Cardinality
Outline for Today

• *Function Composition*
  • Combining functions together.

• *Bijections*
  • A key and important class of functions.

• *Cardinality, Formally*
  • What does it mean for two sets to have the same size?

• *Cantor’s Theorem, Formally*
  • Proving that infinity is not infinity is not infinity.
Function Composition
People

Keith
Wanyue
Lori
Jennie
Akankshita

Places
Cupertino, CA
San Francisco
Redding, CA
Utqiagvik, AK
Palo Alto, CA

Prices
Far Too Much
A King's Ransom
A Modest Amount
More Than You’d Expect

\[ h : People \rightarrow Prices \]
\[ h(x) = g(f(x)) \]
Function Composition

- Suppose that we have two functions $f : A \rightarrow B$ and $g : B \rightarrow C$.

- Notice that the codomain of $f$ is the domain of $g$. This means that we can use outputs from $f$ as inputs to $g$. 

\[ f \circ g : A \rightarrow C \]

\[ f(x) \rightarrow g(f(x)) \]
Function Composition

• Suppose that we have two functions \( f : A \rightarrow B \) and \( g : B \rightarrow C \).

• The \textit{composition of \( f \) and \( g \)}, denoted \( g \circ f \), is a function where
  
  • \( g \circ f : A \rightarrow C \), and
  
  • \((g \circ f)(x) = g(f(x))\).

• A few things to notice:
  
  • The domain of \( g \circ f \) is the domain of \( f \). Its codomain is the codomain of \( g \).
  
  • Even though the composition is written \( g \circ f \), when evaluating \((g \circ f)(x)\), the function \( f \) is evaluated first.

The name of the function is \( g \circ f \). When we apply it to an input \( x \), we write \((g \circ f)(x)\). I don't know why, but that's what we do.
Properties of Composition
**Theorem:** If $f : A \rightarrow B$ is an injection and $g : B \rightarrow C$ is an injection, then the function $g \circ f : A \rightarrow C$ is an injection.
Organizing Our Thoughts
**Theorem:** If \( f : A \rightarrow B \) is an injection and \( g : B \rightarrow C \) is an injection, then the function \( g \circ f : A \rightarrow C \) is an injection.

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| \( f : A \rightarrow B \) is an injection.  
\( \forall x \in A. \forall y \in A. (x \neq y \rightarrow f(x) \neq f(y)) \)  
\( g : B \rightarrow C \) is an injection.  
\( \forall x \in B. \forall y \in B. (x \neq y \rightarrow g(x) \neq g(y)) \)  | \( g \circ f \) is an injection.  
\( \forall a_1 \in A. \forall a_2 \in A. (a_1 \neq a_2 \rightarrow (g \circ f)(a_1) \neq (g \circ f)(a_2)) \) |

We’re assuming these universally-quantified statements, so we won’t introduce any variables for what’s here.

We need to prove this universally-quantified statement. So let’s introduce arbitrarily-chosen values.
**Theorem:** If \( f : A \to B \) is an injection and \( g : B \to C \) is an injection, then the function \( g \circ f : A \to C \) is an injection.

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### What We’re Assuming

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- \( g: B \to C \) is an injection.
  - \( \forall x \in B. \forall y \in B. (x \neq y \rightarrow g(x) \neq g(y)) \)
- \( a_1 \in A \) is arbitrarily-chosen.
- \( a_2 \in A \) is arbitrarily-chosen.
- \( a_1 \neq a_2 \)

### What We Need to Prove

- \( g \circ f \) is an injection.
  - \( \forall a_1 \in A. \forall a_2 \in A. (a_1 \neq a_2 \rightarrow (g \circ f)(a_1) \neq (g \circ f)(a_2)) \)

Now we’re looking at an implication. Let’s **assume** the antecedent and **prove** the consequent.
**Theorem:** If $f : A \rightarrow B$ is an injection and $g : B \rightarrow C$ is an injection, then the function $g \circ f : A \rightarrow C$ is an injection.

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Let’s write this out separately and simplify things a bit. |
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) | $(g \circ f)(a_1) \neq (g \circ f)(a_2)$ |
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- \( g \circ f \) is an injection.
  \[ \forall a_1 \in A. \forall a_2 \in A. (a_1 \neq a_2 \rightarrow (g \circ f)(a_1) \neq (g \circ f)(a_2)) \]
- \( g(f(a_1)) \neq g(f(a_2)) \)
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**Proof:** Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be arbitrary injections.
**Theorem:** If \( f : A \rightarrow B \) is an injection and \( g : B \rightarrow C \) is an injection, then the function \( g \circ f : A \rightarrow C \) is also an injection.

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Since $f$ is injective and $a_1 \neq a_2$, we see that $f(a_1) \neq f(a_2)$. 

![Diagram](image_url)
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Since \( f \) is injective and \( a_1 \neq a_2 \), we see that \( f(a_1) \neq f(a_2) \). Then, since \( g \) is injective and \( f(a_1) \neq f(a_2) \), we see that \( g(f(a_1)) \neq g(f(a_2)) \), as required.
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**Great exercise:** Repeat this proof using the other definition of injectivity.
**Theorem:** If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is also an injection.

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This proof contains no first-order logic syntax (quantifiers, connectives, etc.). It's written in plain English, just as usual.
Theorem: If $f : A \to B$ is a surjection and $g : B \to C$ is a surjection, then the function $g \circ f : A \to C$ is a surjection.

