a computer into a projector.
bool solveProblemA(string input) {
    return solveProblemB(translate(input));
}

Intuition:

Problem A can't be "harder" than problem B, because solving problem B lets us solve problem A.
bool solveProblemA(string input) {
    return solveProblemB(translate(input));
}

- If $A$ and $B$ are problems where it's possible to solve problem $A$ using the strategy shown above*, we write
  
  $A \leq_p B$.

- We say that $A$ is polynomial-time reducible to $B$.

* Assuming that translate runs in polynomial time.
```cpp
bool solveProblemA(string input) {
    return solveProblemB(translate(input));
}
```

- This is a powerful general problem-solving technique. You’ll see it a lot in CS161.
Polynomial-Time Reductions

- If $A \leq_p B$ and $B \in \textbf{P}$, then $A \in \textbf{P}$. 
Polynomial-Time Reductions

- If \( A \leq_p B \) and \( B \in \text{P} \), then \( A \in \text{P} \).
Polynomial-Time Reductions

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Polynomial-Time Reductions

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Polynomial-Time Reductions

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- If $A \leq_p B$ and $B \in \mathbf{NP}$, then $A \in \mathbf{NP}$. 

\[ \mathbf{P} \subseteq \mathbf{NP} \]
This \( \leq_p \) relation lets us rank the relative difficulties of problems in \( \textbf{P} \) and \( \textbf{NP} \).

What else can we do with it?
Time-Out for Announcements!
Please evaluate this course on Axess.

Your feedback makes a difference.
Final Exam Logistics

- Our final exam is a take-home exam that goes out this Friday at 2:30PM and comes due next Thursday (December 9th) at 3:30PM.
- Like the midterms, you can work on the exam for any amount of time in that period.
- Like the midterms, the exam is open-book and open-note, but you cannot communicate with other humans or solicit solutions.
- Unlike the midterms, this exam is designed to take about six hours to complete and covers all topics from the course (PS1 – PS9, plus L00 – L27).
Preparing for the Final

• We’ve posted a gigantic list of cumulative review problems on the course website that you can use to get more practice with whatever topics you’re interested in.

• Our recommendation: Look back over the exams and problem sets and redo any problems that you didn’t really get the first time around.

• Keep the TAs in the loop: stop by office hours to have them review your answers and offer feedback.
Back to CS103!
An Analogy: Running Really Fast
For people $A$ and $B$, we say $A \leq_r B$ if $A$’s top running speed is at most $B$’s top speed.

(Intuitively: $B$ can run at least as fast as $A$.)

We say that person $P$ is **CS103-fast** if 
\[ \forall A \in \text{CS103}. \ A \leq_r P. \]

(How fast are you if you’re CS103-fast?)

We say that person $P$ is **CS103-complete** if 
\[ P \in \text{CS103} \text{ and } P \text{ is CS103-fast.} \]

(How fast are you if you’re CS103-complete?)
For languages $A$ and $B$, we say $A \leq_p B$ if $A$ reduces to $B$ in polynomial time.  
(Intuitively: $B$ is at least as hard as $A$.)

We say that a language $L$ is $\text{NP-hard}$ if
\[ \forall A \in \text{NP}. A \leq_p L. \]
(How hard is a problem that’s NP-hard?)

We say that a language $L$ is $\text{NP-complete}$ if
\[ L \in \text{NP} \text{ and } L \text{ is NP-hard}. \]
(How hard is a problem that’s NP-complete?)
**Intuition:** The \textbf{NP}-complete problems are the hardest problems in \textbf{NP}.

If we can determine how hard those problems are, it would tell us a lot about the \( \mathbf{P} \neq \mathbf{NP} \) question.
The Tantalizing Truth

**Theorem:** If any \( \text{NP} \)-complete language is in \( \text{P} \), then \( \text{P} = \text{NP} \).

**Intuition:** This means the hardest problems in \( \text{NP} \) aren’t actually that hard. We can solve them in polynomial time. So that means we can solve all problems in \( \text{NP} \) in polynomial time.
The Tantalizing Truth

**Theorem:** If any NP-complete language is in P, then \( P = NP \).
The Tantalizing Truth

**Theorem:** If any NP-complete language is in P, then P = NP.
The Tantalizing Truth

Theorem: If any \( \text{NP} \)-complete language is in \( \text{P} \), then \( \text{P} = \text{NP} \).
The Tantalizing Truth

*Theorem:* If any NP-complete language is in P, then P = NP.

