Cardinality

Outline for Today

- ***Bijections***
  - A key and important class of functions.
- ***Cardinality, Formally***
  - What does it mean for two sets to have the same size?
- ***Cantor's Theorem, Formally***
  - Revisiting our Day 1 lecture.
  - *Further exploration:* On the problem set, you'll explore the proof in more depth and see some other applications.
  - *Further reading:* Guide to Cantor's Theorem, on the course website

Bijections

Injections and Surjections

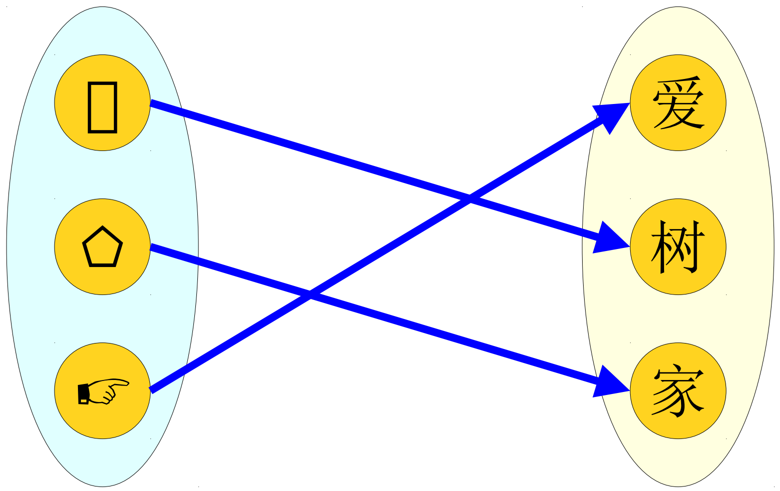
- An injective function associates ***at most*** one element of the domain with each element of the codomain.
- A surjective function associates ***at least*** one element of the domain with each element of the codomain.

Injections and Surjections

- An injective function associates ***at most*** one element of the domain with each element of the codomain.
- A surjective function associates ***at least*** one element of the domain with each element of the codomain.
- ***New!*** A bijective function associates ***exactly one*** element of the domain with each element of the codomain.

Bijections

- A ***bijection*** is a function that is both injective and surjective.
- Intuitively, if $f : A \to B$ is a bijection, then f represents a way of pairing off elements of A and elements of B.



Cardinality Revisited

Cardinality

- Recall *(from our first lecture!)* that the ***cardinality*** of a set is the number of elements it contains.
- If S is a set, we denote its cardinality by $|S|$.

Comparing Cardinality

- Saying two finite sets are equal relies on a definition of "equal" for integers.
  - $|\{1,2\}| = 2 = 2 = |\{3,6\}|$ is true, because **=** is defined for integers
- Defining "equal" for infinite set cardinality can't rely on the integer "=" operator, because infinite values are not integers.

- ***Intuition:*** Two sets have the same cardinality if there's a way to pair off their elements.

Comparing Cardinalities

- Here is the formal definition of what it means for two sets to have the same cardinality:

$|S| = |T|$ if there exists a *bijection* $f : S \to T$

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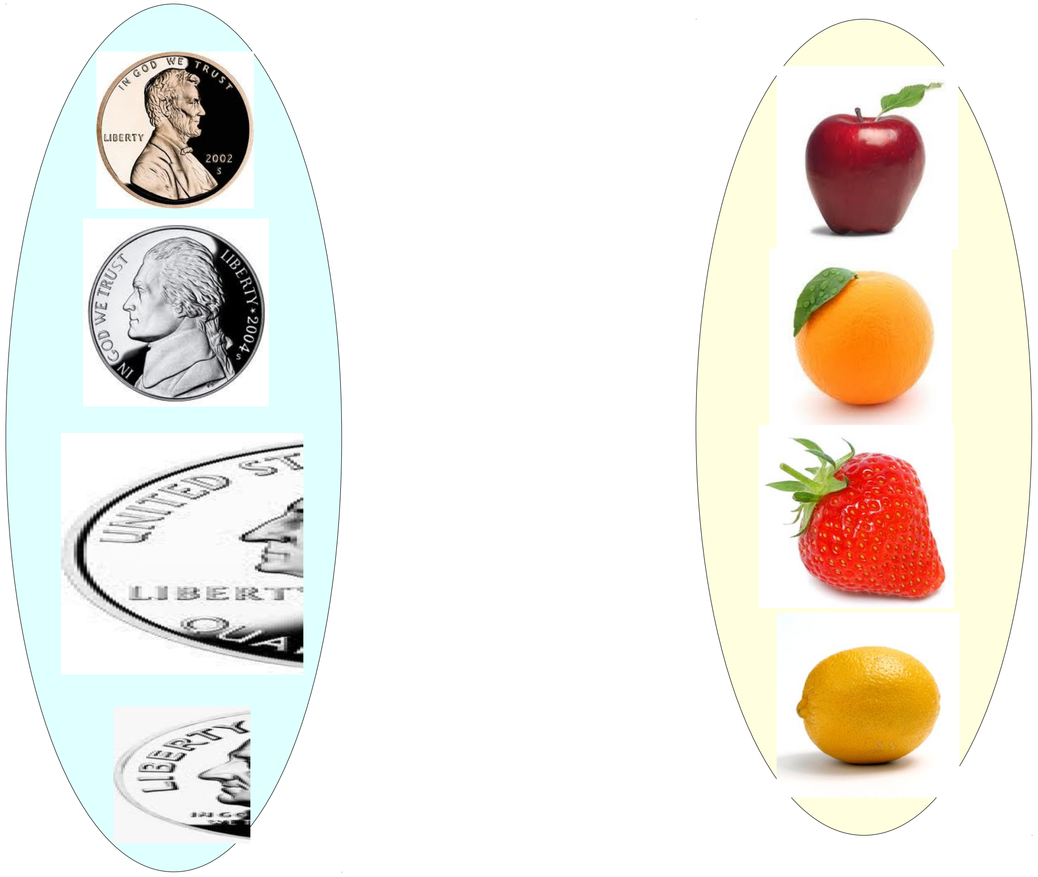

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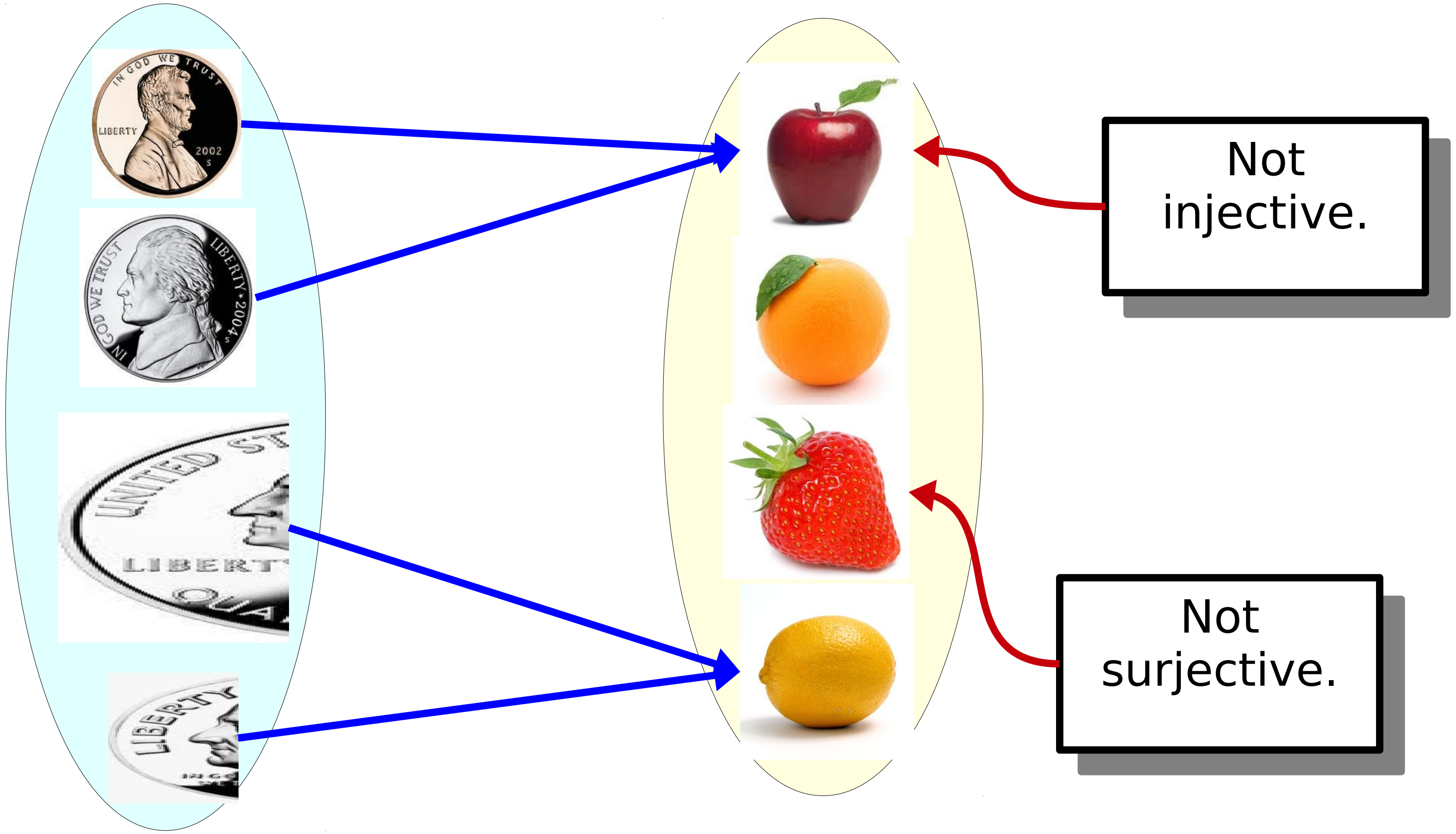

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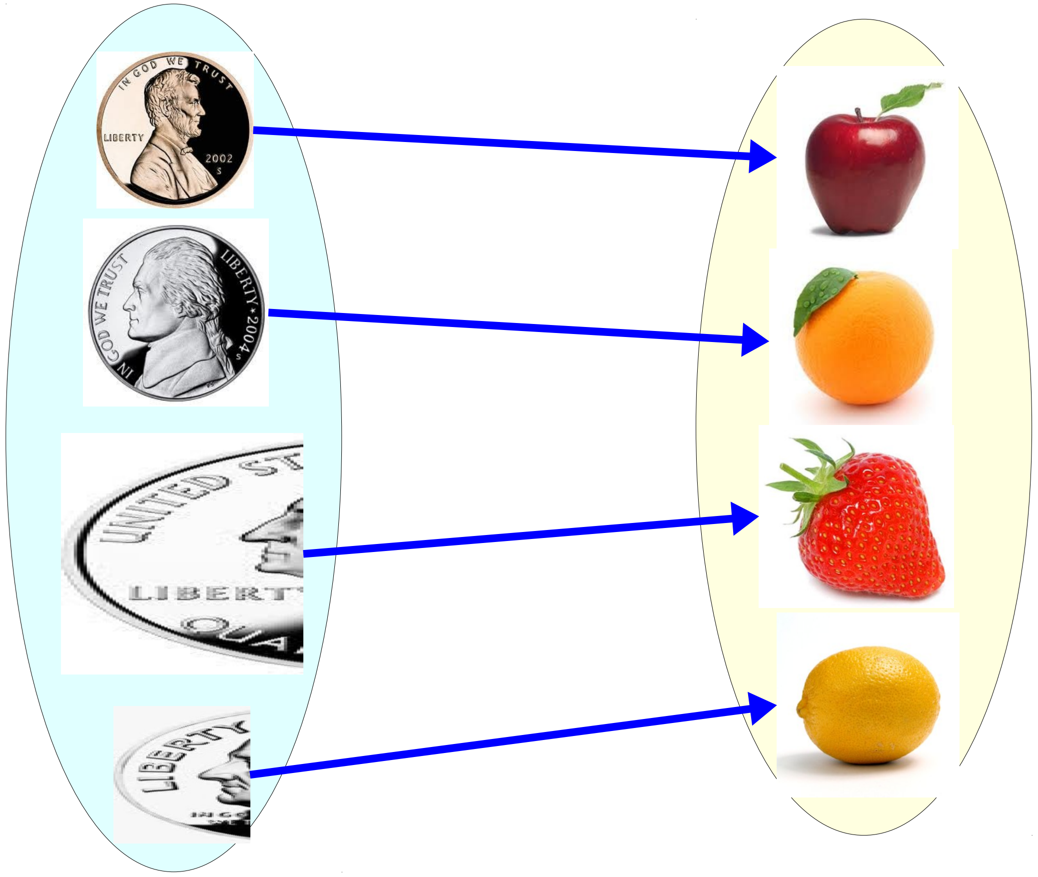

Yay, bijective!

