Context-Free Grammars

Context-Free Grammars

- A ***context-free grammar*** (or ***CFG***) is a new formalism for defining a class of languages.
- ***Goal:*** Give a description of a language by recursively describing the structure of the strings in the language.
- CFGs are best explained by example…

Arithmetic Expressions

- Suppose we want to describe all legal arithmetic expressions using addition, subtraction, multiplication, and division.
- Here is one possible CFG:

```
E → int
E → E Op E
E → (E)
Op → +
Op → -
Op → *
Op → /
```

```
  E
⇒ E Op E
⇒ E Op (E)
⇒ E Op (E Op E)
⇒ E * (E Op E)
⇒ int * (E Op E)
⇒ int * (int Op E)
⇒ int * (int Op int)
⇒ int * (int + int)
```

Arithmetic Expressions

- Suppose we want to describe all legal arithmetic expressions using addition, subtraction, multiplication, and division.
- Here is one possible CFG:

$$
\boxed{
\begin{array}{l}
\mathbf{E} \rightarrow \texttt{int} \\
\mathbf{E} \rightarrow \mathbf{E\ Op\ E} \\
\mathbf{E} \rightarrow \texttt{(}\mathbf{E}\texttt{)} \\
\mathbf{Op} \rightarrow \texttt{+} \\
\mathbf{Op} \rightarrow \texttt{-} \\
\mathbf{Op} \rightarrow \texttt{*} \\
\mathbf{Op} \rightarrow \texttt{/}
\end{array}
}
$$

$$
\begin{array}{l}
\quad \mathbf{E} \\
\Rightarrow \mathbf{E\ Op\ E} \\
\Rightarrow \mathbf{E\ Op}\ \texttt{int} \\
\Rightarrow \texttt{int}\ \mathbf{Op}\ \texttt{int} \\
\Rightarrow \texttt{int / int}
\end{array}
$$

Context-Free Grammars

- Formally, a context-free grammar is a collection of four items:
  - A set of ***nonterminal symbols*** (also called ***variables***),

E → `int`

E → **E Op E**

E → `(`**E**`)`

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  - A set of ***terminal symbols*** (the ***alphabet*** of the CFG)

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Context-Free Grammars

- Formally, a context-free grammar is a collection of four items:
  - A set of ***nonterminal symbols*** (also called ***variables***),
  - A set of ***terminal symbols*** (the ***alphabet*** of the CFG)
  - A set of ***production rules*** saying how each nonterminal can be replaced by a string of terminals and nonterminals, and
  - A ***start symbol*** (which must be a nonterminal) that begins the derivation.

E → `int`

E → **E Op E**

E → `(`**E**`)`

Op → `+`

Op → `-`

Op → `*`

Op → `/`

Some CFG Notation

- In today's slides, capital letters in **Bold Red Uppercase** will represent nonterminals.
  - e.g. **A**, **B**, **C**, **D**
- Lowercase letters in **`blue monospace`** will represent terminals.
  - e.g. **`t`**, **`u`**, **`v`**, **`w`**
- Lowercase Greek letters in ***gray italics*** will represent arbitrary strings of terminals and nonterminals.
  - e.g. $\boldsymbol{\alpha}$, $\boldsymbol{\gamma}$, $\boldsymbol{\omega}$
- You don't need to use these conventions on your own; just make sure whatever you do is readable. ☺

A Notational Shorthand

$$
\begin{aligned}
&\mathbf{E} \to \texttt{int}\\
&\mathbf{E} \to \mathbf{E}\ \mathbf{Op}\ \mathbf{E}\\
&\mathbf{E} \to \texttt{(}\mathbf{E}\texttt{)}\\
&\mathbf{Op} \to \texttt{+}\\
&\mathbf{Op} \to \texttt{-}\\
&\mathbf{Op} \to \texttt{*}\\
&\mathbf{Op} \to \texttt{/}
\end{aligned}
$$

A Notational Shorthand

$$\mathbf{E} \rightarrow \texttt{int} \mid \mathbf{E}\ \mathbf{Op}\ \mathbf{E} \mid \texttt{(}\mathbf{E}\texttt{)}$$

$$\mathbf{Op} \rightarrow \texttt{+} \mid \texttt{-} \mid \texttt{*} \mid \texttt{/}$$

- The vertical bar could be read as “or,” a delimiter separating the options for each nonterminal.