Proof: In the appendix!
Bijections
Injections and Surjections

- An injective function associates \textit{at most} one element of the domain with each element of the codomain.
- A surjective function associates \textit{at least} one element of the domain with each element of the codomain.
- What about functions that associate \textit{exactly one} element of the domain with each element of the codomain?
Bijectioons

- A *bijection* is a function that is both injective and surjective.

- Intuitively, if \( f : A \rightarrow B \) is a bijection, then \( f \) represents a way of pairing off elements of \( A \) and elements of \( B \).
Bijections

• Which of the following are bijections?
  • \( f : \mathbb{R} \to \mathbb{R} \) defined as \( f(x) = x \).
  • \( f : \mathbb{Z} \to \mathbb{R} \) defined as \( f(x) = x \).
  • \( f : \mathbb{R} \to \mathbb{R} \) defined as \( f(x) = 2x + 1 \).
  • \( f : \mathbb{Z} \to \mathbb{Z} \) defined as \( f(x) = 2x + 1 \).

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A bijection is a function that is both injective and surjective.
Bijections

Which of the following are bijections?

- $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = x$.  **Yep!**
- $f : \mathbb{Z} \to \mathbb{R}$ defined as $f(x) = x$.
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  • $f : \mathbb{R} \rightarrow \mathbb{R}$ defined as $f(x) = x$. **Yep!**
  • $f : \mathbb{Z} \rightarrow \mathbb{R}$ defined as $f(x) = x$. **Nope!**
  • $f : \mathbb{R} \rightarrow \mathbb{R}$ defined as $f(x) = 2x + 1$.
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A **bijection** is a function that is both injective and surjective.
Bijective

- Which of the following are bijections?
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  - \( f : \mathbb{Z} \rightarrow \mathbb{R} \) defined as \( f(x) = x \). \text{Nope!}
  - \( f : \mathbb{R} \rightarrow \mathbb{R} \) defined as \( f(x) = 2x + 1 \). \text{Yep!}
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A \textit{bijection} is a function that is both injective and surjective.
A **bijection** is a function that is both injective and surjective.
Cardinality Revisited
Cardinality

- Recall (from our first lecture!) that the **cardinality** of a set is the number of elements it contains.

- If $S$ is a set, we denote its cardinality by $|S|$.

- For finite sets, cardinalities are natural numbers:
  - $|\{1, 2, 3\}| = 3$
  - $|\{100, 200\}| = 2$

- For infinite sets, we introduced **infinite cardinals** to denote the size of sets:
  
  $$|\mathbb{N}| = \aleph_0$$
Defining Cardinality

- It is difficult to give a rigorous definition of what cardinalities actually are.
  - What is 4? What is $\aleph_0$?
  - (Take Math 161 for an answer!)
- Instead, we’ll define cardinality as a relation between two sets rather than an absolute quantity.
- **Intuition:** Two sets have the same cardinality if there’s a way to pair off their elements.
Comparing Cardinalities

Here is the formal definition of what it means for two sets to have the same cardinality:

\[ |S| = |T| \text{ if there exists a bijection } f : S \rightarrow T \]
Comparing Cardinalities

- Here is the formal definition of what it means for two sets to have the same cardinality:

\[ |S| = |T| \text{ if there exists a bijection } f : S \rightarrow T \]
Fun with Cardinality
Terminology Refresher

- Let $a$ and $b$ be real numbers where $a \leq b$.
  
- The notation $[a, b]$ denotes the set of all real numbers between $a$ and $b$, inclusive.
  
  $$[a, b] = \{ x \in \mathbb{R} \mid a \leq x \leq b \}$$

- The notation $(a, b)$ denotes the set of all real numbers between $a$ and $b$, exclusive.
  
  $$(a, b) = \{ x \in \mathbb{R} \mid a < x < b \}$$
Consider the sets $[0, 1]$ and $[0, 2]$.

How do their cardinalities compare?
$f : [0, 1] \rightarrow [0, 2]$

$f(x) = 2x$
**Theorem:** $|[0, 1]| = |[0, 2]|$. 

**Proof:** Consider the function $f : [0, 1] \to [0, 2]$ defined as $f(x) = 2x$. We will prove that $f$ is a bijection.

First, we'll show that $f$ is injective. Pick any $x_1, x_2 \in [0, 1]$ where $f(x_1) = f(x_2)$. We need to show that $x_1 = x_2$. To see this, notice that since $f(x_1) = f(x_2)$, we see that $2x_1 = 2x_2$, which in turn tells us that $x_1 = x_2$, as required.

Finally, we will show that $f$ is surjective. To do so, consider any $y \in [0, 2]$. We need to show there is some $x \in [0, 1]$ where $f(x) = y$.

Let $x = y/2$. Since $y \in [0, 2]$, we know $0 \leq y \leq 2$, and therefore that $0 \leq y/2 \leq 1$. We picked $x = y/2$, so we know that $0 \leq x \leq 1$, which in turn means $x \in [0, 1]$. Moreover, notice that $f(x) = 2x = 2(y/2) = y$, so $f(x) = y$, as required. ■
**Theorem:** \(|[0, 1]| = |[0, 2]|.\)

**Proof:**
**Theorem:** \(|[0, 1]| = |[0, 2]|\).

**Proof:** Consider the function \(f : [0, 1] \rightarrow [0, 2]\) defined as \(f(x) = 2x\).
**Theorem:** \([0, 1] \cong [0, 2]\).

