Graph Theory
Part Two
Outline for Today

- **Walks, Paths, and Reachability**
  - Walking around a graph.

- **Graph Complements**
  - Flipping what’s in a graph.

- **The Teleported Train Problem**
  - A very exciting commute.

- **Appendix: The CBS Theorem**
  - Cardinality meets graph theory!
Recap from Last Time
Graphs and Digraphs

- A **graph** is a pair $G = (V, E)$ of a set of nodes $V$ and set of edges $E$.
  - Nodes can be anything.
  - Edges are *unordered pairs* of nodes. If $\{u, v\} \in E$, then there’s an edge from $u$ to $v$.

- A **digraph** is a pair $G = (V, E)$ of a set of nodes $V$ and set of directed edges $E$.
  - Each edge is represented as the ordered pair $(u, v)$ indicating an edge from $u$ to $v$. 

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Using our Formalisms

• Let $G = (V, E)$ be an (undirected) graph.
• Intuitively, two nodes are adjacent if they're linked by an edge.
• Formally speaking, we say that two nodes $u, v \in V$ are adjacent if we have $\{u, v\} \in E$.
• There isn’t an analogous notion for directed graphs. We usually just say “there’s an edge from $u$ to $v$” as a way of reading $(u, v) \in E$ aloud.
New Stuff!
Walks, Paths, and Reachability
To
SF, Sac, Port, Sea
From

SF, Sac, Port, Sea
But
Mon
LV
Bar
Flag

From

SF, Sac, SLC, Port, Sea

To

SF
Sac
Port
Sea

But

Nog
Phoe

F
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Sea, But, SLC, Port, Sea
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(Sac, SLC, Port, Sac, SLC, Port, Sac)

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What is the length of the longest walk in this graph? Path in this graph? Closed walk? Cycle?
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A graph $G$ is called connected if all pairs of distinct nodes in $G$ are reachable.
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A connected component (or \( CC \)) of \( G \) is a maximal set of mutually reachable nodes.
Fun Facts

- Here’s a collection of useful facts about graphs that you can take as a given.
  
  - **Theorem:** If $G = (V, E)$ is a graph and $u, v \in V$, then there is a path from $u$ to $v$ if and only if there’s a walk from $u$ to $v$.
  
  - **Theorem:** If $G$ is a graph and $C$ is a cycle in $G$, then $C$’s length is at least three and $C$ contains at least three nodes.
  
  - **Theorem:** If $G = (V, E)$ is a graph, then every node in $V$ belongs to exactly one connected component of $G$.
  
  - **Theorem:** If $G = (V, E)$ is a graph, then $G$ is connected if and only if $G$ has exactly one connected component.

- Looking for more practice working with formal definitions? Prove these results!
Graph Complements
Let $G = (V, E)$ be an undirected graph.
The **complement of $G$** is the graph $G^c = (V, E^c)$, where

$$E^c = \{ \{u, v\} \mid u \in V, v \in V, u \neq v, \text{ and } \{u, v\} \notin E \}$$
Graph $G$ isn't connected.
Graph $G^c$ is connected.
Graph \( G \) is connected.
Graph $G^c$ isn’t connected.
**Theorem**: For any graph $G = (V, E)$, at least one of $G$ and $G^c$ is connected.
Proving a Disjunction

• We need to prove the statement

\[ G \text{ is connected} \quad \lor \quad G^c \text{ is connected.} \]

• Here’s a neat observation.
  • If \( G \) is connected, we’re done.
  • Otherwise, \( G \) isn’t connected, and we have to prove that \( G^c \) is connected.

• We will therefore prove

\[ G \text{ is not connected} \quad \rightarrow \quad G^c \text{ is connected.} \]

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Each bubble represents one connected component of $G$.

$G^c$ is connected if, for any distinct nodes $u$ and $v$, there's a path from $u$ to $v$. 
For any graph $G = (V, E)$, if $G$ is not connected, then $G^c$ is connected.
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Proof:
Theorem: If $G = (V, E)$ is a graph, then at least one of $G$ and $G^c$ is connected.

Proof: Let $G = (V, E)$ be an arbitrary graph and assume $G$ is not connected.
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[Further detailed proof text here]
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*Case 1:* $u$ and $v$ are in different connected components of $G$.

*Case 2:* $u$ and $v$ are in the same connected component of $G$. 


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*Case 2:* $u$ and $v$ are in the same connected component of $G$. 

In either case, we find a path from $u$ to $v$ in $G^c$, as required. ■
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Time-Out for Announcements!
Problem Set Two Graded

75th Percentile: 72 / 76 (94%)
50th Percentile: 69 / 76 (91%)
25th Percentile: 63 / 76 (83%)
Midterm Exam Logistics

- Our first midterm exam is next Monday, October 24\(^{th}\), from 7:00PM – 10:00PM.
  - Check the course website for logistics.
- We will have class on Friday, but there’s no lecture on Monday since we have the midterm then.
- If you cannot make this exam time, or if you have OAE accommodations you haven’t shared with us, please let us know ASAP.
Preparing for the Exam

• Make sure to review your feedback on PS1 and PS2.
  • “Make new mistakes.”
  • Come talk to us if you have questions!
• There’s a huge bank of practice problems up on the course website.
  • We had a request to add a question involving code, and that’s now available! Feel free to request other types of problems if you’d like.
• Best of luck – you can do this!
The Center for Teaching and Learning has tutoring support available if you’re looking for extra assistance.

