Mathematical Proofs
Outline for Today

- **How to Write a Proof**
  - Synthesizing definitions, intuitions, and conventions.

- **Proofs on Numbers**
  - Working with odd and even numbers.

- **Universal and Existential Statements**
  - Two important classes of statements.

- **Variable Ownership**
  - Who owns what?
What is a Proof?
Proof as Dialog

- A mathematical proof is a dialog between two parties: a **proof writer** and a **proof reader**.
  - The **proof writer** knows a mathematical fact.
  - The **proof reader** is honest but skeptical.
- The proof writer’s job is to take the reader on a journey from ignorance to understanding.
What terms are used in this proof?
What do they formally mean?

What does this theorem mean?
Why, intuitively, should it be true?

What is the standard format for writing a proof?
What are the techniques for doing so?
Writing our First Proof
**Theorem:** If $n$ is an even integer, then $n^2$ is even.
Conventions

What terms are used in this proof? What do they formally mean?

Definitions

Intuitions

What does this theorem mean? Why, intuitively, should it be true?

Conventions

What is the standard format for writing a proof? What are the techniques for doing so?
Theorem: If $n$ is an even integer, then $n^2$ is even.
An integer $n$ is called **even** if there is an integer $k$ where $n = 2k$. 
Theorem: If $n$ is an even integer, then $n^2$ is even.
Conventions

What terms are used in this proof? What do they formally mean?

Definitions

What does this theorem mean? Why, intuitively, should it be true?

Intuitions

What is the standard format for writing a proof? What are the techniques for doing so?
Let’s Try Some Examples!

Theorem: If $n$ is an even integer, then $n^2$ is even.
Let’s Draw Some Pictures!

**Theorem:** If $n$ is an even integer, then $n^2$ is even.
Conventions

What terms are used in this proof? What do they formally mean?

What does this theorem mean? Why, intuitively, should it be true?

What is the standard format for writing a proof? What are the techniques for doing so?
Our First Proof!

**Theorem:** If \( n \) is an even integer, then \( n^2 \) is even.

**Proof:** Assume \( n \) is an even integer. We want to show that \( n^2 \) is even.

Since \( n \) is even, there is some integer \( k \) such that \( n = 2k \). This means that

\[
\begin{align*}
    n^2 &= (2k)^2 \\
         &= 4k^2 \\
         &= 2(2k^2).
\end{align*}
\]

From this, we see that there is an integer \( m \) (namely, \( 2k^2 \)) where \( n^2 = 2m \). Therefore, \( n^2 \) is even, which is what we wanted to show. ■
Our First Proof!

**Theorem:** If $n$ is an even integer, then $n^2$ is even.

**Proof:** Assume $n$ is an even integer. We want to show that $n^2$ is even.

Since $n$ is even, there is some integer $k$ such that $n = 2k$. This means that

$$n^2 = (2k)^2 = 4k^2 = 2(2k^2).$$

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Our First Proof!

Theorem: If $n$ is an even integer, then $n^2$ is even.

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$$n^2 = (2k)^2 = 4k^2 = 2(2k^2).$$

From this, we see that there is an integer $m$ (namely, $2k^2$) where $n^2 = 2m$. Therefore, $n^2$ is even, which is what we wanted to show. ■

To prove a statement of the form

"If $P$ is true, then $Q$ is true,"

start by asking the reader to assume that $P$ is true.
Our First Proof!

**Theorem:** If \( n \) is an even integer, then \( n^2 \) is even.

**Proof:** Assume \( n \) is an even integer. We want to show that \( n^2 \) is even.

Since \( n \) is even, there is some integer \( k \) such that \( n = 2k \). This means that

\[
\begin{align*}
\text{From this, we assume } P \text{ is true, then need to show that } Q \text{ is true. Here, we're telling the reader where we're headed.}
\end{align*}
\]
Our First Proof!

**Theorem:** If $n$ is an even integer, then $n^2$ is even.

**Proof:** Assume $n$ is an even integer. We want to show that $n^2$ is even.

Since $n$ is even, there is some integer $k$ such that $n = 2k$. This means that

$$n^2 = (2k)^2 = 4k^2 = 2(2k^2).$$

From this, we see that there is an integer $m$ (namely, $2k^2$) where $n^2 = 2m$. Therefore, $n^2$ is even, which is what we wanted to show. ■

This is the definition of an even integer. We need to use this definition to make this proof rigorous.
Our First Proof!

