

CS103, AUTUMN 2026



Lecture 07: **Functions**

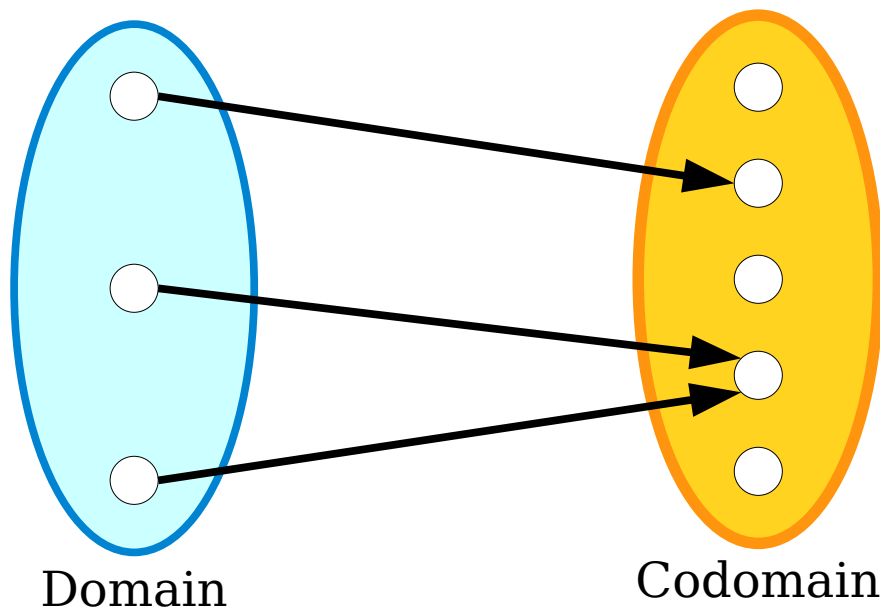
Part 2 of 2

Functions

Part 2

- 1. Recap from Last Time**
2. A Proof About Birds
3. Assuming vs. Proving
4. Leveling Up the Assume-Prove Table
5. Connecting Function Types
6. The Two-Column Proof Organizer
7. Announcements
8. Function Composition
9. Properties of Composition: A Proof
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Recap from Last Time