Derivations

$\mathbf{E} \rightarrow \mathbf{E}\ \mathbf{Op}\ \mathbf{E} \mid \texttt{int} \mid (\mathbf{E})$
$\mathbf{Op} \rightarrow \texttt{+} \mid \texttt{*} \mid \texttt{-} \mid \texttt{/}$

$$
\begin{aligned}
& \mathbf{E} \\
\Rightarrow\ & \mathbf{E\ Op\ E} \\
\Rightarrow\ & \mathbf{E\ Op}\ \texttt{(}\mathbf{E}\texttt{)} \\
\Rightarrow\ & \mathbf{E\ Op}\ \texttt{(}\mathbf{E\ Op\ E}\texttt{)} \\
\Rightarrow\ & \mathbf{E}\ \texttt{*}\ \texttt{(}\mathbf{E\ Op\ E}\texttt{)} \\
\Rightarrow\ & \texttt{int * (}\mathbf{E\ Op\ E}\texttt{)} \\
\Rightarrow\ & \texttt{int * (int}\ \mathbf{Op\ E}\texttt{)} \\
\Rightarrow\ & \texttt{int * (int}\ \mathbf{Op}\ \texttt{int)} \\
\Rightarrow\ & \texttt{int * (int + int)}
\end{aligned}
$$

- A sequence of steps where nonterminals are replaced by the right-hand side of a production is called a ***derivation***.
- If string $\boldsymbol{\alpha}$ derives string $\boldsymbol{\omega}$, we write $\boldsymbol{\alpha} \Rightarrow^* \boldsymbol{\omega}$.
- In the example on the left, we see $\mathbf{E} \Rightarrow^* \texttt{int * (int + int)}$.

The Language of a Grammar

- If G is a CFG with alphabet Σ and start symbol **S**, then the ***language of G*** is the set

$$\mathscr{L}(G) = \{\ \boldsymbol{\omega} \in \Sigma^* \mid \mathbf{S} \Rightarrow^* \boldsymbol{\omega}\ \}$$

- That is, $\mathscr{L}(G)$ is the set of strings of terminals derivable from the start symbol.

If G is a CFG with alphabet Σ and start symbol **S**, then the ***language of G*** is the set

$$\mathscr{L}(G) = \{ \omega \in \Sigma^* \mid \mathbf{S} \Rightarrow^* \omega \}$$

Consider the following CFG G over $\Sigma = \{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}\}$:

$$\mathbf{S} \rightarrow \mathbf{Sa} \mid \mathbf{dT}$$

$$\mathbf{T} \rightarrow \mathbf{bTb} \mid \mathbf{c}$$

How many of the following strings are in $\mathscr{L}(G)$?