**Theorem:** If \( n \) is an even integer, then \( n^2 \) is even.

**Proof:** Assume \( n \) is an even integer. We want to show that \( n^2 \) is even.

Since \( n \) is even, we know there is some integer \( k \) such that \( n = 2k \). This means that

\[
\begin{align*}
  n^2 &= (2k)^2 \\
     &= 4k^2 \\
     &= 2(2k^2).
\end{align*}
\]

From this, we see that there is an integer \( m \) (namely, \( 2k^2 \)) where \( n^2 = 2m \). Therefore, \( n^2 \) is even, which is what we wanted to show. ■

Notice how we use the value of \( k \) that we obtained above. Giving names to quantities, allows us to manipulate them. This is similar to variables in programs.
Our First Proof!

**Theorem:** If \( n \) is an even integer, then \( n^2 \) is even.

**Proof:** Assume \( n \) is an even integer. We want to show that \( n^2 \) is even.

Since \( n \) is even, there is some integer \( k \) such that \( n = 2k \).

This means that \( n^2 = (2k)^2 = 4k^2 = 2(2k^2) \).

From this, we see that there is an integer \( m \) (namely, \( 2k^2 \)) where \( n^2 = 2m \). Therefore, \( n^2 \) is even, which is what we wanted to show. ■
Our First Proof!

**Theorem:** If \( n \) is an even integer, then \( n^2 \) is even.

**Proof:** Assume \( n \) is an even integer. We want to show that \( n^2 \) is even.

Since \( n \) is even, there is some integer \( k \) such that \( n = 2k \). This means that

\[
\begin{align*}
n^2 &= (2k)^2 \\
&= 4k^2 \\
&= 2(2k^2)
\end{align*}
\]

From this, we see that there is an integer \( m \) (namely, \( 2k^2 \)) where \( n^2 = 2m \). Therefore, \( n^2 \) is even, which is what we wanted to show. ■

Hey, that's what we said we were going to do! We're done.
Our Next Proof
**Theorem:** For any integers $m$ and $n$, if $m$ and $n$ are odd, then $m + n$ is even.
Conventions

What terms are used in this proof?
What do they formally mean?

Definitions

What does this theorem mean?
Why, intuitively, should it be true?

Intuitions

What is the standard format for writing a proof?
What are the techniques for doing so?
Theorem: For any integers $m$ and $n$, if $m$ and $n$ are odd, then $m + n$ is even.
An integer $n$ is called *odd* if there is an integer $k$ where $n = 2k + 1$. 
Going forward, we’ll assume the following:

1. Every integer is either even or odd.
2. No integer is both even and odd.
**Theorem**: For any integers $m$ and $n$, if $m$ and $n$ are odd, then $m + n$ is even.
Conventions

What terms are used in this proof? What do they formally mean?

What does this theorem mean? Why, intuitively, should it be true?

What is the standard format for writing a proof? What are the techniques for doing so?
Let’s Try Some Examples!

**Theorem:** For any integers $m$ and $n$, if $m$ and $n$ are odd, then $m+n$ is even.
Let’s Do Some Math!

\[
\begin{array}{c|c|c}
2k+1 & k & 1 \\
\hline
& & r \\
\hline
2r+1 & & \\
\end{array}
\]

\[(2k+1) + (2r+1) = 2(k + r + 1)\]

**Theorem:** For any integers \(m\) and \(n\), if \(m\) and \(n\) are odd, then \(m+n\) is even.
What terms are used in this proof? What do they formally mean?

What does this theorem mean? Why, intuitively, should it be true?

Conventions

What is the standard format for writing a proof? What are the techniques for doing so?
**Theorem:** For any integers $m$ and $n$, if $m$ and $n$ are odd, then $m + n$ is even.