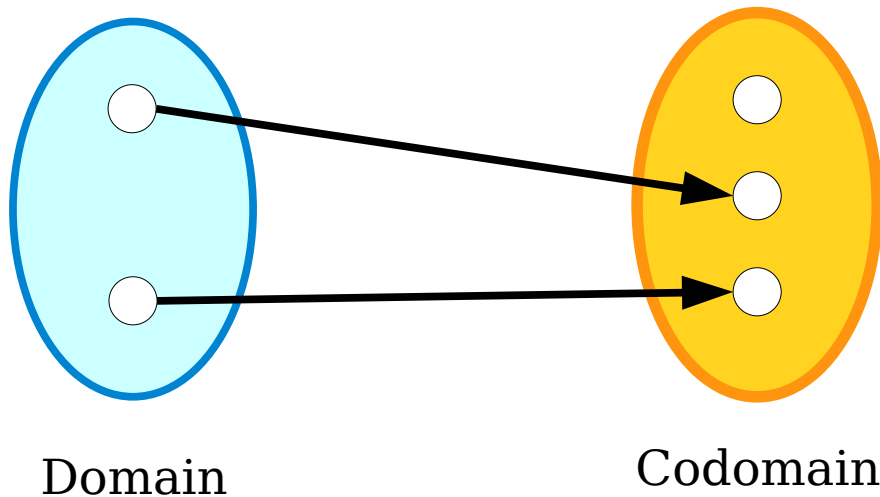


*Is it a **function**?* **Yes!**

*Is it an **injection**?* **No.**

*Is it a **surjection**?* **No.**

Recap from Last Time

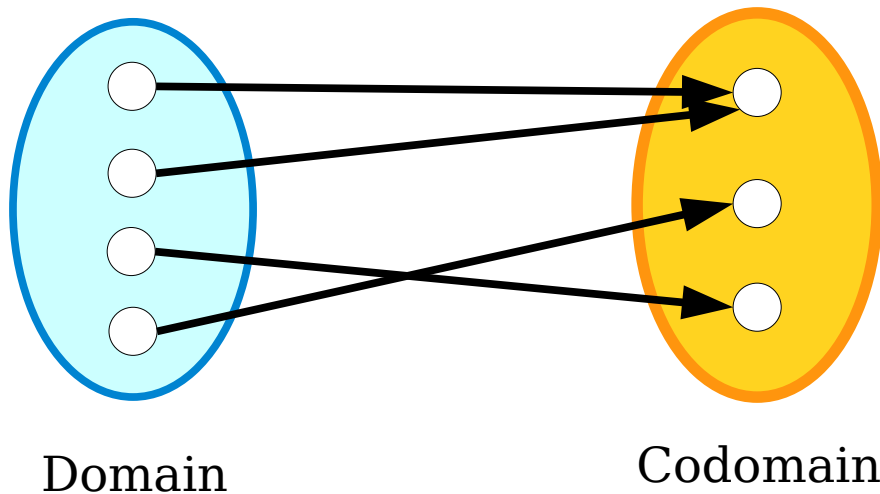


*Is it a **function**?* **Yes!**

*Is it an **injection**?* **Yes!**

*Is it a **surjection**?* **No.**

Recap from Last Time



*Is it a **function**?* **Yes!**

*Is it an **injection**?* **No.**

*Is it a **surjection**?* **Yes!**

Recap from Last Time

Injection: $\forall a_1 \in A. \forall a_2 \in A. (a_1 \neq a_2 \rightarrow f(a_1) \neq f(a_2))$

If the inputs are different, the outputs are different

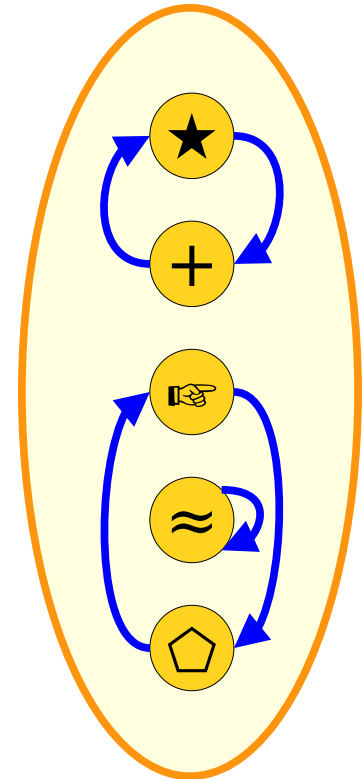
- Can also define with the contrapositive!

Surjection: $\forall b \in B. \exists a \in A. f(a) = b$

“For every possible output, there's an input that produces it.”

Involution: $\forall x \in A. f(f(x)) = x$

“Applying f twice is equivalent to not applying f at all.”



The Assume-Prove Table

		To prove that this is true...
$\forall x. A$		Have the reader pick an arbitrary x . We then prove A is true for that choice of x .
$\exists x. A$		Find an x where A is true. Then prove A is true for that specific choice of x .
$A \rightarrow B$		Assume A is true, then prove B is true.
$A \wedge B$		Prove A . Also prove B .
$A \vee B$		Either prove $\neg A \rightarrow B$ or prove $\neg B \rightarrow A$. <i>(Why does this work?)</i>
$A \leftrightarrow B$		Prove $A \rightarrow B$ and $B \rightarrow A$.
$\neg A$		Simplify the negation, then consult this table on the result.

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Theorem: If all birds have feathers,
then all herons have feathers.

A Proof About Birds

Theorem: If all birds have feathers, then all herons have feathers.

Given the predicates

Bird(*b*), which says *b* is a bird;

Heron(*h*), which says *h* is a heron; and

Feathers(*x*), which says *x* has feathers,

translate the theorem into first-order logic.

$$(\forall b. (Bird(b) \rightarrow Feathers(b))) \rightarrow (\forall h. (Heron(h) \rightarrow Feathers(h)))$$

All birds
have feathers.

All herons
have feathers.

A Proof About Birds

Theorem: If all birds have feathers, then all herons have feathers.

Proof: Assume that all birds have feathers. We will show that all herons have feathers.

Consider an arbitrary bird b . Since b is a bird, b has feathers. *[And now we're stuck! We are interested in herons, but b might not be one. It could be a hummingbird, for example!]*

$(\forall b. (Bird(b) \rightarrow Feathers(b))) \rightarrow (\forall h. (Heron(h) \rightarrow Feathers(h)))$

All birds
have feathers.

All herons
have feathers.

A Proof About Birds

Theorem: If all birds have feathers, then all herons have feathers.

Proof: Assume that all birds have feathers. We will show that all herons have feathers.

Consider an arbitrary heron h . We will show that h has feathers. To do so, note that since h is a heron we know h is a bird. Therefore, by our earlier assumption, h has feathers. ■

$$(\forall b. (Bird(b) \rightarrow Feathers(b))) \rightarrow (\forall h. (Heron(h) \rightarrow Feathers(h)))$$

All birds
have feathers.

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Proving vs. Assuming

- In the context of a proof, you will need to assume some statements and prove others.
 - Here, we **assumed** all birds have feathers.
 - Here, we **proved** all herons have feathers.
- Statements behave differently based on whether you're assuming or proving them.

$(\forall b. (Bird(b) \rightarrow Feathers(b))) \rightarrow (\forall h. (Heron(h) \rightarrow Feathers(h)))$

We never introduce a variable b .

