First-Order Logic

Part Two

Recap from Last Time

What is First-Order Logic?

- ***First-order logic*** is a logical system for reasoning about properties of objects.
- Augments the logical connectives from propositional logic with
  - ***predicates*** that describe properties of objects,
  - ***functions*** that map objects to one another, and
  - ***quantifiers*** that allow us to reason about many objects at once.

Some rabbit is cute.

$$\exists m.\ (Rabbit(m) \land Cute(m))$$

∃ is the ***existential quantifier*** and says "for some choice of m, the following is true."

New Stuff!

The Aristotelian Forms

"All *A*s are *B*s"

$$\forall x.\ (A(x) \to B(x))$$

"Some *A*s are *B*s"

$$\exists x.\ (A(x) \land B(x))$$

"No *A*s are *B*s"

$$\forall x.\ (A(x) \to \neg B(x))$$

"Some *A*s aren't *B*s"

$$\exists x.\ (A(x) \land \neg B(x))$$

It is worth committing these patterns to memory. We'll be using them throughout the day and they form the backbone of many first-order logic translations.

The Art of Translation

Using the predicates

- $Person(p)$, which states that p is a person, and
- $Loves(x, y)$, which states that x loves y,

write a sentence in first-order logic that means "every person loves someone else."

Every person loves someone else

Every person loves some other person

Every person p loves some other person

Every person p loves some other person

“All As are Bs”

$$\forall x.\ (A(x) \rightarrow B(x))$$

Every person p loves some other person

"All *A*s are *B*s"

$$\forall x.\ (A(x) \to B(x))$$

$\forall p.\ (Person(p) \to$

p loves some other person

$)$

"All *A*s are *B*s"

$\boldsymbol{\forall x.\ (A(x) \to B(x))}$

$\forall p. \ (Person(p) \rightarrow$

p loves some other person

$)$

$\forall p. \ (Person(p) \rightarrow$

there is some other person that p loves

$)$

$\forall p. \ (Person(p) \rightarrow$

there is a person other than p that p loves

$)$

$\forall p.\ (Person(p) \to$

there is a person q, other than p, where p loves q

$)$

$\forall p.\ (Person(p) \rightarrow$

there is a person q, other than p, where

p loves q

$)$

$\forall p.\ (Person(p) \rightarrow$

there is a person q, other than p, where

p loves q

$)$

"Some *A*s are *B*s"

$\exists x.\ (A(x) \land B(x))$

$\forall p.\ (Person(p) \rightarrow$

$\quad \exists q.\ (Person(q)\ \wedge$*, other than p, where*

p loves q

$\quad)$

$)$

"Some *A*s are *B*s"

$$\boldsymbol{\exists x.\ (A(x)\ \wedge\ B(x))}$$

$\forall p.\ (Person(p) \rightarrow$

$\quad \exists q.\ (Person(q)\ \land$ *, other than p, where*

p loves q

$\quad)$

$)$

$$
\begin{aligned}
&\forall p.\ (Person(p) \rightarrow \\
&\quad \exists q.\ (Person(q) \land p \neq q \land \\
&\qquad \textit{p loves q} \\
&\quad) \\
&)
\end{aligned}
$$

$$
\begin{aligned}
&\forall p.\ (Person(p) \to \\
&\quad \exists q.\ (Person(q) \land p \neq q \land \\
&\qquad Loves(p,\ q) \\
&\quad) \\
&)
\end{aligned}
$$

Using the predicates

- $Person(p)$, which states that p is a person, and
- $Loves(x, y)$, which states that x loves y,

write a sentence in first-order logic that means “there is a person that everyone else loves.”

There is a person that everyone else loves

There is a person p where everyone else loves p

There is a person p where everyone else loves p

"Some *A*s are *B*s"

$$\exists x.\ (A(x) \land B(x))$$

$\exists p.\ (Person(p)\ \land$

everyone else loves p

$)$

"Some *A*s are *B*s"

$\boldsymbol{\exists x.\ (A(x)\ \land\ B(x))}$

$\exists p. \, (Person(p) \, \wedge$

everyone else loves p

$)$

$\exists p. \ (Person(p) \ \land$

every other person q loves p

$)$

$\exists p. \ (Person(p) \ \wedge$

every person q, other than p, loves p

$)$

$\exists p. \, (Person(p) \, \wedge$

every person q, other than p, loves p

$)$

"All As are Bs"

$\boldsymbol{\forall x. \, (A(x) \rightarrow B(x))}$

$\exists p.\ (Person(p)\ \land$

$\quad \forall q.\ (Person(q)\ \land\ p \neq q\ \to$

$\qquad$ *q loves p*

$\quad)$

$)$

"All *A*s are *B*s"

$\forall x.\ (A(x) \to B(x))$

$$
\begin{aligned}
&\exists p.\ (Person(p)\ \land \\
&\quad \forall q.\ (Person(q)\ \land\ p \neq q \rightarrow \\
&\qquad \textit{q loves p} \\
&\quad) \\
&)
\end{aligned}
$$

$$
\begin{array}{l}
\exists p.\ (Person(p)\ \land \\
\quad \forall q.\ (Person(q)\ \land\ p \neq q \rightarrow \\
\qquad Loves(q, p) \\
\quad) \\
)
\end{array}
$$

Combining Quantifiers

- Most interesting statements in first-order logic require a combination of quantifiers.
- Example: "Every person loves someone else"

For every person…	$\forall p.\ (Person(p) \to$
… there is another person …	$\quad \exists q.\ (Person(q) \land p \neq q \land$
… they love	$\qquad Loves(p, q)$
	$\quad)$
	$)$

Combining Quantifiers

- Most interesting statements in first-order logic require a combination of quantifiers.
- Example: "There is someone everyone else loves."

There is a person…	$\exists p. \ (Person(p) \land$
… that everyone else …	$\quad \forall q. \ (Person(q) \land p \neq q \rightarrow$
… loves.	$\quad\quad Loves(q, p)$
	$\quad)$
	$)$

For Comparison

For every person…

… there is another person …

… they love

$$
\begin{aligned}
&\forall p.\ (Person(p) \to \\
&\quad \exists q.\ (Person(q) \land p \neq q \land \\
&\qquad Loves(p, q) \\
&\quad) \\
&)
\end{aligned}
$$

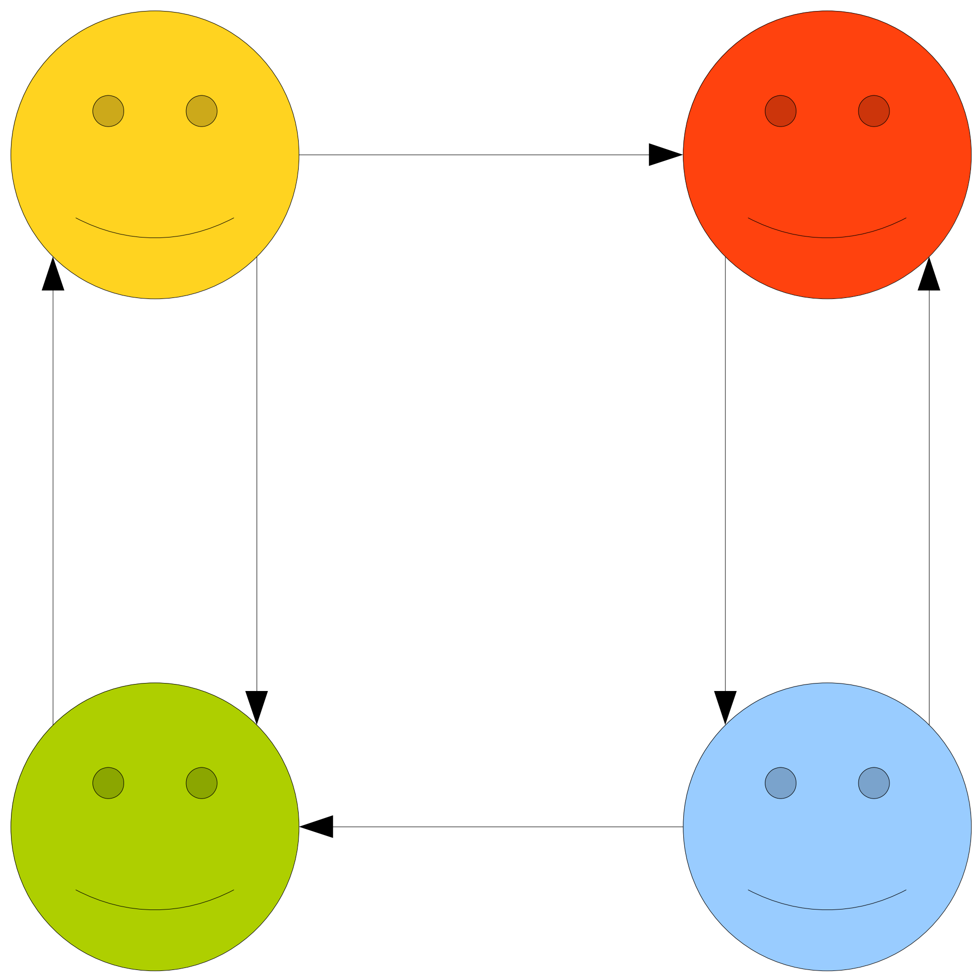
There is a person…

… that everyone else …

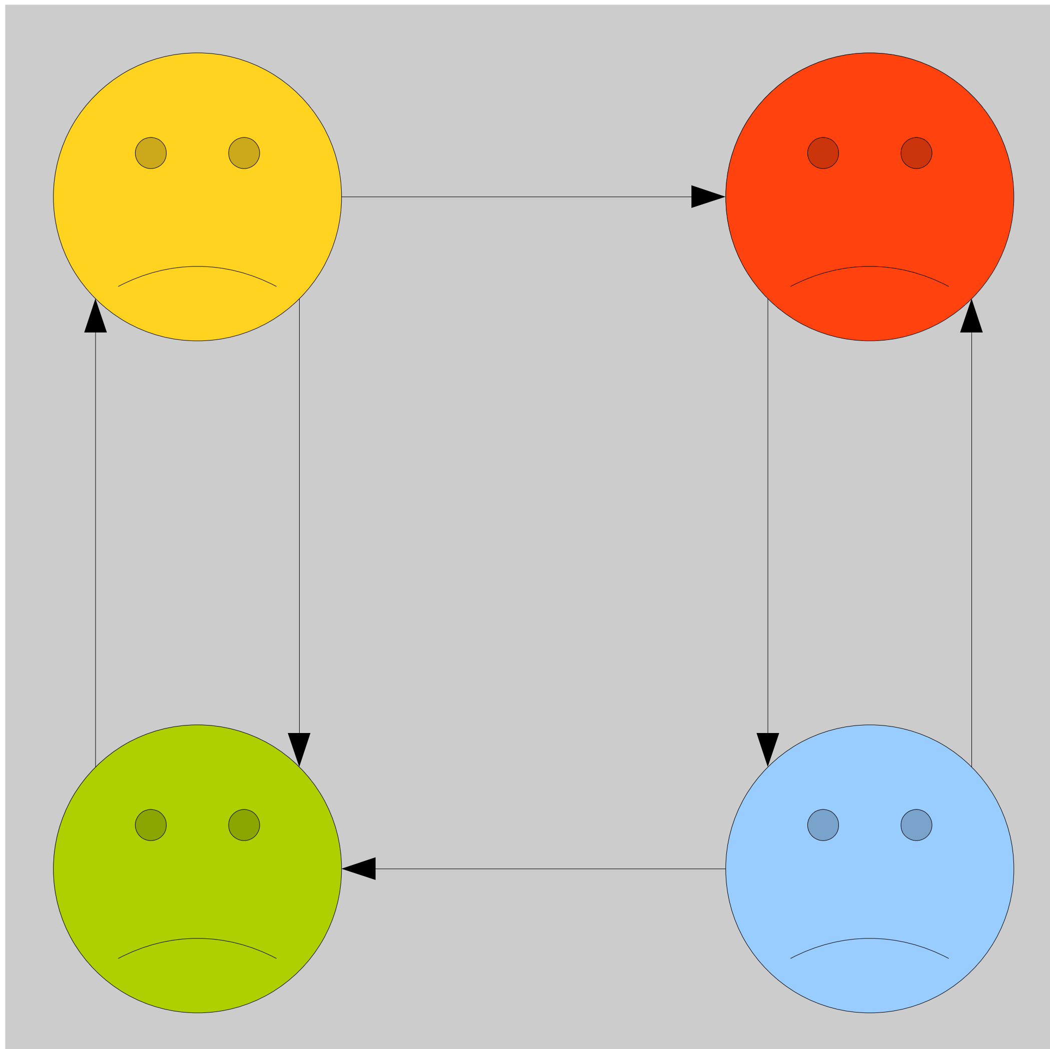
… loves.

$$
\begin{aligned}
&\exists p.\ (Person(p) \land \\
&\quad \forall q.\ (Person(q) \land p \neq q \to \\
&\qquad Loves(q, p) \\
&\quad) \\
&)
\end{aligned}
$$