**Proof:** Consider the function \(f : [0, 1] \to [0, 2]\) defined as \(f(x) = 2x\). We will prove that \(f\) is a bijection.
**Theorem:** $|[0, 1]| = |[0, 2]|$.

**Proof:** Consider the function $f : [0, 1] \rightarrow [0, 2]$ defined as $f(x) = 2x$. We will prove that $f$ is a bijection.

First, we’ll show that $f$ is injective.
**Theorem:** \([0, 1] = [0, 2]\).

**Proof:** Consider the function \(f : [0, 1] \to [0, 2]\) defined as \(f(x) = 2x\). We will prove that \(f\) is a bijection.

First, we’ll show that \(f\) is injective. Pick any \(x_1, x_2 \in [0, 1]\) where \(f(x_1) = f(x_2)\).
**Theorem:** $|[0, 1]| = |[0, 2]|$.

**Proof:** Consider the function $f : [0, 1] \to [0, 2]$ defined as $f(x) = 2x$. We will prove that $f$ is a bijection.

First, we’ll show that $f$ is injective. Pick any $x_1, x_2 \in [0, 1]$ where $f(x_1) = f(x_2)$. We need to show that $x_1 = x_2$.
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**Proof:** Consider the function \(f : [0, 1] \rightarrow [0, 2]\) defined as \(f(x) = 2x\). We will prove that \(f\) is a bijection.

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Theorem: $|[0, 1]| = |[0, 2]|$.

Proof: Consider the function $f : [0, 1] \rightarrow [0, 2]$ defined as $f(x) = 2x$. We will prove that $f$ is a bijection.

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Finally, we will show that $f$ is surjective.
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**Theorem:** $|[0, 1]| = |[0, 2]|$.

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Let $x = \frac{y}{2}$.
**Theorem:** $|[0, 1]| = |[0, 2]|$.

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Let $x = y/2$. Since $y \in [0, 2]$, we know $0 \leq y \leq 2$, and therefore that $0 \leq y/2 \leq 1$. 


**Theorem:** \(|[0, 1]| = |[0, 2]|.\)

**Proof:** Consider the function \(f : [0, 1] \to [0, 2]\) defined as \(f(x) = 2x\). We will prove that \(f\) is a bijection.

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Let \(x = \frac{y}{2}\). Since \(y \in [0, 2]\), we know \(0 \leq y \leq 2\), and therefore that \(0 \leq \frac{y}{2} \leq 1\). We picked \(x = \frac{y}{2}\), so we know that \(0 \leq x \leq 1\), which in turn means \(x \in [0, 1]\).
Theorem: \(|[0, 1]| = |[0, 2]|.\)

Proof: Consider the function \(f : [0, 1] \rightarrow [0, 2]\) defined as \(f(x) = 2x\). We will prove that \(f\) is a bijection.

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Let \(x = \frac{y}{2}\). Since \(y \in [0, 2]\), we know \(0 \leq y \leq 2\), and therefore that \(0 \leq \frac{y}{2} \leq 1\). We picked \(x = \frac{y}{2}\), so we know that \(0 \leq x \leq 1\), which in turn means \(x \in [0, 1]\). Moreover, notice that

\[
 f(x) = 2x = 2\left(\frac{y}{2}\right) = y,
\]

so \(f(x) = y\), as required.
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Let $x = \frac{y}{2}$. Since $y \in [0, 2]$, we know $0 \leq y \leq 2$, and therefore that $0 \leq \frac{y}{2} \leq 1$. We picked $x = \frac{y}{2}$, so we know that $0 \leq x \leq 1$, which in turn means $x \in [0, 1]$. Moreover, notice that $f(x) = 2x = 2(\frac{y}{2}) = y$, so $f(x) = y$, as required. ■
\[ f : [0, 1] \rightarrow [0, 2] \]

\[ f(x) = 2x \]
$f : [0, 1] \rightarrow [0, 3]$

$f(x) = 3x$
\[ f : [0, 1] \rightarrow [0, 137] \]
\[ f(x) = 137x \]
Cardinality (how many points there are) is different than mass (how much those points weigh). Look into measure theory if to learn more!

\[ f : [0, 1] \rightarrow [0, k] \]
\[ f(x) = kx \]
And one more example, just for funzies.
Put a Ring On It

\[ f : (-\pi/2, \pi/2) \to \mathbb{R} \]
\[ f(x) = \tan x \]
\[ |(-\pi/2, \pi/2)| = |\mathbb{R}| \]
Facts About Cardinality

• **Theorem:** For any set $A$, we have $|A| = |A|$.
  • **Proof Idea:** Define a bijection from $A$ to itself. Specifically, pick $f(x) = x$.
  • **Proof:** Appendix!

• **Theorem:** For any sets $A$, $B$, and $C$, if $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$.
  • **Proof Idea:** We begin with bijections $f : A \rightarrow B$ and $g : B \rightarrow C$. Compose them to get a bijection $g \circ f : A \rightarrow C$.
  • **Proof:** Appendix!