Fun with Cardinality

Terminology Refresher

- Let a and b be real numbers where $a \leq b$.
- The notation **[*a*, *b*]** denotes the set of all real numbers between a and b, inclusive.

$$[a, b] = \{ x \in \mathbb{R} \mid a \leq x \leq b \}$$

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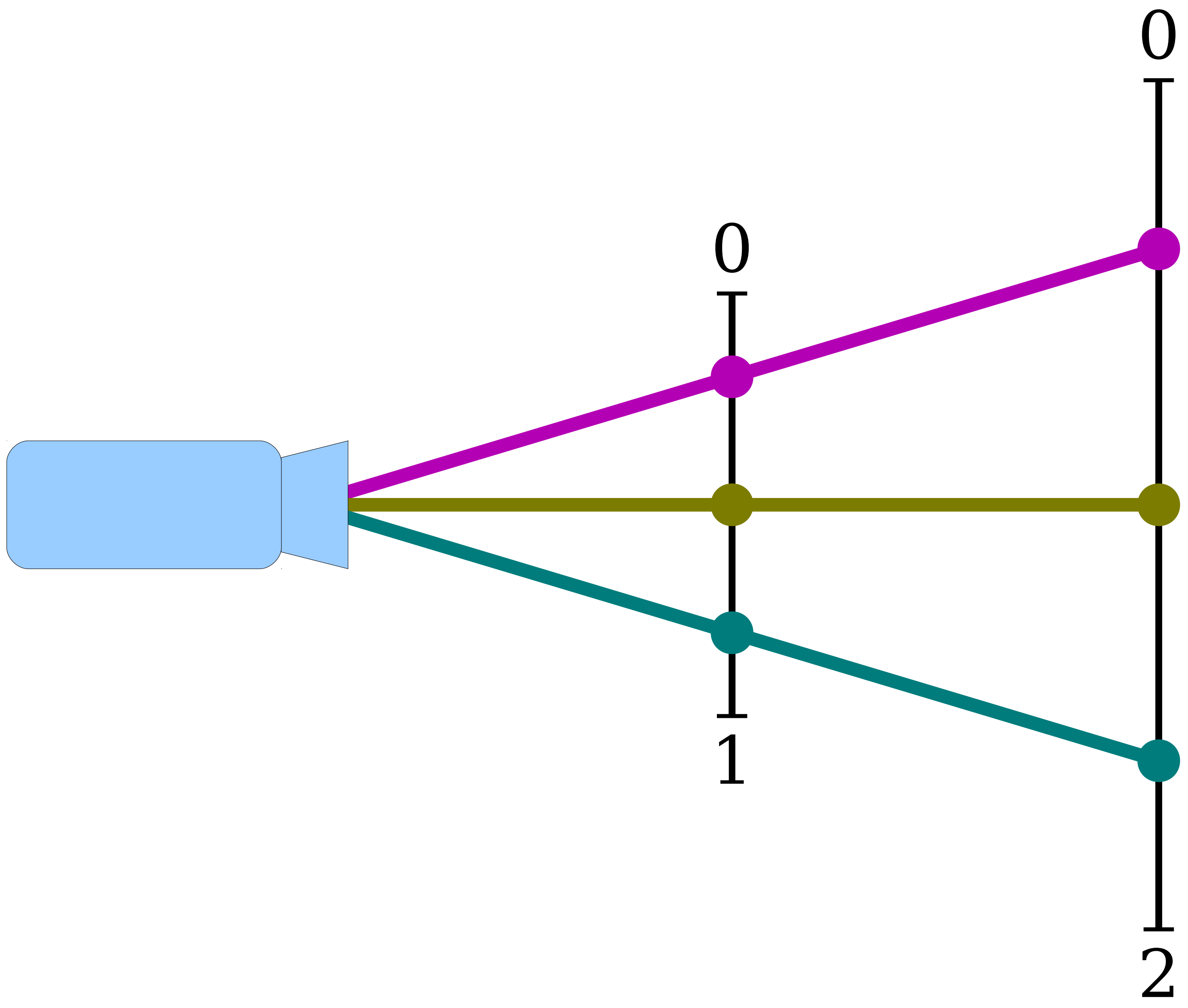
$$(a, b) = \{ x \in \mathbb{R} \mid a < x < b \}$$

Consider the sets $[0, 1]$ and $[0, 2]$.

How do their cardinalities compare?



$$f : [0, 1] \to [0, 2]$$

$$f(x) = 2x$$

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$$f(x) = 2x = 2(y/2)$$

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When defining something we claim is a function, the convention is to prove that it obeys the domain/codomain rules. For whatever reason, there isn't a convention of showing that it's deterministic. Ah, tradition. 😃

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Some Properties of Cardinality

Theorem: For any set A, we have $|A| = |A|$.

Proof: Consider any set A, and let $f : A \to A$ be the function defined as $f(x) = x$. We will prove that f is a bijection.

First, we'll show that f is a well-defined function. To see this, note that for any $x \in A$, we have $f(x) = x \in A$, as needed.

Next, we'll show that f is injective. Pick any $x_1, x_2 \in A$ where $f(x_1) = f(x_2)$. We need to show that $x_1 = x_2$. Since $f(x_1) = f(x_2)$, we see by definition of f that $x_1 = x_2$, as required.

Finally, we'll show that f is surjective. Consider any $y \in A$. We will prove that there is some $x \in A$ where $f(x) = y$. Pick $x = y$. Then $x \in A$ (since $y \in A$) and $f(x) = x = y$, as required. ■

Theorem: If A, B, and C are sets where $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$.

Proof: Consider any sets A, B, and C where $|A| = |B|$ and $|B| = |C|$. We need to prove that $|A| = |C|$. To do so, we need to show that there is a bijection from A to C.

Since $|A| = |B|$, we know that there is a some bijection $f : A \to B$. Similarly, since $|B| = |C|$ we know that there is at least one bijection $g : B \to C$.

Consider the function $g \circ f : A \to C$. Since g and f are bijections and the composition of two bijections is a bijection, we see that $g \circ f$ is a bijection from A to C. Thus $|A| = |C|$, as required. ■

Cantor’s Theorem Revisited

Cantor's Theorem

- In our very first lecture, we sketched out a proof of ***Cantor's theorem***, which says that

If S is a set, then $|S| < |\wp(S)|$.

- Today, we finally have the tools to more formally prove that result, or more specifically, this version:

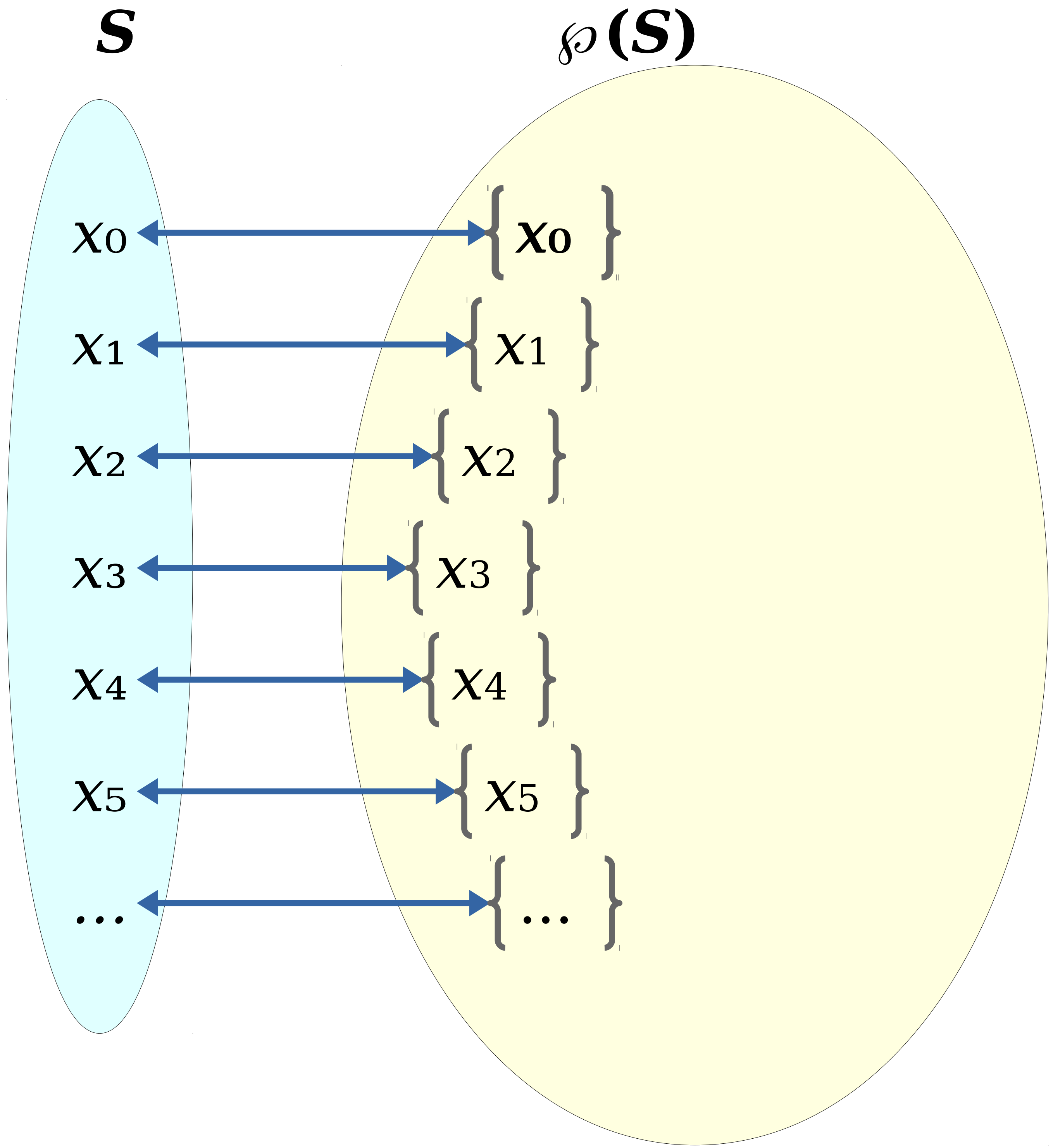
If S is a set, then $|S| \neq |\wp(S)|$.