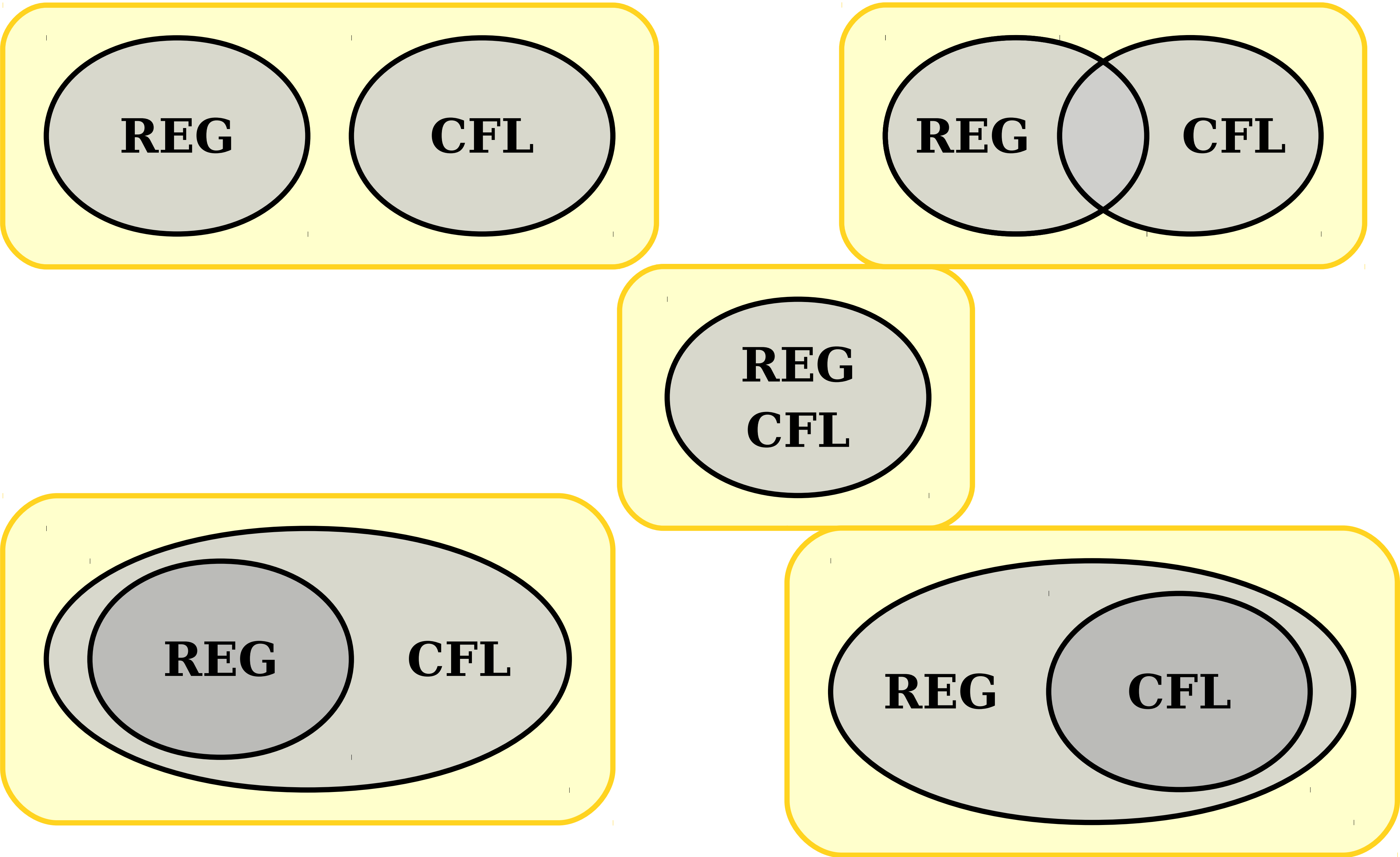
dc **dca** **cad**

bcb **dTaa**

Context-Free Languages

- A language L is called a ***context-free language*** (or CFL) if there is a CFG G such that $L = \mathscr{L}(G)$.
- Questions:
  - What languages are context-free?
  - How are context-free and regular languages related?

Five Possibilities



From Regexes to CFGs

- CFGs consist purely of production rules of the form **A** → $\boldsymbol{\omega}$. They do not have the regular expression operators * or ∪.
- However, we can convert regular expressions to CFGs as follows:

S → **`a*b`**

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$$\mathbf{S} \to \mathbf{A}\mathtt{b}$$
$$\mathbf{A} \to \mathbf{A}\mathtt{a} \mid \varepsilon$$

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$$\mathbf{X} \to \mathtt{b} \mid \mathtt{c^*}$$

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X → **b | c***

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- However, we can convert regular expressions to CFGs as follows:

S → **`a`X**

X → **`b`** | **`c*`**

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$$\mathbf{C} \rightarrow \mathbf{C}\mathtt{c} \mid \varepsilon$$

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X → `b` | `c*`
C → **C**`c` | `ε`