We introduce a variable h almost immediately.

Proving vs. Assuming

- To **prove** the universally-quantified statement

$$\forall x. P(x)$$

we introduce a new variable x representing some arbitrarily-chosen value.

- Then, we prove that $P(x)$ is true for that variable x .
- That's why we introduced a variable h in this proof representing a heron.

$$(\forall b. (Bird(b) \rightarrow Feathers(b))) \rightarrow (\forall h. (Heron(h) \rightarrow Feathers(h)))$$

We never introduce a variable b .

We introduce a variable h almost immediately.

Proving vs. Assuming

- If we **assume** the statement

$$\forall x. P(x)$$

we **do not** introduce a variable x .

- Rather, if we find a relevant value z somewhere else in the proof, we can conclude that $P(z)$ is true.
- That's why we didn't introduce a variable b in our proof, and why we concluded that h , our heron, have feathers.

$$(\forall b. (Bird(b) \rightarrow Feathers(b))) \rightarrow (\forall h. (Heron(h) \rightarrow Feathers(h)))$$

We never introduce a variable b .

We introduce a variable h almost immediately.

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The Assume-Prove Table

	If you assume this is true...	To prove that this is true...
$\forall x. A$	Initially, do nothing . Once you find a z through other means, you can state it has property A .	Have the reader pick an arbitrary x . We then prove A is true for that choice of x .
$\exists x. A$	Introduce a variable x into your proof that has property A .	Find an x where A is true. Then prove A is true for that specific choice of x .
$A \rightarrow B$	Initially, do nothing . Once you know A is true, you can conclude B is also true.	Assume A is true, then prove B is true.
$A \wedge B$	Assume A . Also assume B .	Prove A . Also prove B .
$A \vee B$	Consider two cases. Case 1: A is true. Case 2: B is true.	Either prove $\neg A \rightarrow B$ or prove $\neg B \rightarrow A$. <i>(Why does this work?)</i>
$A \leftrightarrow B$	Assume $A \rightarrow B$ and $B \rightarrow A$.	Prove $A \rightarrow B$ and $B \rightarrow A$.
$\neg A$	Simplify the negation, then consult this table on the result.	Simplify the negation, then consult this table on the result.

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Types of Functions

- We now have three special types of functions:
 - ***involutions***, functions that undo themselves;
 - ***injections***, functions where different inputs go to different outputs; and
 - ***surjections***, functions that cover their whole codomain.
- ***Question:*** How do these three classes of functions relate to one another?

Theorem: For any function $f : A \rightarrow A$,
if f is an involution, then f is surjective.

$$\underbrace{(\forall x \in A. f(f(x)) = x)}_{\substack{\uparrow \\ f \text{ is an} \\ \text{involution.}}} \rightarrow \underbrace{(\forall b \in A. \exists a \in A. f(a) = b)}_{\substack{\uparrow \\ f \text{ is} \\ \text{surjective.}}}$$

Theorem: For any function $f : A \rightarrow A$, if f is an involution, then f is surjective.

$$(\forall x \in A. f(f(x)) = x) \rightarrow (\forall b \in A. \exists a \in A. f(a) = b)$$

Assume this.

Prove this.

If you **assume** this is true...

Initially, **do nothing**. Once you find a z through other means, you can state it has property A .

Theorem: For any function $f : A \rightarrow A$, if f is an involution, then f is surjective.

$$\underbrace{(\forall x \in A. f(f(x)) = x)}_{\uparrow} \rightarrow \underbrace{(\forall b \in A. \exists a \in A. f(a) = b)}_{\uparrow}$$

Assume this.

Prove this.

Since we're assuming this, we aren't going to pick a specific choice of x right now. Instead, we're going to keep an eye out for something to apply this fact to.

Proof Outline

1. Assume f is an involution.

Theorem: For any function $f : A \rightarrow A$, if f is an involution, then f is surjective.

$$(\forall x \in A. f(f(x)) = x) \rightarrow (\forall b \in A. \exists a \in A. f(a) = b)$$

Ass

There's a universal quantifier up front. Since we're proving this, we'll pick an arbitrary $b \in A$.

Prove this.

Proof Outline

1. Assume f is an involution.
2. Pick an arbitrary $b \in A$.

Theorem: For any function $f : A \rightarrow A$, if f is an involution, then f is surjective.

$$(\forall x \in A. f(f(x)) = x) \rightarrow (\forall b \in A. \exists a \in A. f(a) = b)$$