Bijection and Cardinality

- If we think this is true for some set S:

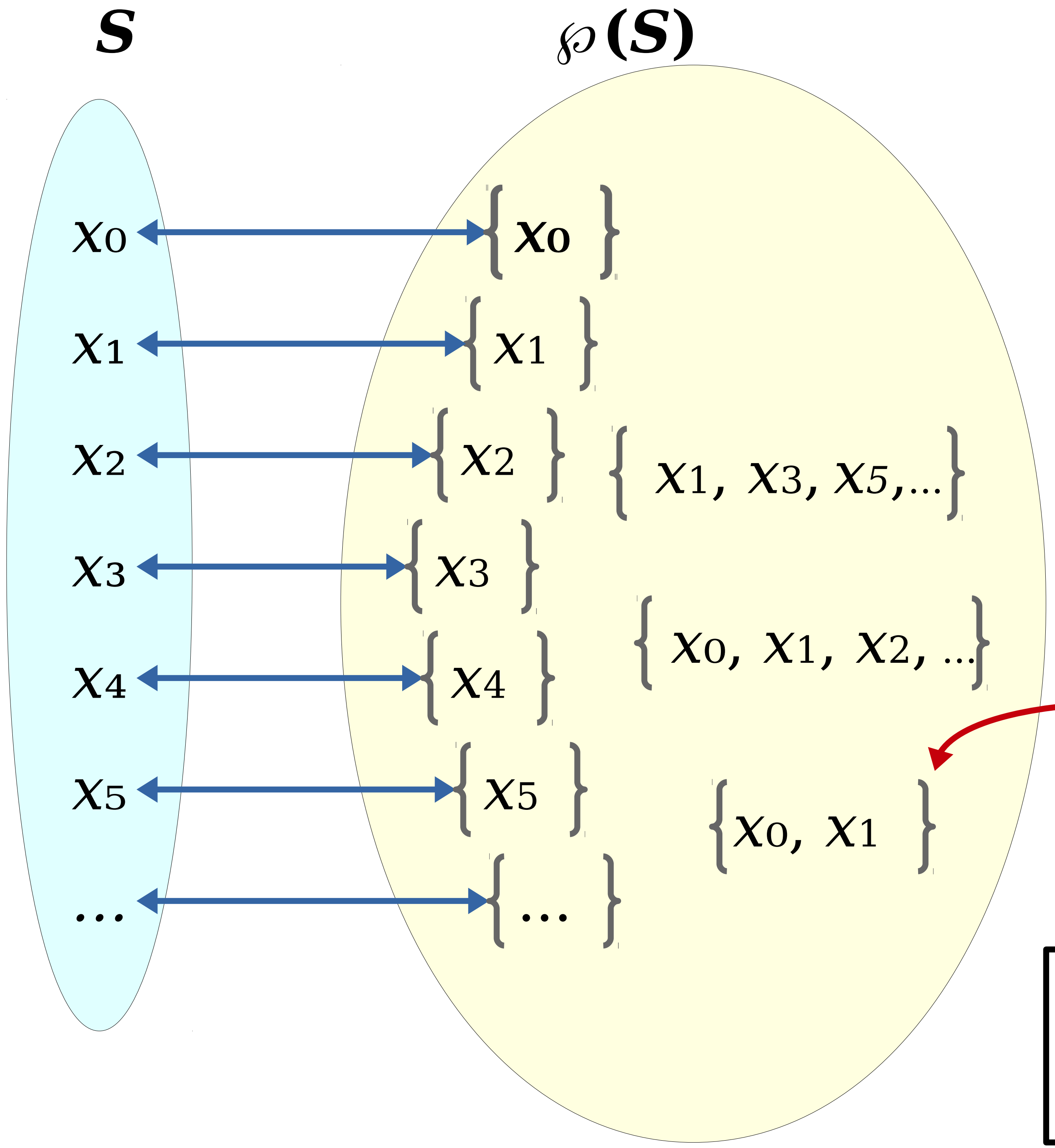
$$|\boldsymbol{S}| \neq |\wp(\boldsymbol{S})|$$

- Then we're saying we don't believe that there exists a bijection between $\boldsymbol{S}$ and $\wp(\boldsymbol{S})$.
- Let's explore one example function from $\boldsymbol{S}$ to $\wp(\boldsymbol{S})$.
  - (remember: we aren't expecting that this can be a bijection)



This is a drawing of a function $f : S \to \wp(S)$, *where* $f(x) = \{x\}$. In other words, we map each element onto a singleton set containing just itself.

This function is injective.



This is a drawing of a function $f : S \to \wp(S)$, where $f(x) = \{x\}$.
In other words, we map each element onto a singleton set containing just itself.

Not surjective.

(As we expected, this *f* is not bijective.)

Bijection and Cardinality

- Ok we found one function $f : S \to \wp(S)$, where $f(x) = \{x\}$, and showed that this function is not bijective.
- **Question:** Have we proved this?

$$|S| \neq |\wp(S)|$$

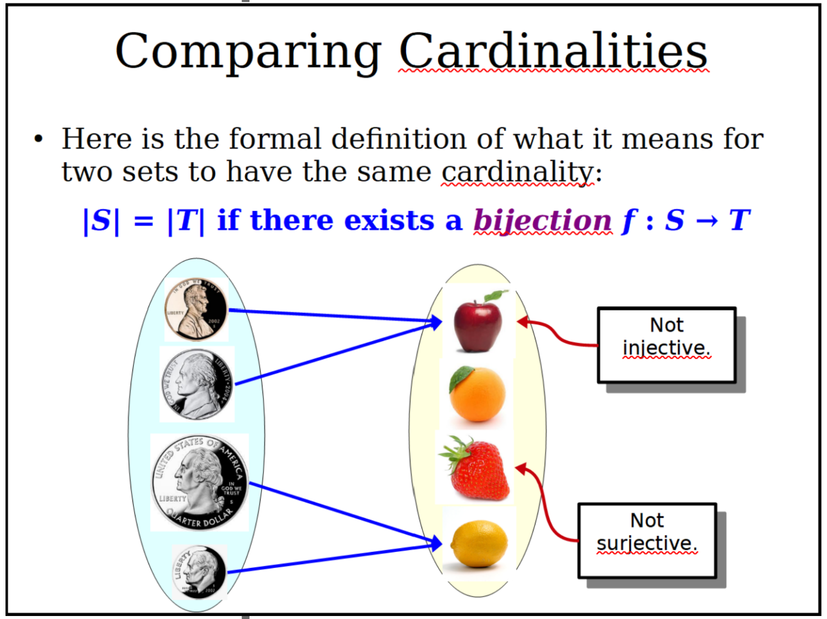
- Why or why not?

Bijection and Cardinality

- Ok we found one function $f : S \to \wp(S)$, where $f(x) = \{x\}$, and showed that this function is not bijective.
- **Question:** Have we proved this?

$$|S| \neq |\wp(S)|$$

- **Answer:** No, because there could be some other function that is bijective.
- Remember our coins/fruit slide from earlier!



If S is a set, then $|S| \neq |\wp(S)|$.

- What would be a rigorous way to approach this?
  1) Show that the function $f : S \to \wp(S)$, where $f(x) = \{x\}$ is not bijective.
  2) Pick an arbitrary function $f : S \to \wp(S)$, and show f is not injective.
  3) Pick an arbitrary function $f : S \to \wp(S)$, and show f is not surjective.

The Roadmap

- We're going to prove this statement:

$$\text{If } S \text{ is a set, then } |S| \neq |\wp(S)|.$$

- Here's how this will work:
  - Pick an arbitrary set S.
  - Pick an **arbitrary** function $f : S \to \wp(S)$.
  - Show that f is not surjective using a diagonal argument.
  - Conclude that there are no bijections from S to $\wp(S)$.
  - Conclude that $|S| \neq |\wp(S)|$.

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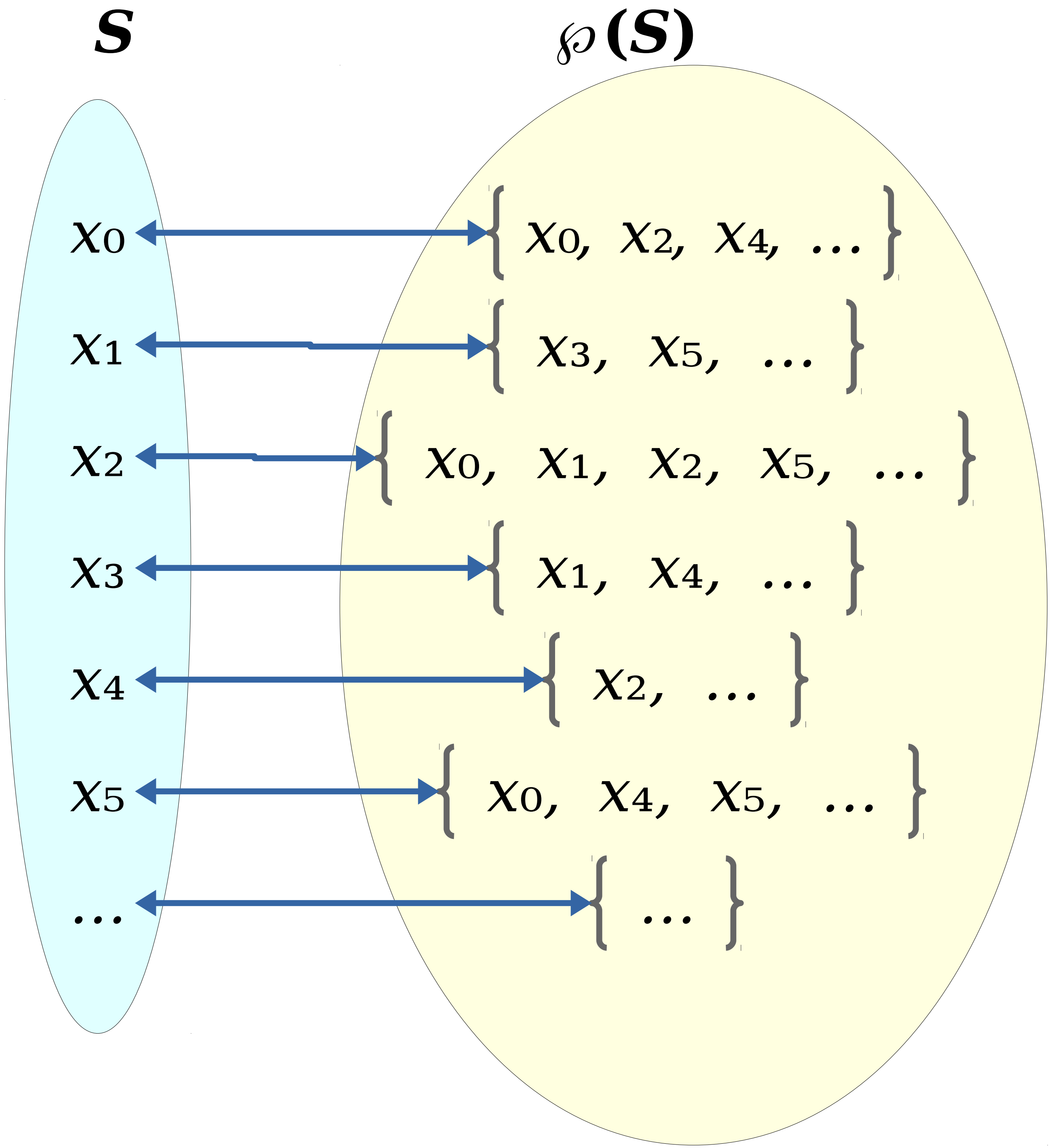
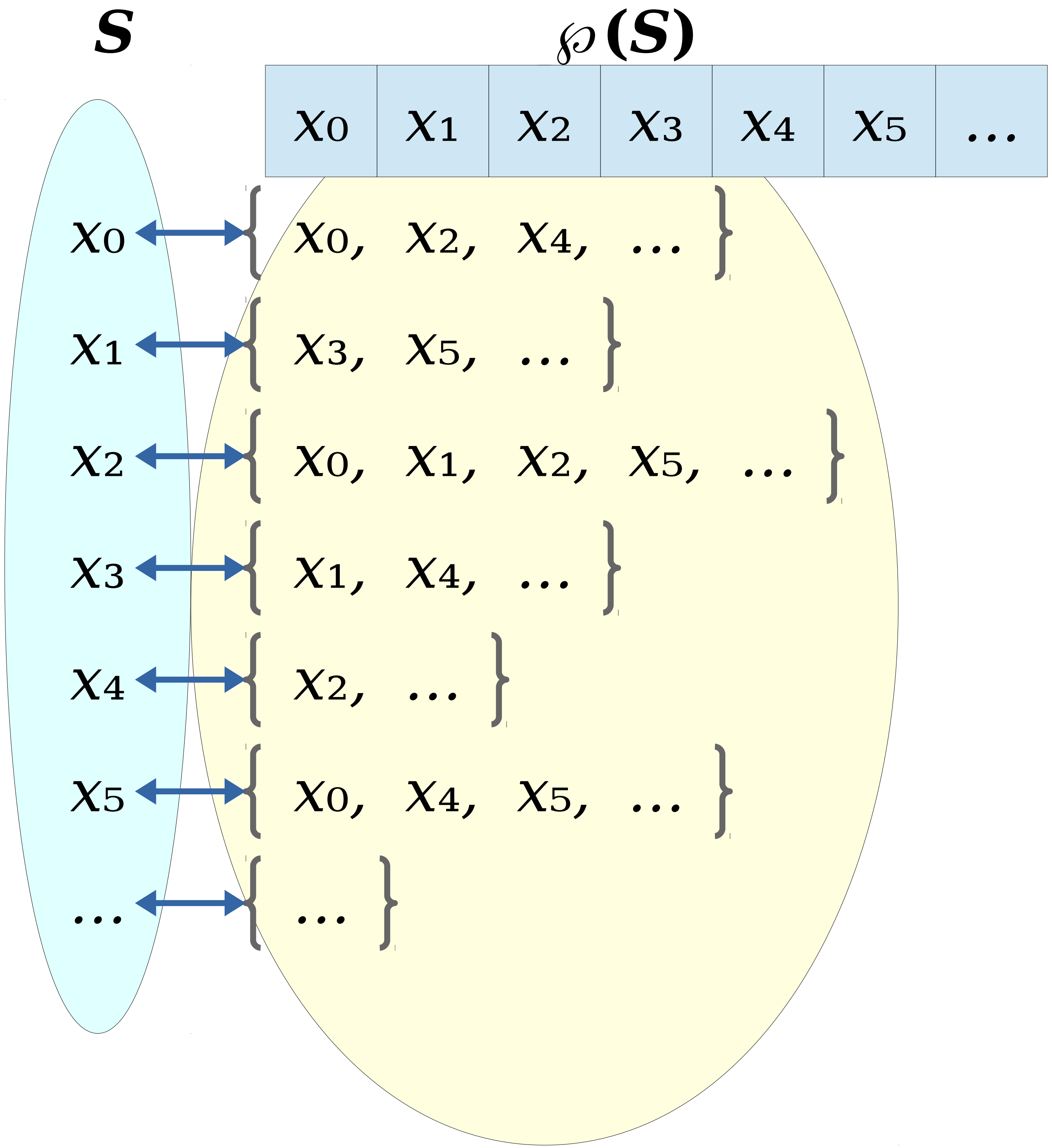