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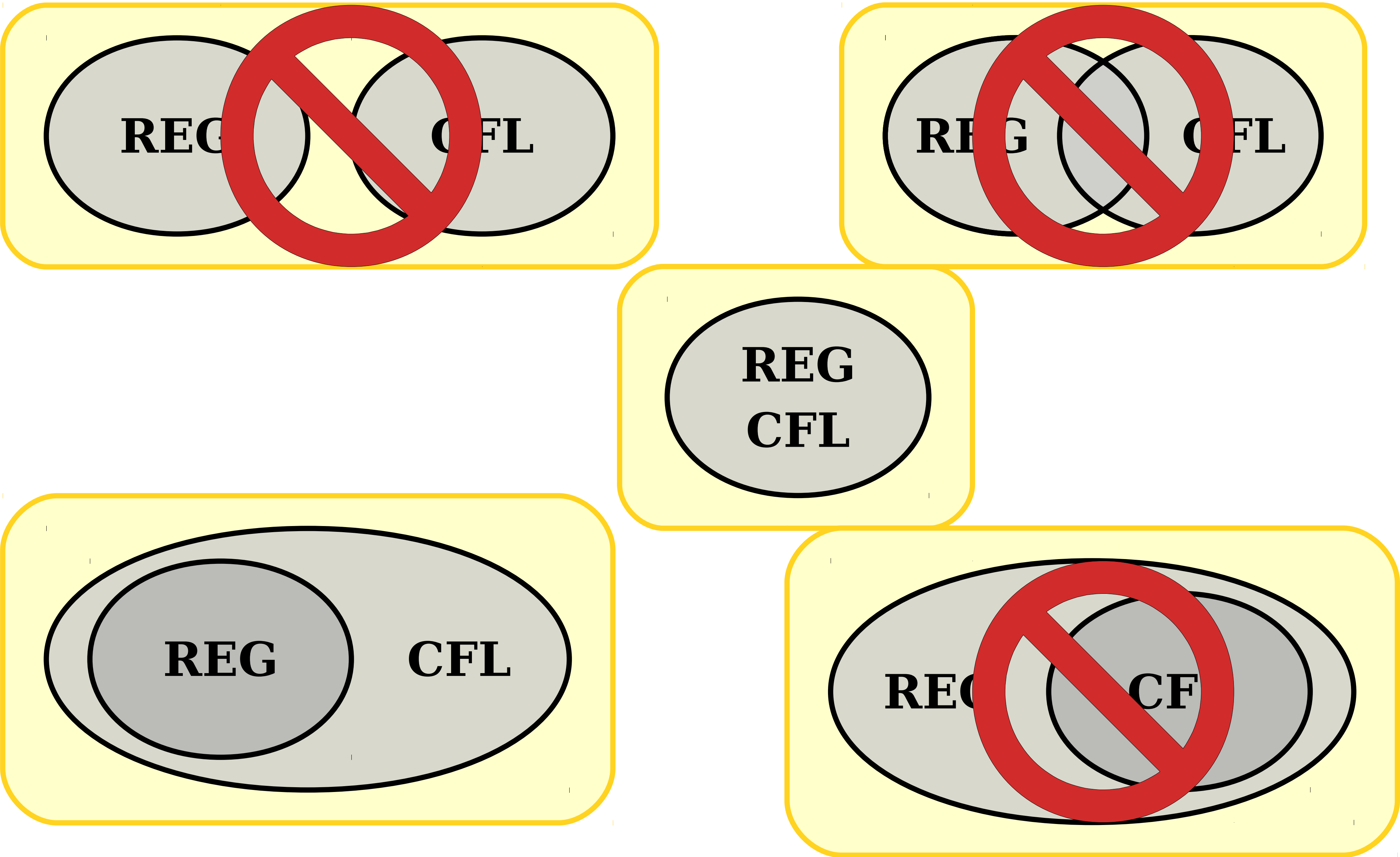
$$\mathbf{S} \rightarrow \mathtt{a}\mathbf{X}$$
$$\mathbf{X} \rightarrow \mathtt{b} \mid \mathbf{C}$$
$$\mathbf{C} \rightarrow \mathbf{C}\mathtt{c} \mid \varepsilon$$

Regular Languages and CFLs

- ***Theorem:*** Every regular language is context-free.
- ***Proof Idea:*** Use the construction from the previous slides to convert a regular expression for L into a CFG for L. ■
- ***Problem Set 8 Exercise:*** Instead, show how to convert a DFA/NFA into a CFG.

Two Possibilities



The Language of a Grammar

- Consider the following CFG G:

$$\mathbf{S} \rightarrow \mathtt{a}\mathbf{S}\mathtt{b} \mid \varepsilon$$

- What strings can this generate?