Every Person Loves Someone Else

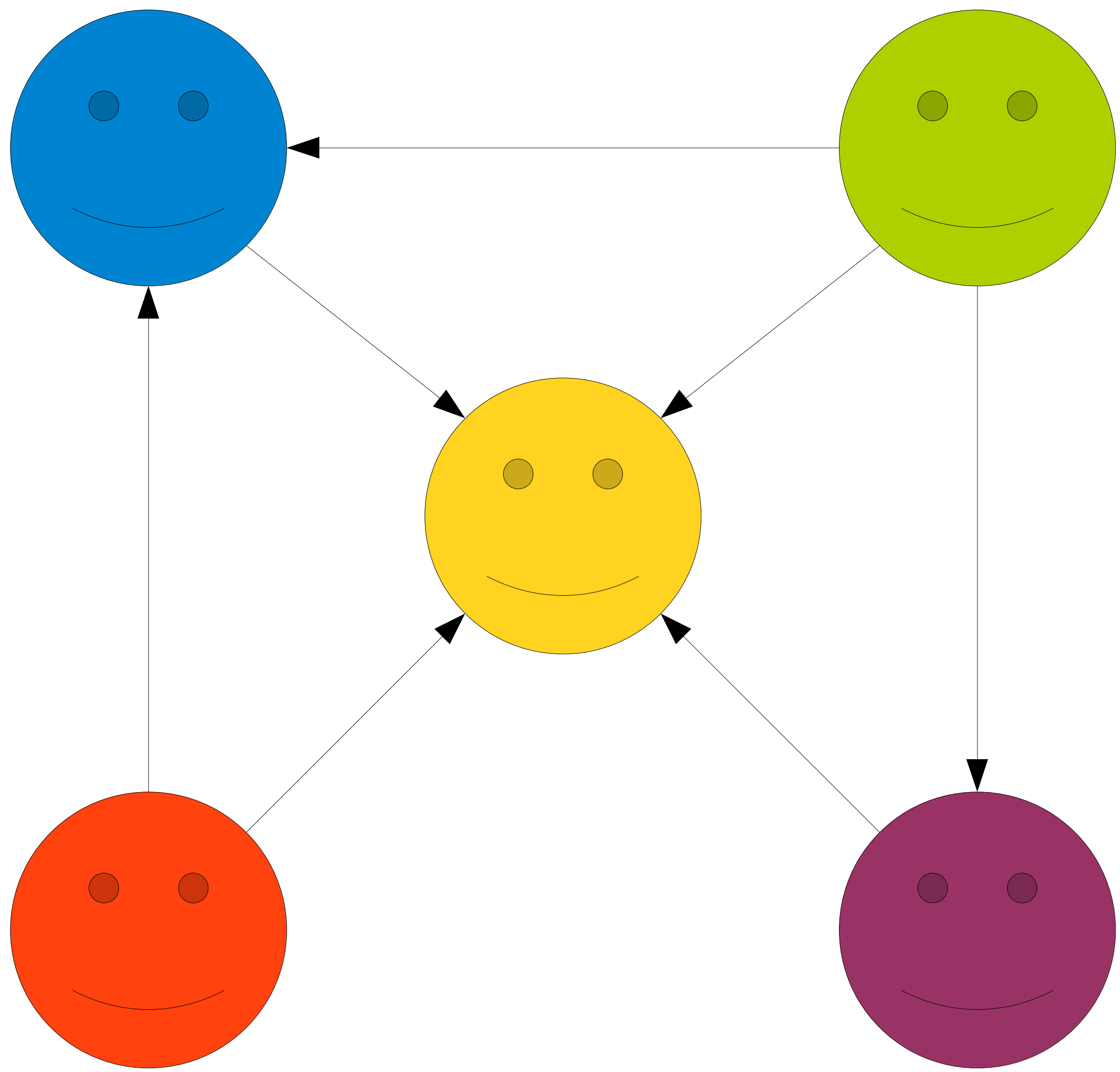


Every Person Loves Someone Else

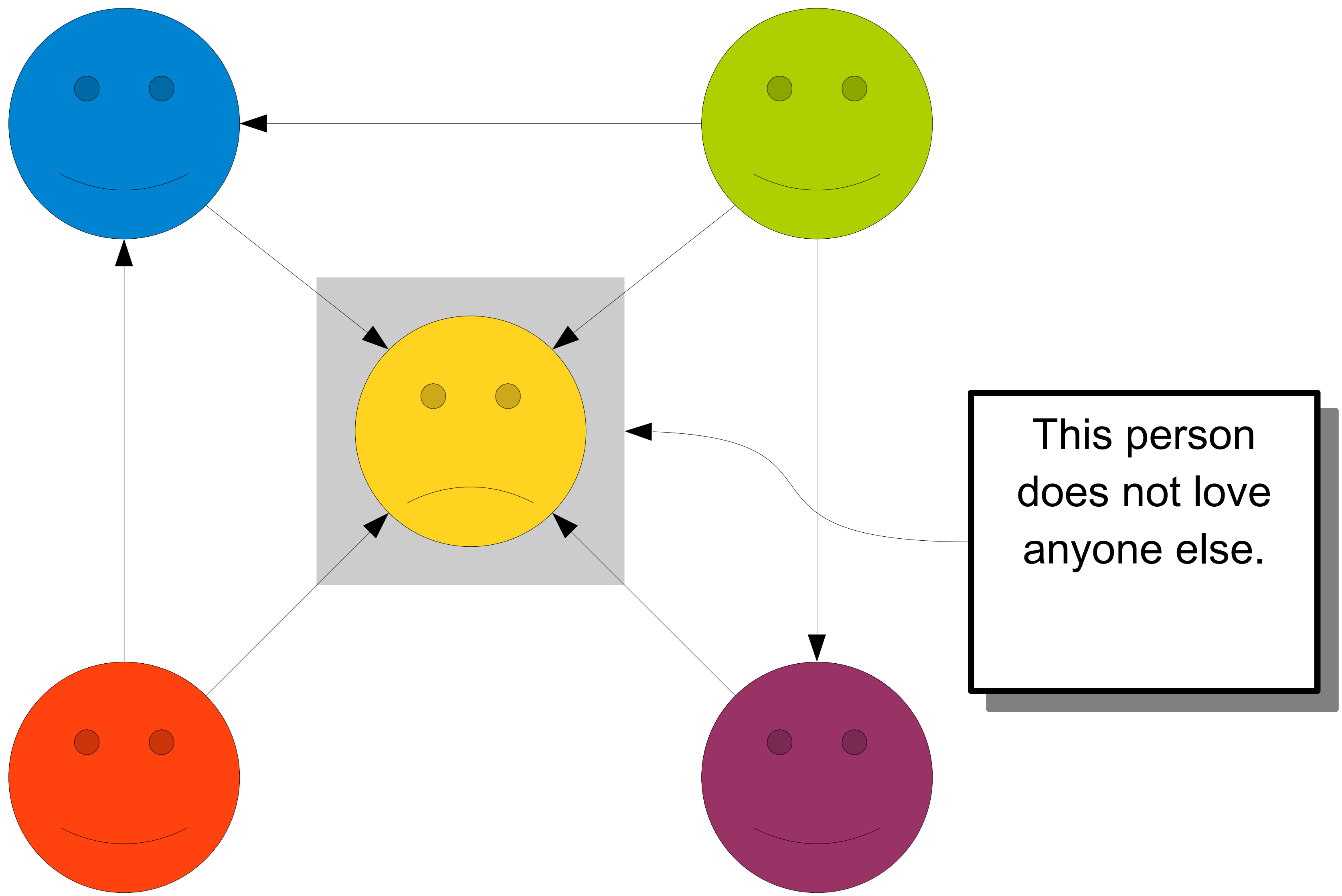


No one here is universally loved.

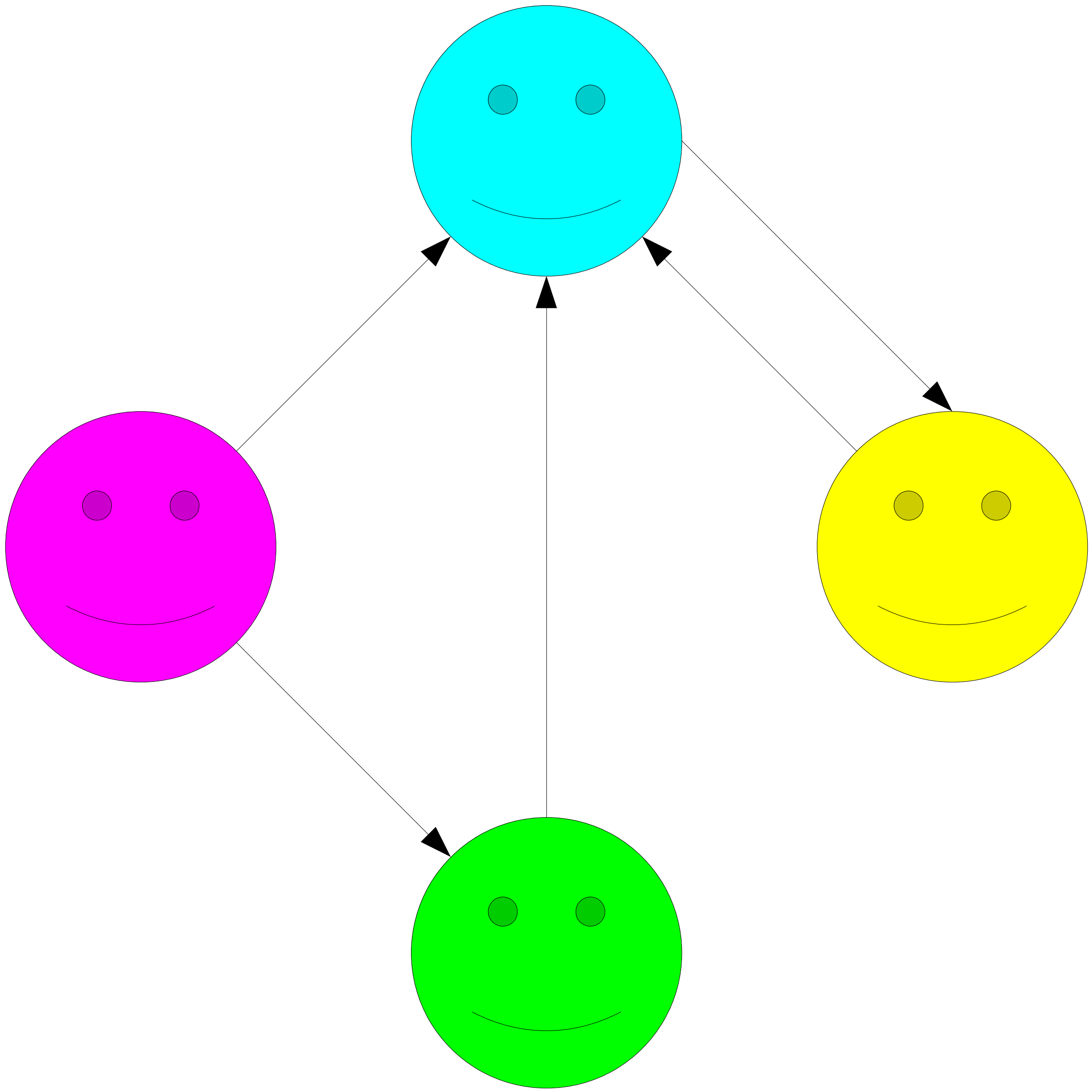
There is Someone Everyone Else Loves



There is Someone Everyone Else Loves



Every Person Loves Someone Else ***and*** There is Someone Everyone Else Loves



For every person…	$\forall p.\ (Person(p) \rightarrow$
… there is another person …	$\quad \exists q.\ (Person(q) \land p \neq q \land$
… they love	$\quad\quad Loves(p, q)$
	$\quad)$
	$)$
and	$\boldsymbol{\land}$
There is a person…	$\exists p.\ (Person(p) \land$
… that everyone else …	$\quad \forall q.\ (Person(q) \land p \neq q \rightarrow$
… loves.	$\quad\quad Loves(q, p)$
	$\quad)$
	$)$

Quantifier Ordering

- The statement

$$\forall x.\ \exists y.\ P(x, y)$$

  means "for any choice of x, there's some choice of y where $P(x, y)$ is true."
- The choice of y can be different every time and can depend on x.

Quantifier Ordering

- The statement

$$\exists x.\ \forall y.\ P(x, y)$$

  means "there is some x where for any choice of y, we get that $P(x, y)$ is true."
- Since the inner part has to work for any choice of y, this places a lot of constraints on what x can be.

Order matters when mixing existential and universal quantifiers!

Set Translations

Using the predicates

- $Set(S)$, which states that S is a set, and
- $x \in y$, which states that x is an element of y,

write a sentence in first-order logic that means "the empty set exists."

Using the predicates

- $Set(S)$, which states that S is a set, and
- $x \in y$, which states that x is an element of y,

write a sentence in first-order logic that means "the empty set exists."

First-order logic doesn't have set operators or symbols "built in." If we only have the predicates given above, how might we describe this?

The empty set exists.

There is some set S that is empty.

$\exists S.\ (Set(S)\ \land$

S is empty.

$)$

$\exists S.\ (Set(S)\ \wedge$

there are no elements in S

$)$

$\exists S.\ (Set(S)\ \wedge$

$\neg$ *there is an element in S*

$)$

$\exists S.\ (Set(S)\ \land$

$\neg$*there is an element x in S*

$)$

$$
\begin{aligned}
&\exists S.\ (Set(S)\ \land \\
&\qquad \neg\exists x.\ x \in S \\
&)
\end{aligned}
$$

$$\exists S.\ (Set(S) \wedge \neg \exists x.\ x \in S)$$

$$\exists S.\ (Set(S) \land \neg \exists x.\ x \in S)$$

$\exists S.\ (Set(S) \land$

there are no elements in S

$)$

$$\exists S.\ (Set(S) \wedge \neg \exists x.\ x \in S)$$

$\exists S.\ (Set(S) \wedge$

every object does not belong to S

$)$

$$\exists S.\ (Set(S) \land \neg \exists x.\ x \in S)$$

$\exists S.\ (Set(S) \land$

every object x does not belong to S

$)$

$$\exists S.\ (Set(S) \wedge \neg\exists x.\ x \in S)$$

$$\exists S.\ (Set(S) \wedge$$
$$\quad \forall x.\ x \notin S$$
$$)$$

$$\exists S.\ (Set(S) \land \neg \exists x.\ x \in S)$$

$$\exists S.\ (Set(S) \land \forall x.\ x \notin S)$$

$$\exists S. \ (Set(S) \land \neg \exists x. \ x \in S)$$

$$\exists S. \ (Set(S) \land \forall x. \ x \notin S)$$

Both of these translations are correct. Just like in propositional logic, there are many different equivalent ways of expressing the same statement in first-order logic.

$$\exists S.\ (Set(S) \land \neg\exists x.\ x \in S)$$

$$\exists S.\ (Set(S) \land \forall x.\ x \notin S)$$

$$\exists S.\ (Set(S) \land \neg\exists x.\ x \in S)$$

$$\exists S.\ (Set(S) \land \forall x.\ x \notin S)$$

Why can we switch which quantifier we're using here?

Mechanics: Negating Statements

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	For all choices of x, $P(x)$	For some choice of x, $\neg P(x)$
$\exists x.\ P(x)$	For some choice of x, $P(x)$	For all choices of x, $\neg P(x)$
$\forall x.\ \neg P(x)$	For all choices of x, $\neg P(x)$	For some choice of x, $P(x)$
$\exists x.\ \neg P(x)$	For some choice of x, $\neg P(x)$	For all choices of x, $P(x)$

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	When is this true?	When is this false?
$\forall x.\ P(x)$	For all choices of x, $P(x)$	**For some choice of x, $\neg P(x)$**
$\exists x.\ P(x)$	For some choice of x, $P(x)$	For all choices of x, $\neg P(x)$
$\forall x.\ \neg P(x)$	For all choices of x, $\neg P(x)$	For some choice of x, $P(x)$
$\exists x.\ \neg P(x)$	**For some choice of x, $\neg P(x)$**	For all choices of x, $P(x)$

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$\forall x.\ \neg P(x)$	For all choices of x, $\neg P(x)$	For some choice of x, $P(x)$
$\exists x.\ \neg P(x)$	**For some choice of x, $\neg P(x)$**	For all choices of x, $P(x)$

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$\forall x.\ \neg P(x)$	For all choices of x, $\neg P(x)$	For some choice of x, $P(x)$
$\exists x.\ \neg P(x)$	For some choice of x, $\neg P(x)$	For all choices of x, $P(x)$

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$\forall x.\ \neg P(x)$	**For all choices of x, $\neg P(x)$**	For some choice of x, $P(x)$
$\exists x.\ \neg P(x)$	For some choice of x, $\neg P(x)$	For all choices of x, $P(x)$

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	For all choices of x, $P(x)$	$\exists x.\ \neg P(x)$
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$\forall x.\ \neg P(x)$	For all choices of x, $\neg P(x)$	For some choice of x, $P(x)$
$\exists x.\ \neg P(x)$	For some choice of x, $\neg P(x)$	For all choices of x, $P(x)$

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	For all choices of x, $P(x)$	$\exists x.\ \neg P(x)$
$\exists x.\ P(x)$	**For some choice of x, $P(x)$**	$\forall x.\ \neg P(x)$
$\forall x.\ \neg P(x)$	For all choices of x, $\neg P(x)$	**For some choice of x, $P(x)$**
$\exists x.\ \neg P(x)$	For some choice of x, $\neg P(x)$	For all choices of x, $P(x)$

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	For all choices of x, $P(x)$	$\exists x.\ \neg P(x)$
$\exists x.\ P(x)$	**For some choice of x, $P(x)$**	$\forall x.\ \neg P(x)$
$\forall x.\ \neg P(x)$	For all choices of x, $\neg P(x)$	$\exists x.\ P(x)$
$\exists x.\ \neg P(x)$	For some choice of x, $\neg P(x)$	For all choices of x, $P(x)$

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	For all choices of x, $P(x)$	$\exists x.\ \neg P(x)$
$\exists x.\ P(x)$	For some choice of x, $P(x)$	$\forall x.\ \neg P(x)$
$\forall x.\ \neg P(x)$	For all choices of x, $\neg P(x)$	$\exists x.\ P(x)$
$\exists x.\ \neg P(x)$	For some choice of x, $\neg P(x)$	For all choices of x, $P(x)$

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	**For all choices of *x*, *P*(*x*)**	$\exists x.\ \neg P(x)$
$\exists x.\ P(x)$	For some choice of x, $P(x)$	$\forall x.\ \neg P(x)$
$\forall x.\ \neg P(x)$	For all choices of x, $\neg P(x)$	$\exists x.\ P(x)$
$\exists x.\ \neg P(x)$	For some choice of x, $\neg P(x)$	**For all choices of *x*, *P*(*x*)**

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	**For all choices of *x*, *P*(*x*)**	$\exists x.\ \neg P(x)$
$\exists x.\ P(x)$	For some choice of x, $P(x)$	$\forall x.\ \neg P(x)$
$\forall x.\ \neg P(x)$	For all choices of x, $\neg P(x)$	$\exists x.\ P(x)$
$\exists x.\ \neg P(x)$	For some choice of x, $\neg P(x)$	$\forall x.\ P(x)$

An Extremely Important Table

	When is this true?	When is this false?
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$\exists x.\ P(x)$	For some choice of x, $P(x)$	$\forall x.\ \neg P(x)$
$\forall x.\ \neg P(x)$	For all choices of x, $\neg P(x)$	$\exists x.\ P(x)$
$\exists x.\ \neg P(x)$	For some choice of x, $\neg P(x)$	$\forall x.\ P(x)$

Negating First-Order Statements

- Use the equivalences

$$\neg\forall x.\ A \quad \textit{is equivalent to} \quad \exists x.\ \neg A$$

$$\neg\exists x.\ A \quad \textit{is equivalent to} \quad \forall x.\ \neg A$$

  to negate quantifiers.
- Mechanically:
  - Push the negation across the quantifier.
  - Change the quantifier from ∀ to ∃ or vice-versa.
- Use techniques from propositional logic to negate connectives.