• **Theorem:** For any sets $A$ and $B$, if $|A| = |B|$, then $|B| = |A|$.
  • **Proof Idea:** Start with a bijection $f : A \rightarrow B$ and look at its inverse, the function $f^{-1} : B \rightarrow A$, which is also a bijection.
  • **Proof:** Appendix!
Time-Out for Announcements!
Problem Set Three

- Problem Set Two was due today at 2:30PM.
- Problem Set Three goes out today. It’s due next Friday at 2:30PM.
  - Play around with functions, set cardinality, and the nature of infinity!
- As always, ping us if you need help working on this one: post on EdStem or stop by office hours.
Extra Practice Problems

- We’ve posted a bank of 24 cumulative practice problems to the course website, along with solutions.
- Feel free to use these as an extra study resource to get more reps with the material.
- You can also use this to study for the midterm if you want to get a jump on that. More details on Monday.
Alternate Exams

• We are working on finalizing alternate exam times for the first midterm exam.

• If you have an OAE letter and haven’t yet contacted us, please do so ASAP so we can reserve space for you.

• If you have an academic conflict during the normal exam time, please let us know as soon as possible.
Your Questions
“What is your affinity for the number 137?”

It’s a fun example of a “nothing up my sleeve number” that has a fun historical backstory.
Back to CS103!
Unequal Cardinalities

• Recall: \(|A| = |B|\) if the following statement is true:

\[
\text{There exists a bijection } f : A \to B
\]

• What does it mean for \(|A| \neq |B|\) to be true?

Every function \(f : A \to B\) is not a bijection.

• This is a strong statement! To prove \(|A| \neq |B|\), we need to show that no possible function from \(A\) to \(B\) can be injective and surjective.
Unequal Cardinalities

• Recall: $|A| = |B|$ if the following statement is true:

\[ \text{There exists a bijection } f : A \rightarrow B \]

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Unequal Cardinalities

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    There exists a bijection $f : A \rightarrow B$

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  no possible function from $A$ to $B$ can be injective and surjective.
Unequal Cardinalities

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There exists a bijection \( f : A \rightarrow B \)

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Unequal Cardinalities

- Recall: \(|A| = |B|\) if the following statement is true:
  
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  Every function \(f : A \rightarrow B\) is not a bijection.

- This is a strong statement! To prove \(|A| \neq |B|\), we need to show that no possible function from \(A\) to \(B\) can be injective and surjective.
Cantor’s Theorem Revisited
Cantor’s Theorem

• In our very first lecture, we sketched out a proof of *Cantor’s theorem*, which says that

   \[ \text{If } S \text{ is a set, then } |S| < |\mathcal{P}(S)|. \]

• That proof was visual and pretty hand-wavy. Let’s see if we can go back and formalize it!
Where We’re Going

• Today, we’re going to formally prove the following result:

   **If S is a set, then |S| ≠ |℘(S)|.**

• We’ve released an online Guide to Cantor’s Theorem, which will go into way more depth than what we’re going to see here.

• The goal for today will be to see how to start with our picture and turn it into something rigorous.

• On the problem set, you’ll explore the proof in more depth and see some other applications.
The Roadmap

• We’re going to prove this statement:
  If \( S \) is a set, then \( |S| \neq |\wp(S)| \).

• Here’s how this will work:
  • Pick an arbitrary set \( S \).
  • Pick an arbitrary function \( f : S \to \wp(S) \).
  • Show that \( f \) is not surjective using a diagonal argument.
  • Conclude that there are no bijections from \( S \) to \( \wp(S) \).
  • Conclude that \( |S| \neq |\wp(S)| \).
The Roadmap

We’re going to prove this statement:

If $S$ is a set, then $|S| \neq |\mathcal{P}(S)|$.

Here’s how this will work:

- Pick an arbitrary set $S$.
- Pick an arbitrary function $f : S \to \mathcal{P}(S)$.
- Show that $f$ is not surjective using a diagonal argument.

Conclude that there are no bijections from $S$ to $\mathcal{P}(S)$. Conclude that $|S| \neq |\mathcal{P}(S)|$. 
This is a drawing of our function $f : S \rightarrow \mathcal{P}(S)$. 

\begin{align*}
X_0 & \leftrightarrow \left\{ X_0, X_2, X_4, \ldots \right\} \\
X_1 & \leftrightarrow \left\{ X_3, X_5, \ldots \right\} \\
X_2 & \leftrightarrow \left\{ X_0, X_1, X_2, X_5, \ldots \right\} \\
X_3 & \leftrightarrow \left\{ X_1, X_4, \ldots \right\} \\
X_4 & \leftrightarrow \left\{ X_2, \ldots \right\} \\
X_5 & \leftrightarrow \left\{ X_0, X_4, X_5, \ldots \right\} \\
\ldots & \leftrightarrow \left\{ \ldots \right\}
\end{align*}
This is a drawing of our function \( f : S \rightarrow \mathcal{P}(S) \).
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This is a drawing of our function \( f : S \to \mathcal{P}(S) \).
This is a drawing of our function $f : S \to \mathcal{P}(S)$.

Flip this set. Swap what’s included and what’s excluded.
Which element is paired with this set?

This is a drawing of our function $f: S \rightarrow \wp(S)$. 
This is a drawing of our function $f : S \to \mathcal{P}(S)$.

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Which element is paired with this set?
This is a drawing of our function $f : S \to \wp(S)$.  