**Proof:** Consider any arbitrary integers $m$ and $n$ where $m$ and $n$ are odd. We need to show that $m + n$ is even.

Since $m$ is odd, we know that there is an integer $k$ where

$$m = 2k + 1. \quad (1)$$

Similarly, because $n$ is odd there must be some integer $r$ such that

$$n = 2r + 1. \quad (2)$$

By adding equations (1) and (2) we learn that

$$m + n = 2k + 1 + 2r + 1$$
$$= 2k + 2r + 2$$
$$= 2(k + r + 1). \quad (3)$$

Equation (3) tells us that there is an integer $s$ (namely, $k + r + 1$) such that $m + n = 2s$. Therefore, we see that $m + n$ is even, as required. ■
**Theorem:** For any integers $m$ and $n$, if $m$ and $n$ are odd, then $m + n$ is even.

**Proof:** Consider any arbitrary integers $m$ and $n$ where $m$ and $n$ are odd. We need to show that $m + n$ is even.

Since $m$ is odd, there is an integer $k$ such that $m = 2k + 1$. (1)

Similarly, because $n$ is odd there must be some integer $r$ such that $n = 2r + 1$. (2)

By adding equations (1) and (2) we learn that

$$m + n = 2k + 1 + 2r + 1 = 2k + 2r + 2 = 2(k + r + 1).$$ (3)

Equation (3) tells us that there is an integer $s$ (namely, $k + r + 1$) such that $m + n = 2s$. Therefore, we see that $m + n$ is even, as required. ■

We ask the reader to make an arbitrary choice. Rather than specifying what $m$ and $n$ are, we’re signaling to the reader that they could, in principle, supply any choices of $m$ and $n$ that they’d like.

By letting the reader pick $m$ and $n$ arbitrarily, anything we prove about $m$ and $n$ will generalize to all possible choices for those values.
**Theorem:** For any integers $m$ and $n$, if $m$ and $n$ are odd, then $m + n$ is even.

**Proof:** Consider any arbitrary integers $m$ and $n$ where $m$ and $n$ are odd. We need to show that $m + n$ is even.

Since $m$ is odd, we know that there is an integer $k$ where $m = 2k + 1$. (1)

Similarly, because $n$ is odd there must be some integer $r$ such that $n = 2r + 1$. (2)

By adding equations (1) and (2) we learn that $m + n = 2k + 1 + 2r + 1 = 2k + 2r + 2 = 2(k + r + 1)$. (3)

Equation (3) tells us that there is an integer $s$ (namely, $k + r + 1$) such that $m + n = 2s$. Therefore, we see that $m + n$ is even, as required. ■

To prove a statement of the form “If $P$ is true, then $Q$ is true,” start by asking the reader to assume that $P$ is true.
Theorem: For any integers $m$ and $n$, if $m$ and $n$ are odd, then $m + n$ is even.

Proof: Consider any arbitrary integers $m$ and $n$ where $m$ and $n$ are odd. We need to show that $m + n$ is even.

Since $m$ is odd, we know that there is an integer $k$ such that $m = 2k + 1$.

Similarly, because $n$ is odd there must be some integer $r$ such that $n = 2r + 1$.

By adding equations (1) and (2) we learn that

$$m + n = (2k + 1) + (2r + 1) = 2k + 2r + 2 = 2(k + r + 1).$$

Equation (3) tells us that there is an integer $s$ (namely, $k + r + 1$) such that $m + n = 2s$. Therefore, we see that $m + n$ is even, as required. ■

To prove a statement of the form “If $P$ is true, then $Q$ is true,” after assuming $P$ is true, you need to show that $Q$ is true.
**Theorem:** For any integers \( m \) and \( n \), if \( m \) and \( n \) are odd, then \( m + n \) is even.

**Proof:** Consider any arbitrary integers \( m \) and \( n \) where \( m \) and \( n \) are odd. We need to show that \( m + n \) is even.

Since \( m \) is odd, we know that there is an integer \( k \) where

\[ m = 2k + 1. \quad (1) \]

Similarly, because \( n \) is odd there must be some integer \( r \) such that

\[ n = 2r + 1. \quad (2) \]

By adding equations (1) and (2) we learn that

\[ m + n = 2k + 1 + 2r + 1 \]
\[ = 2k + 2r + 2 \]
\[ = 2(k + r + 1). \quad (3) \]

Equation (3) tells us that there is an integer \( s \) (namely, \( k + r + 1 \)) such that \( m + n = 2s \). Therefore, we see that \( m + n \) is even, as required. ■
Theorem: For any integers $m$ and $n$, if $m$ and $n$ are odd, then $m + n$ is even.