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$$\boxed{\mathbf{S}}$$

The Language of a Grammar

- Consider the following CFG G:

$$\mathbf{S} \rightarrow \mathbf{aSb} \mid \varepsilon$$

- What strings can this generate?

a	S	b

The Language of a Grammar

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$$\boxed{\mathbf{a}} \quad \boxed{\mathbf{S}} \quad \boxed{\mathbf{b}}$$

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- What strings can this generate?

a	a	S	b	b

The Language of a Grammar

- Consider the following CFG G:

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- What strings can this generate?

$$\boxed{\mathtt{a}}\boxed{\mathtt{a}} \quad \boxed{\mathbf{S}} \quad \boxed{\mathtt{b}}\boxed{\mathtt{b}}$$

The Language of a Grammar

- Consider the following CFG G:

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- What strings can this generate?

a	a	a	S	b	b	b

The Language of a Grammar

- Consider the following CFG G:

$$S \rightarrow aSb \mid \varepsilon$$

- What strings can this generate?

a	a	a	S	b	b	b

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- Consider the following CFG *G*:

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- What strings can this generate?

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- Consider the following CFG *G*:

$$\mathbf{S} \rightarrow \mathtt{a}\mathbf{S}\mathtt{b} \mid \varepsilon$$

- What strings can this generate?

a	a	a	a

b	b	b	b

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- Consider the following CFG G:

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- What strings can this generate?

a	a	a	a	b	b	b	b

The Language of a Grammar

- Consider the following CFG G:

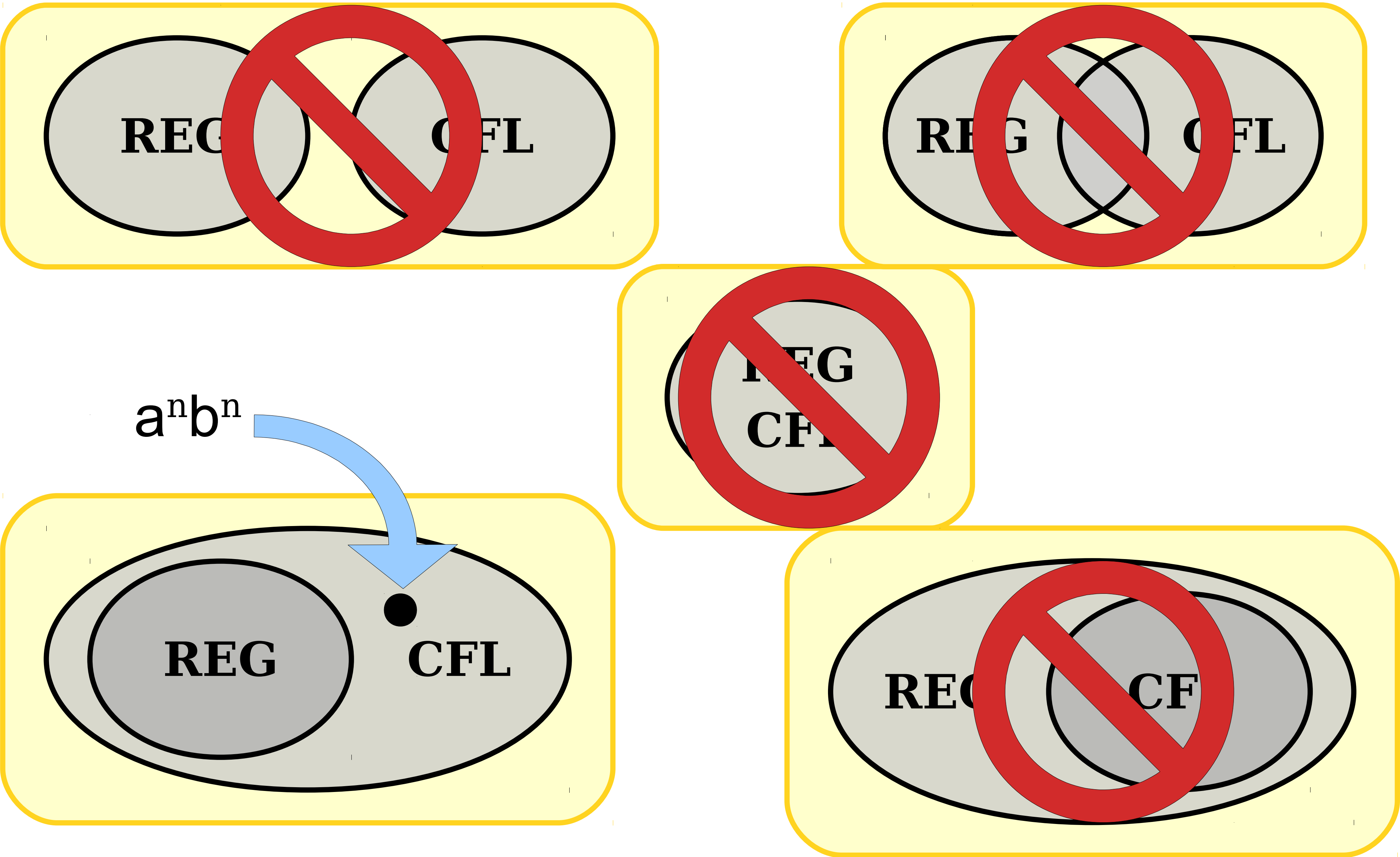
$$\mathbf{S} \rightarrow \mathtt{a}\mathbf{S}\mathtt{b} \mid \varepsilon$$