Taking a Negation

$$\forall x.\ \exists y.\ Loves(x, y)$$

("Everyone loves someone.")

$$\neg\forall x.\ \exists y.\ Loves(x, y)$$
$$\exists x.\ \neg\exists y.\ Loves(x, y)$$
$$\exists x.\ \forall y.\ \neg Loves(x, y)$$

("There's someone who doesn't love anyone.")

Two Useful Equivalences

- The following equivalences are useful when negating statements in first-order logic:

$$\neg(p \land q) \quad \textit{is equivalent to} \quad p \to \neg q$$

$$\neg(p \to q) \quad \textit{is equivalent to} \quad p \land \neg q$$

- These identities are useful when negating statements involving quantifiers.
  - $\land$ is used in existentially-quantified statements.
  - $\to$ is used in universally-quantified statements.
- When pushing negations across quantifiers, we *strongly recommend* using the above equivalences to keep $\to$ with $\forall$ and $\land$ with $\exists$.

Negating Quantifiers

- What is the negation of the following statement, which says "there is a cute puppy"?

$$\boldsymbol{\exists x. (Puppy(x) \land Cute(x))}$$

- We can obtain it as follows:

$$\neg \exists x. (Puppy(x) \land Cute(x))$$

$$\forall x. \neg (Puppy(x) \land Cute(x))$$

$$\forall x. (Puppy(x) \to \neg Cute(x))$$

- This says "no puppy is cute."
- Do you see why this is the negation of the original statement from both an intuitive and formal perspective?

$$\exists S.\ (Set(S) \land \forall x.\ \neg(x \in S))$$

("There is a set with no elements.")

$$\neg\exists S.\ (Set(S) \land \forall x.\ \neg(x \in S))$$
$$\forall S.\ \neg(Set(S) \land \forall x.\ \neg(x \in S))$$
$$\forall S.\ (Set(S) \to \neg\forall x.\ \neg(x \in S))$$
$$\forall S.\ (Set(S) \to \exists x.\ \neg\neg(x \in S))$$
$$\forall S.\ (Set(S) \to \exists x.\ x \in S)$$

("Every set contains at least one element.")

Restricted Quantifiers

Quantifying Over Sets

- The notation

$$\forall x \in S.\ P(x)$$

  means “for any element x of set S, $P(x)$ holds.” (It’s vacuously true if S is empty.)

- The notation

$$\exists x \in S.\ P(x)$$

  means “there is an element x of set S where $P(x)$ holds.” (It’s false if S is empty.)

Quantifying Over Sets

- The syntax

$$\forall x \in S.\ P(x)$$

$$\exists x \in S.\ P(x)$$

  is allowed for quantifying over sets.
- In CS103, feel free to use these restricted quantifiers, but please do not use variants of this syntax.
- For example, don't do things like this:

⚠ $\forall x$ with $P(x)$. $Q(x)$ ⚠

⚠ $\forall y$ such that $P(y) \land Q(y)$. $R(y)$. ⚠

⚠ $\exists P(x)$. $Q(x)$ ⚠

Expressing Uniqueness

Using the predicate

- $WayToFindOut(w)$, which states that w is a way to find out,

write a sentence in first-order logic that means “there is only one way to find out.”

There is only one way to find out.

Something is a way to find out, and nothing else is.

Some thing w is a way to find out, and nothing else is.

Some thing w is a way to find out, and nothing besides w is a way to find out

$\exists w.\ (WayToFindOut(w)\ \wedge$

nothing besides w is way to find out

$)$

$\exists w.\ (WayToFindOut(w)\ \land$

anything that isn't w isn't a way to find out

$)$

$\exists w.\ (\mathit{WayToFindOut}(w)\ \wedge$

any thing x that isn't w isn't a way to find out

$)$

$$\exists w.\ (\mathit{WayToFindOut}(w)\ \land$$
$$\quad \forall x.\ (x \neq w \rightarrow \textit{x isn't a way to find out})$$
$$)$$

$$
\begin{aligned}
&\exists w.\ (\mathit{WayToFindOut}(w)\ \land \\
&\qquad \forall x.\ (x \neq w \rightarrow \neg \mathit{WayToFindOut}(x)) \\
&)
\end{aligned}
$$

$$
\begin{aligned}
&\exists w.\ (\mathit{WayToFindOut}(w) \land \\
&\quad \forall x.\ (x \neq w \to \neg \mathit{WayToFindOut}(x)) \\
&)
\end{aligned}
$$

$$
\begin{aligned}
&\exists w.\ (\mathit{WayToFindOut}(w)\ \land \\
&\quad \forall x.\ (\mathit{WayToFindOut}(x) \rightarrow x = w) \\
&)
\end{aligned}
$$

$$
\begin{aligned}
&\exists w.\ (\mathit{WayToFindOut}(w)\ \land \\
&\quad \forall x.\ (\mathit{WayToFindOut}(x) \rightarrow x = w) \\
&)
\end{aligned}
$$

Expressing Uniqueness

- To express the idea that there is exactly one object with some property, we write that
  - there exists at least one object with that property, and that
  - there are no other objects with that property.
- You sometimes see a special "uniqueness quantifier" used to express this:

$$\exists! x.\ P(x)$$

- For the purposes of CS103, please do not use this quantifier. We want to give you more practice using the regular ∀ and ∃ quantifiers.

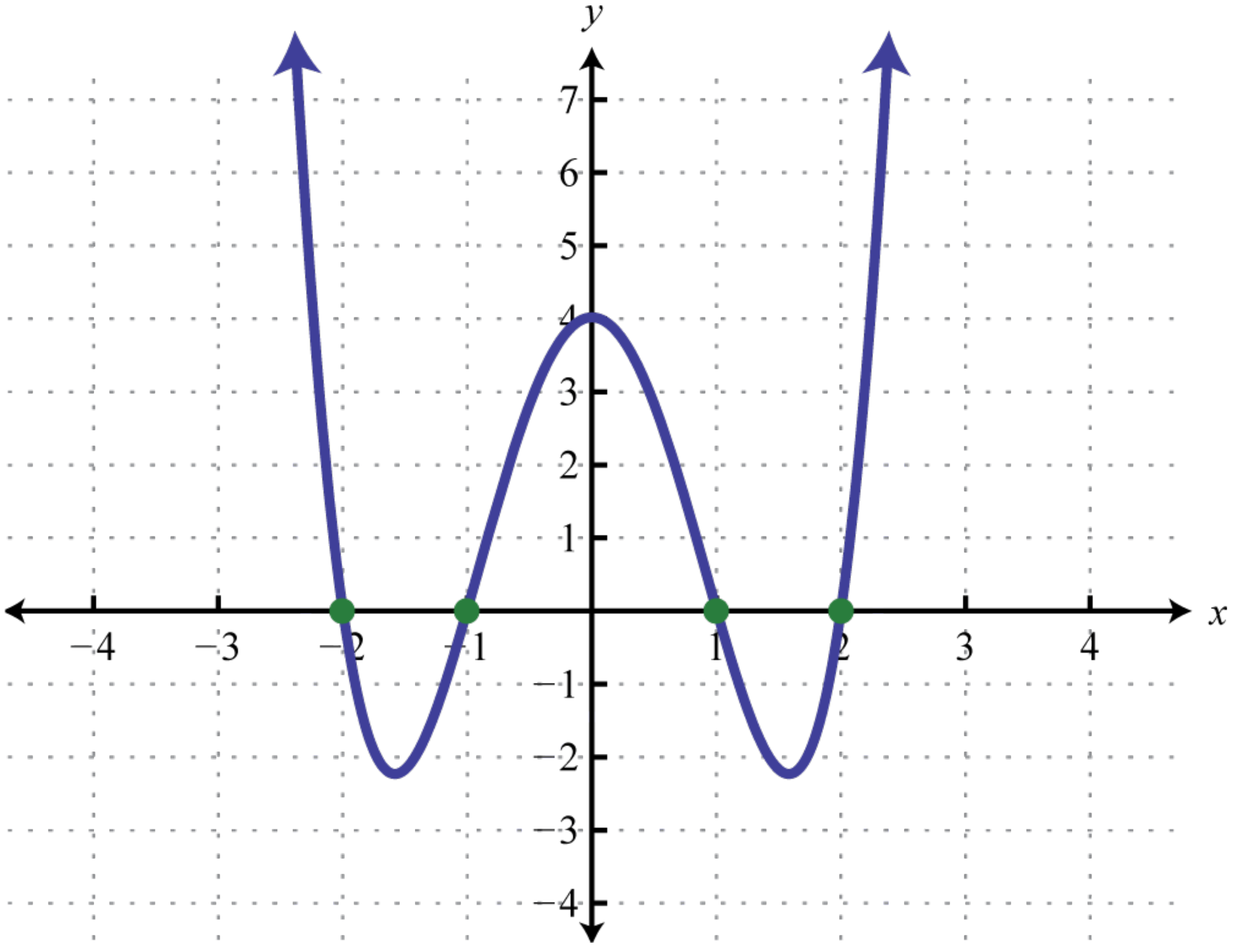
Functions

Outline for Today

- ***What is a Function?***
  - It’s more nuanced than you might expect.
- ***Domains and Codomains***
  - Where functions start, and where functions end.
- ***Defining a Function***
  - Expressing transformations compactly.
- ***Special Classes of Functions***
  - Useful types of functions you’ll encounter IRL.
- ***Proofs on First-Order Definitions***
  - A key skill.

What is a function?

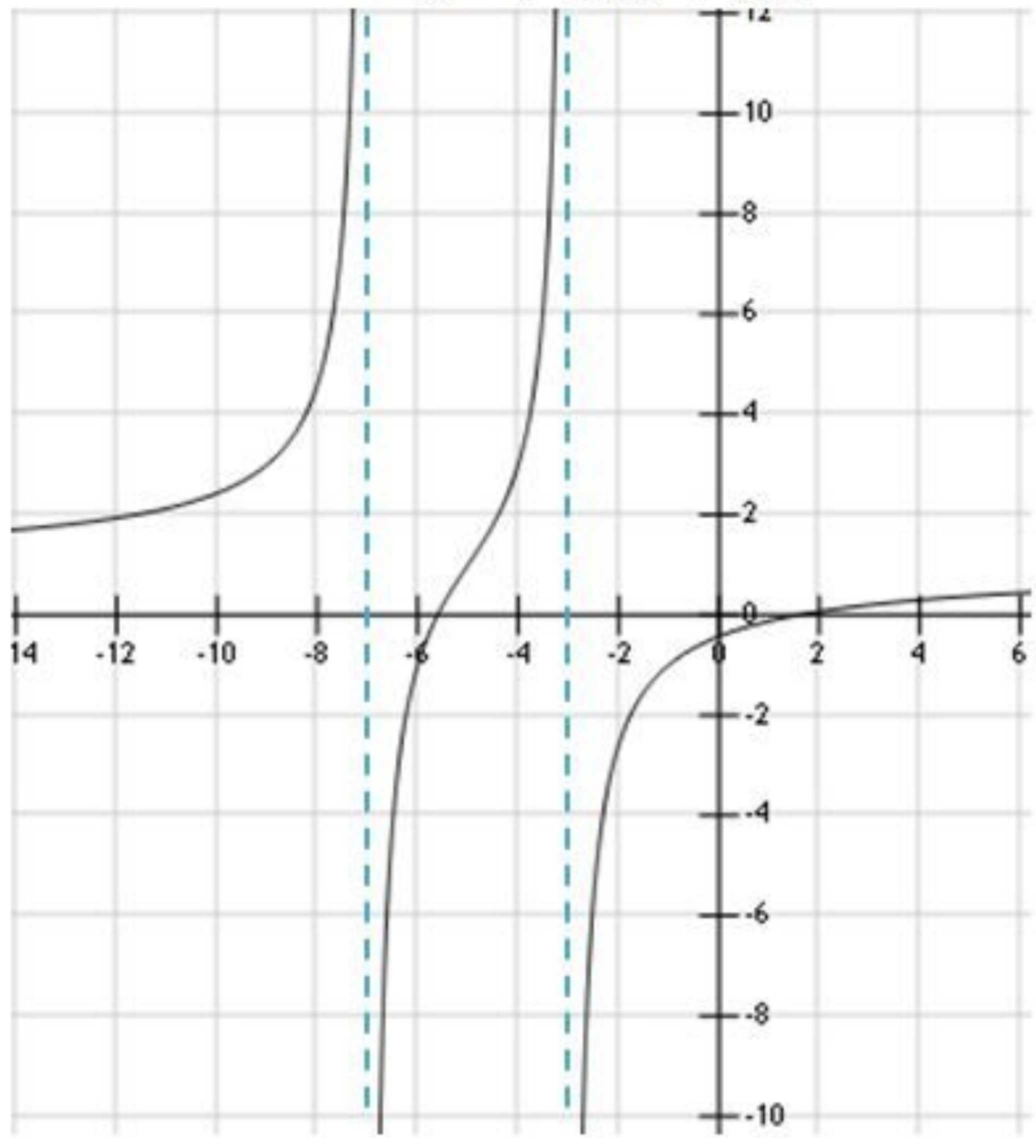
Functions, High-School Edition



$$f(x) = x^4 - 5x^2 + 4$$

source: https://saylordotorg.github.io/text_intermediate-algebra/section_07/6aaf3a5ab540885474d58855068b64ce.png

$$f(x) = \frac{x^2 + 4x - 9}{x^2 + 10x + 21}$$



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Functions, High-School Edition

- In high school, functions are usually given as objects of the form

$$f(x) = \frac{x^3 + 3x^2 + 15x + 7}{1 - x^{137}}$$

- What does a function do?
  - It takes in as input a real number.
  - It outputs a real number
  - … except when there are vertical asymptotes or other discontinuities, in which case the function doesn't output anything.

Functions, CS Edition

```
int flipUntil(int n) {
  int numHeads = 0;
  int numTries = 0;

  while (numHeads < n) {
    if (randomBoolean()) {
      numHeads++;
    }
    numTries++;
  }

  return numTries;
}
```

Functions, CS Edition

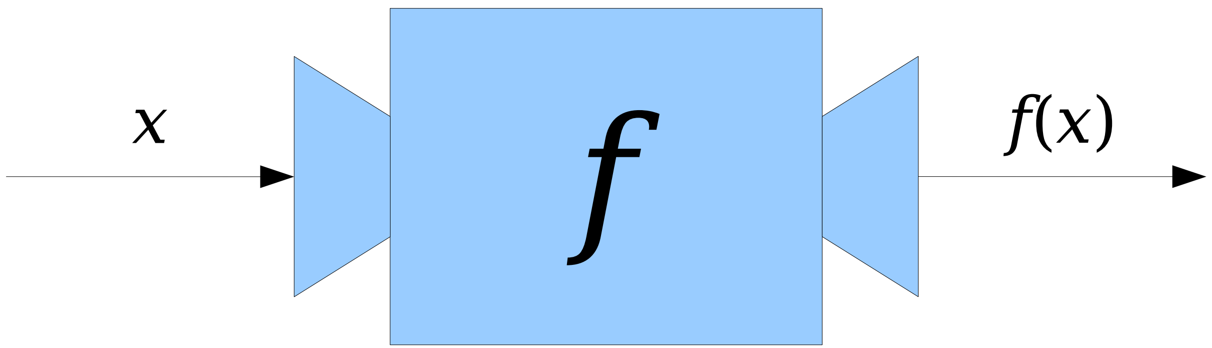
- In programming, functions
  - might take in inputs,
  - might return values,
  - might have side effects,
  - might never return anything,
  - might crash, and
  - might return different values when called multiple times.