Proof: Consider any arbitrary integers $m$ and $n$ where $m$ and $n$ are odd. We need to show that $m + n$ is even.

Since $m$ is odd, we know that there is an integer $k$ where

$$m = 2k + 1.$$  \hspace{1cm} (1)

Similarly, because $n$ is odd there must be some integer $r$ such that

$$n = 2r + 1.$$  \hspace{1cm} (2)

By adding equations (1) and (2) we learn that

$$m + n = 2k + 1 + 2r + 1 = 2k + 2r + 2 = 2(k + r + 1).$$ \hspace{1cm} (3)

Equation (3) tells us that there is an integer $s$ (namely, $k + r + 1$) such that $m + n = 2s$. Therefore, we see that $m + n$ is even, as required. $\blacksquare$

This is a complete sentence! Proofs are expected to be written in complete sentences, so you’ll often use punctuation at the end of formulas.

We recommend using the “mugga mugga” test – if you read a proof and replace all the mathematical notation with “mugga mugga,” what comes back should be a valid sentence.
Some Little Exercises

• Here’s a list of other theorems that are true about odd and even numbers:
  
  • **Theorem:** The sum and difference of any two even numbers is even.
  
  • **Theorem:** The sum and difference of an odd number and an even number is odd.
  
  • **Theorem:** The product of any integer and an even number is even.
  
  • **Theorem:** The product of any two odd numbers is odd.

• Going forward, we’ll just take these results for granted. Feel free to use them in the problem sets.

• If you’d like to practice the techniques from today, try your hand at proving these results!
Universal and Existential Statements
**Theorem:** For any odd integer $n$, there exist integers $r$ and $s$ where $r^2 - s^2 = n$. 
Conventions

What terms are used in this proof? What do they formally mean?

Definitions

What does this theorem mean? Why, intuitively, should it be true?

Intuitions

What is the standard format for writing a proof? What are the techniques for doing so?
**Theorem:** For any odd integer $n$, there exist integers $r$ and $s$ where $r^2 - s^2 = n$.

This result is true for every possible choice of odd integer $n$. It'll work for $n = 1$, $n = 137$, $n = 103$, etc.
Theorem: For any odd integer \( n \), there exist integers \( r \) and \( s \) where \( r^2 - s^2 = n \).

We aren’t saying this is true for every choice of \( r \) and \( s \). Rather, we’re saying that somewhere out there are choices of \( r \) and \( s \) where this works.
Universal vs. Existential Statements

- A *universally-quantified statement* is a statement of the form

  For all \( x \), [some-property] holds for \( x \).

- We've seen how to prove these statements.

- An *existentially-quantified statement* is a statement of the form

  There is some \( x \) where [some-property] holds for \( x \).

- How do you prove an existentially-quantified statement?
Proving an Existential Statement

- Over the course of the quarter, we will see several different ways to prove an existentially-quantified statement of the form

  \textit{There is an x where [some-property] holds for x.}

- \textit{Simplest approach:} Search far and wide, find an x that has the right property, then show why your choice is correct.
What terms are used in this proof? What do they formally mean?

Conventions

What is the standard format for writing a proof? What are the techniques for doing so?

What does this theorem mean? Why, intuitively, should it be true?
Let's Try Some Examples!

\[
\begin{align*}
1 &= 2 \cdot 0 + 1 &= 1^2 - 0^2 \\
3 &= 2 \cdot 1 + 1 &= 2^2 - 1^2 \\
5 &= 2 \cdot 2 + 1 &= 3^2 - 2^2 \\
7 &= 2 \cdot 3 + 1 &= 4^2 - 3^2 \\
9 &= 2 \cdot 4 + 1 &= 5^2 - 4^2
\end{align*}
\]

**Theorem:** For any odd integer \( n \), there exist integers \( r \) and \( s \) where \( r^2 - s^2 = n \).
Let’s Draw Some Pictures!

\[ (k+1)^2 - k^2 = 2k+1 \]

**Theorem:** For any odd integer \( n \), there exist integers \( r \) and \( s \) where \( r^2 - s^2 = n \).
What terms are used in this proof? What do they formally mean?