For this proof, we pick an ***arbitrary*** *function $f : S \to \wp(S)$. We don't know what f looks like, so this drawing just has some "random" values as examples of what the f might look like.*



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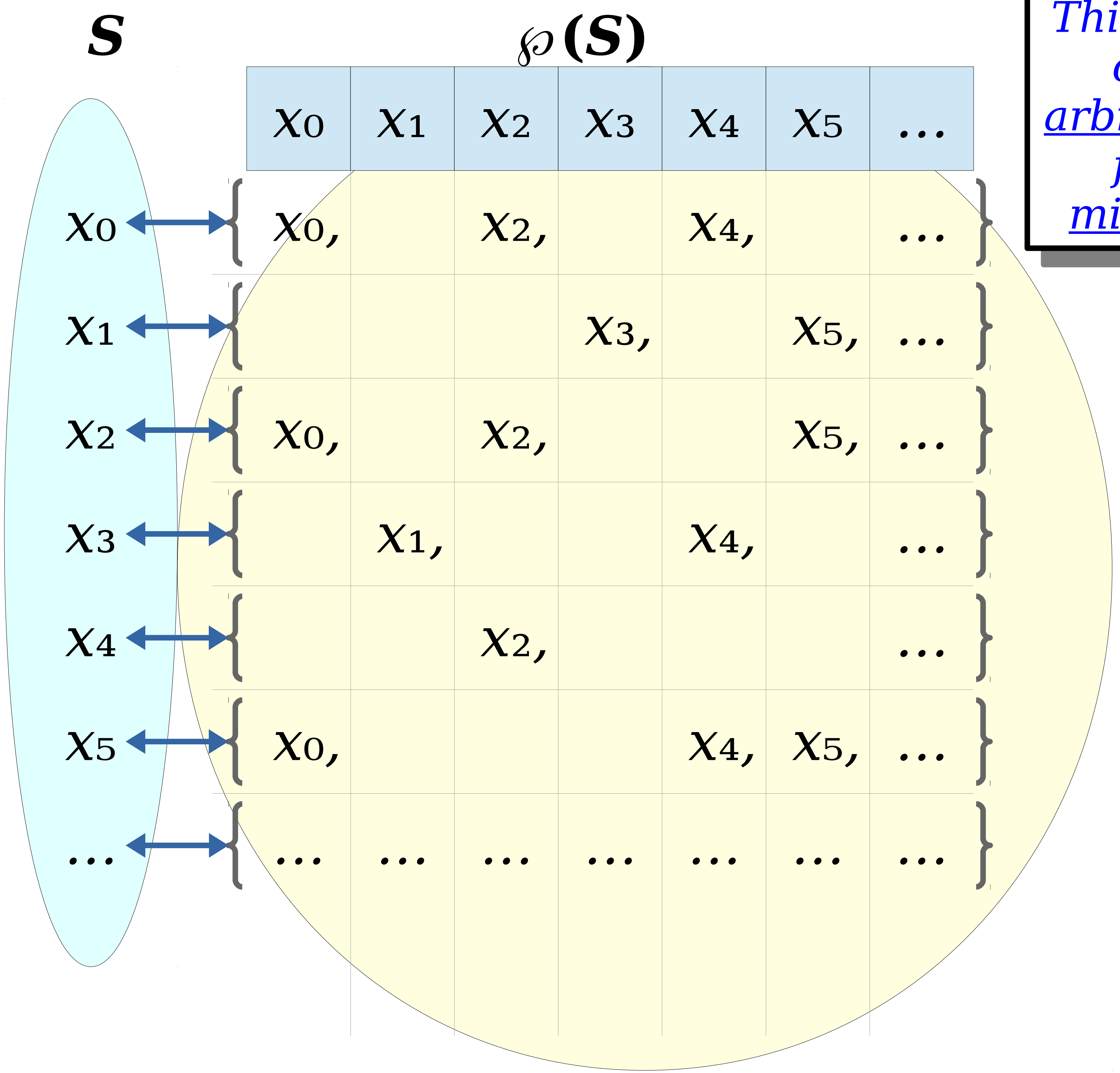

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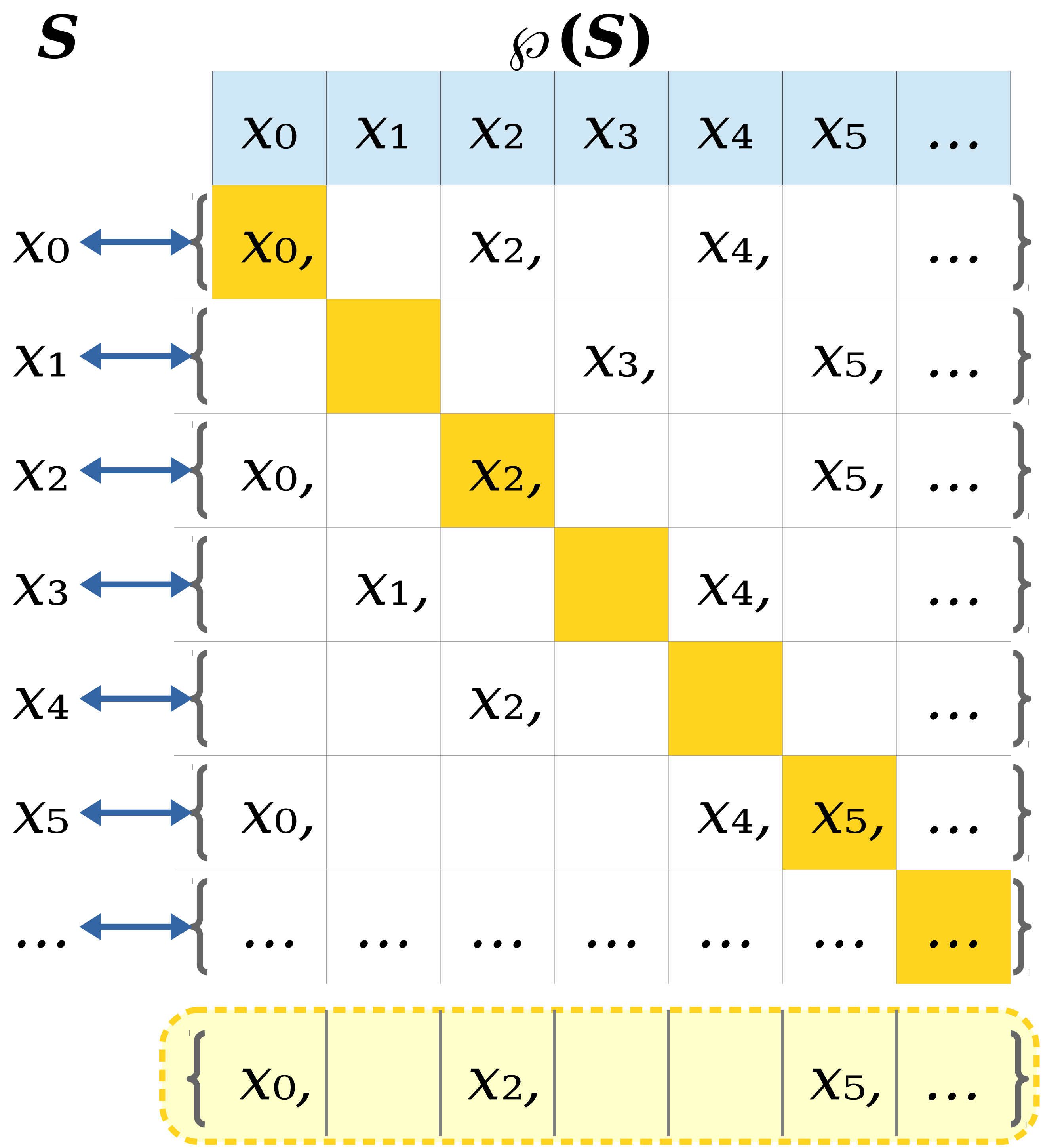