What's Common?

- Although high-school math functions and CS functions are pretty different, they have two key aspects in common:
  - They take in inputs.
  - They produce outputs.
- In math, we like to keep things easy, so that's pretty much how we're going to define a function.

High-Level Intuition:

A function is an object f that takes in exactly one input x and produces exactly one output $f(x)$.



(This is not definition. It's just to help you build and intuition.)

High School versus CS Functions

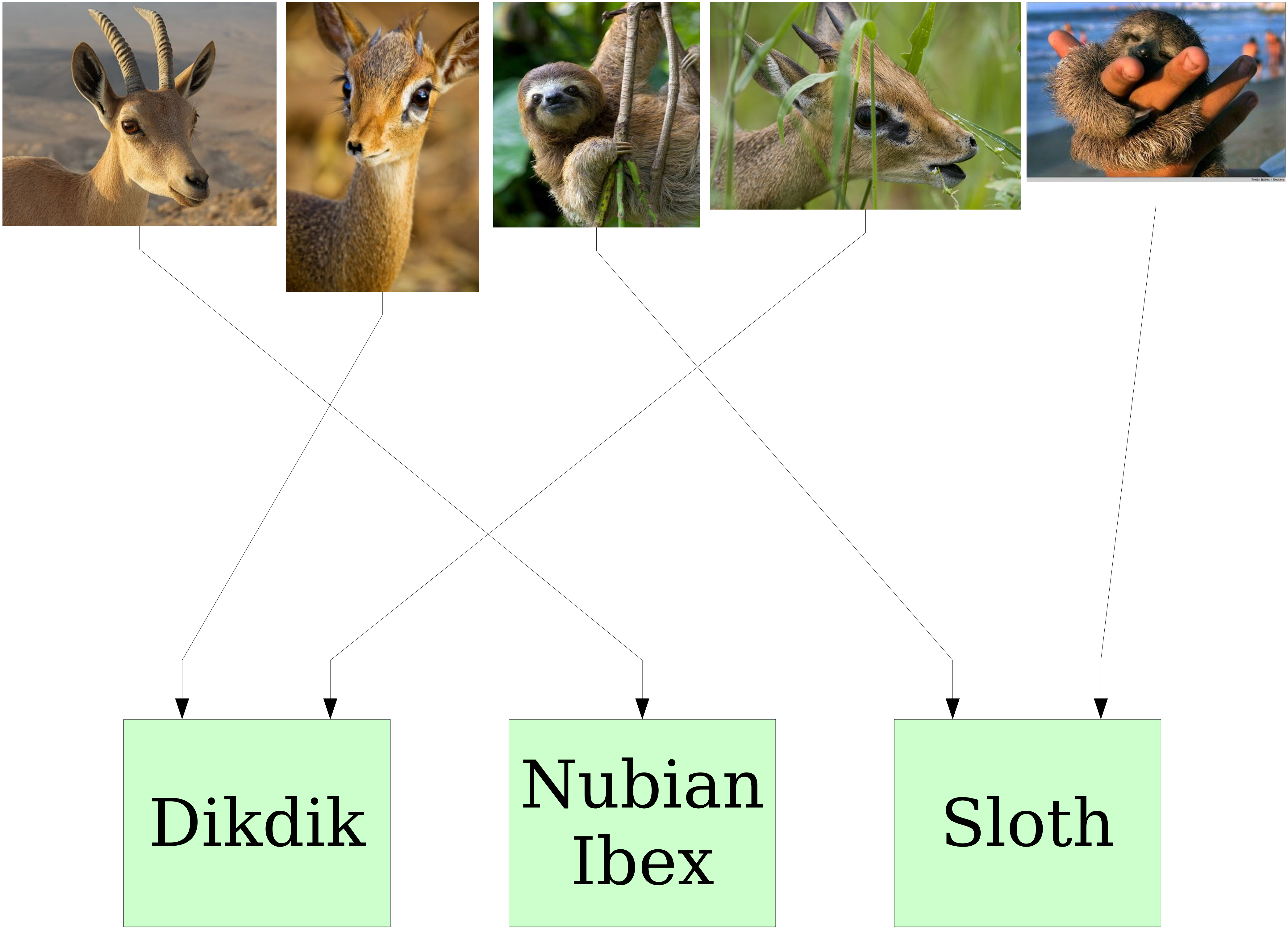
- In high school, functions usually were given by a rule:

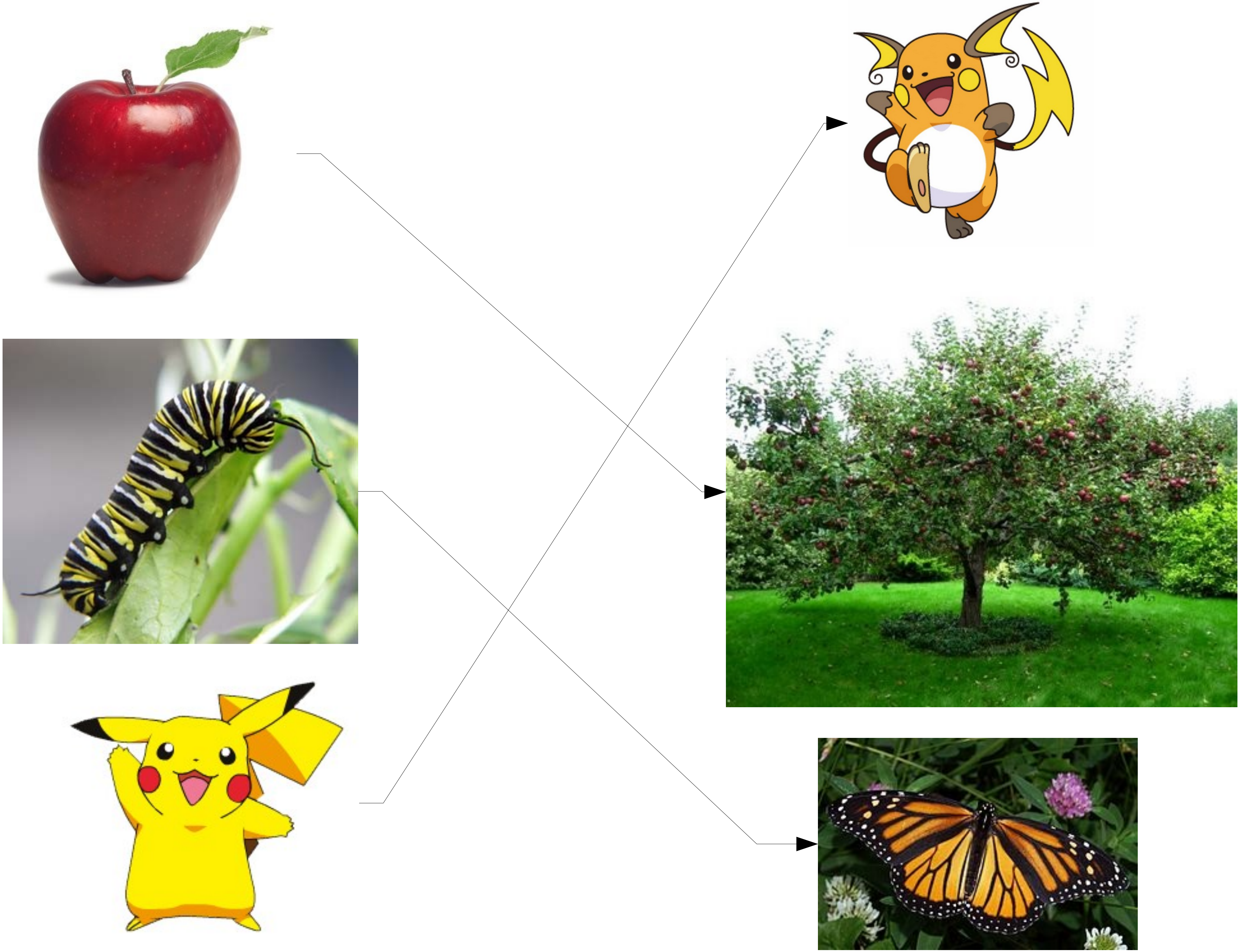
$$f(x) = 4x + 15$$

- In CS, functions are usually given by code:

```
int factorial(int n) {
    int result = 1;
    for (int i = 1; i <= n; i++) {
        result *= i;
    }
    return result;
}
```

- What sorts of functions are we going to allow from a mathematical perspective?


Dikdik
Nubian Ibex
Sloth



… but also …

$$f(x) = x^2 + 3x - 15$$

In mathematics, functions are ***deterministic***.

That is, given the same input, a function must always produce the same output.

The following is a perfectly valid piece of C++ code, but it's not a valid function under our definition:

```
int randomNumber(int numOutcomes) {
    return rand() % numOutcomes;
}
```

One Challenge

$$f(x) = x^2 + 2x + 5$$

$$f(x) = x^2 + 2x + 5$$

$$f(\ 3\ ) = 3^2 + 3 \cdot 2 + 5 = 20$$

$$f(x) = x^2 + 2x + 5$$

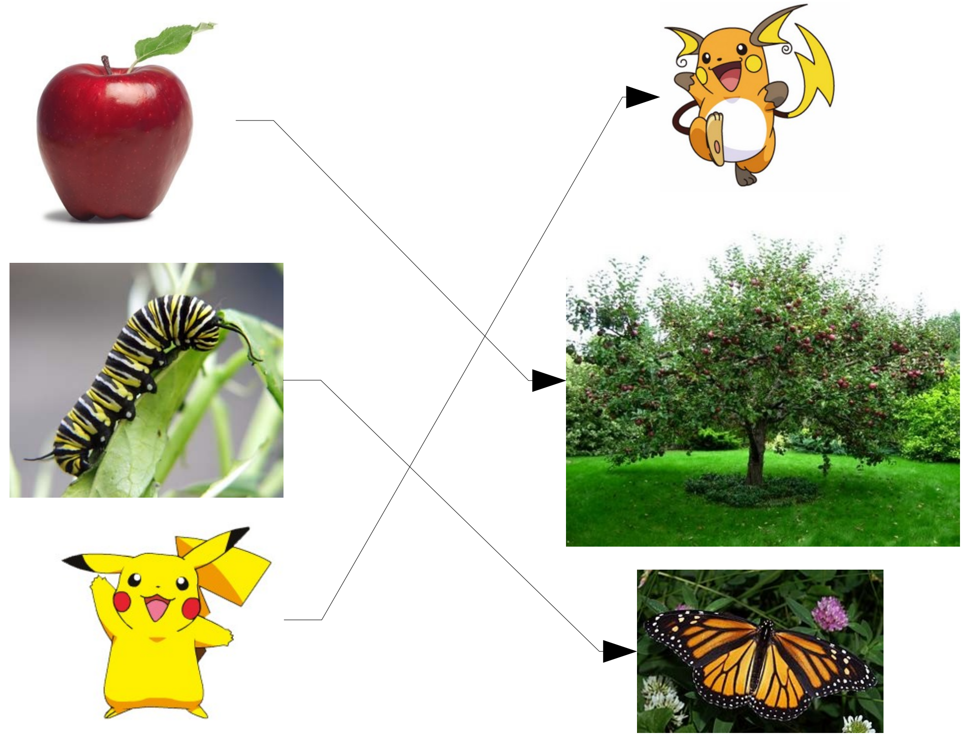
$$f(\ 3\ ) = 3^2 + 3 \cdot 2 + 5 = 20$$

$$f(\ 0\ ) = 0^2 + 0 \cdot 2 + 5 = 5$$

$$f(x) = x^2 + 2x + 5$$

$$f(\ 3\ ) = 3^2 + 3 \cdot 2 + 5 = 20$$
$$f(\ 0\ ) = 0^2 + 0 \cdot 2 + 5 = 5$$

$f($  $) = \ldots ?$



$f($  $) =$ 

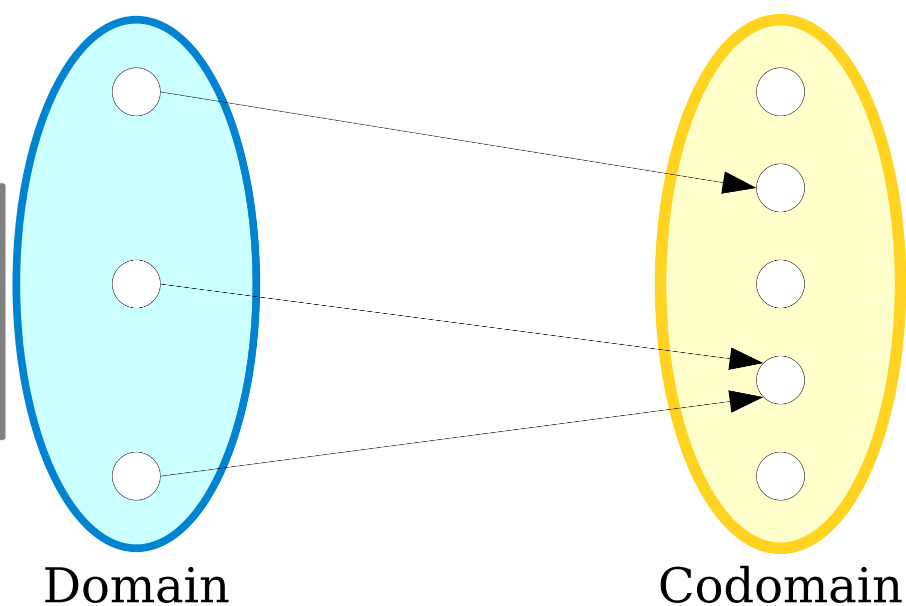
$f(137) = \ldots?$

We need to make sure we can't apply functions to meaningless inputs.

Domains and Codomains

- Every function *f* has two sets associated with it: its ***domain*** and its ***codomain***.
- A function *f* can only be applied to elements of its domain. For any *x* in the domain, *f*(*x*) belongs to the codomain.

The function must be defined for every element of the domain.



The output of the function must always be in the codomain, but not all elements of the codomain must be produced as outputs.

Domains and Codomains

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- A function *f* can only be applied to elements of its domain. For any *x* in the domain, *f*(*x*) belongs to the codomain.