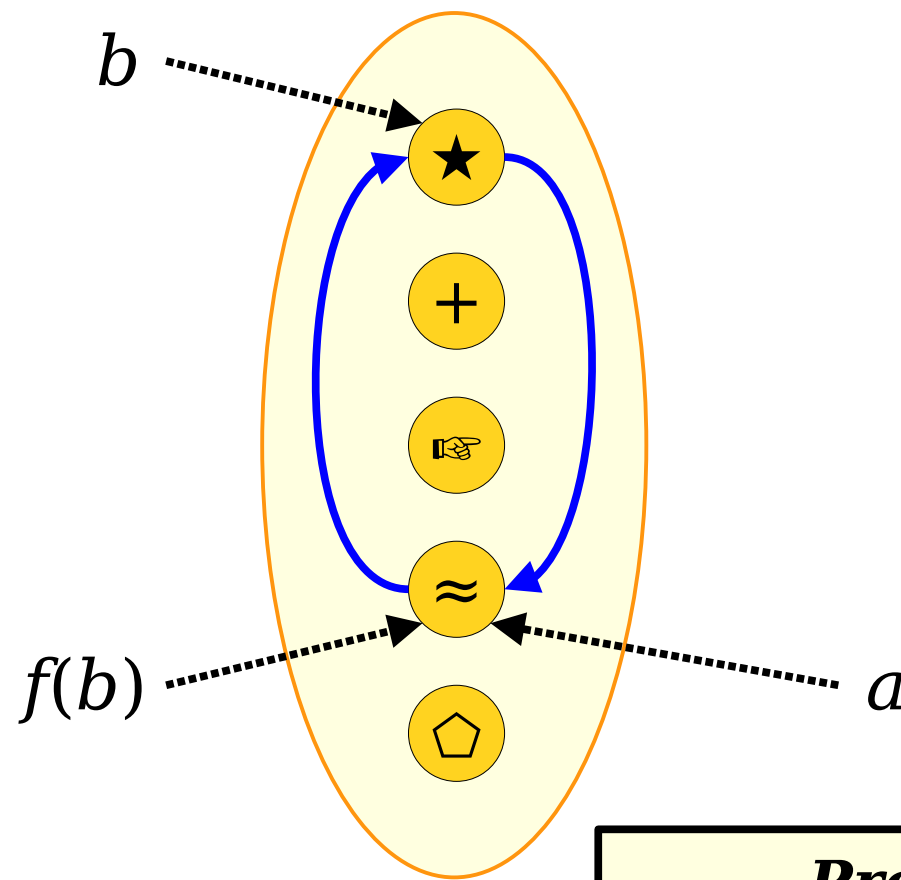
Now, we hit an existential quantifier. Since we're proving this, we need to find a choice of $a \in A$ where this is true.

Prove this.

Proof Outline

1. Assume f is an involution.
2. Pick an arbitrary $b \in A$.
3. Give a choice of $a \in A$ where $f(a) = b$.

Theorem: For any function $f : A \rightarrow A$, if f is an involution, then f is surjective.



Proof Outline

1. Assume f is an involution.
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Proof: Pick any involution $f : A \rightarrow A$. We will prove that f is surjective. To do so, pick an arbitrary $b \in A$. We need to show that there is an $a \in A$ where $f(a) = b$.

Specifically, pick $a = f(b)$. This means that $f(a) = f(f(b))$, and since f is an involution we know that $f(f(b)) = b$. Putting this together, we see that $f(a) = b$, which is what we needed to show. ■

Proof Outline

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This proof contains no first-order logic syntax (quantifiers, connectives, etc.). It's written in plain English, just as usual.

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Then f is injective.

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Then f is injective.

What We're Assuming

$f : A \rightarrow A$ is an involution.

$$\forall z \in A. f(f(z)) = z.$$

We're *assuming* this universally-quantified statement, so we won't introduce a variable for what's here.

What We Need to Prove

f is injective.

$$\forall a_1 \in A. \forall a_2 \in A. (f(a_1) = f(a_2) \rightarrow a_1 = a_2)$$

We need to *prove* this universally-quantified statement. so let's introduce arbitrarily-chosen values.

Theorem: Let $f : A \rightarrow A$ be an involution.
Then f is injective.

What We're Assuming

$f : A \rightarrow A$ is an involution.

$$\forall z \in A. f(f(z)) = z.$$

$$a_1 \in A$$

$$a_2 \in A$$

What We Need to Prove

f is injective.

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What We Need to Prove

f is injective.

$$\forall a_1 \in A. \forall a_2 \in A. (f(a_1) = f(a_2) \rightarrow a_1 = a_2)$$

We need to prove this **implication**. So we **assume the antecedent** and **prove the consequent**.

Theorem: Let $f : A \rightarrow A$ be an involution.
Then f is injective.