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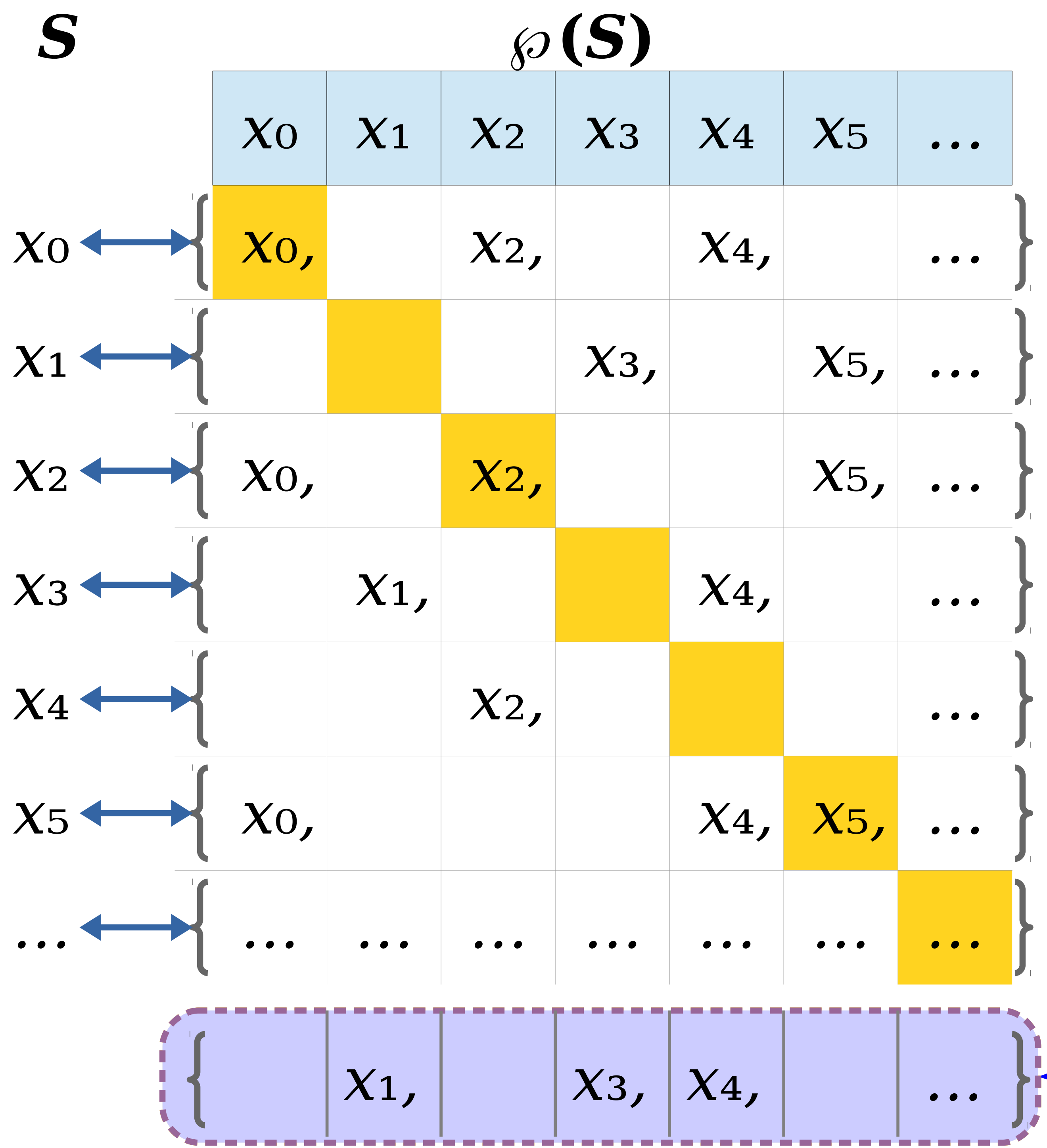
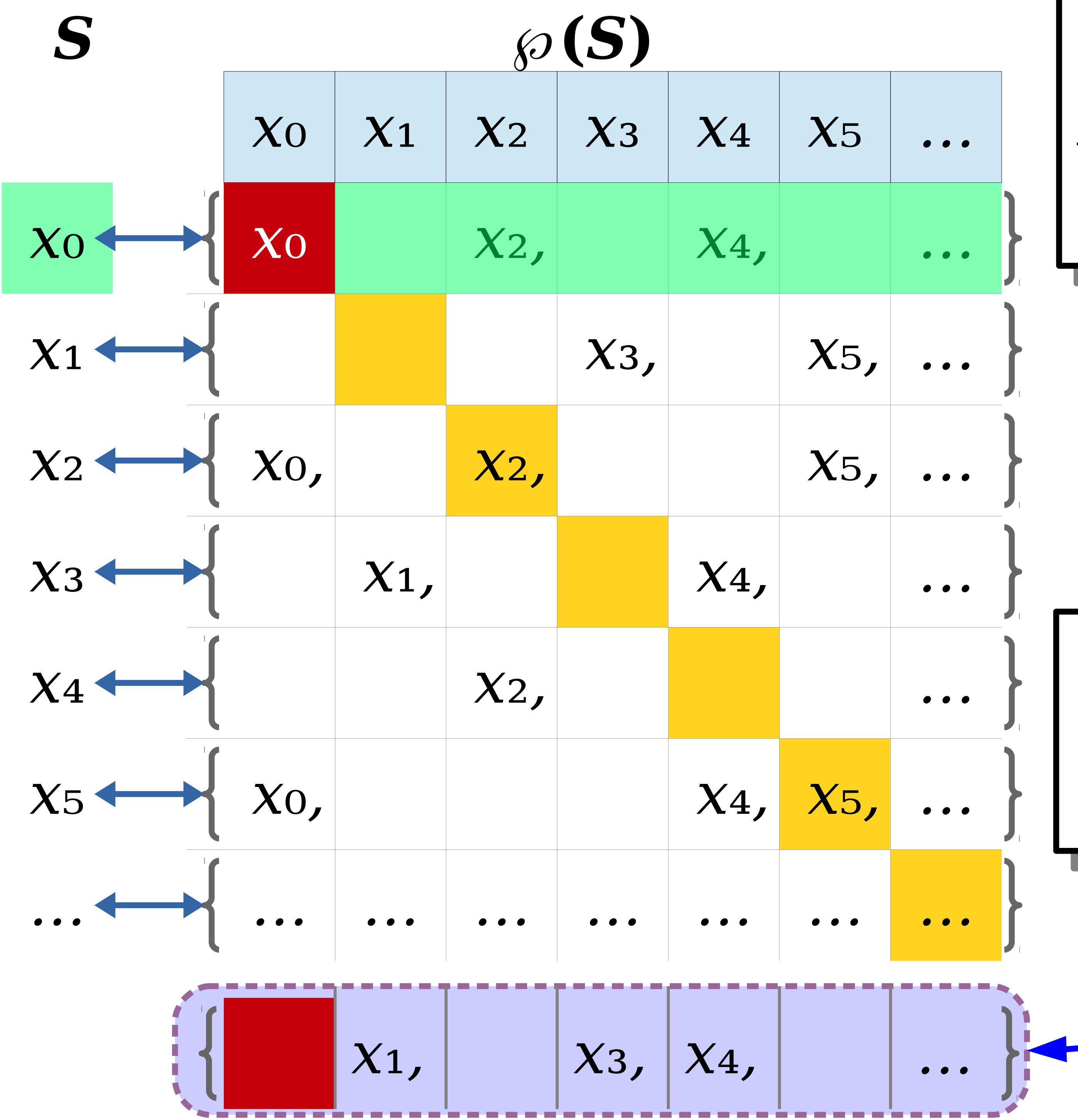


This is a drawing of what our ***arbitrary*** *function $f : S \to \wp(S)$* ***might*** *look like.*

"Flip" this set. Swap what's included and what's excluded.



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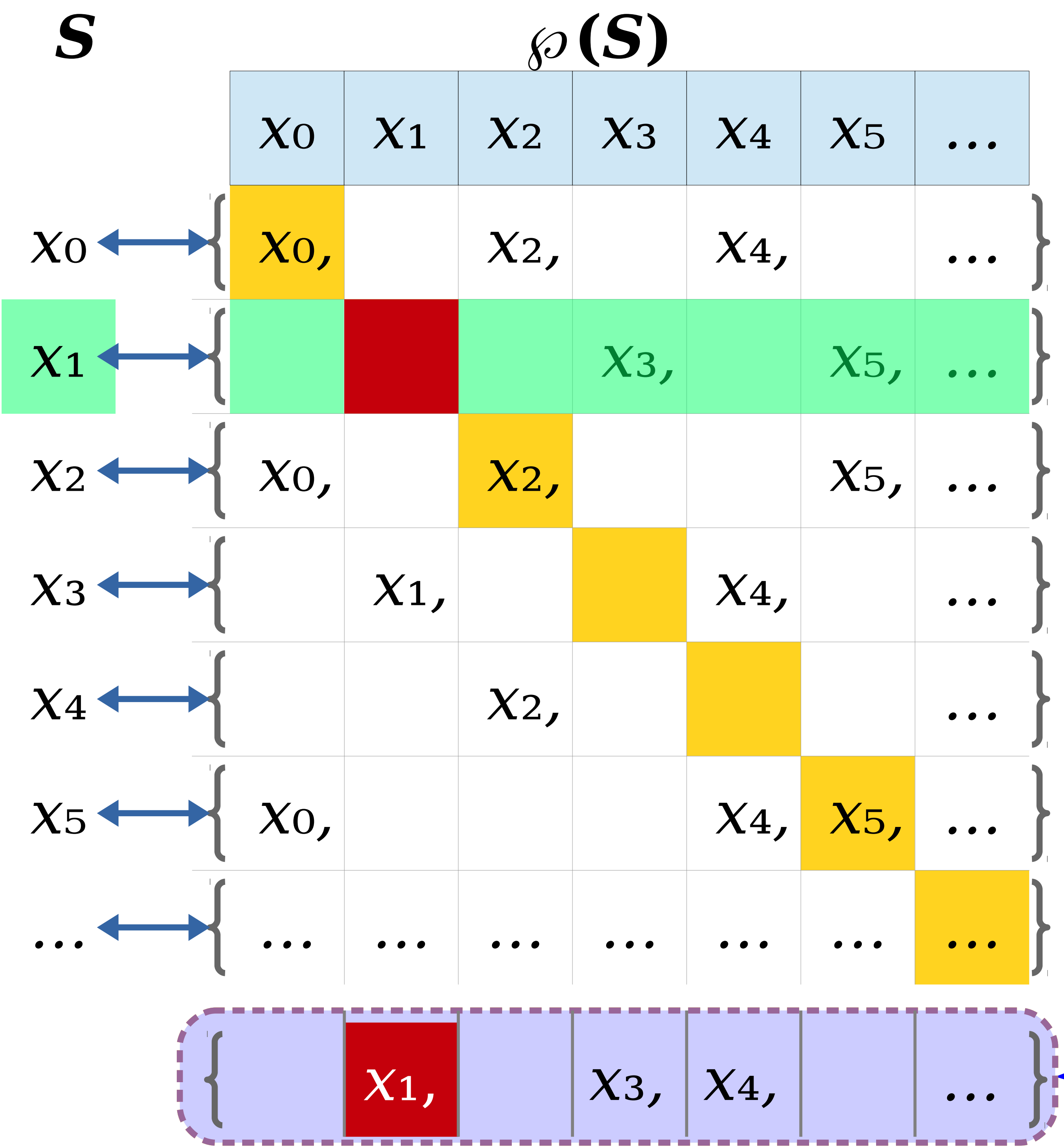
Which element is paired with this set?