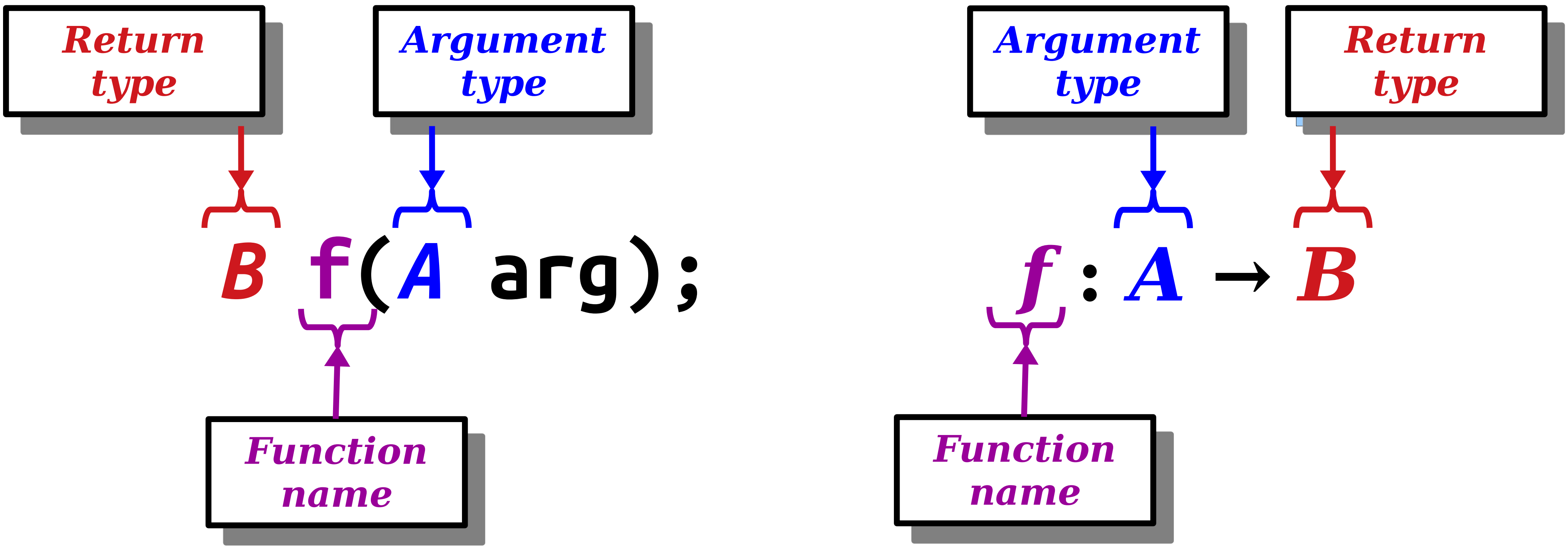
The ***domain*** of this function is ℝ. Any real number can be provided as input.

The ***codomain*** of this function is ℝ. Everything produced is a real number, but not all real numbers can be produced.

```
double absoluteValueOf(double x) {
    if (x >= 0) {
        return x;
    } else {
        return -x;
    }
}
```

Domains and Codomains

- If f is a function whose domain is A and whose codomain is B, we write $\boldsymbol{f : A \to B}$.
- Think of this like a "function prototype" in C++.



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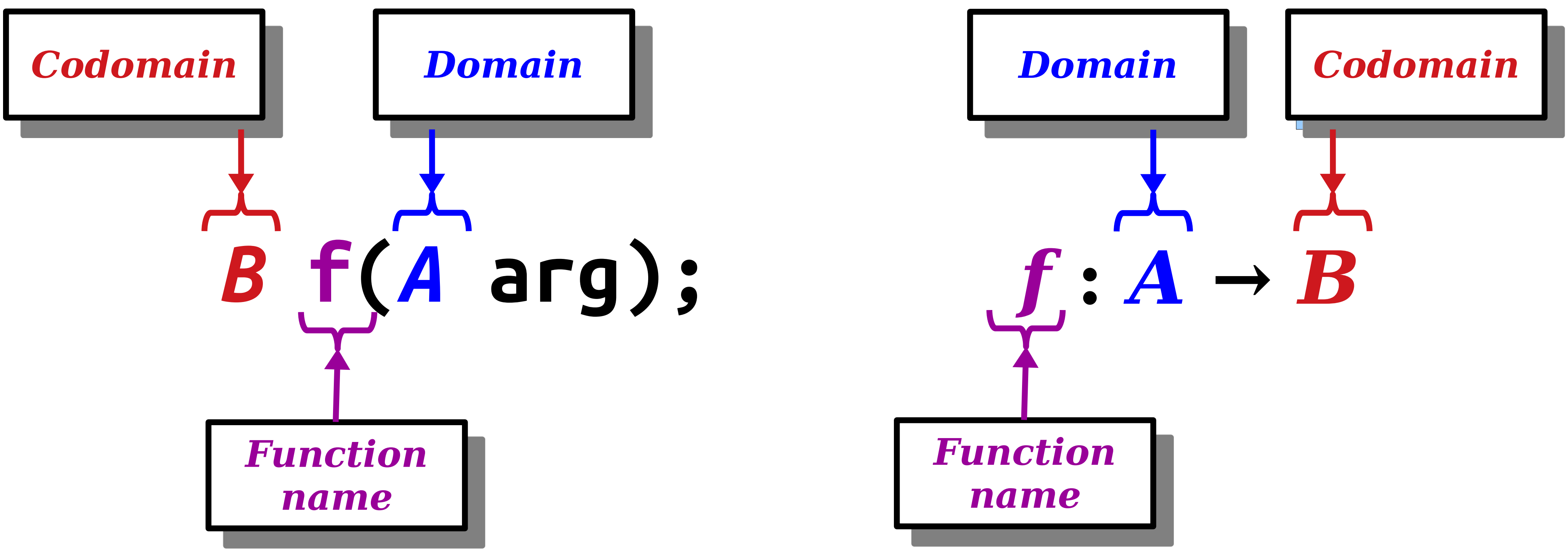


The Official Rules for Functions

- Formally speaking, we say that $f : A \to B$ if the following two rules hold.
- First, f must be obey its domain/codomain rules:

$$\forall a \in A.\ \exists b \in B.\ f(a) = b$$

(*"Every input in A maps to some output in B."*)

- Second, f must be deterministic:

$$\forall a_1 \in A.\ \forall a_2 \in A.\ (a_1 = a_2 \to f(a_1) = f(a_2))$$

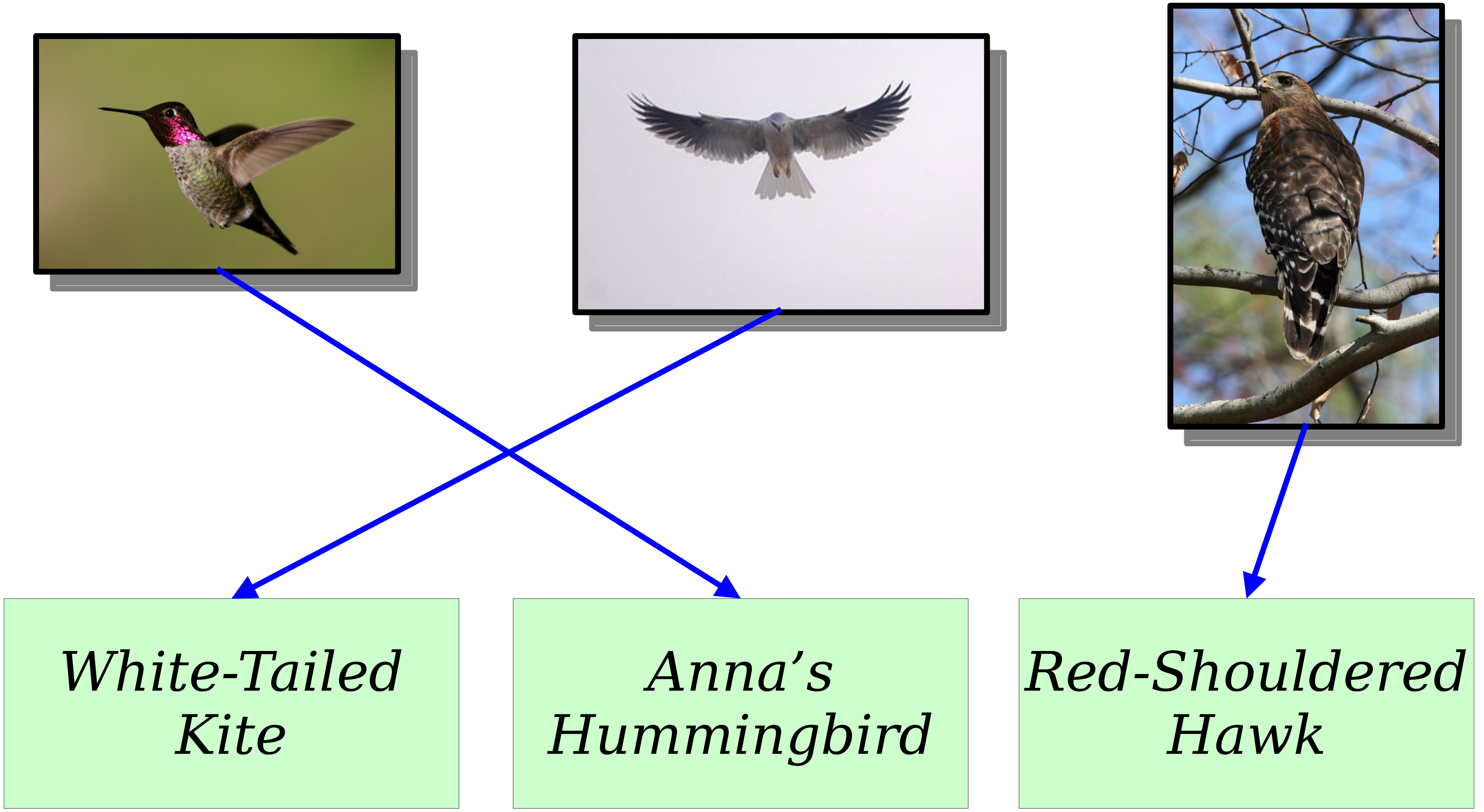
(*"Equal inputs produce equal outputs."*)

- If you're ever curious about whether something is a function, look back at these rules and check! For example:
  - Can a function have an empty domain?
  - Can a function have an empty codomain?

Defining Functions

Defining Functions

- To define a function, you need to
  - specify the domain,
  - specify the codomain, and
  - give a ***rule*** used to evaluate the function.
- All three pieces are necessary.
  - We need to domain to know what the function can be applied to.
  - We need to codomain to know what the output space is.
  - We need the rule to be able to evaluate the function.
- There are many ways to do this. Let’s go over a few examples.



Functions can be defined as a ***picture***.
Draw the domain and codomain explicitly.
Then, add arrows to show the outputs.

$$f : \mathbb{Z} \to \mathbb{Z}, \text{ where}$$

$$f(x) = x^2 + 3x - 15$$

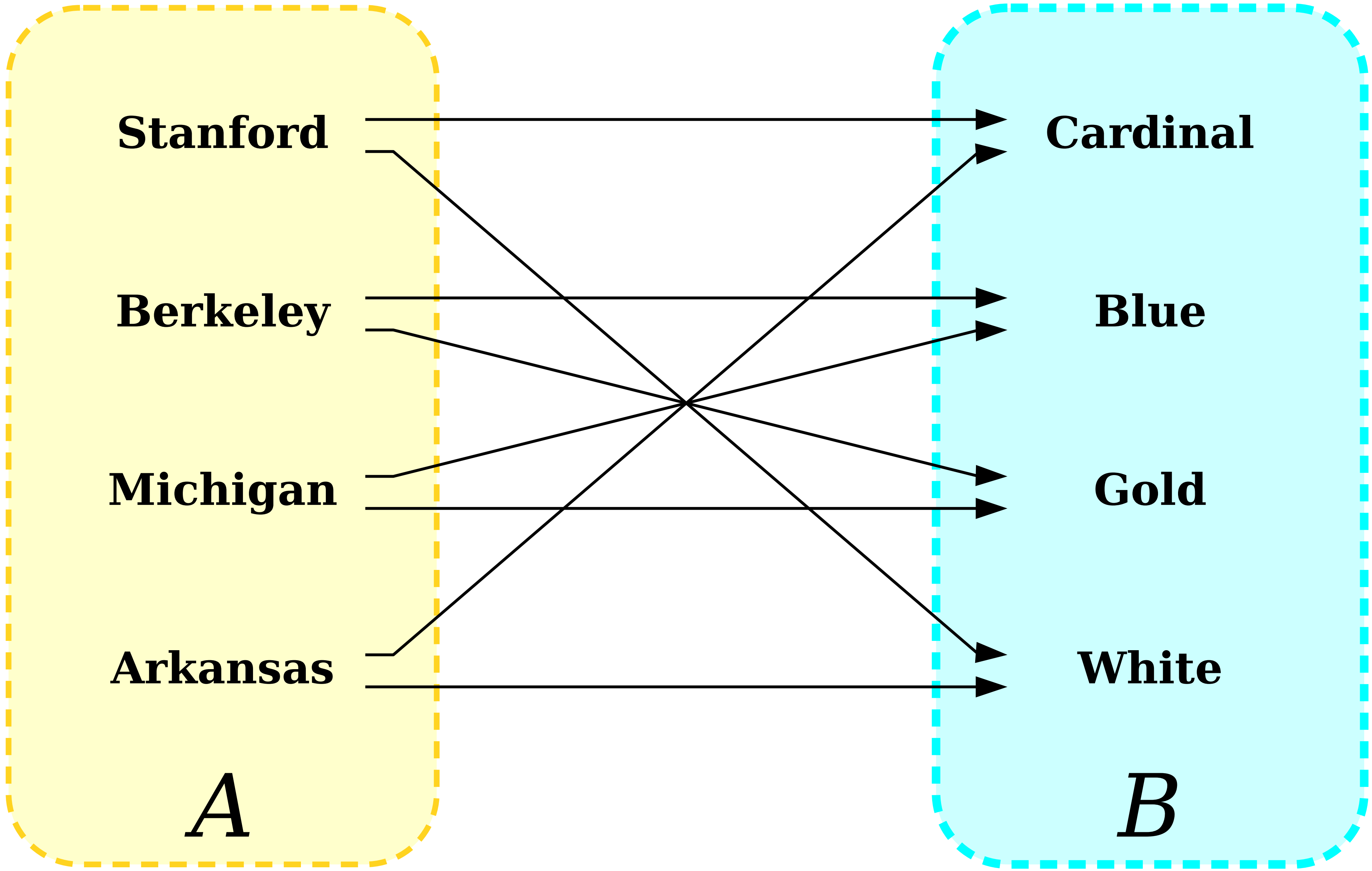
Functions can be defined as a ***rule***.
Be sure to explicitly state what the domain and codomain are!

$$f : \mathbb{Z} \to \mathbb{N}, \text{ where}$$

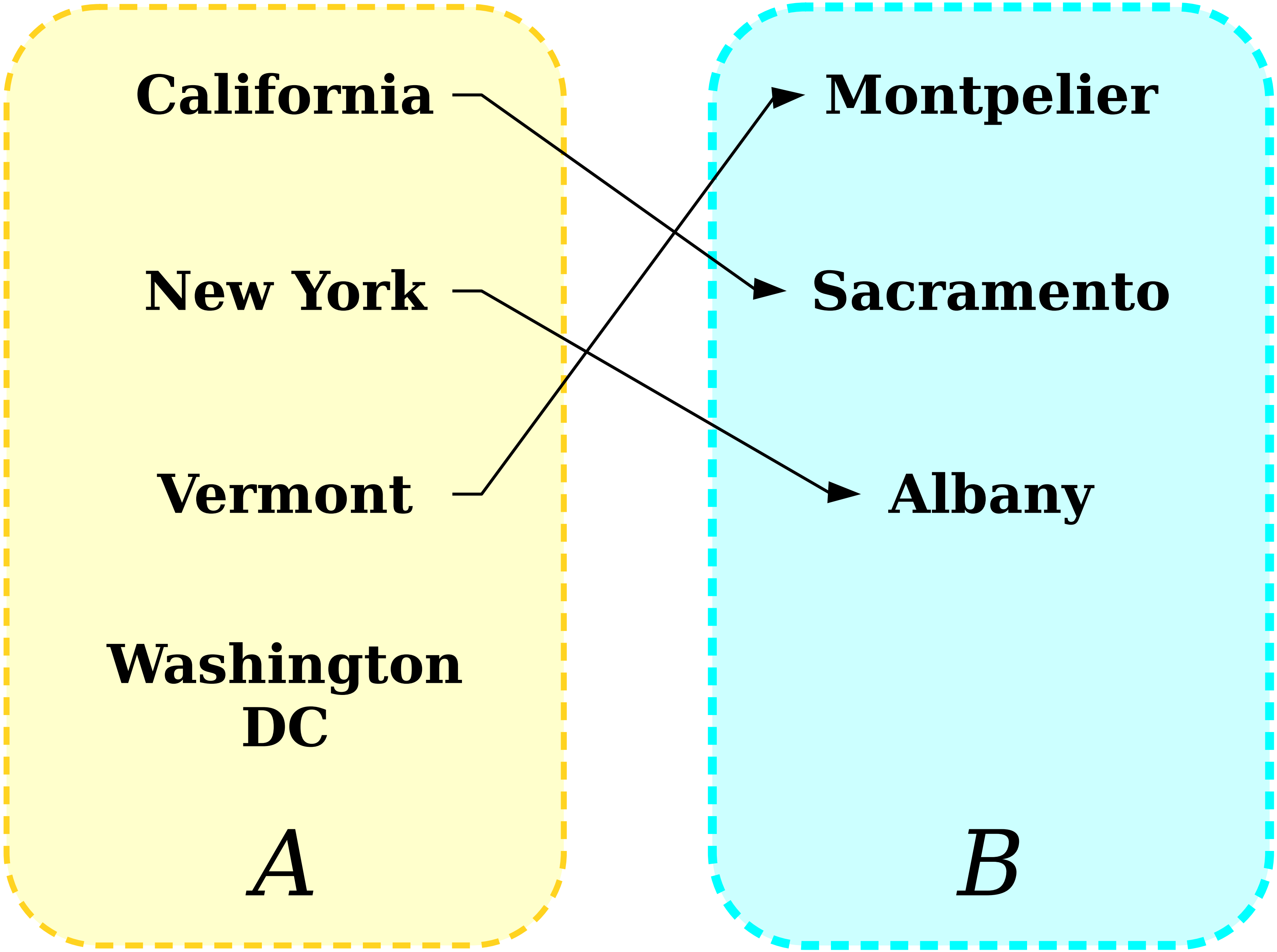
$$f(n) = \begin{cases} n & \text{if } n \geq 0 \\ -n & \text{if } n \leq 0 \end{cases}$$

Some rules are given ***piecewise***. We select which rule to apply based on the conditions on the right. (Just make sure at least one condition applies and that all applicable conditions give the same result!)