What We're Assuming

$f : A \rightarrow A$ is an involution.

$$\forall z \in A. f(f(z)) = z.$$

$$a_1 \in A$$

$$a_2 \in A$$

$$f(a_1) = f(a_2)$$

$$f(f(a_1)) = f(f(a_2))$$

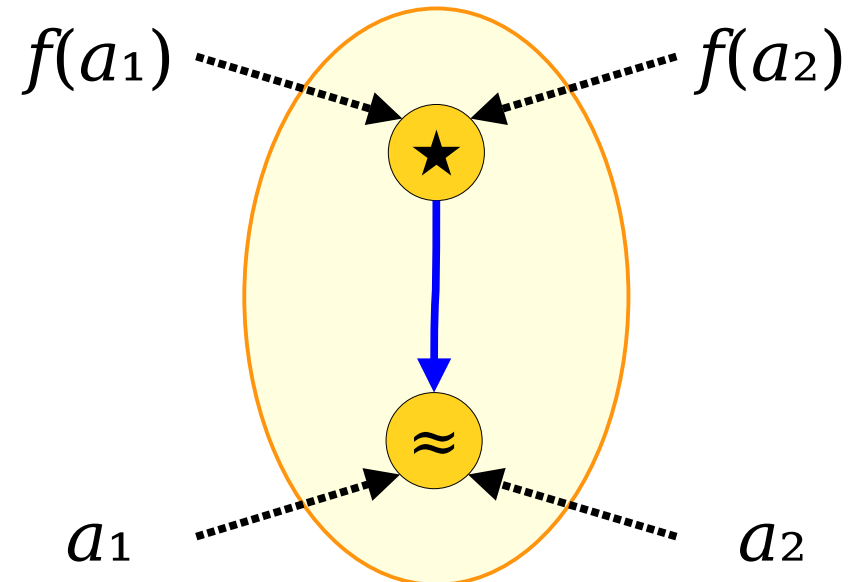
$$f(f(a_1)) = a_1$$

$$f(f(a_2)) = a_2$$

What We Need to Prove

f is injective.

$$\forall a_1 \in A. \forall a_2 \in A. (f(a_1) = f(a_2) \rightarrow a_1 = a_2)$$



Theorem: Let $f : A \rightarrow A$ be an involution. Then f is injective.

Proof: Choose any $a_1, a_2 \in A$ where $f(a_1) = f(a_2)$. We need to show that $a_1 = a_2$.

Since $f(a_1) = f(a_2)$, we know that $f(f(a_1)) = f(f(a_2))$. Because f is an involution, we see $a_1 = f(f(a_1))$ and that $f(f(a_2)) = a_2$. Putting this together, we see that

$$a_1 = f(f(a_1)) = f(f(a_2)) = a_2,$$

so $a_1 = a_2$, as needed. ■

This proof contains no first-order logic syntax (quantifiers, connectives, etc.). It's written in plain English, just as usual.

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Announcements

- ***Recitations***

- The recitation dashboard is open!
- Please check your recitation opt-in / opt-out status by Sunday at 11:59 PM.

- ***Left-handed desk form***

- due Monday of next week

- ***Attendance opt-out form***

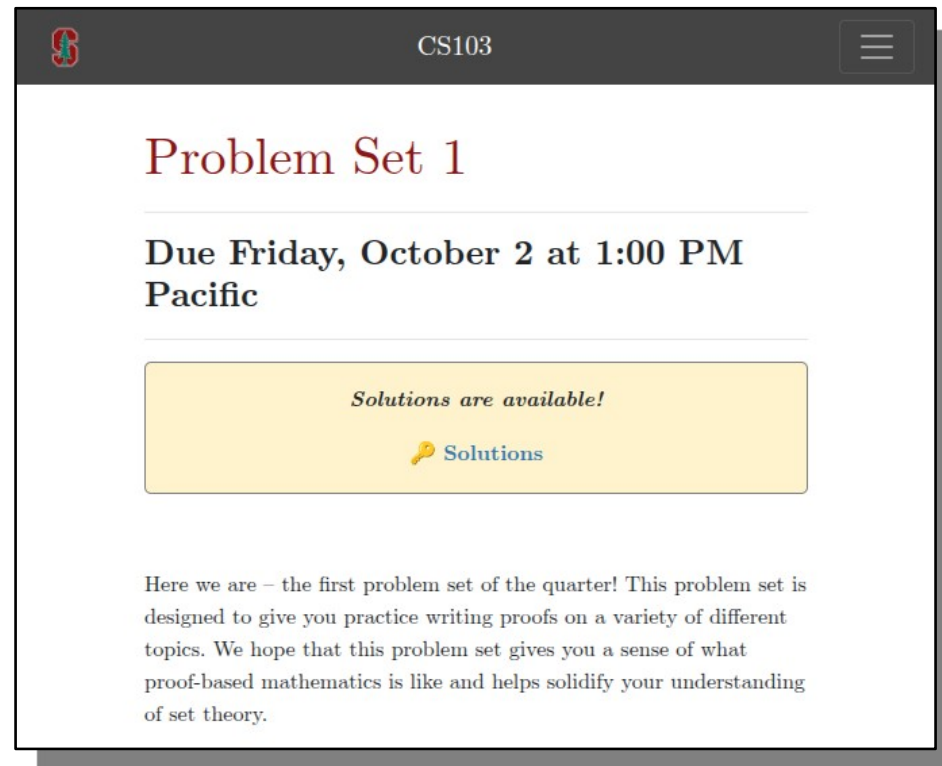
- available sometime next Monday, due next Friday at 11:59 PM

- ***Problem Set Submissions***

- ~~Partners should each submit coding portion individually, but submit just one copy of the written portion.~~

Problem Set One Graded

- TAs have finished grading Problem Set One!
- Grades and feedback are up on the Gradescope.
- Solutions posted on course website (via PS1 page). Check them out!