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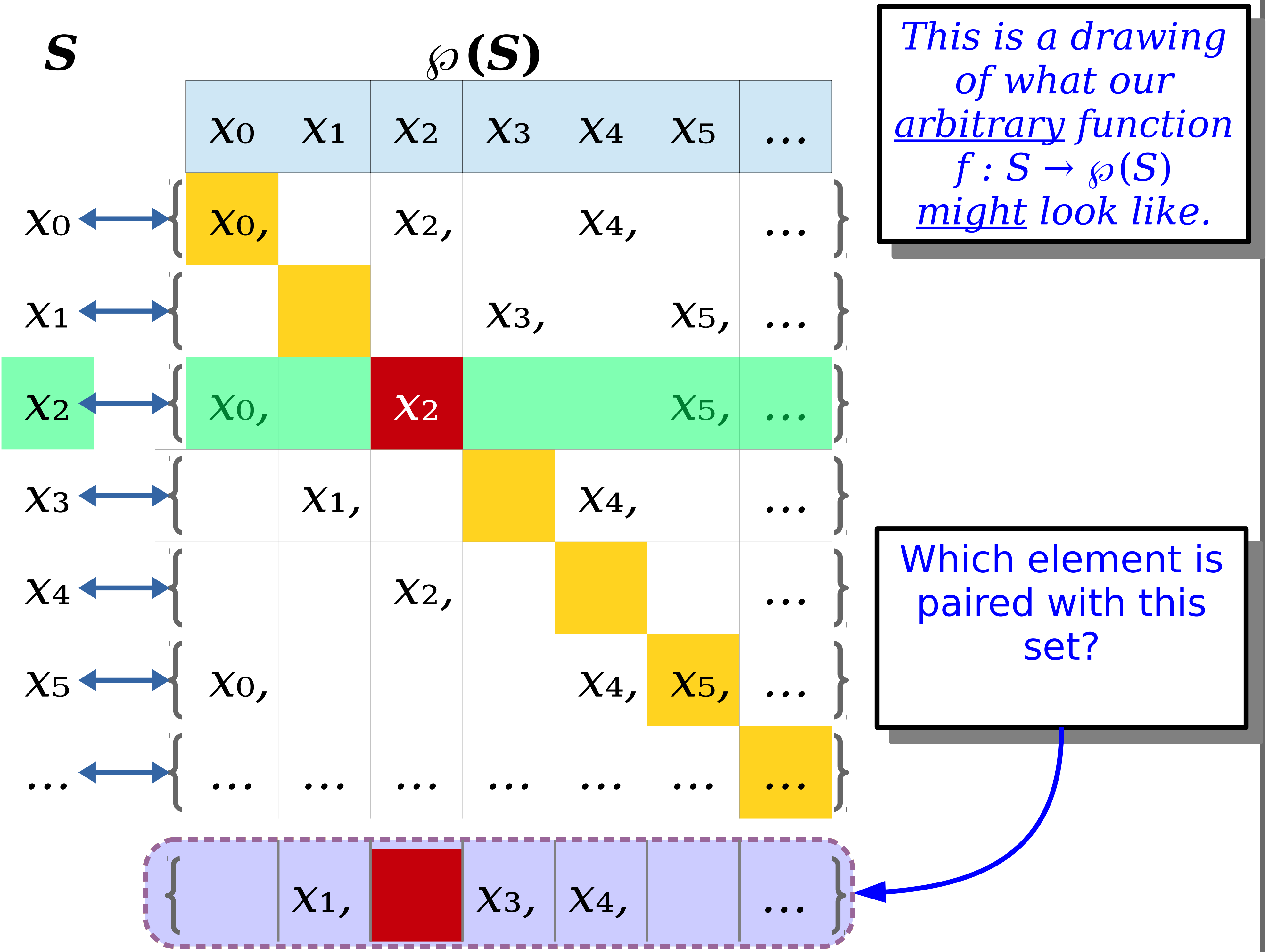
S
℘(S)
x0 x1 x2 x3 x4 x5 ...
x0 { x0, x2, x4, ... }
x1 { x3, x5, ... }
x2 { x0, x2 x5, ... }
x3 { x1, x4, ... }
x4 { x2, ... }
x5 { x0, x4, x5, ... }
... { ... ... }
{ x1, x3, x4, ... }
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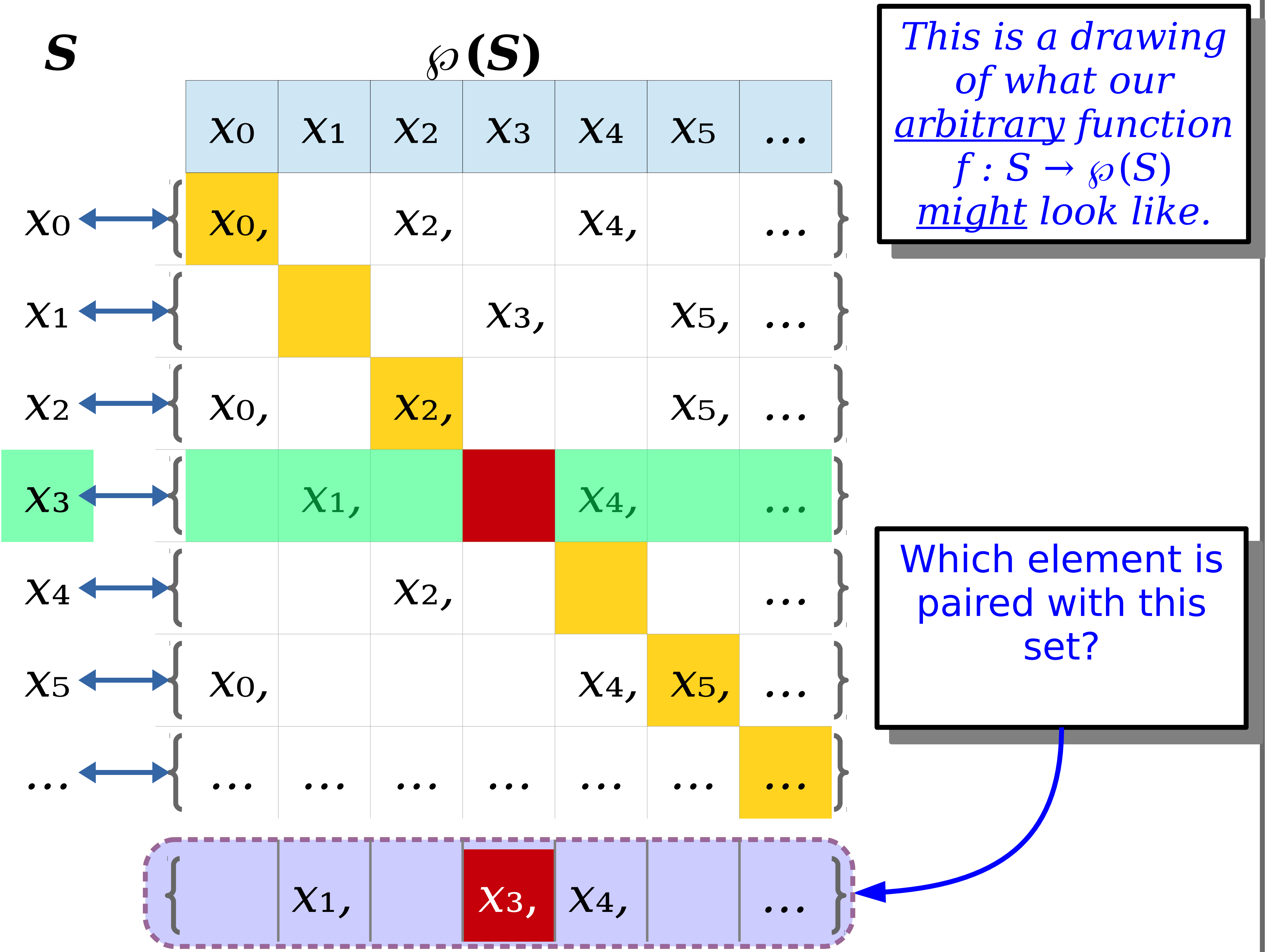
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℘(S)
x0 x1 x2 x3 x4 x5 ...
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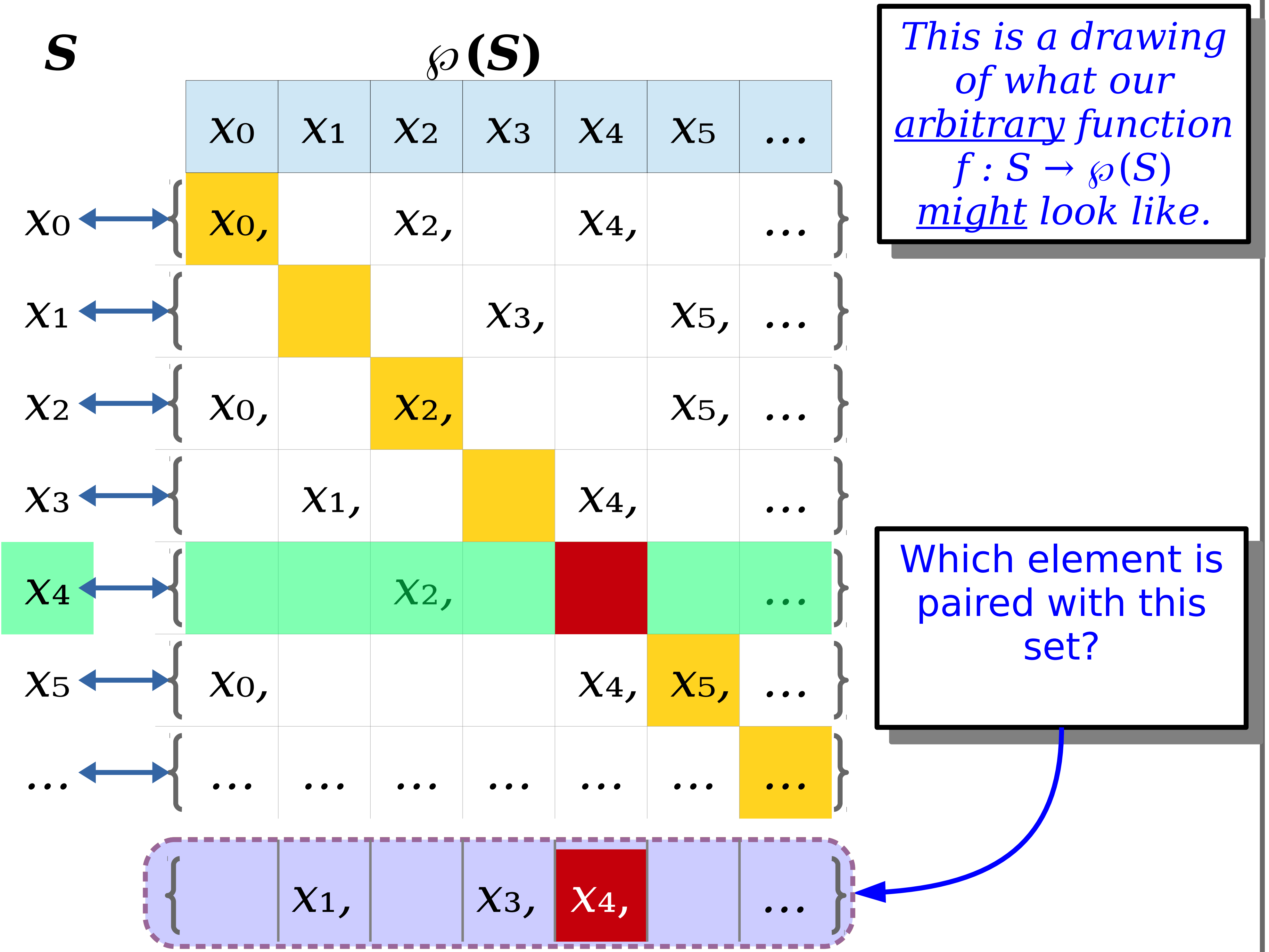

S
℘(S)
x0 x1 x2 x3 x4 x5 …
x0 { x0, x2, x4, … }
x1 { x3, x5, … }
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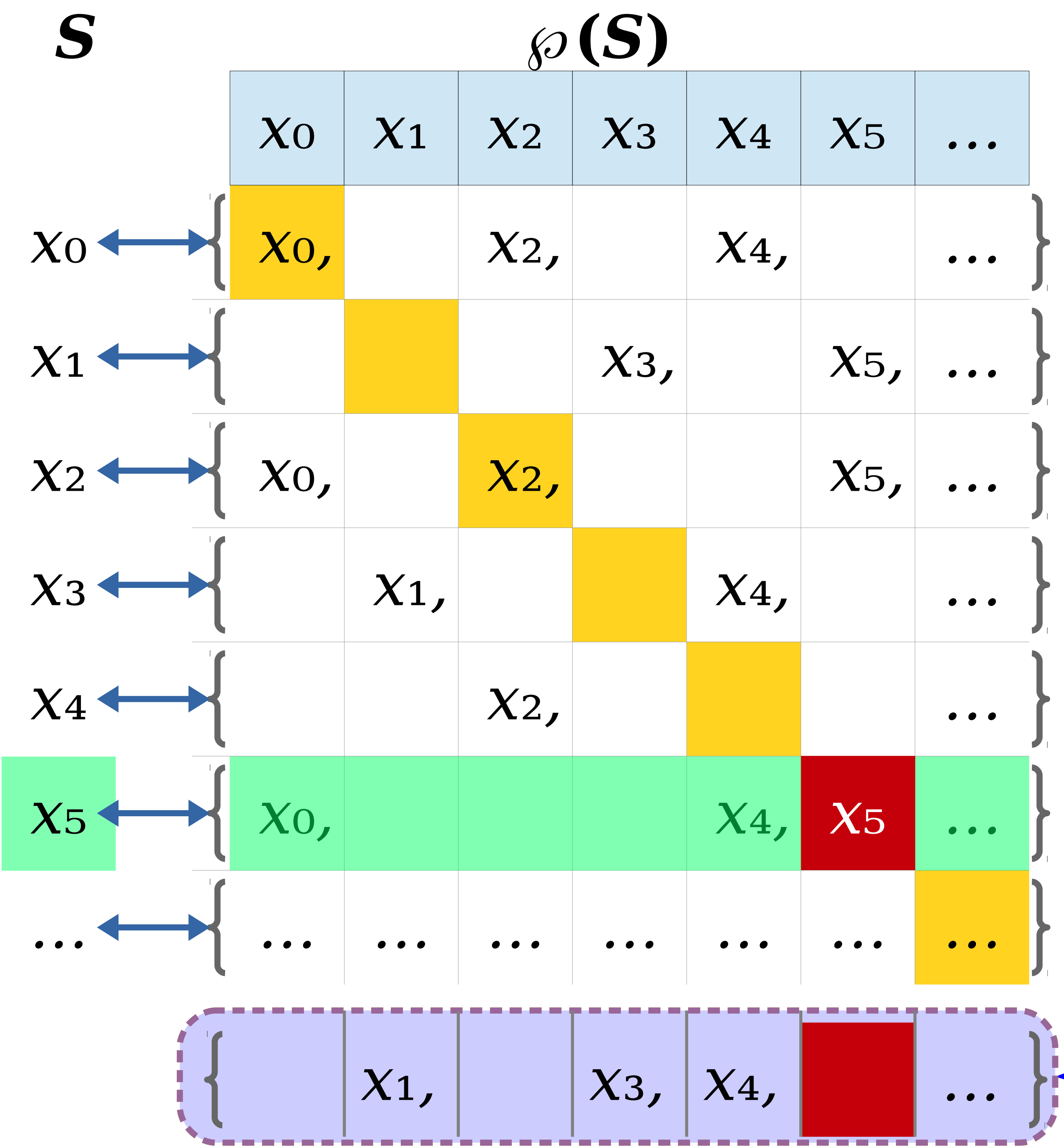