$P = \text{NP}$
The Tantalizing Truth

**Theorem:** If any NP-complete language is in P, then P = NP.

**Proof:** Suppose that L is NP-complete and L ∈ P. Now consider any arbitrary NP problem A. Since L is NP-complete, we know that A \leq_p L. Since L ∈ P and A \leq_p L, we see that A ∈ P. Since our choice of A was arbitrary, this means that NP ⊆ P, so P = NP. ■
The Tantalizing Truth

**Theorem:** If any $\text{NP}$-complete language is not in $\text{P}$, then $\text{P} \neq \text{NP}$.

**Intuition:** This means the hardest problems in $\text{NP}$ are so hard that they can’t be solved in polynomial time. So the hardest problems in $\text{NP}$ aren’t in $\text{P}$, meaning $\text{P} \neq \text{NP}$. 
The Tantalizing Truth

**Theorem:** If any \textbf{NP}-complete language is not in \textbf{P}, then \textbf{P} ≠ \textbf{NP}.

**Proof:** Suppose that \( L \) is an \textbf{NP}-complete language not in \textbf{P}. Since \( L \) is \textbf{NP}-complete, we know that \( L \in \textbf{NP} \). Therefore, we know that \( L \in \textbf{NP} \) and \( L \notin \textbf{P} \), so \( \textbf{P} \neq \textbf{NP} \). \( \blacksquare \)
How do we even know NP-complete problems exist in the first place?
Satisfiability

• A propositional logic formula $\varphi$ is called **satisfiable** if there is some assignment to its variables that makes it evaluate to true.
  • $p \land q$ is satisfiable.
  • $p \land \neg p$ is unsatisfiable.
  • $p \rightarrow (q \land \neg q)$ is satisfiable.

• An assignment of true and false to the variables of $\varphi$ that makes it evaluate to true is called a **satisfying assignment**.
SAT

• The *boolean satisfiability problem* (*SAT*) is the following:

  Given a propositional logic formula $\varphi$, is $\varphi$ satisfiable?

• Formally:

  \[
  SAT = \{ \langle \varphi \rangle \mid \varphi \text{ is a satisfiable PL formula} \} \]
**Theorem (Cook-Levin):** SAT is \textbf{NP}-complete.

**Proof Idea:** To see that SAT \textbf{\textit{\textbullet}} \textbf{NP}, show how to make a polynomial-time verifier for it. Key idea: have the certificate be a satisfying assignment.

To show that SAT is \textbf{NP}-hard, given a polynomial-time verifier \textit{V} for an arbitrary \textbf{NP} language \textit{L}, for any string \textit{w} you can construct a polynomially-sized formula \textit{\varphi(w)} that says “there is a certificate \textit{c} where \textit{V} accepts \langle \textit{w}, \textit{c} \rangle.” This formula is satisfiable if and only if \textit{w} \textbf{\textit{\textbullet}} \textbf{L}, so deciding whether the formula is satisfiable decides whether \textit{w} is in \textit{L}. ■-ish

**Proof:** Take CS154!
Why All This Matters

- Resolving $P \neq NP$ is equivalent to just figuring out how hard SAT is.

$$\text{SAT} \in P \iff P = \text{NP}$$

- We've turned a huge, abstract, theoretical problem about solving problems versus checking solutions into the concrete task of seeing how hard one problem is.

- You can get a sense for how little we know about algorithms and computation given that we can't yet answer this question!
Why All This Matters

• You will almost certainly encounter NP-hard problems in practice – they're everywhere!

• If a problem is NP-hard, then there is no known algorithm for that problem that
  • is efficient on all inputs,
  • always gives back the right answer, and
  • runs deterministically.

• Useful intuition: If you need to solve an NP-hard problem, you will either need to settle for an approximate answer, an answer that's likely but not necessarily right, or have to work on really small inputs.
Sample NP-Hard Problems

- **Computational biology:** Given a set of genomes, what is the most probable evolutionary tree that would give rise to those genomes? *(Maximum parsimony problem)*

- **Game theory:** Given an arbitrary perfect-information, finite, two-player game, who wins? *(Generalized geography problem)*

- **Operations research:** Given a set of jobs and workers who can perform those tasks in parallel, can you complete all the jobs within some time bound? *(Job scheduling problem)*

- **Machine learning:** Given a set of data, find the simplest way of modeling the statistical patterns in that data *(Bayesian network inference problem)*

- **Medicine:** Given a group of people who need kidneys and a group of kidney donors, find the maximum number of people who can receive transplants. *(Cycle cover problem)*

- **Systems:** Given a set of processes and a number of processors, find the optimal way to assign those tasks so that they complete as soon as possible *(Processor scheduling problem)*
**Coda:** What if $\mathsf{P} \neq \mathsf{NP}$ is resolved?
Next Time

- Why All This Matters
- Where to Go from Here
- A Final “Your Questions”
- Parting Words!