- What strings can this generate?

a	a	a	a	b	b	b	b

$$\mathscr{L}(G) = \{\ \mathtt{a}^n\mathtt{b}^n \mid n \in \mathbb{N}\ \}$$

One Possibility



Why the Extra Power?

- Why do CFGs have more power than regular expressions?
- ***Intuition:*** Derivations of strings have unbounded "memory."

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a	**S**	**b**

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S → **aSb** | **ε**

a	**a**	**S**	**b**	**b**

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a	a	a	a	S	b	b	b	b

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$$\mathbf{S} \rightarrow \mathtt{a}\mathbf{S}\mathtt{b} \mid \varepsilon$$

`a`	`a`	`a`	`a`		`b`	`b`	`b`	`b`

Why the Extra Power?

- Why do CFGs have more power than regular expressions?
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$$S \rightarrow aSb \mid \varepsilon$$

a	a	a	a	b	b	b	b

Designing CFGs

- Like designing DFAs, NFAs, and regular expressions, designing CFGs is a craft.
- When thinking about CFGs:
  - ***Think recursively:*** Build up bigger structures from smaller ones.
  - ***Have a construction plan:*** Know in what order you will build up the string.
  - ***Store information in nonterminals:*** Have each nonterminal correspond to some useful piece of information.

Designing CFGs

- Let $\Sigma = \{\mathbf{a}, \mathbf{b}\}$ and let $L = \{w \in \Sigma^* \mid w$ is a palindrome $\}$
- We can design a CFG for L by thinking inductively:
  - Base case: ε, **a**, and **b** are palindromes.
  - If ω is a palindrome, then **a**ω**a** and **b**ω**b** are palindromes.
  - No other strings are palindromes.

$$\mathbf{S} \to \boldsymbol{\varepsilon} \mid \mathbf{a} \mid \mathbf{b} \mid \mathbf{aSa} \mid \mathbf{bSb}$$

Designing CFGs

- Let $\Sigma = \{$ **(**, **)** $\}$ and let $L = \{w \in \Sigma^* \mid w$ is a string of balanced parentheses $\}$
- Some sample strings in L:

(())()

(())(())(())()

((((()))(())))

ε

()()

Designing CFGs

- Let $\Sigma = \{$ **(**, **)** $\}$ and let $L = \{ w \in \Sigma^* \mid w$ is a string of balanced parentheses $\}$
- Let's think about this recursively.
  - Base case: the empty string is a string of balanced parentheses.
  - Recursive step: Look at the closing parenthesis that matches the first open parenthesis.