Problem Set One Graded

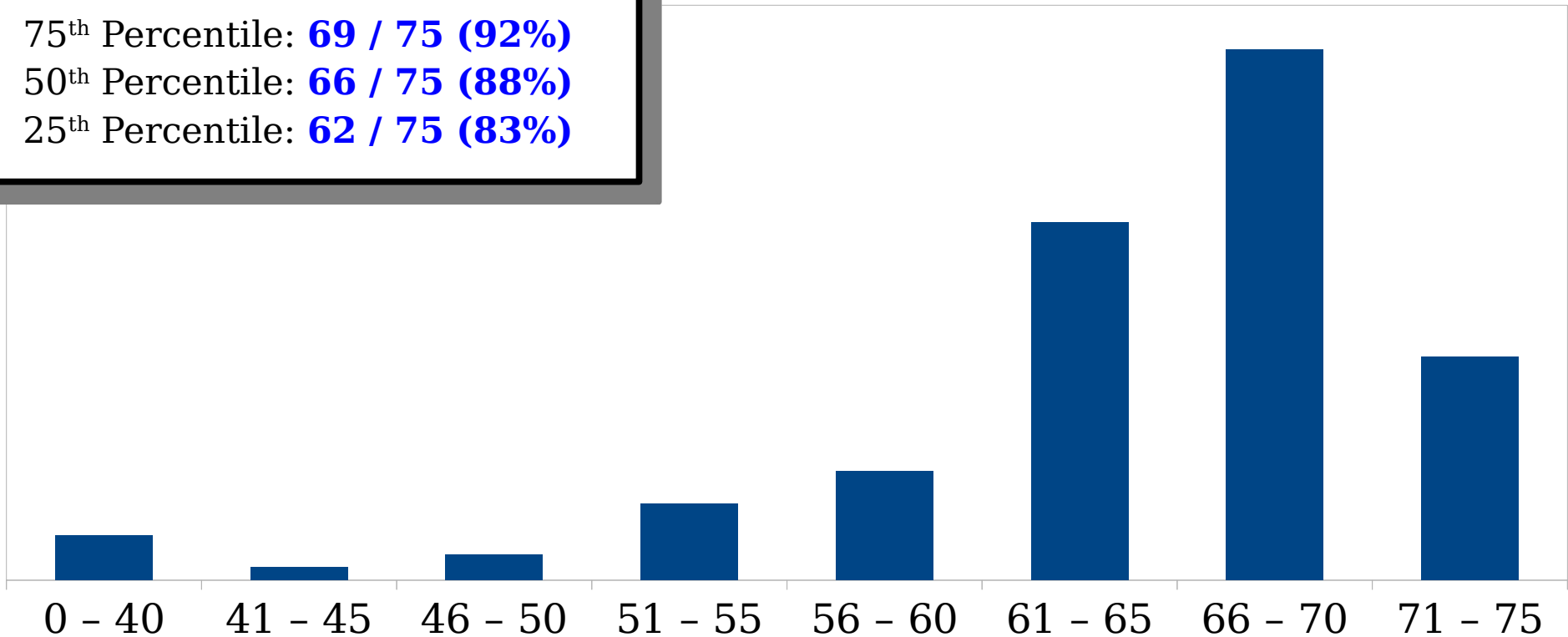
- TAs have finished grading Problem Set One!
- Grades and feedback are up on the Gradescope.
- Solutions posted on course website (via PS1 page). Check them out!
 - You'll get to see examples of polished written proofs.
 - We may have solved the problem differently than you, and this will give you more perspectives to use.
 - Each problem has a "Why We Asked This Question" section, which gives some context.