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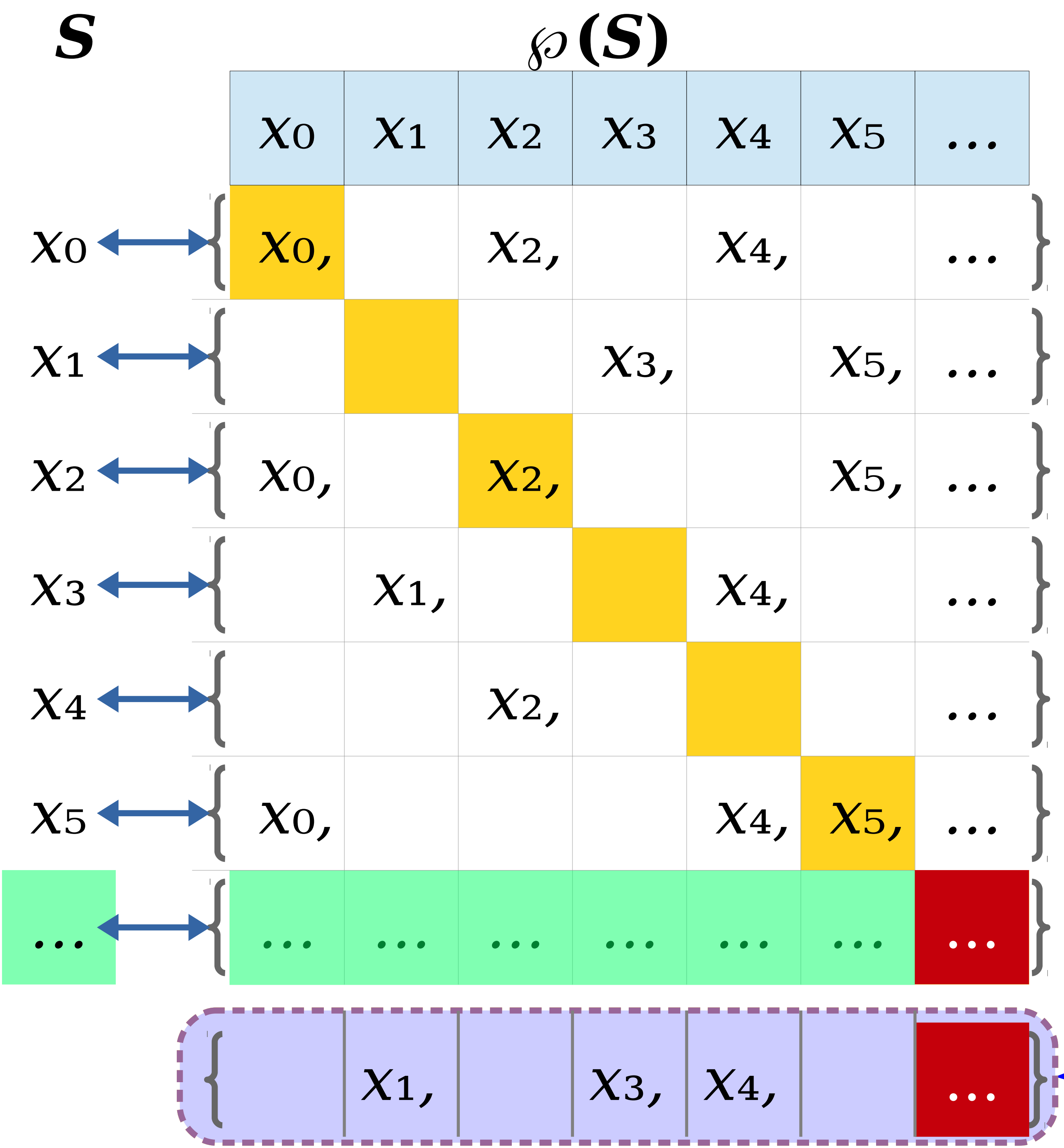
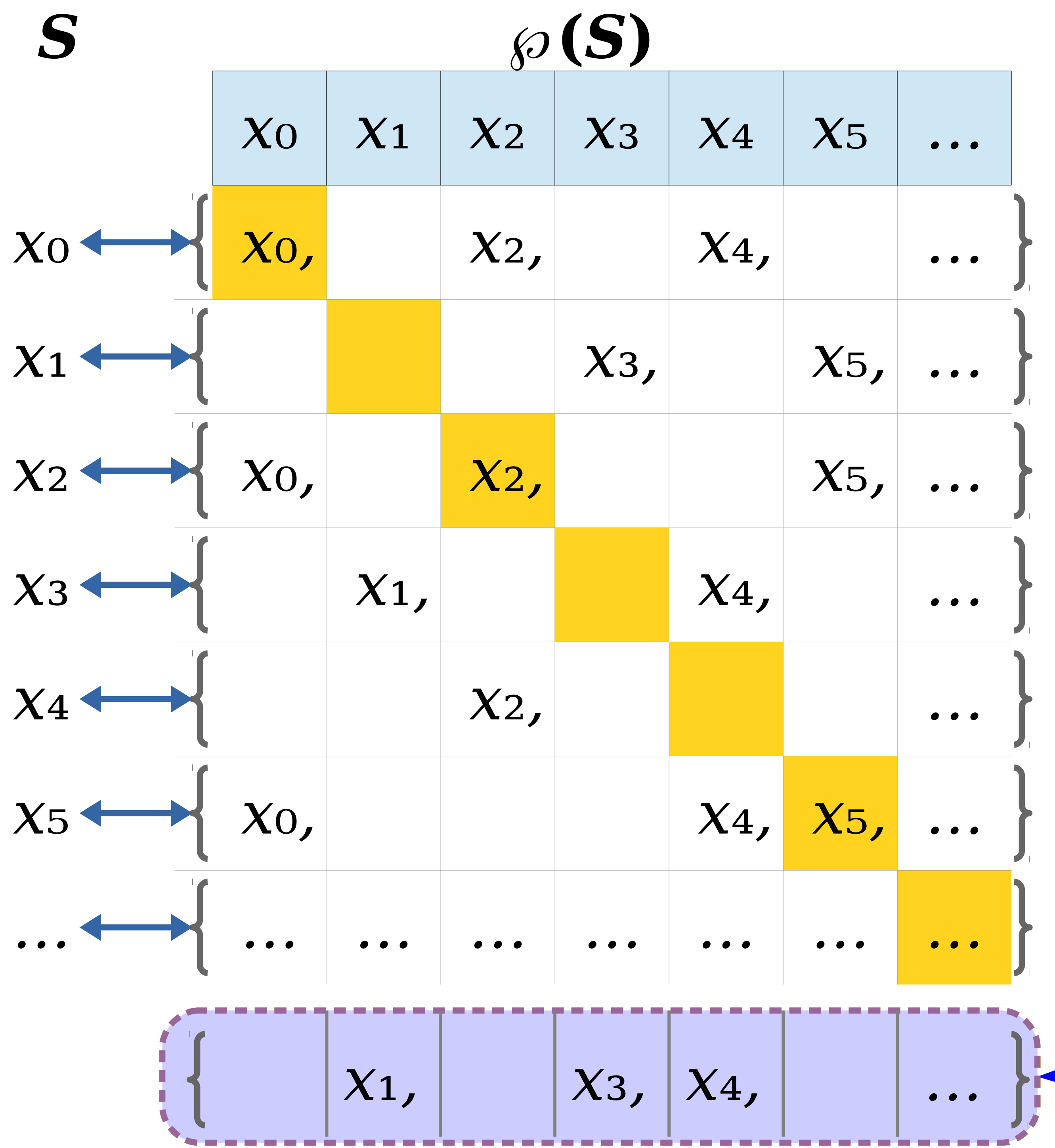
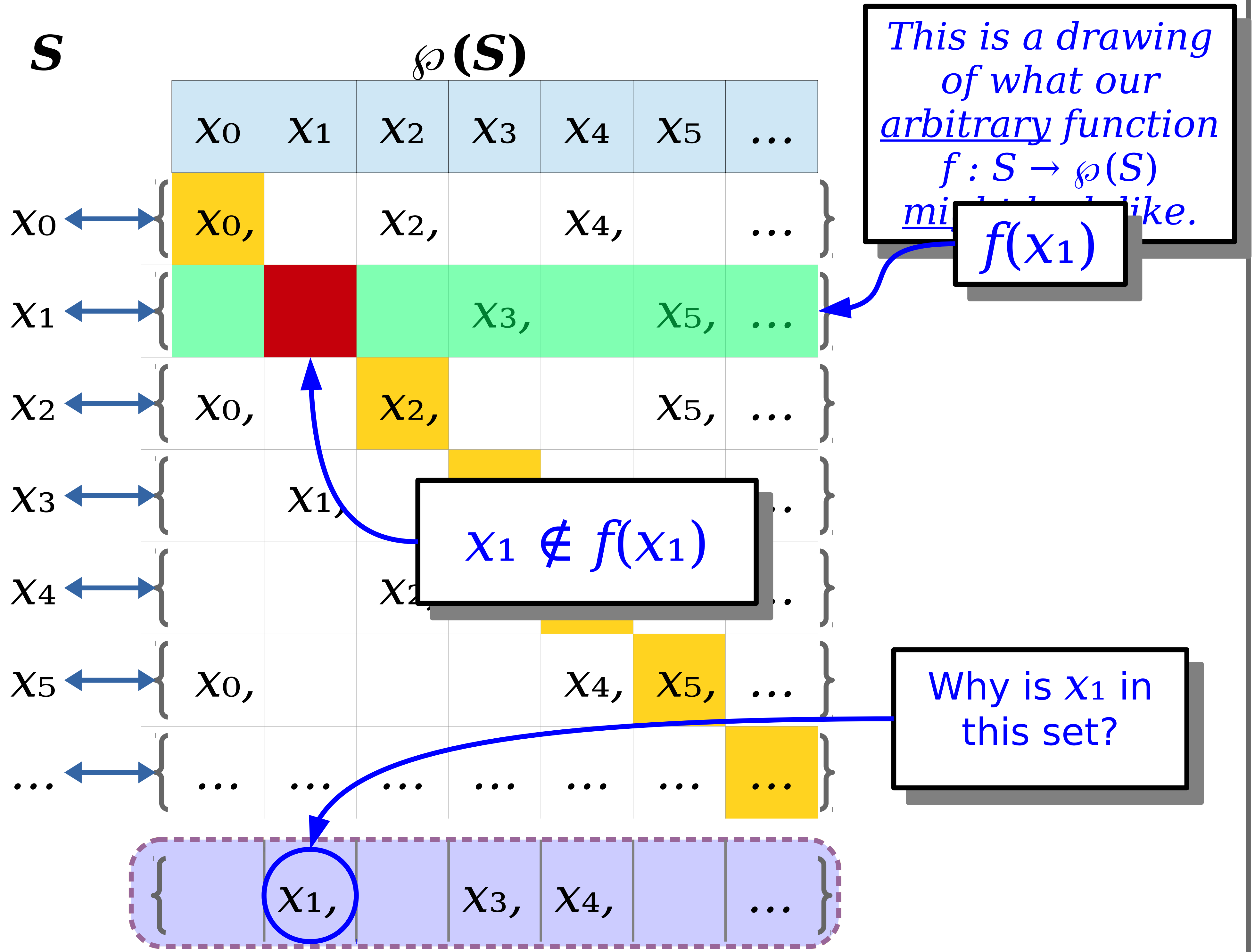


Which element is paired with this set?



This is a drawing of what our arbitrary function $f : S \to \wp(S)$ might look like.

What set is this?