What set is this?
Why is $x_1$ in this set?

This is a drawing of our function $f : S \to \wp(S)$. 

$f(x_1)$
This is a drawing of our function $f : S \rightarrow \mathcal{P}(S)$.

Why is $x_3$ in this set?

- $x_3 \notin f(x_3)$
- $f(x_3) = \{x_1, x_3, x_4, \ldots\}$
- $x_3 \notin f(x_3)$
This is a drawing of our function \( f : S \rightarrow \wp(S) \).

Why is \( x_4 \) in this set?

\[ x_4 \notin f(x_4) \]
This is a drawing of our function $f : S \rightarrow \mathcal{P}(S)$.

Why isn't $x_0$ in this set?

$x_0 \in f(x_0)$
This is a drawing of our function $f : S \to \wp(S)$.

Why isn't $x_2$ in this set?

$x_2 \in f(x_2)$

$f(x_2)$
Why isn’t $x_5$ in this set?

This is a drawing of our function $f : S \to \wp(S)$. 

$$f(x_5) \quad \text{why isn’t} \quad x_5 \text{ in this set?}$$
This is a drawing of our function \( f : S \to \mathcal{P}(S) \).

If \( x \notin f(x) \), include \( x \) in the set.
If \( x \in f(x) \), exclude \( x \) from the set.
This is a drawing of our function $f : S \rightarrow \mathcal{P}(S)$.

If $x \notin f(x)$, include $x$ in the set.
If $x \in f(x)$, exclude $x$ from the set.

Define $D = \{ x \in S \mid x \notin f(x) \}$.
The Diagonal Set

• For any set $S$ and function $f : S \rightarrow \mathcal{P}(S)$, we can define a set $D$ as follows:

$$D = \{ x \in S \mid x \notin f(x) \}$$

(“The set of all elements $x$ where $x$ is not an element of the set $f(x)$.”)

• This is a formalization of the set we found in the previous picture.

• Using this choice of $D$, we can formally prove that no function $f : S \rightarrow \mathcal{P}(S)$ is a bijection.
Theorem: If $S$ is a set, then $|S| \neq |\mathcal{P}(S)|$. 
**Theorem:** If $S$ is a set, then $|S| \neq |\mathcal{P}(S)|$.

**Proof:**
**Theorem:** If $S$ is a set, then $|S| \neq |\wp(S)|$.

**Proof:** Let $S$ be an arbitrary set.
**Theorem:** If $S$ is a set, then $|S| \neq |\mathcal{P}(S)|$.

**Proof:** Let $S$ be an arbitrary set. We will prove that $|S| \neq |\mathcal{P}(S)|$ by showing that there are no bijections from $S$ to $\mathcal{P}(S)$. 

To do so, choose an arbitrary function $f : S \to \mathcal{P}(S)$. We will prove that $f$ is not surjective. Starting with $f$, we define the set $D = \{ x \in S | x \notin f(x) \}$.

We will show that there is no $y \in S$ such that $f(y) = D$. To do so, we proceed by contradiction. Suppose that there is some $y \in S$ such that $f(y) = D$. From the definition of $D$, we know that $y \in D$ if and only if $y \notin f(y)$.

Since we know $f(y) = D$, the above statement means that $y \in D$ if and only if $y \notin D$. This is impossible. We have reached a contradiction, so our assumption must have been wrong. Therefore, there is no $y \in S$ such that $f(y) = D$, so $f$ is not surjective. ■
**Theorem:** If $S$ is a set, then $|S| \neq |\wp(S)|$.

**Proof:** Let $S$ be an arbitrary set. We will prove that $|S| \neq |\wp(S)|$ by showing that there are no bijections from $S$ to $\wp(S)$. To do so, choose an arbitrary function $f : S \to \wp(S)$. Starting with $f$, we define the set $D = \{ x \in S | x \notin f(x) \}$. We will show that there is no $y \in S$ such that $f(y) = D$. To do so, we proceed by contradiction. Suppose that there is some $y \in S$ such that $f(y) = D$. From the definition of $D$, we know that $y \in D$ if and only if $y \notin f(y)$. Since we know $f(y) = D$, the above statement means that $y \in D$ if and only if $y \notin D$. This is impossible. We have reached a contradiction, so our assumption must have been wrong. Therefore, there is no $y \in S$ such that $f(y) = D$, so $f$ is not surjective. ■
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Proof: Let $S$ be an arbitrary set. We will prove that $|S| \neq |\wp(S)|$ by showing that there are no bijections from $S$ to $\wp(S)$. To do so, choose an arbitrary function $f : S \to \wp(S)$. We will prove that $f$ is not surjective.
**Theorem:** If $S$ is a set, then $|S| \neq |\wp(S)|$.

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Starting with $f$, we define the set

$$D = \{ x \in S \mid x \notin f(x) \}.$$
**Theorem:** If $S$ is a set, then $|S| \neq |\wp(S)|$.

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**Theorem:** If $S$ is a set, then $|S| \neq |\wp(S)|$.

**Proof:** Let $S$ be an arbitrary set. We will prove that $|S| \neq |\wp(S)|$ by showing that there are no bijections from $S$ to $\wp(S)$. To do so, choose an arbitrary function $f : S \rightarrow \wp(S)$. We will prove that $f$ is not surjective.