Problem Set One Graded

75th Percentile: **69 / 75 (92%)**

50th Percentile: **66 / 75 (88%)**

25th Percentile: **62 / 75 (83%)**



Tips for reading a grading distribution:

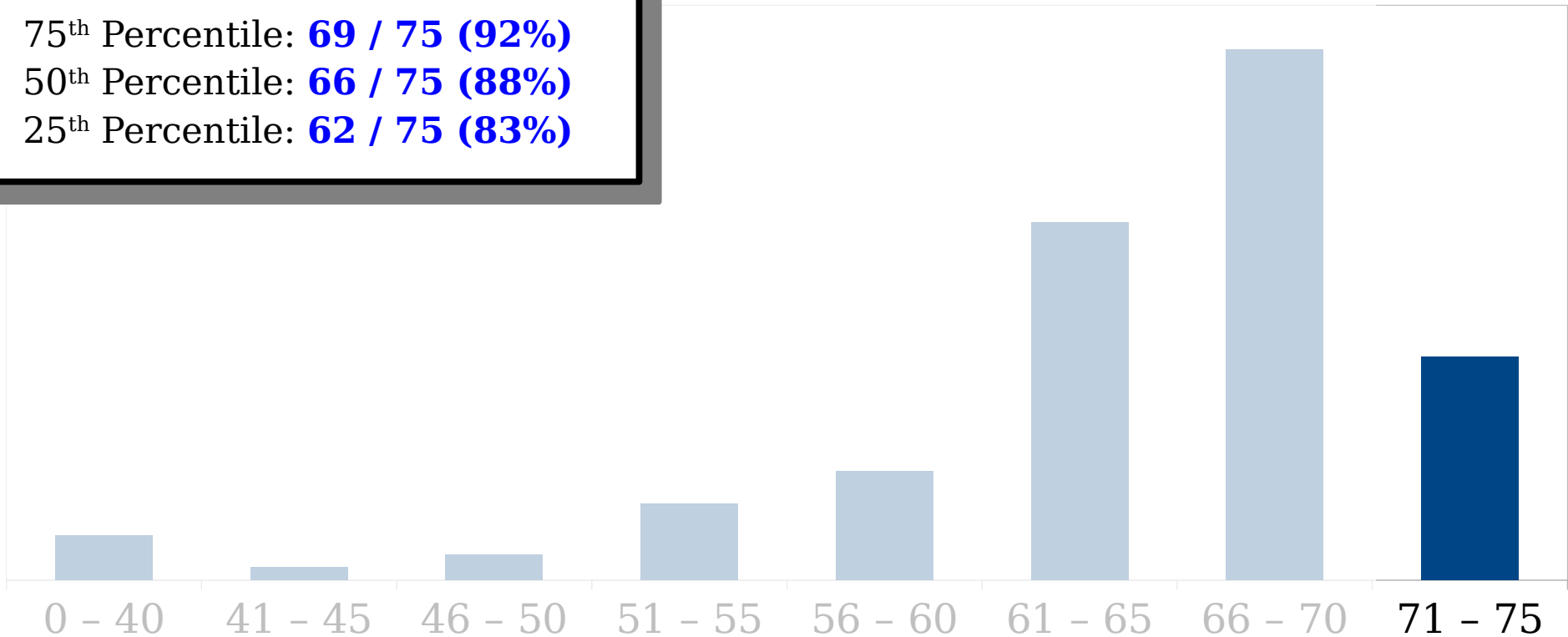
1. Standard deviations are *unhelpful and discouraging*. Ignore them.
2. The average score is a *unhelpful*. Ignore it.
3. Raw scores are *unhelpful and discouraging*. Ignore them.

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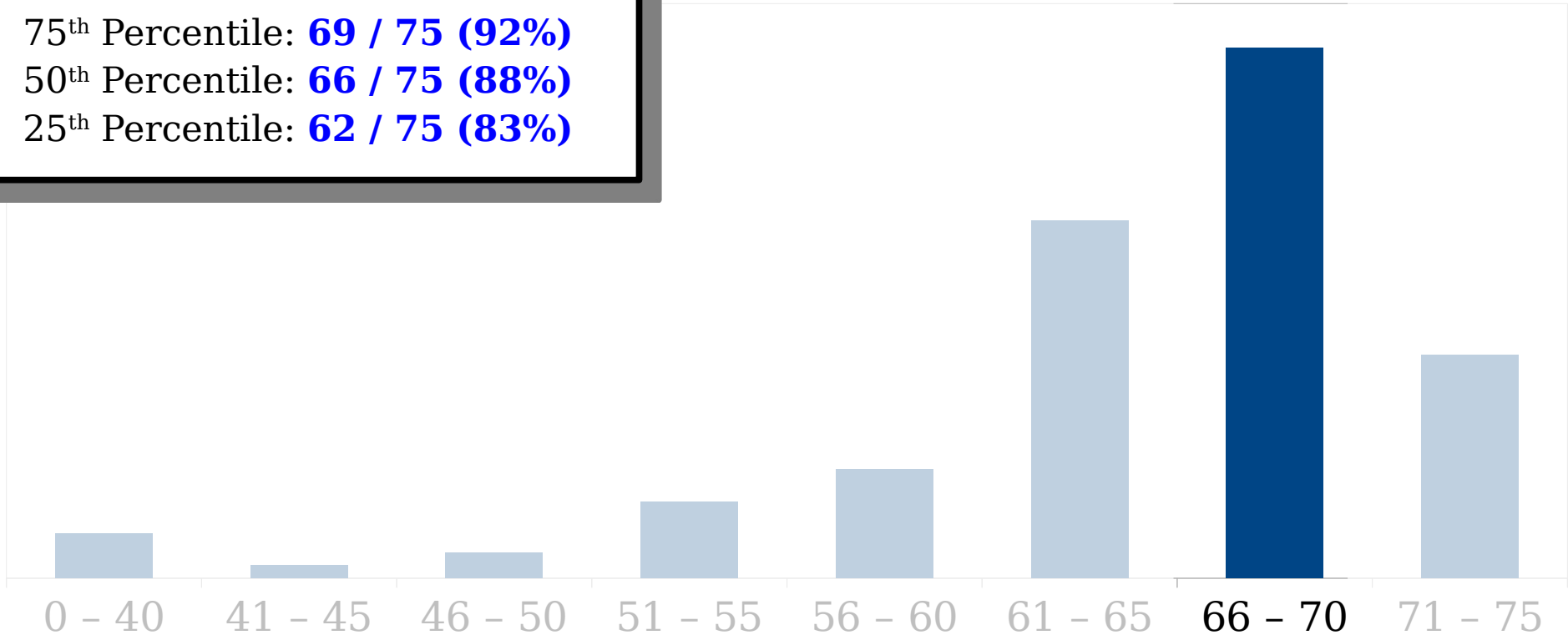
"Great job! Look over your feedback for some tips on how to tweak things for next time."