What does this theorem mean? Why, intuitively, should it be true?

Conventions

What is the standard format for writing a proof? What are the techniques for doing so?
Theorem: For any odd integer $n$, there exist integers $r$ and $s$ where $r^2 - s^2 = n$.

Proof: Let $n$ be an arbitrary odd integer. We will show that there exist integers $r$ and $s$ where $r^2 - s^2 = n$.

Since $n$ is odd, we know there is an integer $k$ where $n = 2k + 1$. Now, let $r = k+1$ and $s = k$. Then we see that

\[
  r^2 - s^2 = (k+1)^2 - k^2 \\
  = k^2 + 2k + 1 - k^2 \\
  = 2k + 1 \\
  = n.
\]

This means that $r^2 - s^2 = n$, which is what we needed to show. ■
**Theorem:** For any odd integer $n$, there exist integers $r$ and $s$ where $r^2 - s^2 = n$.

**Proof:** Let $n$ be an arbitrary odd integer. We will show that there exist integers $r$ and $s$ where $r^2 - s^2 = n$.

Since $n$ is odd, we know there is an integer $k$ where $n = 2k + 1$. Now, let $r = k + 1$ and $s = k$. Then we see that

$$r^2 - s^2 = (k + 1)^2 - k^2 = 2k + 1 = n.$$

This means that $r^2 - s^2 = n$, which is what we needed to show. ■
**Theorem:** For any odd integer $n$, there exist integers $r$ and $s$ where $r^2 - s^2 = n$.

**Proof:** Let $n$ be an arbitrary odd integer. We will show that there exist integers $r$ and $s$ where $r^2 - s^2 = n$.

Since $n$ is odd, we know there is an integer $k$ where $n = 2k + 1$. Now, let $r = k+1$ and $s = k$.

We see that

$$r^2 - s^2 = (k+1)^2 - k^2$$
$$= k^2 + 2k + 1 - k^2$$
$$= 2k + 1$$
$$= n.$$

This means that $r^2 - s^2 = n$, which is what we needed to show. ■

As always, it’s helpful to write out what we need to demonstrate with the rest of the proof.
**Theorem:** For any odd integer $n$, there exist integers $r$ and $s$ where $r^2 - s^2 = n$.

**Proof:** Let $n$ be an arbitrary odd integer. We will show that there exist integers $r$ and $s$ where $r^2 - s^2 = n$.

Since $n$ is odd, we know there is an integer $k$ where $n = 2k + 1$. **Now, let** $r = k+1$ **and** $s = k$. **Then we see that**

\[
\begin{align*}
  r^2 - s^2 &= (k+1)^2 - k^2 \\
           &= k^2 + 2k + 1 - k^2 \\
           &= 2k + 1 \\
           &= n.
\end{align*}
\]

This means that $r^2 - s^2 = n$, which is what we needed to show. ■
**Theorem:** For any odd integer \( n \), there exist integers \( r \) and \( s \) where \( r^2 - s^2 = n \).

**Proof:** Let \( n \) be an arbitrary odd integer. We will show that there exist integers \( r \) and \( s \) where \( r^2 - s^2 = n \).

Since \( n \) is odd, we know there is an integer \( k \) where \( n = 2k + 1 \). Now, let \( r = k+1 \) and \( s = k \). Then we see that

\[
r^2 - s^2 = (k+1)^2 - k^2
= k^2 + 2k + 1 - k^2
= 2k + 1
= n.
\]

This means that \( r^2 - s^2 = n \), which is what we needed to show. ■
Who Owns What?
**Theorem:** For any odd integer $n$, there exist integers $r$ and $s$ where $r^2 - s^2 = n$.

**Proof:** Let $n$ be an arbitrary odd integer. We will show that there exist integers $r$ and $s$ where $r^2 - s^2 = n$.

Since $n$ is odd, we can write $n = 2k + 1$ for some integer $k$. Now, let $r = k + 1$ and $s = k$. Then we see that

$$r^2 - s^2 = (k+1)^2 - k^2 = k^2 + 2k + 1 - k^2 = 2k + 1 = n.$$

This means that $r^2 - s^2 = n$, which is what we needed to show. ■

To prove a universal statement about a variable $n$, ask the person reading the proof to make the choice for $n$. You (the proof writer) don’t get to pick its value.
**Theorem:** For any odd integer \( n \), there exist integers \( r \) and \( s \) where \( r^2 - s^2 = n \).