Some Nuances



Is this a function from A to B?



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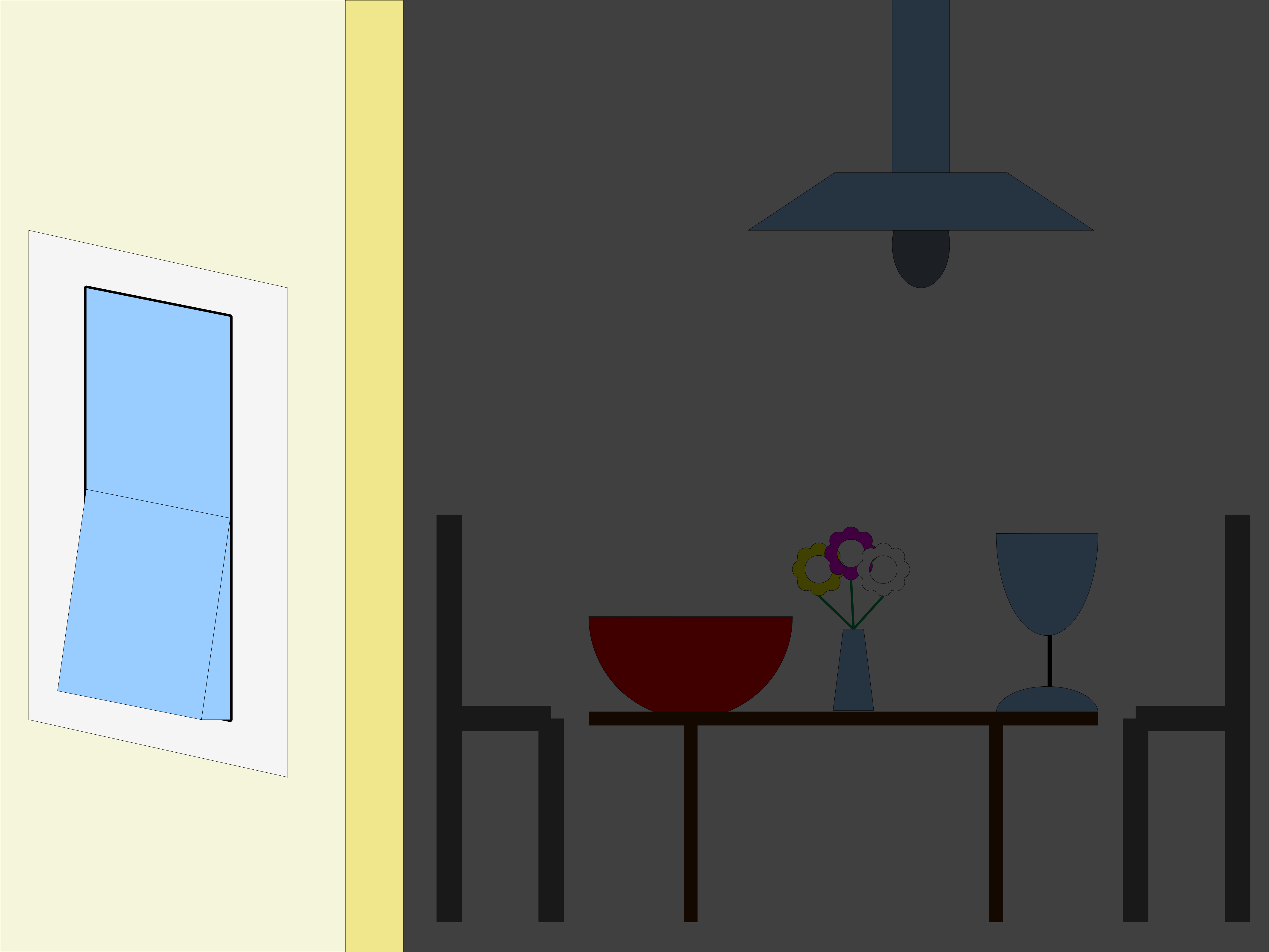
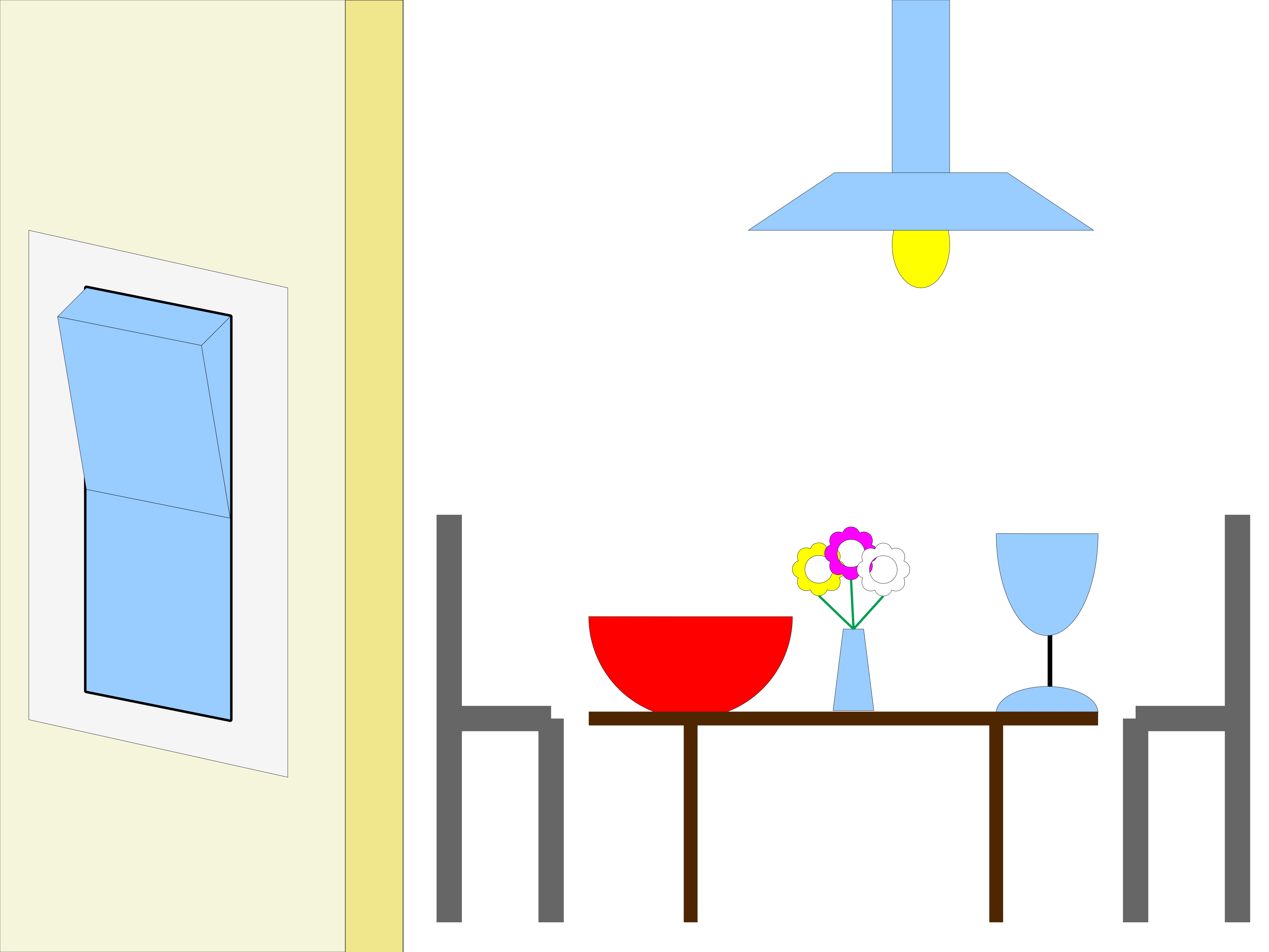
Special Types of Functions





Undoing by Doing Again

- Some operations invert themselves. For example:
  - Flipping a switch twice is the same as not flipping it at all.
  - In first-order logic, $\neg\neg A$ is equivalent to A.
  - In algebra, $-(-x) = x$.
  - In set theory, $(A \Delta B) \Delta B = A$. *(Yes, really!)*
- Operations with these properties are surprisingly useful in CS theory and come up in a bunch of contexts.
  - Storing compressed approximations of sets (XOR filters).
  - Theoretically unbreakable encryption (one-time pads).
  - Transmitting a large file to multiple receivers (fountain codes).

Involutions

- A function $f : A \to A$ from a set back to itself is called an ***involution*** if the following first-order logic statement is true about f:

$$\forall x \in A.\ f(f(x)) = x.$$

("Applying f twice is equivalent to not applying f at all.")

- Involutions have lots of interesting properties. Let's explore them and see what we can find.

Involutions

- Which of the following are involutions?
  - $f : \mathbb{Z} \to \mathbb{Z}$ defined as $f(x) = x$.
  - $f : \mathbb{Z} \to \mathbb{Z}$ defined as $f(x) = -x$.
  - $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = {}^1/_x$.
  - $f : \mathbb{N} \to \mathbb{N}$ defined as follows:

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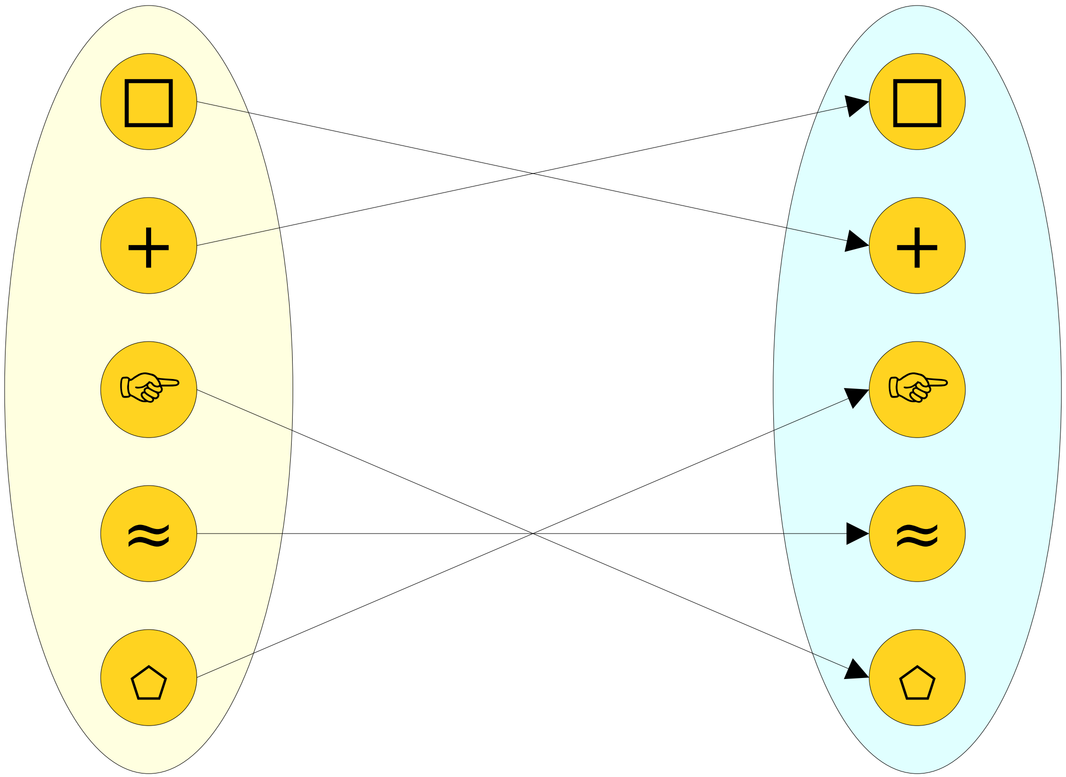
Involutions, Visually



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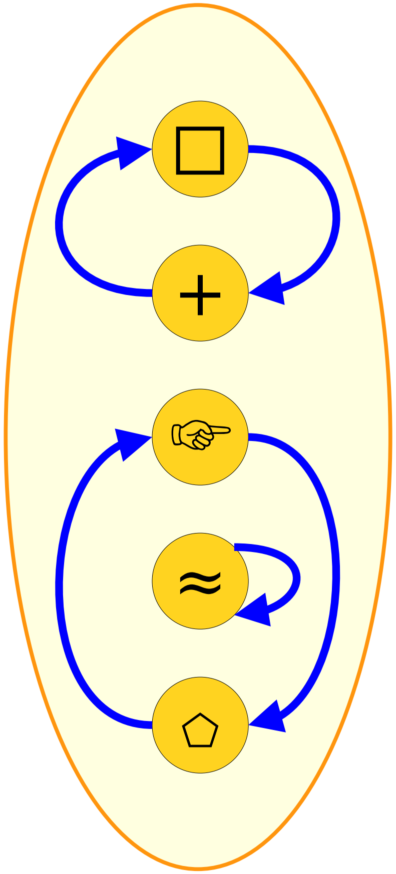

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Proofs on Involutions

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is an involution.

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Case 1: n is even.

Case 2: n is odd.

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Case 2: n is odd. Then $f(n) = n - 1$, which is even. Then we see that $f(f(n)) = f(n - 1) = (n - 1) + 1 = n$.

Theorem: The function $f : \mathbb{Z} \to \mathbb{Z}$ defined as

$$f(n) = \begin{cases} n+1 & \text{if } n \text{ is even} \\ n-1 & \text{if } n \text{ is odd} \end{cases}$$

is an involution.

Proof: Pick some $n \in \mathbb{Z}$. We need to show that $f(f(n)) = n$. To do so, we consider two cases.