S
℘(S)
x0 x1 x2 x3 x4 x5 …
x0 { x0, x2, x4, … }
x1 { x3, x5, … }
x2 { x0, x2, x5, … }
x3 { x1, … }
x4 { x2, … }
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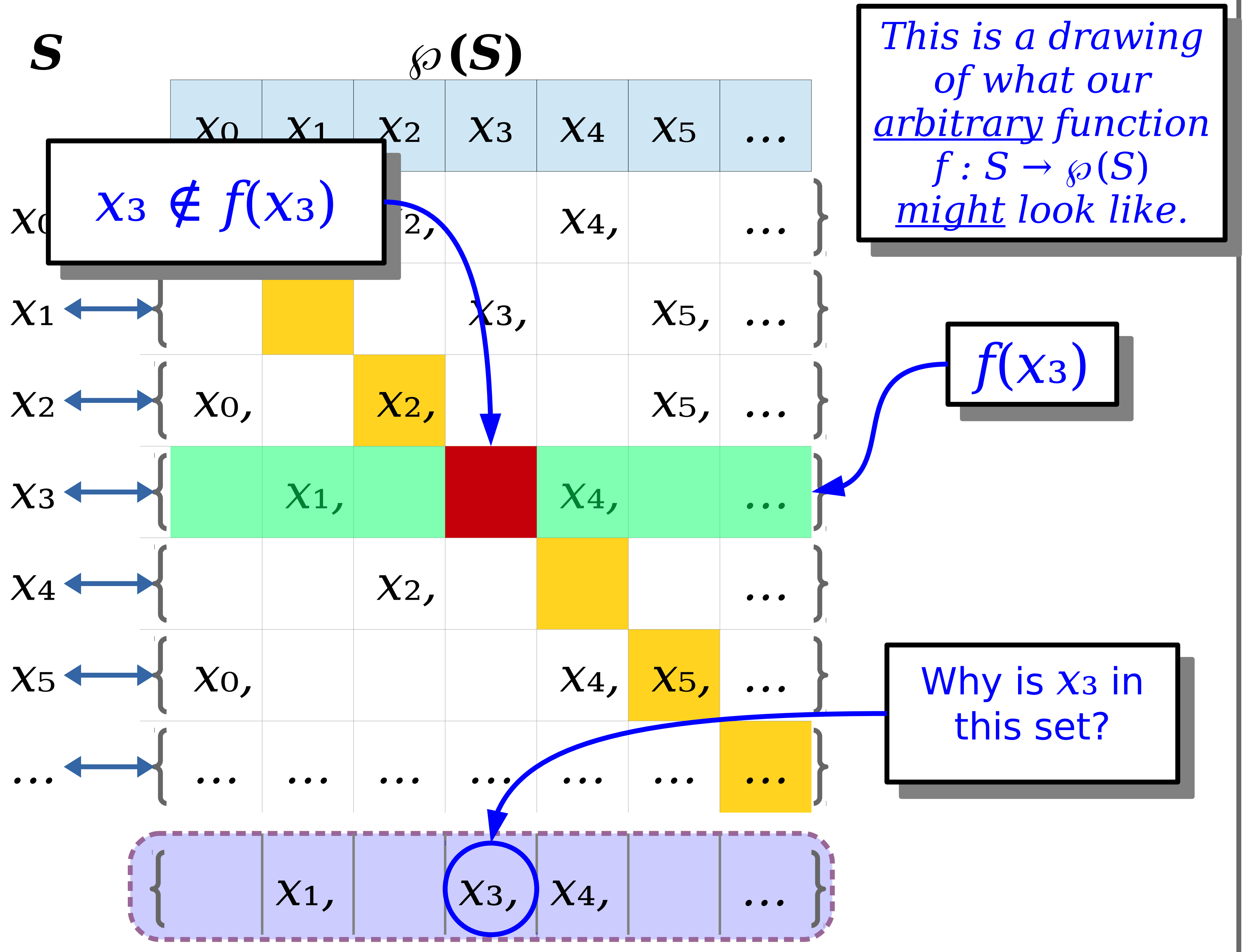
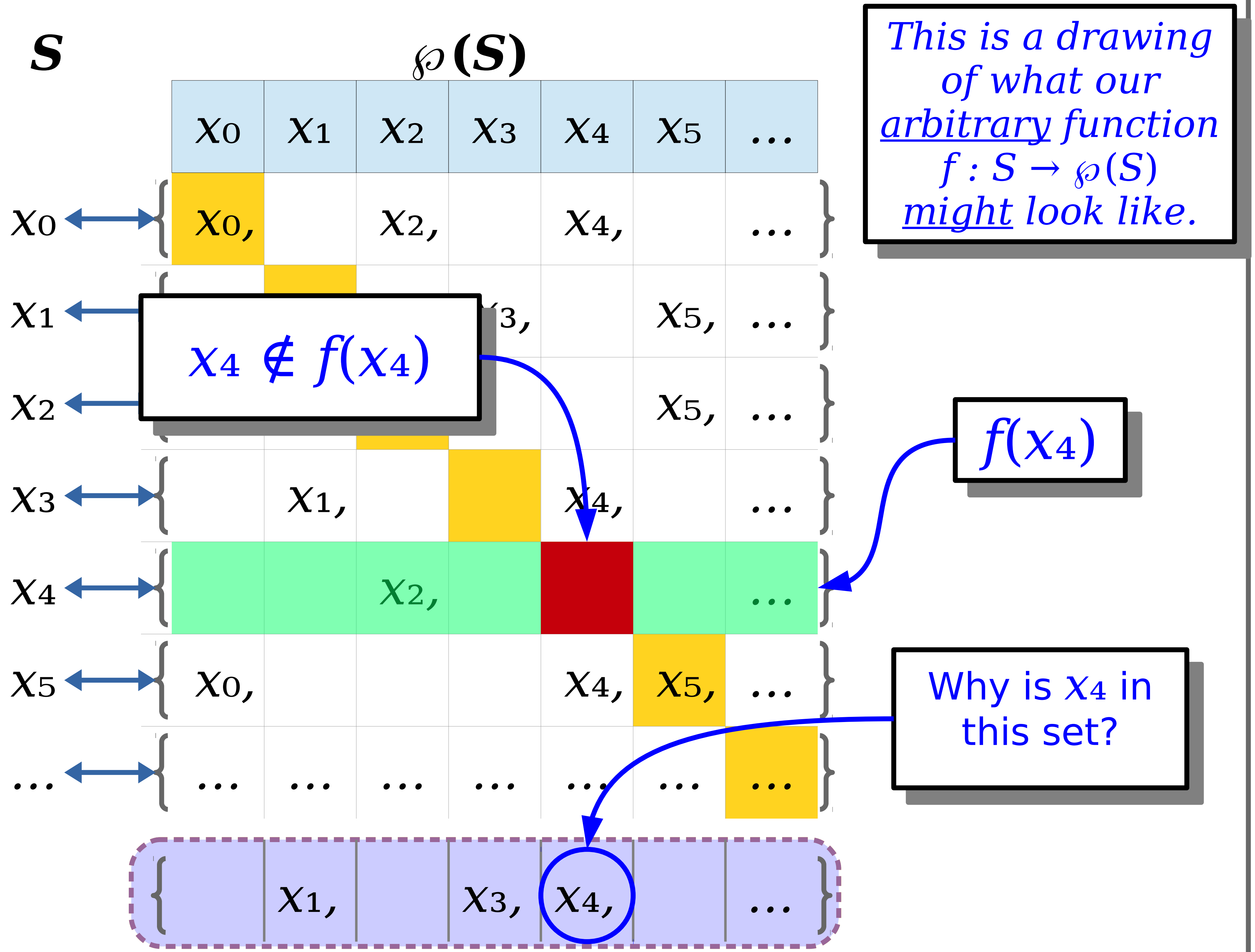
… { … … … … … … … }
{ x1, x3, x4, … }
This is a drawing of what our arbitrary function f : S → ℘(S)
mi
like.
f(x1)
x1 ∉ f(x1)
Why is x1 in this set?


S
℘(S)
x0 x1 x2 x3 x4 x5 …
x0 … 2, x4, …
x1 x3, x5, …
x2 x0, x2, x5, …
x3 x1, x4, …
x4 x2, …
x5 x0, x4, x5, …
… … … … … … … … …
{ x1, x3, x4, … }
x3 ∉ f(x3)
This is a drawing of what our arbitrary function f : S → ℘(S) might look like.
f(x3)
Why is x3 in this set?


S
℘(S)
x0 x1 x2 x3 x4 x5 …
x0 { x0, x2, x4, … }
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{ x1, x3, x4, … }
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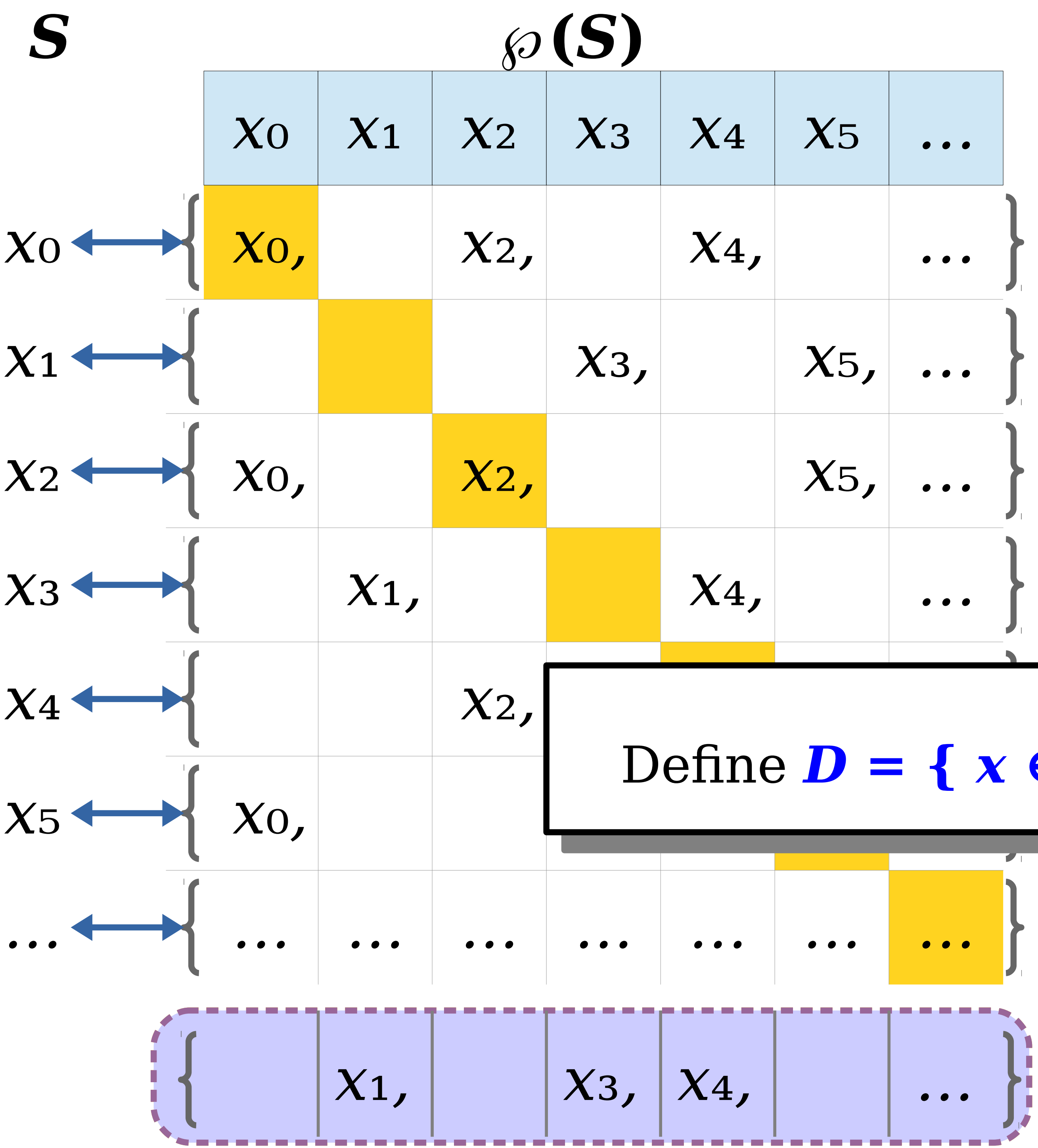
x4 ∉ f(x4)
f(x4)
Why is x4 in this set?



This is a drawing of what our <u>arbitrary</u> function $f : S \to \wp(S)$ <u>might</u> look like.

Define $\boldsymbol{D = \{\ x \in S \mid x \notin f(x)\ \}}$

The Diagonal Set

- For any set S and function $f : S \to \wp(S)$, we can define a set D as follows:

$$\boldsymbol{D = \{\ x \in S \mid x \notin f(x)\ \}}$$

("The set of all elements x where x is not an element of the set f(x).")

- This is a formalization of the set we found in the previous picture.
- Using this choice of D, we can formally prove that no function $f : S \to \wp(S)$ is a bijection.

Theorem: If S is a set, then $|S| \neq |\wp(S)|$.

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Next Time

- ***Graphs***
  - A ubiquitous, expressive, and flexible abstraction!
- ***Properties of Graphs***
  - Building high-level structures out of lower-level ones!