**Proof:** Let \( n \) be an arbitrary odd integer. We will show that there exist integers \( r \) and \( s \) where \( r^2 - s^2 = n \).

Since \( n \) is odd, we know there is an integer \( k \) where \( n = 2k + 1 \). **Now, let \( r = k+1 \) and \( s = k \).** Then we see that

\[
 r^2 - s^2 = (k+1)^2 - k^2 = 2k + 1 = n.
\]

This means that \( r^2 - s^2 = n \), which is what we needed to show. ■
Check the appendix to this slide deck for more about who gets to choose values.
Time-Out for Announcements!
Working in Pairs

- Problem Set Zero is due this Friday at 2:30PM. It must be completed individually.
- After that, the remaining problem sets can be done individually or in pairs.
- We have advice about how to work effectively in pairs up on the course website – check the “Guide to Partners.”
- Want to work in a pair, but don’t know who to work with? Fill out this Google form and we’ll connect you with a partner on Friday.
CURIS Poster Session

- CURIS is the CS department’s undergraduate research program. It’s a great way to get involved in research!
- There’s a CURIS poster session showcasing work from the summer going on from 3PM – 5PM Friday in the Engineering Quad. Feel free to stop on by!
- Interested in seeing what research projects are open right now? Visit https://curis.stanford.edu.
- Have questions about research or how CURIS works? Email the CURIS mentors, PhD students who answer questions about research: curis-mentors@cs.stanford.edu
Qt Creator Help Session

- The lovely CS106B staff have invited all y’all to join them for a Qt Creator Help Session this evening if you’re having trouble getting Qt Creator up and running on your system.

- Runs **7:00PM - 9:00PM** on the third floor of the Durand Building.

- SCPD students – please reach out to us if you need help setting things up. We’ll do our best to help out.
Back to CS103!
Theorem: If \( n \) is an integer, then \( \lceil \frac{n}{2} \rceil + \lfloor \frac{n}{2} \rfloor = n \).
What terms are used in this proof?
What do they formally mean?

What does this theorem mean?
Why, intuitively, should it be true?

What is the standard format for writing a proof?
What are the techniques for doing so?
Floors and Ceilings

- The notation \( \lceil x \rceil \) represents the **ceiling** of \( x \), the smallest integer greater than or equal to \( x \).
  - What is \( \lceil 1 \rceil \)? What’s \( \lceil 1.2 \rceil \)? What’s \( \lceil -1.2 \rceil \)?
  - **Intuition:** Start at \( x \) on the number line, then move to the right until you hit a tick mark.

- The notation \( \lfloor x \rfloor \) represents is the **floor** of \( x \), the largest integer less than or equal to \( x \).
  - What is \( \lfloor 1 \rfloor \)? What’s \( \lfloor 1.2 \rfloor \)? What’s \( \lfloor -1.2 \rfloor \)?
  - **Intuition:** Start at \( x \) on the number line, then move to the left until you hit a tick mark.
Conventions

What terms are used in this proof? What do they formally mean?

What does this theorem mean? Why, intuitively, should it be true?

What is the standard format for writing a proof? What are the techniques for doing so?
Theorem: If \( n \) is an integer, then \( \lceil \frac{n}{2} \rceil + \lfloor \frac{n}{2} \rfloor = n \).
Let’s Draw Some Pictures!

\[ n = 2k \]

**Theorem:** If \( n \) is an integer, then \( \lceil \frac{n}{2} \rceil + \lfloor \frac{n}{2} \rfloor = n \).
Theorem: If $n$ is an integer, then $\lceil \frac{n}{2} \rceil + \lfloor \frac{n}{2} \rfloor = n$. 

Let's Draw Some Pictures!

$n = 2k + 1$
Conventions

What terms are used in this proof? What do they formally mean?

What does this theorem mean? Why, intuitively, should it be true?

What is the standard format for writing a proof? What are the techniques for doing so?
**Theorem:** If \( n \) is an integer, then \( \lceil \frac{n}{2} \rceil + \lfloor \frac{n}{2} \rfloor = n \).