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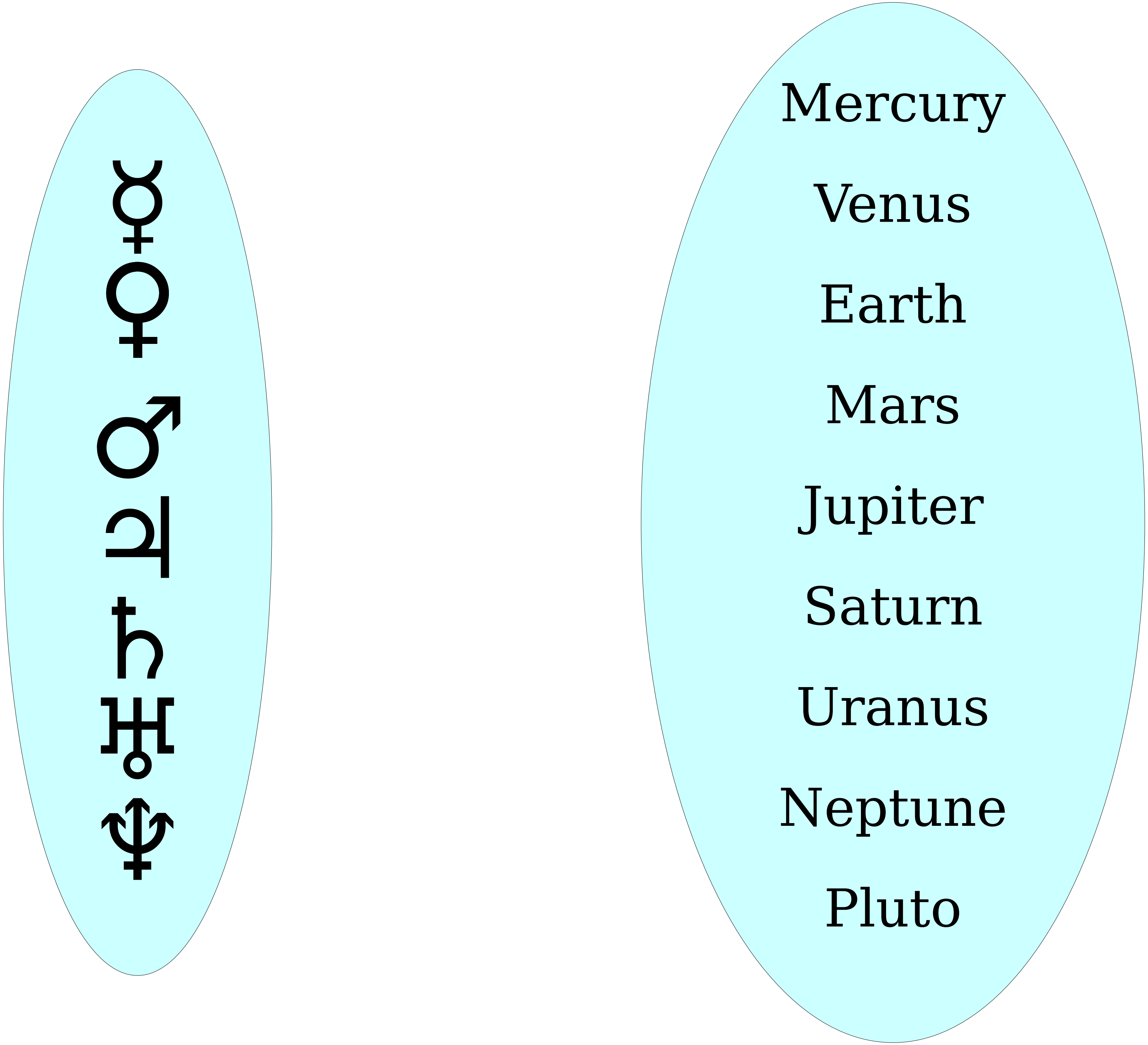
Another Class of Functions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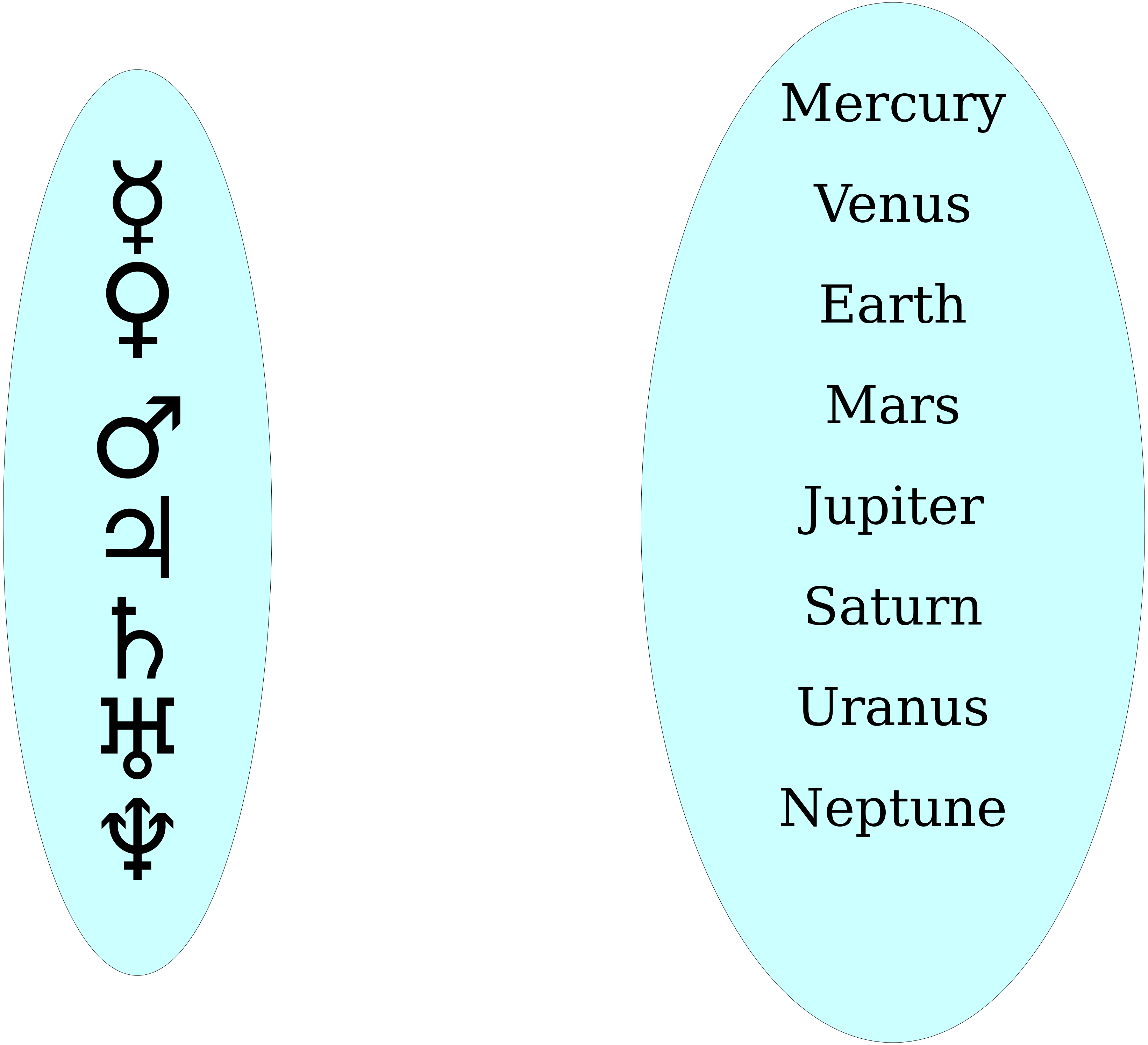
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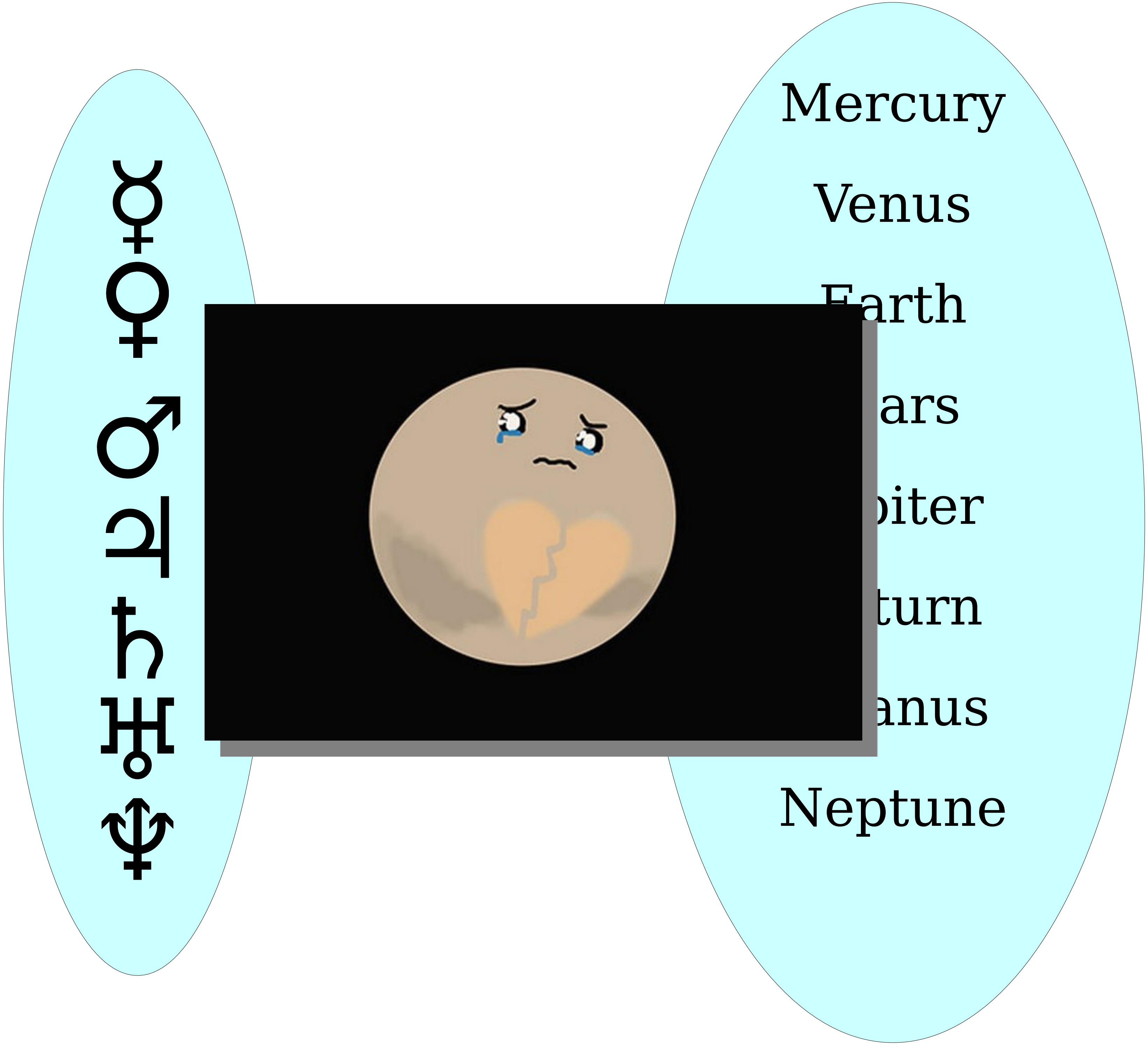
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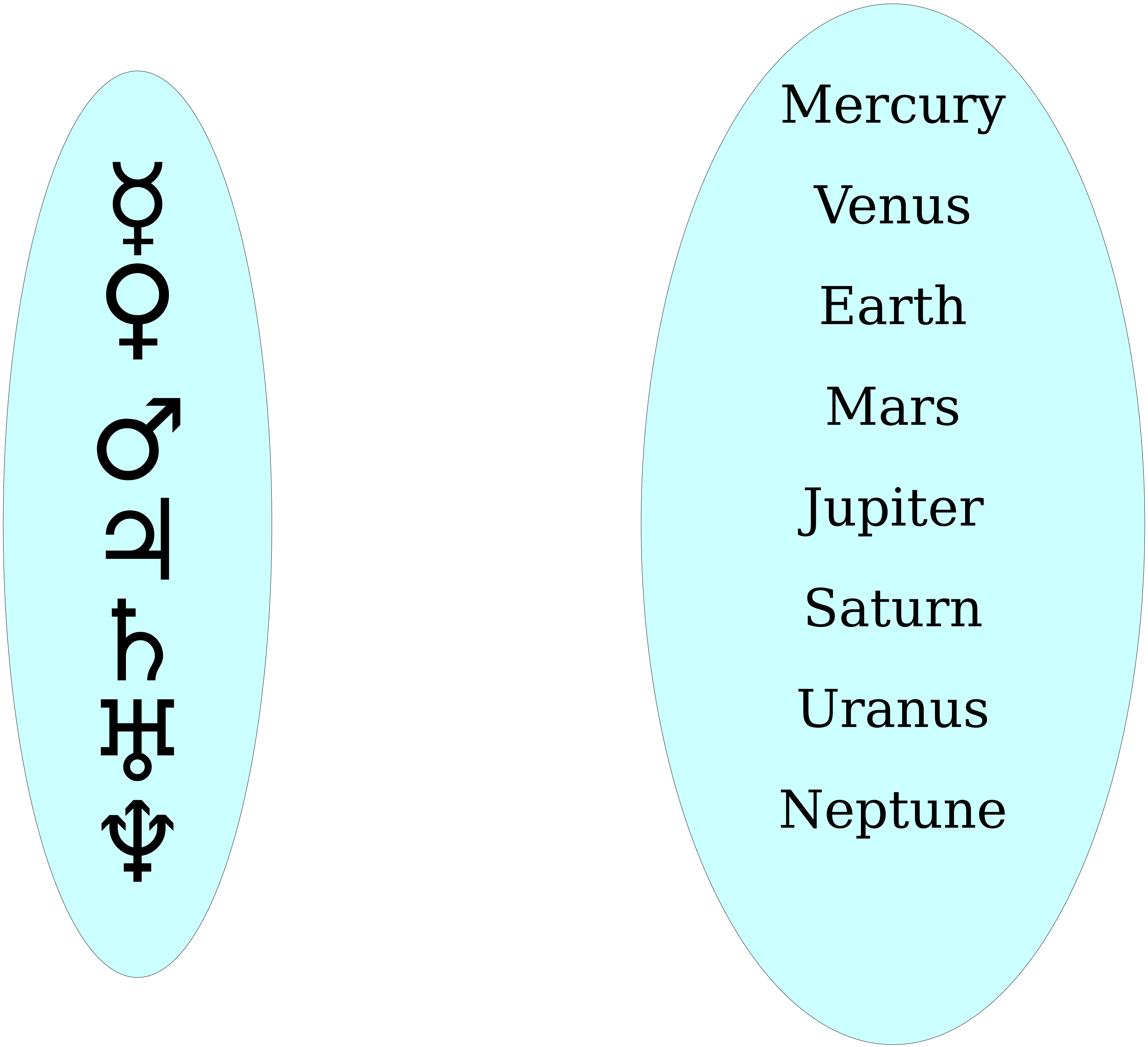
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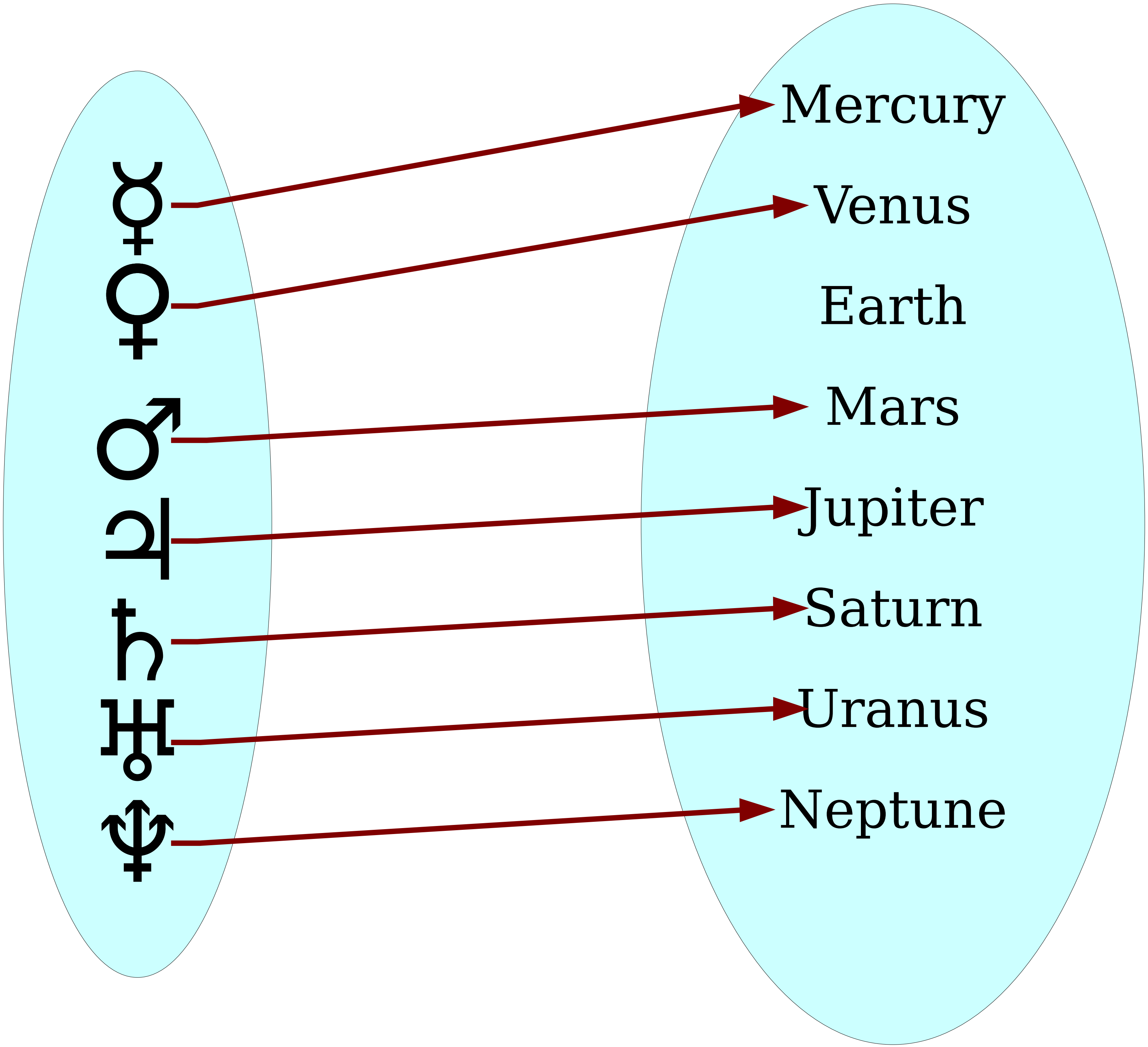
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Injective Functions