Starting with $f$, we define the set

$$D = \{ x \in S \mid x \notin f(x) \}.$$

We will show that there is no $y \in S$ such that $f(y) = D$. To do so, we proceed by contradiction. Suppose that there is some $y \in S$ such that $f(y) = D$. From the definition of $D$, we know that

$$y \in D \text{ if and only if } y \notin f(y).$$
**Theorem:** If $S$ is a set, then $|S| \neq |\varnothing(S)|$.

**Proof:** Let $S$ be an arbitrary set. We will prove that $|S| \neq |\varnothing(S)|$ by showing that there are no bijections from $S$ to $\varnothing(S)$. To do so, choose an arbitrary function $f : S \rightarrow \varnothing(S)$. We will prove that $f$ is not surjective.

Starting with $f$, we define the set

$$D = \{ x \in S \mid x \notin f(x) \}.$$

We will show that there is no $y \in S$ such that $f(y) = D$. To do so, we proceed by contradiction. Suppose that there is some $y \in S$ such that $f(y) = D$. From the definition of $D$, we know that

$$y \in D \text{ if and only if } y \notin f(y).$$

Since we know $f(y) = D$, the above statement means that

$$y \in D \text{ if and only if } y \notin D.$$
**Theorem:** If \( S \) is a set, then \(|S| \neq |\wp(S)|\).

**Proof:** Let \( S \) be an arbitrary set. We will prove that \(|S| \neq |\wp(S)|\) by showing that there are no bijections from \( S \) to \( \wp(S) \). To do so, choose an arbitrary function \( f : S \to \wp(S) \). We will prove that \( f \) is not surjective.

Starting with \( f \), we define the set

\[
D = \{ x \in S \mid x \notin f(x) \}.
\]

We will show that there is no \( y \in S \) such that \( f(y) = D \). To do so, we proceed by contradiction. Suppose that there is some \( y \in S \) such that \( f(y) = D \). From the definition of \( D \), we know that

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y \in D \text{ if and only if } y \notin f(y).
\]

Since we know \( f(y) = D \), the above statement means that

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\]
**Theorem:** If $S$ is a set, then $|S| \neq |\wp(S)|$.

**Proof:** Let $S$ be an arbitrary set. We will prove that $|S| \neq |\wp(S)|$ by showing that there are no bijections from $S$ to $\wp(S)$. To do so, choose an arbitrary function $f : S \rightarrow \wp(S)$. We will prove that $f$ is not surjective.

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**Theorem:** If $S$ is a set, then $|S| \neq |\mathcal{P}(S)|$.

**Proof:** Let $S$ be an arbitrary set. We will prove that $|S| \neq |\mathcal{P}(S)|$ by showing that there are no bijections from $S$ to $\mathcal{P}(S)$. To do so, choose an arbitrary function $f : S \to \mathcal{P}(S)$. We will prove that $f$ is not surjective.

Starting with $f$, we define the set
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D = \{ x \in S \mid x \notin f(x) \}.
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We will show that there is no $y \in S$ such that $f(y) = D$. To do so, we proceed by contradiction. Suppose that there is some $y \in S$ such that $f(y) = D$. From the definition of $D$, we know that
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Since we know $f(y) = D$, the above statement means that
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This is impossible.
**Theorem:** If $S$ is a set, then $|S| \neq |\wp(S)|$.

**Proof:** Let $S$ be an arbitrary set. We will prove that $|S| \neq |\wp(S)|$ by showing that there are no bijections from $S$ to $\wp(S)$. To do so, choose an arbitrary function $f : S \to \wp(S)$. We will prove that $f$ is not surjective.

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Starting with $f$, we define the set

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We will show that there is no $y \in S$ such that $f(y) = D$. To do so, we proceed by contradiction. Suppose that there is some $y \in S$ such that $f(y) = D$. From the definition of $D$, we know that

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Since we know $f(y) = D$, the above statement means that

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This is impossible. We have reached a contradiction, so our assumption must have been wrong. Therefore, there is no $y \in S$ such that $f(y) = D$, so $f$ is not surjective. \[\blacksquare\]
The Big Recap

- We define equal cardinality in terms of bijections between sets.
- Lots of different sets of infinite size have the same cardinality.
- Although the syntax of set cardinality makes it look like we’re working with quantities, we are really working in terms of bijections between sets.
- Cantor’s theorem can be formalized in terms of surjectivity.
Next Time

• **Graphs**
  • A ubiquitous, expressive, and flexible abstraction!

• **Properties of Graphs**
  • Building high-level structures out of lower-level ones!
Appendix: More Function Proofs
**Proof 1:** Composing surjections yields a surjection.
**Theorem:** If \( f : A \to B \) is surjective and \( g : B \to C \) is surjective, then \( g \circ f : A \to C \) is also surjective.
**Theorem:** If \( f : A \to B \) is surjective and \( g : B \to C \) is surjective, then \( g \circ f : A \to C \) is also surjective.

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Theorem: If $f : A \to B$ is surjective and $g : B \to C$ is surjective, then $g \circ f : A \to C$ is also surjective.

Proof: Let $f : A \to B$ and $g : B \to C$ be arbitrary surjections.
**Theorem:** If \( f : A \to B \) is surjective and \( g : B \to C \) is surjective, then \( g \circ f : A \to C \) is also surjective.