**Proof:** Let \( n \) be an integer. We will show that \( \lceil \frac{n}{2} \rceil + \lfloor \frac{n}{2} \rfloor = n \). To do so, we consider two cases:

**Case 1:** \( n \) is even. This means there is an integer \( k \) such that \( n = 2k \). Some algebra then tells us that

\[
\left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{n}{2} \right\rceil = \left\lfloor \frac{2k}{2} \right\rfloor + \left\lceil \frac{2k}{2} \right\rceil = \lfloor k \rfloor + \lceil k \rceil = 2k = n.
\]

**Case 2:** \( n \) is odd. Then there’s an integer \( k \) where \( n = 2k + 1 \), and

\[
\left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{n}{2} \right\rceil = \left\lfloor \frac{2k+1}{2} \right\rfloor + \left\lceil \frac{2k+1}{2} \right\rceil = \left\lfloor \frac{k+1}{2} \right\rfloor + \left\lceil \frac{k+1}{2} \right\rceil = (k+1) + k = 2k+1 = n.
\]

In either case, we see that \( \lceil \frac{n}{2} \rceil + \lfloor \frac{n}{2} \rfloor = n \), as required. ■
To Recap
Writing a good proof requires a blend of definitions, intuitions, and conventions.
An integer $n$ is **even** if there is an integer $k$ where $n = 2k$.

An integer $n$ is **odd** if there is an integer $k$ where $n = 2k + 1$.

Definitions tell us what we need to do in a proof. Many proofs directly reference these definitions.
Building intuition for results requires creativity, trial, and error.
- Prove universal statements by making arbitrary choices.
- Prove existential statements by making concrete choices.
- Prove "If $P$, then $Q$" by assuming $P$ and proving $Q$.
- Write in complete sentences.
- Number sub-formulas when referring to them.
- Summarize what was shown in proofs by cases.
- Articulate your start and end points.

Mathematical proofs have established conventions that increase rigor and readability.
Your Action Items

• **Read “Guide to ∈ and ⊆.”**
  • You’ll want to have a handle on how these concepts are related, and on how they differ.

• **Read “Guide to Proofs.”**
  • This resource covers proofwriting strategies and conventions and is an essential complement to this lecture.

• **Read “Guide to Partners.”**
  • It’s all about how to work effectively in pairs. Mull this over so you’re ready to go for Problem Set 1.

• **Finish and submit Problem Set 0.**
  • Don’t put this off until the last minute!
Next Time

- **Indirect Proofs**
  - How do you prove something without actually proving it?

- **Mathematical Implications**
  - What exactly does “if $P$, then $Q$” mean?

- **Proof by Contrapositive**
  - A helpful technique for proving implications.

- **Proof by Contradiction**
  - Proving something is true by showing it can't be false.
Appendix: *Proofs as Dialogs*
Proofs as a Dialog

Let $n$ be an arbitrary odd integer.

Since $n$ is an odd integer, there is an integer $k$ such that $n = 2k + 1$.

Now, let $z = k - 34$. 
Proofs as a Dialog

Let $n$ be an arbitrary odd integer.

Since $n$ is an odd integer, there is an integer $k$ such that $n = 2k + 1$.

Now, let $z = k - 34$. 

$n = 137$

Reader Picks
Proofs as a Dialog

Let $n$ be an arbitrary odd integer.

Since $n$ is an odd integer, there is an integer $k$ such that $n = 2k + 1$.

Now, let $z = k - 34$. 

Proof Writer (You)

Proof Reader

Reader Picks

$n = 137$

$\text{Neither} \text{ Picks}$

$k = 68$
Proofs as a Dialog

Let \( n \) be an arbitrary odd integer.

Since \( n \) is an odd integer, there is an integer \( k \) such that \( n = 2k + 1 \).

Now, let \( z = k - 34 \).
Proofs as a Dialog

Let $n$ be an arbitrary odd integer.

Since $n$ is an odd integer, there is an integer $k$ such that $n = 2k + 1$.

Now, let $z = k - 34$. 