- A function $f : A \to B$ is called ***injective*** (or ***one-to-one***) if the following statement is true about f:

$$\forall a_1 \in A.\ \forall a_2 \in A.\ (a_1 \neq a_2 \to f(a_1) \neq f(a_2))$$

(*"If the inputs are different, the outputs are different."*)

- The following first-order definition is equivalent *(why?)* and is often useful in proofs.

$$\forall a_1 \in A.\ \forall a_2 \in A.\ (f(a_1) = f(a_2) \to a_1 = a_2)$$

(*"If the outputs are the same, the inputs are the same."*)

- A function with this property is called an ***injection***.
- How does this compare to our second rule for functions?

Injections

- Let 👬👫👫👫👫 be the set of all CS103 students. Which of the following are injective?
  1) f : 👫 $\to \mathbb{N}$ where $f(x)$ is x's Stanford ID number.
  2) f : 🧑🤝👫 $\to$ 🌎, where 🌎 is the set of all countries and $f(x)$ is x's country of birth.
  3) f : 🧑🤝👫 $\to$ 💬, where 💬 is the set of all given (first) names, where $f(x)$ is x's given (first) name.

Respond at pollev.com/robynreiss

A function $f : A \to B$ is ***injective*** if either statement is true:

$$\forall x_1 \in A.\ \forall x_2 \in A.\ (x_1 \neq x_2 \to f(x_1) \neq f(x_2))$$
$$\forall x_1 \in A.\ \forall x_2 \in A.\ (f(x_1) = f(x_2) \to x_1 = x_2)$$

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- Let be the set of all CS103 students. Which of the following are injective?

✔ *1)* $f : \quad \to \mathbb{N}$ where $f(x)$ is x's Stanford ID number.

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Theorem: Let $f : \mathbb{N} \to \mathbb{N}$ be defined as $f(n) = 2n + 7$. Then f is injective.

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Good exercise: Repeat this proof using the other definition of injectivity!

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$A \to B$	Assume A is true, then prove B is true.	

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Theorem: Let $f : \mathbb{Z} \to \mathbb{N}$ be defined as $f(x) = x^4$. Then f is not injective.

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What does it mean for f to be injective?

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Therefore, we need to find $x_1, x_2 \in \mathbb{Z}$ such that $x_1 \neq x_2$, but $f(x_1) = f(x_2)$. Can we do that?

Injective Functions

Theorem: Let $f : \mathbb{Z} \to \mathbb{N}$ be defined as $f(x) = x^4$. Then f is not injective.

Proof: We will prove that there exist integers x_1 and x_2 such that $x_1 \neq x_2$, but $f(x_1) = f(x_2)$.

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Let $x_1 = -1$ and $x_2 = +1$.

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Let $x_1 = -1$ and $x_2 = +1$. Notice that

$$f(x_1) = f(-1) = (-1)^4 = 1$$

and

$$f(x_2) = f(1) = 1^4 = 1,$$

so $f(x_1) = f(x_2)$ even though $x_1 \neq x_2$, as required. ■

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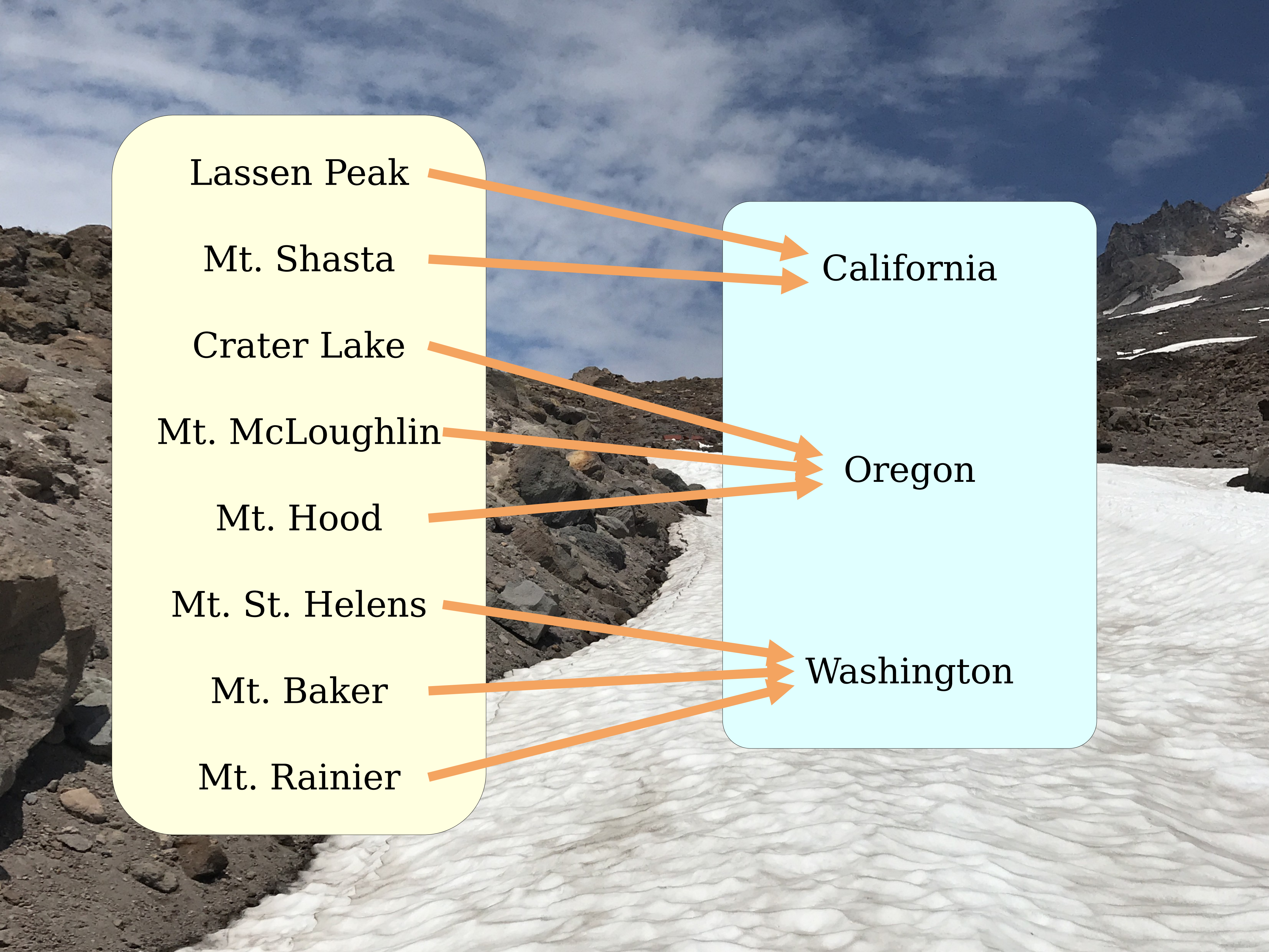
Another Class of Functions


Lassen Peak
Mt. Shasta
Crater Lake
Mt. McLoughlin
Mt. Hood
Mt. St. Helens
Mt. Baker
Mt. Rainier
California
Oregon
Washington

Surjective Functions

- A function $f : A \to B$ is called ***surjective*** (or ***onto***) if this first-order logic statement is true about f:

$$\forall b \in B.\ \exists a \in A.\ f(a) = b$$

(*"For every output, there's an input that produces it."*)

- A function with this property is called a ***surjection***.
- How does this compare to our first rule of functions?

Surjective Functions

Theorem: Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = 2x$. Then $f(x)$ is surjective.

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Therefore, we'll choose an arbitrary $y \in \mathbb{R}$, then prove that there is some $x \in \mathbb{R}$ where $f(x) = y$.

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Let $x = y / 2$.

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$$f(x) = f(y / 2)$$

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Surjective Functions

Theorem: Let $g : \mathbb{N} \to \mathbb{N}$ be defined as $g(n) = 2n$. Then $g(x)$ is not surjective.

Question: What do we need to do to prove that g is not surjective? Try taking the definition of surjectivity and then negating it.

Respond at pollev.com/robynreiss

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What does it mean for g to be surjective?

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$$\exists n \in \mathbb{N}.\ \neg\exists m \in \mathbb{N}.\ g(m) = n$$
$$\exists n \in \mathbb{N}.\ \forall m \in \mathbb{N}.\ g(m) \neq n$$

Therefore, we need to find a natural number n where, regardless of which m we pick, we have $g(m) \neq n$.

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Proof: Let $n = 137$.

Our overall goal is to prove

$$\exists n \in \mathbb{N}.\ \forall m \in \mathbb{N}.\ g(m) \neq n.$$

We just made our choice of n. Therefore, we need to prove

$$\forall m \in \mathbb{N}.\ g(m) \neq n.$$

We'll therefore pick an arbitrary $m \in \mathbb{N}$, then prove that $g(m) \neq n$.

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A Proof About Birds



Theorem: If all birds can fly, then all herons can fly.

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Given the predicates

$Bird(b)$, which says b is a bird;
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Consider an arbitrary bird b. Since b is a bird, b can fly. *[and now we're stuck! we are interested in herons, but b might not be one. It could be a hummingbird, for example!]*

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Proving vs. Assuming

- In the context of a proof, you will need to assume some statements and prove others.
  - Here, we ***assumed*** all birds can fly.
  - Here, we ***proved*** all herons can fly.
- Statements behave differently based on whether you're assuming or proving them.

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Proving vs. Assuming

- To ***prove*** the universally-quantified statement

$$\forall x.\ P(x)$$

  we introduce a new variable x representing some arbitrarily-chosen value.
- Then, we prove that $P(x)$ is true for that variable x.
- That's why we introduced a variable h in this proof representing a heron.

$$\underbrace{(\forall b.\ (\mathit{Bird}(b) \to \mathit{CanFly}(b)))}_{\text{We never introduce a variable } b.} \to \underbrace{(\forall h.\ (\mathit{Heron}(h) \to \mathit{CanFly}(h)))}_{\text{We introduce a variable } h \text{ almost immediately.}}$$

Proving vs. Assuming

- If we ***assume*** the statement

$$\forall x.\ P(x)$$

  we ***do not*** introduce a variable x.
- Rather, if we find a relevant value z somewhere else in the proof, we can conclude that $P(z)$ is true.
- That's why we didn't introduce a variable b in our proof, and why we concluded that h, our heron, can fly.

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$\neg A$	Simplify the negation, then consult this table on the result.	

	To ***prove*** that this is true…	If you ***assume*** this is true…
$\forall x.\ A$	Have the reader pick an arbitrary x. We then prove A is true for that choice of x.	Initially, ***do nothing***. Once you find a z through other means, you can state it has property A.
$\exists x.\ A$	Find an x where A is true. Then prove that A is true for that specific choice of x.	
$A \to B$	Assume A is true, then prove B is true.	Initially, ***do nothing***. Once you know A is true, you can conclude B is also true.
$A \land B$	Prove A. Then prove B.	
$A \lor B$	Either prove $\neg A \to B$ or prove $\neg B \to A$. *(Why does this work?)*	
$A \leftrightarrow B$	Prove $A \to B$ and $B \to A$.	
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	To ***prove*** that this is true…	If you ***assume*** this is true…
$\forall x.\ A$	Have the reader pick an arbitrary x. We then prove A is true for that choice of x.	Initially, ***do nothing***. Once you find a z through other means, you can state it has property A.
$\exists x.\ A$	Find an x where A is true. Then prove that A is true for that specific choice of x.	Introduce a variable x into your proof that has property A.
$A \to B$	Assume A is true, then prove B is true.	Initially, ***do nothing***. Once you know A is true, you can conclude B is also true.
$A \land B$	Prove A. Then prove B.	Assume A. Then assume B.
$A \lor B$	Either prove $\neg A \to B$ or prove $\neg B \to A$. *(Why does this work?)*	
$A \leftrightarrow B$	Prove $A \to B$ and $B \to A$.	
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	To ***prove*** that this is true…	If you ***assume*** this is true…
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$\exists x.\ A$	Find an x where A is true. Then prove that A is true for that specific choice of x.	Introduce a variable x into your proof that has property A.
$A \to B$	Assume A is true, then prove B is true.	Initially, ***do nothing***. Once you know A is true, you can conclude B is also true.
$A \land B$	Prove A. Then prove B.	Assume A. Then assume B.
$A \lor B$	Either prove $\neg A \to B$ or prove $\neg B \to A$. *(Why does this work?)*	Consider two cases. Case 1: A is true. Case 2: B is true.
$A \leftrightarrow B$	Prove $A \to B$ and $B \to A$.	Assume $A \to B$ and $B \to A$.
$\neg A$	Simplify the negation, then consult this table on the result.	Simplify the negation, then consult this table on the result.

Recap from Today

- A ***function*** takes in an element of a ***domain*** and maps it to an element of a ***codomain***. Functions must be deterministic.
- Definitions are often given in first-order logic, and the structure of a first-order logic statement dictates the structure of a proof.
- ***Involutions***, ***injections***, and ***surjections*** are specific classes of functions that have nice properties.

Next Time

- ***Connecting Function Types***
  - Involutions, injections, and surjections are related to one another. How?
- ***Function Composition***
  - Sequencing functions together.