Want to learn more? Check out their website or pick up a bookmark from up front.
Back to CS103!
The Teleported Train Problem
These are *teleporters*. Anything entering a teleporter from the left side emerges from the right side of the paired teleporter.
It took a while, but eventually the train reached the end of the track.
Will the train reach the end of the track? Or will it get stuck in a loop?
The train gets trapped if it starts here and only moves right.
Can You Trap the Train?

- The train always drives to the right.
- The train starts just before the first teleporter.
- Teleporters always link in pairs.
- Teleporters can’t stack on top of one another.
- Teleporters can’t appear at or after the end point.
- You can use as many teleporters as you’d like.
\[ s \rightarrow A_1 \rightarrow C_1 \rightarrow B_1 \rightarrow E_1 \rightarrow E_2 \rightarrow C_2 \rightarrow A_2 \rightarrow D_1 \rightarrow B_2 \rightarrow D_2 \rightarrow f \]
The Teleporter Digraph

- Each line of teleporters gives rise to a directed graph.
  - Each node in the graph represents a segment.
  - Each edge represents following a teleporter.
- That digraph consists of paths and cycles.
- **Question:** Why does the digraph look like this?
The Teleporter Digraph

- In a directed graph, the **indegree** of a node is the number of edges entering that node. The **outdegree** of a node is the number of edges leaving that node.
- Notice anything about the indegrees and outdegrees of this digraph?
The Teleporter Digraph

- Let $G = (V, E)$ be a digraph where each node’s indegree is at most one and each node’s outdegree is at most one.

- **Theorem:** Any walk starting at a node of indegree zero is also a path.
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This node now has indegree two.
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**Theorem:** Let $G = (V, E)$ be a directed graph where each node has indegree at most one and outdegree at most one. Consider any walk $T$ beginning at a node $v_0$ of indegree zero. Then $T$ is a path.

**Proof:** Suppose for the sake of contradiction that $T$ is not a path, meaning that it contains a repeated node. List the nodes in $T$, stopping just before we list the first repeated node. Label the nodes found this way as $v_0, v_1, v_2, v_3, \ldots, v_k$.

Nodes $v_0, v_1, \ldots,$ and $v_k$ are distinct because we've stopped just before revisiting a node. We also know that the next node in the walk (call it $r$) is a repeated node, with $(v_k, r)$ being a directed edge in $E$. We now ask: which earlier node is $r$ equal to?

**Case 1:** $r = v_0$. This means that $(v_k, v_0)$ is a directed edge, which is impossible because $v_0$ has indegree zero.

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In either case we've reached a contradiction, so our assumption must have been wrong. Thus $T$ is a path. ■
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Trapping the Train
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The train begins before the first teleporter, so the start node has indegree zero.
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The only node of outdegree zero is the one after the last teleporter, where the goal is.
Theorem: It is impossible to trap the train if it starts before the first teleporter.
**Theorem:** It is not possible to trap the train in the Teleported Train Problem.

**Proof:** Consider any arrangement of teleporters. We will prove that the train makes it to the end without getting stuck in a loop.

Divide the train track into segments denoting the ranges between two teleporters or between a teleporter and the start/end of the track. From these segments, construct a directed graph whose nodes are the segments and where there’s an edge from a segment $S_1$ to a segment $S_2$ if, upon reaching the end of segment $S_1$, the train teleports to the start of segment $S_2$.

We claim that every node in this graph has indegree at most one and outdegree at most one. To see this, pick any segment. If that segment begins with a teleporter, then it has one incoming edge that originates at the segment that ends with the paired teleporter. If that segment ends with a teleporter, then it has one outgoing edge to the start of the segment with the paired teleporter.

Now, consider the walk traced out by the train from the starting segment. That segment has indegree zero because it does not begin with a teleporter, so by our previous theorem this walk is a path. There are only finitely many segments and our path never revisits one, so eventually the path ends at a node with outdegree zero. The only node with this property is the end segment, so the train eventually reaches the end of the track. ■
Recap for Today

- We can use walks and closed walks to travel around a graph. Walks and closed walks that don’t repeat nodes or edges are called paths and cycles, respectively.

- The complement of a graph is a graph formed by toggling which edges are included and which are excluded. At least one of a graph and its complement will always be connected.

- The indegree and outdegree of a node in a digraph are the number of edges entering or leaving the node, respectively.

- Digraphs where the indegree and outdegree of each node are at most one break apart into isolated paths and cycles.

- You can’t trap a train on a track with teleporters, unless there’s a teleporter behind the train.
Next Time

• **The Pigeonhole Principle**
  • A simple, powerful, versatile theorem.