Proof Writer (You) 

Proof Reader 

$n = 137$ 

Reader Picks 

$k = 68$ 

Neither Picks 

$z = 34$ 

Writer Picks 

Proof Writer (You) 

Proof Reader
Let \( n \) be an arbitrary odd integer. Since \( n \) is an odd integer, there is an integer \( k \) such that \( n = 2k + 1 \). Now, let \( z = k - 34 \). Each of these variables has a distinct, assigned value. Each variable was either picked by the reader, picked by the writer, or has a value that can be determined from other variables.
Who Owns What?

- The **reader** chooses and owns a value if you use wording like this:
  - Pick a natural number \( n \).
  - Consider some \( n \in \mathbb{N} \).
  - Fix a natural number \( n \).
  - Let \( n \) be a natural number.

- The **writer** (you) chooses and owns a value if you use wording like this:
  - Let \( r = n + 1 \).
  - Pick \( s = n \).

- **Neither** of you chooses a value if you use wording like this:
  - Since \( n \) is even, we know there is some \( k \in \mathbb{Z} \) where \( n = 2k \).
  - Because \( n \) is odd, there must be some integer \( k \) where \( n = 2k + 1 \).
Proofs as a Dialog

Let \( x \) be an arbitrary even integer.

Then for any even \( x \), we know that \( x+1 \) is odd.
Proofs as a Dialog

Let \( x \) be an arbitrary even integer.

Then for any even \( x \), we know that \( x + 1 \) is odd.

Proof Writer (You)

\[ x = 242 \]

 Reader Picks

Proof Reader
Proofs as a Dialog

Let $x$ be an arbitrary even integer.

Then for any even $x$, we know that $x+1$ is odd.
Proofs as a Dialog

Let $x$ be an arbitrary even integer.

Since $x$ is even, we know that $x+1$ is odd.

Proof Writer (You)
Proof Reader
Proofs as a Dialog

Let $x$ be an arbitrary even integer.

Since $x$ is even, we know that $x+1$ is odd.

$x = 242$

Reader Picks
Let $x$ be an arbitrary even integer.

Since $x$ is even, we know that $x+1$ is odd.
Every variable needs a value.

Avoid talking about “all x” or “every x” when manipulating something concrete.

To prove something is true for any choice of a value for x, let the reader pick x.
Once you’ve said something like

Let \( x \) be an integer.
Consider an arbitrary \( x \in \mathbb{Z} \).
Pick any \( x \).

Do not say things like the following:

This means that for any \( x \in \mathbb{Z} \) ... 
So for all \( x \in \mathbb{Z} \) ...
Proofs as a Dialog

Pick two integers \( m \) and \( n \) where \( m+n \) is odd.

Let \( n = 1 \), which means that \( m+1 \) is odd.
Proofs as a Dialog

Pick two integers $m$ and $n$ where $m+n$ is odd.

Let $n = 1$, which means that $m+1$ is odd.

Proof Writer (You)

$m = 103$

Reader Picks

$n = 166$

Reader Picks

Proof Reader
Proofs as a Dialog

Pick two integers \( m \) and \( n \) where \( m+n \) is odd.

Let \( n = 1 \), which means that \( m+1 \) is odd.

Proof Writer (You)

Proof Reader

\( m = 103 \)

\( n = 166 \)

Reader Picks

Hold on! I already chose a value for \( n \)!
Proofs as a Dialog

Let $n = 1$.

Pick any integer $m$ where $m+1$ is odd.
Proofs as a Dialog

Let $n = 1$. Pick any integer $m$ where $m+1$ is odd.

Proof Writer (You)

Proof Reader

⚠ ⚠

$n = 1$  

Writer Picks

Proof Writer (You)

Proof Reader
Proofs as a Dialog

Let $n = 1$.

Pick any integer $m$ where $m+1$ is odd.

Proof Writer (You)

Proof Reader

$m = 166$

Reader Picks

$n = 1$

Writer Picks

Proof Writer (You)

Proof Reader
Proofs as a Dialog

Let \( n = 1 \).

Pick any integer \( m \) where \( m+1 \) is odd.

Do we even need \( n \) here?

\[ m = 166 \]

\( n = 1 \)
Proofs as a Dialog

Pick any integer $m$ where $m+1$ is odd.

$m = 166$

Reader Picks
Be mindful of who owns what variable.

Don’t change something you don’t own.

You don’t always need to name things, especially if they already have a name.