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**Theorem:** If $f : A \to B$ is surjective and $g : B \to C$ is surjective, then $g \circ f : A \to C$ is also surjective.

**Proof:** Let $f : A \to B$ and $g : B \to C$ be arbitrary surjections. We will prove that the function $g \circ f : A \to C$ is also surjective.

What does it mean for $g \circ f : A \to C$ to be surjective?
**Theorem:** If \( f : A \to B \) is surjective and \( g : B \to C \) is surjective, then \( g \circ f : A \to C \) is also surjective.

**Proof:** Let \( f : A \to B \) and \( g : B \to C \) be arbitrary surjections. We will prove that the function \( g \circ f : A \to C \) is also surjective.

What does it mean for \( g \circ f : A \to C \) to be surjective?

\[ \forall c \in C. \exists a \in A. (g \circ f)(a) = c \]
**Theorem:** If $f : A \to B$ is surjective and $g : B \to C$ is surjective, then $g \circ f : A \to C$ is also surjective.

**Proof:** Let $f : A \to B$ and $g : B \to C$ be arbitrary surjections. We will prove that the function $g \circ f : A \to C$ is also surjective.

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$$\forall c \in C. \exists a \in A. (g \circ f)(a) = c$$

Therefore, we'll choose an arbitrary $c \in C$ and prove that there is some $a \in A$ such that $(g \circ f)(a) = c$. 

**Theorem:** If $f : A \to B$ is surjective and $g : B \to C$ is surjective, then $g \circ f : A \to C$ is also surjective.

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$$\forall c \in C. \exists a \in A. (g \circ f)(a) = c$$

Therefore, we’ll choose an arbitrary $c \in C$ and prove that there is some $a \in A$ such that $(g \circ f)(a) = c$. 
**Theorem:** If \( f : A \rightarrow B \) is surjective and \( g : B \rightarrow C \) is surjective, then \( g \circ f : A \rightarrow C \) is also surjective.

**Proof:** Let \( f : A \rightarrow B \) and \( g : B \rightarrow C \) be arbitrary surjections. We will prove that the function \( g \circ f : A \rightarrow C \) is also surjective.

What does it mean for \( g \circ f : A \rightarrow C \) to be surjective?

\[
\forall c \in C. \exists a \in A. \ (g \circ f)(a) = c
\]

Therefore, we’ll choose an arbitrary \( c \in C \) and prove that there is some \( a \in A \) such that \( (g \circ f)(a) = c \).
**Theorem:** If $f : A \to B$ is surjective and $g : B \to C$ is surjective, then $g \circ f : A \to C$ is also surjective.

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Therefore, we'll choose an arbitrary $c \in C$ and prove that there is some $a \in A$ such that $(g \circ f)(a) = c$. 
**Theorem:** If \( f : A \rightarrow B \) is surjective and \( g : B \rightarrow C \) is surjective, then \( g \circ f : A \rightarrow C \) is also surjective.

**Proof:** Let \( f : A \rightarrow B \) and \( g : B \rightarrow C \) be arbitrary surjections. We will prove that the function \( g \circ f : A \rightarrow C \) is also surjective. To do so, we will prove that for any \( c \in C \), there is some \( a \in A \) such that \( (g \circ f)(a) = c \).
**Theorem:** If $f : A \to B$ is surjective and $g : B \to C$ is surjective, then $g \circ f : A \to C$ is also surjective.

**Proof:** Let $f : A \to B$ and $g : B \to C$ be arbitrary surjections. We will prove that the function $g \circ f : A \to C$ is also surjective. To do so, we will prove that for any $c \in C$, there is some $a \in A$ such that $(g \circ f)(a) = c$. Equivalently, we will prove that for any $c \in C$, there is some $a \in A$ such that $g(f(a)) = c$. 

![Diagram showing surjections between sets A, B, and C.](image)
**Theorem:** If \( f : A \to B \) is surjective and \( g : B \to C \) is surjective, then \( g \circ f : A \to C \) is also surjective.

**Proof:** Let \( f : A \to B \) and \( g : B \to C \) be arbitrary surjections. We will prove that the function \( g \circ f : A \to C \) is also surjective. To do so, we will prove that for any \( c \in C \), there is some \( a \in A \) such that \( (g \circ f)(a) = c \). Equivalently, we will prove that for any \( c \in C \), there is some \( a \in A \) such that \( g(f(a)) = c \).

Consider any \( c \in C \).
**Theorem:** If $f : A \rightarrow B$ is surjective and $g : B \rightarrow C$ is surjective, then $g \circ f : A \rightarrow C$ is also surjective.

**Proof:** Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be arbitrary surjections. We will prove that the function $g \circ f : A \rightarrow C$ is also surjective. To do so, we will prove that for any $c \in C$, there is some $a \in A$ such that $(g \circ f)(a) = c$. Equivalently, we will prove that for any $c \in C$, there is some $a \in A$ such that $g(f(a)) = c$.

Consider any $c \in C$. Since $g : B \rightarrow C$ is surjective, there is some $b \in B$ such that $g(b) = c$. 

\[ a \rightarrow f \rightarrow b \rightarrow g \rightarrow c \]
**Theorem:** If $f : A \to B$ is surjective and $g : B \to C$ is surjective, then $g \circ f : A \to C$ is also surjective.