• **Graph Theory Party Tricks**
  • Applying math to graphs of people!

• **A Little Movie Puzzle**
  • Who watched what?
Appendix: The Cantor-Bernstein-Schroeder Theorem
Theorem (Cantor-Bernstein-Schroeder):
Let $S$ and $T$ be sets. If $|S| \leq |T|$ and $|T| \leq |S|$, then $|S| = |T|$.

(This was first proven by Richard Dedekind.)
**Theorem (Cantor-Bernstein-Schroeder):**
Let $S$ and $T$ be sets. If there is an injection $f : S \to T$ and an injection $g : T \to S$, then there is a bijection $h : S \to T$. 
The open interval $(0, 1)$

The closed interval $[0, 1]$

\[ f(x) = \frac{x}{2} + \frac{1}{4} \]

**Theorem (Cantor-Bernstein-Schroeder):** Let $S$ and $T$ be sets. If there is an injection $f : S \rightarrow T$ and an injection $g : T \rightarrow S$, then there is a bijection $h : S \rightarrow T$. 
The open interval $(0, 1)$

The closed interval $[0, 1]$}

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\[ g(x) = \frac{x}{2} + \frac{1}{4} \]
The open interval \((0, 1)\)

The closed interval \([0, 1]\)

**Theorem (Cantor-Bernstein-Schroeder):** Let \(S\) and \(T\) be sets. If there is an injection \(f : S \rightarrow T\) and an injection \(g : T \rightarrow S\), then there is a bijection \(h : S \rightarrow T\).
Theorem (Cantor-Bernstein-Schroeder):

Let $S$ and $T$ be sets. If there is an injection $f : S \to T$ and an injection $g : T \to S$, then there is a bijection $h : S \to T$.

$f : \mathbb{N} \to \mathbb{N}^2$
$g : \mathbb{N}^2 \to \mathbb{N}$

$f(n) = (0, n)$
$g(m, n) = 2^m \cdot 3^n$

These functions are injective.

**Challenge**: Find a bijection $h : \mathbb{N} \to \mathbb{N}^2$.

**Theorem (Cantor-Bernstein-Schroeder)**: Let $S$ and $T$ be sets. If there is an injection $f : S \to T$ and an injection $g : T \to S$, then there is a bijection $h : S \to T$. 
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**Theorem (Cantor-Bernstein-Schröder):** Let $S$ and $T$ be sets. If there is an injection $f : S \rightarrow T$ and an injection $g : T \rightarrow S$, then there is a bijection $h : S \rightarrow T$. 

Blue lines represent the injection $f : S \rightarrow T$. 

Red lines represent the injection $g : T \rightarrow S$. 

The blue lines and red lines illustrate the mappings of elements from $S$ to $T$ and from $T$ to $S$, respectively.
Theorem (Cantor-Bernstein-Schroeder): Let $S$ and $T$ be sets. If there is an injection $f : S \rightarrow T$ and an injection $g : T \rightarrow S$, then there is a bijection $h : S \rightarrow T$. 

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**Theorem (Cantor-Bernstein-Schroeder):** Let $S$ and $T$ be sets. If there is an injection $f : S \to T$ and an injection $g : T \to S$, then there is a bijection $h : S \to T$. 

Blue lines represent the injection $f : S \to T$
Red lines represent the injection $g : T \to S$
Blue lines represent the injection $f : S \rightarrow T$

Red lines represent the injection $g : T \rightarrow S$

Every node in this (possibly infinite) digraph has outdegree one and indegree at most one. Therefore, the digraph consists of a mix of paths and cycles.

**Theorem (Cantor-Bernstein-Schroeder):** Let $S$ and $T$ be sets. If there is an injection $f : S \rightarrow T$ and an injection $g : T \rightarrow S$, then there is a bijection $h : S \rightarrow T$. 
Blue lines represent the injection \( f : S \to T \)
Red lines represent the injection \( g : T \to S \)

For nodes within a cycle, define the bijection from \( S \) to \( T \) to be “follow the blue arrows.”

**Theorem (Cantor-Bernstein-Schröder):** Let \( S \) and \( T \) be sets. If there is an injection \( f : S \to T \) and an injection \( g : T \to S \), then there is a bijection \( h : S \to T \).
For nodes in a path starting at a red node, have the bijection from $S$ to $T$ be “follow the red arrows in reverse.”

**Blue lines** represent the injection $f : S \to T$

**Red lines** represent the injection $g : T \to S$

**Theorem (Cantor-Bernstein-Schröder):** Let $S$ and $T$ be sets. If there is an injection $f : S \to T$ and an injection $g : T \to S$, then there is a bijection $h : S \to T$. 
For nodes in any other path, have the bijection from $S$ to $T$ be "follow the blue arrows."

**Theorem (Cantor-Bernstein-Schroeder):** Let $S$ and $T$ be sets. If there is an injection $f : S \to T$ and an injection $g : T \to S$, then there is a bijection $h : S \to T$. 

*Blue lines* represent the injection $f : S \to T$

*Red lines* represent the injection $g : T \to S$