((()(()))(()))(())((()))

Designing CFGs

- Let Σ = { (,) } and let L = {w ∈ Σ* | w is a string of balanced parentheses }
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(()(()))(()) (())((()))

Designing CFGs

- Let $\Sigma = \{ (,) \}$ and let $L = \{ w \in \Sigma^* \mid w \text{ is a string of balanced parentheses} \}$
- Let's think about this recursively.
  - Base case: the empty string is a string of balanced parentheses.
  - Recursive step: Look at the closing parenthesis that matches the first open parenthesis. Removing the first parenthesis and the matching parenthesis forms two new strings of balanced parentheses.

$$\mathbf{S} \rightarrow \mathbf{(S)S} \mid \boldsymbol{\varepsilon}$$

Designing CFGs

- Let $\Sigma = \{\mathtt{a}, \mathtt{b}\}$ and let $L = \{w \in \Sigma^* \mid w$ has the same number of **a**'s and **b**'s $\}$

How many of the following CFGs have language L?

S → **aSb** | **bSa** | **ε**	**S** → **abS** | **baS** | **ε**
S → **abSba** | **baSab** | **ε**	**S** → **SbaS** | **SabS** | **ε**

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S → abSba | baSab | ε	**S → SbaS | SabS | ε**

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How many of the following CFGs have language L?

S → aSb | bSa | ε

S → abS | baS | ε

S → abSba | baSab | ε

S → SbaS | SabS | ε

Designing CFGs: A Caveat

- When designing a CFG for a language, make sure that it
  - generates all the strings in the language and
  - never generates a string outside the language.
- The first of these can be tricky – make sure to test your grammars!

CFG Caveats II

- Is the following grammar a CFG for the language $\{ \mathbf{a}^n\mathbf{b}^n \mid n \in \mathbb{N} \}$?

$$\mathbf{S} \to \mathbf{a}\mathbf{S}\mathbf{b}$$

- What strings in $\{\mathbf{a}, \mathbf{b}\}^*$ can you derive?
  - Answer: ***None!***
- What is the language of the grammar?
  - Answer: $\emptyset$
- When designing CFGs, make sure your recursion actually terminates!

Designing CFGs

- When designing CFGs, remember that each nonterminal can be expanded out independently of the others.
- Let $\Sigma = \{\mathbf{a}, \stackrel{?}{=}\}$ and let $L = \{\mathbf{a}^n\stackrel{?}{=}\mathbf{a}^n \mid n \in \mathbb{N}\}$.
- Is the following a CFG for L?

$$\mathbf{S} \to \mathbf{X}\stackrel{?}{=}\mathbf{X}$$

$$\mathbf{X} \to \mathbf{aX} \mid \boldsymbol{\varepsilon}$$

$$\begin{aligned}
&\mathbf{S} \\
\Rightarrow\ &\mathbf{X}\stackrel{?}{=}\mathbf{X} \\
\Rightarrow\ &\mathbf{aX}\stackrel{?}{=}\mathbf{X} \\
\Rightarrow\ &\mathbf{aaX}\stackrel{?}{=}\mathbf{X} \\
\Rightarrow\ &\mathbf{aa}\stackrel{?}{=}\mathbf{X} \\
\Rightarrow\ &\mathbf{aa}\stackrel{?}{=}\mathbf{aX} \\
\Rightarrow\ &\mathbf{aa}\stackrel{?}{=}\mathbf{a}
\end{aligned}$$

Finding a Build Order

- Let Σ = {**a**, ≟} and let L = {**a**^{n}≟**a**^{n} | $n \in \mathbb{N}$ }.
- To build a CFG for L, we need to be more clever with how we construct the string.
  - If we build the strings of **a**'s independently of one another, then we can't enforce that they have the same length.
  - ***Idea:*** Build both strings of **a**'s at the same time.
- Here's one possible grammar based on that idea:

S → ≟ | **aSa**

S
⇒ **aSa**
⇒ **aaSaa**
⇒ **aaaSaaa**
⇒ **aaa≟aaa**

Function Prototypes

- Let Σ = {**void**, **int**, **double**, **name**, **(**, **)**, **,**, **;**}.
- Let's write a CFG for C-style function prototypes!
- Examples:
  - `void name(int name, double name);`
  - `int name();`
  - `int name(double name);`
  - `int name(int, int name, int);`
  - `void name(void);`

Function Prototypes

- Here's one possible grammar:
  - **S** → **Ret** **`name`** **`(`Args`);`**
  - **Ret** → **Type** | **`void`**
  - **Type** → **`int`** | **`double`**
  - **Args** → **ε** | **`void`** | **ArgList**
  - **ArgList** → **OneArg** | **ArgList**, **OneArg**
  - **OneArg** → **Type** | **Type** **`name`**
- Fun question to think about: what changes would you need to make to support pointer types?