**Proof:** Let $f : A \to B$ and $g : B \to C$ be arbitrary surjections. We will prove that the function $g \circ f : A \to C$ is also surjective. To do so, we will prove that for any $c \in C$, there is some $a \in A$ such that $(g \circ f)(a) = c$. Equivalently, we will prove that for any $c \in C$, there is some $a \in A$ such that $g(f(a)) = c$.

Consider any $c \in C$. Since $g : B \to C$ is surjective, there is some $b \in B$ such that $g(b) = c$. Similarly, since $f : A \to B$ is surjective, there is some $a \in A$ such that $f(a) = b$. 
**Theorem:** If \( f : A \to B \) is surjective and \( g : B \to C \) is surjective, then \( g \circ f : A \to C \) is also surjective.

**Proof:** Let \( f : A \to B \) and \( g : B \to C \) be arbitrary surjections. We will prove that the function \( g \circ f : A \to C \) is also surjective. To do so, we will prove that for any \( c \in C \), there is some \( a \in A \) such that \((g \circ f)(a) = c\). Equivalently, we will prove that for any \( c \in C \), there is some \( a \in A \) such that \( g(f(a)) = c \).

Consider any \( c \in C \). Since \( g : B \to C \) is surjective, there is some \( b \in B \) such that \( g(b) = c \). Similarly, since \( f : A \to B \) is surjective, there is some \( a \in A \) such that \( f(a) = b \). Then we see that

\[
g(f(a)) = g(b) = c,
\]
which is what we needed to show.
**Theorem:** If \( f : A \to B \) is surjective and \( g : B \to C \) is surjective, then \( g \circ f : A \to C \) is also surjective.

**Proof:** Let \( f : A \to B \) and \( g : B \to C \) be arbitrary surjections. We will prove that the function \( g \circ f : A \to C \) is also surjective. To do so, we will prove that for any \( c \in C \), there is some \( a \in A \) such that \((g \circ f)(a) = c\). Equivalently, we will prove that for any \( c \in C \), there is some \( a \in A \) such that \( g(f(a)) = c \).

Consider any \( c \in C \). Since \( g : B \to C \) is surjective, there is some \( b \in B \) such that \( g(b) = c \). Similarly, since \( f : A \to B \) is surjective, there is some \( a \in A \) such that \( f(a) = b \). Then we see that

\[
g(f(a)) = g(b) = c,
\]

which is what we needed to show. ■
Proof 2: $|A| = |A|$. 

**Theorem:** For any set $A$, we have $|A| = |A|$.

**Proof:** Consider any set $A$, and let $f : A \to A$ be the function defined as $f(x) = x$. We will prove that $f$ is a bijection.

First, we’ll show that $f$ is injective. Pick any $x_1, x_2 \in A$ where $f(x_1) = f(x_2)$. We need to show that $x_1 = x_2$. Since $f(x_1) = f(x_2)$, we see by definition of $f$ that $x_1 = x_2$, as required.

Next, we’ll show that $f$ is surjective. Consider any $y \in A$. We will prove that there is some $x \in A$ where $f(x) = y$. Pick $x = y$. Then $x \in A$ (since $y \in A$) and $f(x) = x = y$, as required. ■
\textbf{Proof 3:} \(|A| = |B|\) and \(|B| = |C|\) means that \(|A| = |C|\).
Theorem: If $A$, $B$, and $C$ are sets where $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$.

Proof: Consider any sets $A$, $B$, and $C$ where $|A| = |B|$ and $|B| = |C|$. We need to prove that $|A| = |C|$. To do so, we need to show that there is a bijection from $A$ to $C$.

Since $|A| = |B|$, we know that there is a some bijection $f : A \to B$. Similarly, since $|B| = |C|$ we know that there is at least one bijection $g : B \to C$.

Consider the function $g \circ f : A \to C$. Since $g$ and $f$ are bijections, their composition is a bijection. Thus $g \circ f$ is a bijection from $A$ to $C$, so $|A| = |C|$, as required. ■
Proof 4: \( |A| = |B| \) means \( |B| = |A| \).
Inverse Functions

• If \( f : A \rightarrow B \) is a function, the \textit{inverse of } \( f \), denoted \( f^{-1} \), is a function \( f^{-1} : B \rightarrow A \) where the following is true:

\[
\forall a \in A. \ \forall b \in B. \ (f(a) = b \iff f^{-1}(b) = a)
\]

• \textbf{Theorem:} A function \( f \) has an inverse if and only if \( f \) is a bijection.
  • It’s worth thinking about why this is! This isn’t obvious, but it makes sense with some tinkering.

• \textbf{Theorem:} If \( f \) is a bijection, then so is \( f^{-1} \).
  • Intuition: \( f^{-1} \) has \( f \) as its inverse, so the above theorem says that \( f^{-1} \) must be a bijection.

• These are great exercises if you’re up for a challenge!
**Theorem:** If $A$ and $B$ are sets where $|A| = |B|$, then $|B| = |A|$.

**Proof:** Pick any sets $A$ and $B$ where $|A| = |B|$. We need to show that $|B| = |A|$.

Since $|A| = |B|$, we know that there is a bijection $f : A \to B$. Therefore, the function $f^{-1} : B \to A$ exists and is a bijection. Thus there is a bijection from $B$ to $A$, so we conclude that $|B| = |A|$.