Summary of CFG Design Tips

- Look for recursive structures where they exist: they can help guide you toward a solution.
- Keep the build order in mind – often, you'll build two totally different parts of the string concurrently.
  - Usually, those parts are built in opposite directions: one's built left-to-right, the other right-to-left.
- Use different nonterminals to represent different structures.

Applications of Context-Free Grammars

CFGs for Programming Languages

```
BLOCK  →  STMT
       |  { STMTS }

STMTS  →  ε
       |  STMT STMTS

STMT   →  EXPR;
       |  if (EXPR) BLOCK
       |  while (EXPR) BLOCK
       |  do BLOCK while (EXPR);
       |  BLOCK
       |  …

EXPR   →  identifier
       |  constant
       |  EXPR + EXPR
       |  EXPR – EXPR
       |  EXPR * EXPR
       |  …
```

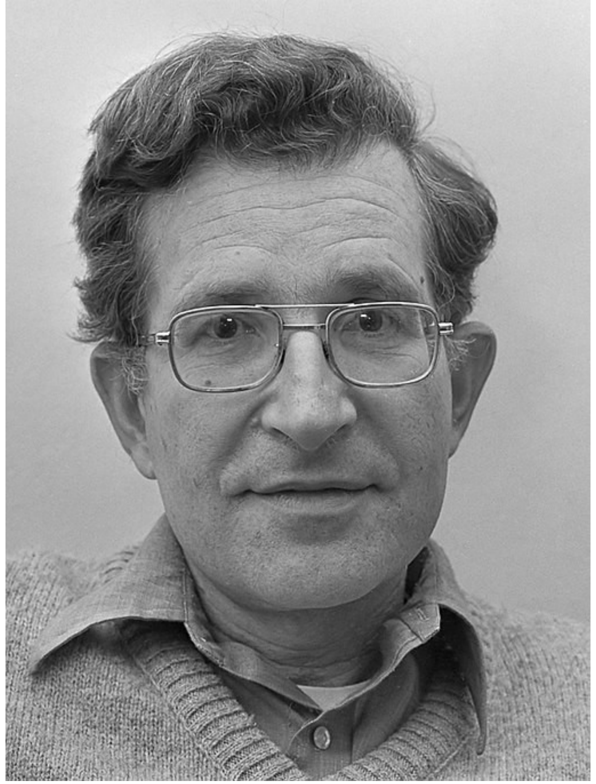
Grammars in Compilers

- One of the key steps in a compiler is figuring out what a program “means.”
- This is usually done by defining a grammar showing the high-level structure of a programming language.
- There are certain classes of grammars (LL(1) grammars, LR(1) grammars, LALR(1) grammars, etc.) for which it's easy to figure out how a particular string was derived.
- Tools like `yacc` or `bison` automatically generate parsers from these grammars.
- Curious to learn more? Take CS143!

Natural Language Processing

- By building context-free grammars for actual languages and applying statistical inference, it's possible for a computer to recover the likely meaning of a sentence.
  - In fact, CFGs were first called ***phrase-structure grammars*** and were introduced by Noam Chomsky in his seminal work *Syntactic Structures*.
  - They were then adapted for use in the context of programming languages, where they were called ***Backus-Naur forms***.
- Stanford's CoreNLP project is one place to look for an example of this.
- Want to learn more? Take CS124 or CS224N!

Biography Minute: Noam Chomsky



PC: Hans Peters / Anefo (via Wikimedia)

- Invented CFGs!
- Helped found entire fields of linguistics and cognitive science
- Today, perhaps more broadly known for politics than linguistics
  - Anti-capitalism, anti-imperialism, anti-war
  - Drawing on linguistics expertise, wrote extensively on state propaganda (*Manufacturing Consent*)

Next Time

- ***Turing Machines***
  - What does a computer with unbounded memory look like?
  - How would you program it?