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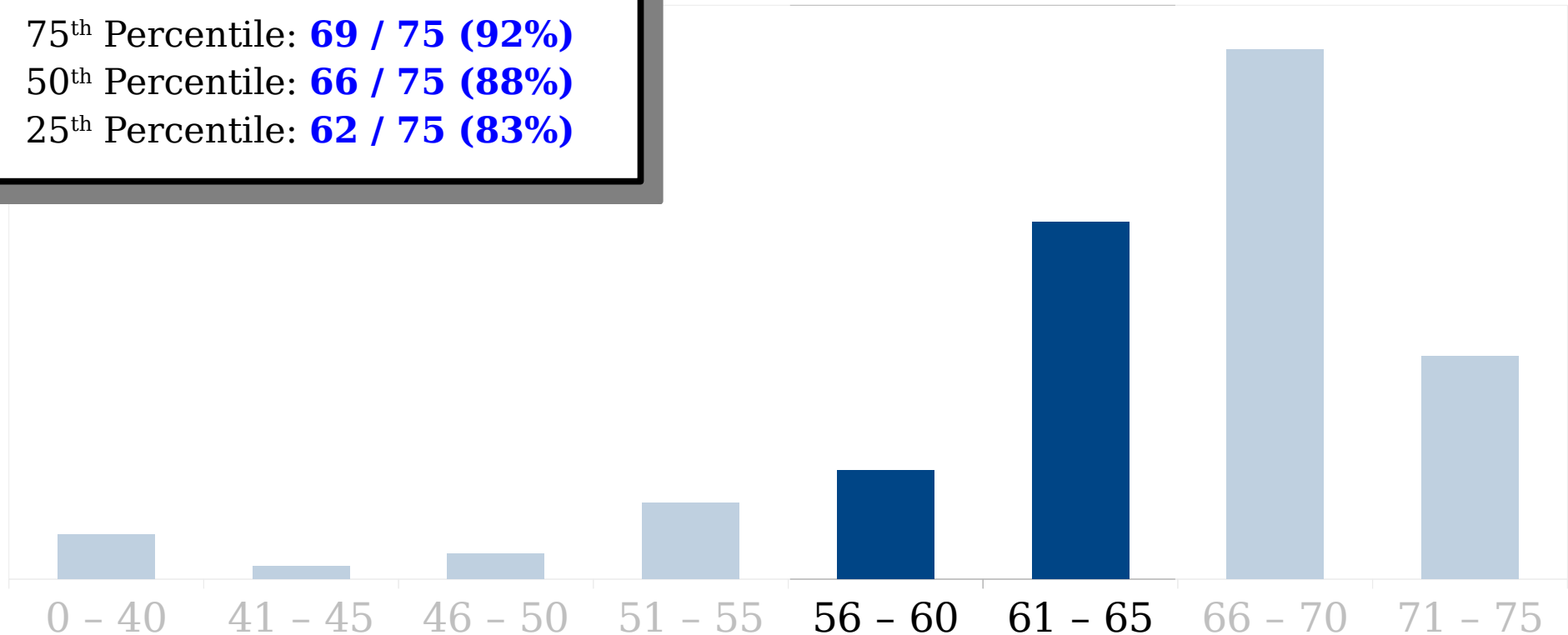
"You're almost there! Review the feedback on your submission and see what to focus on for next time."

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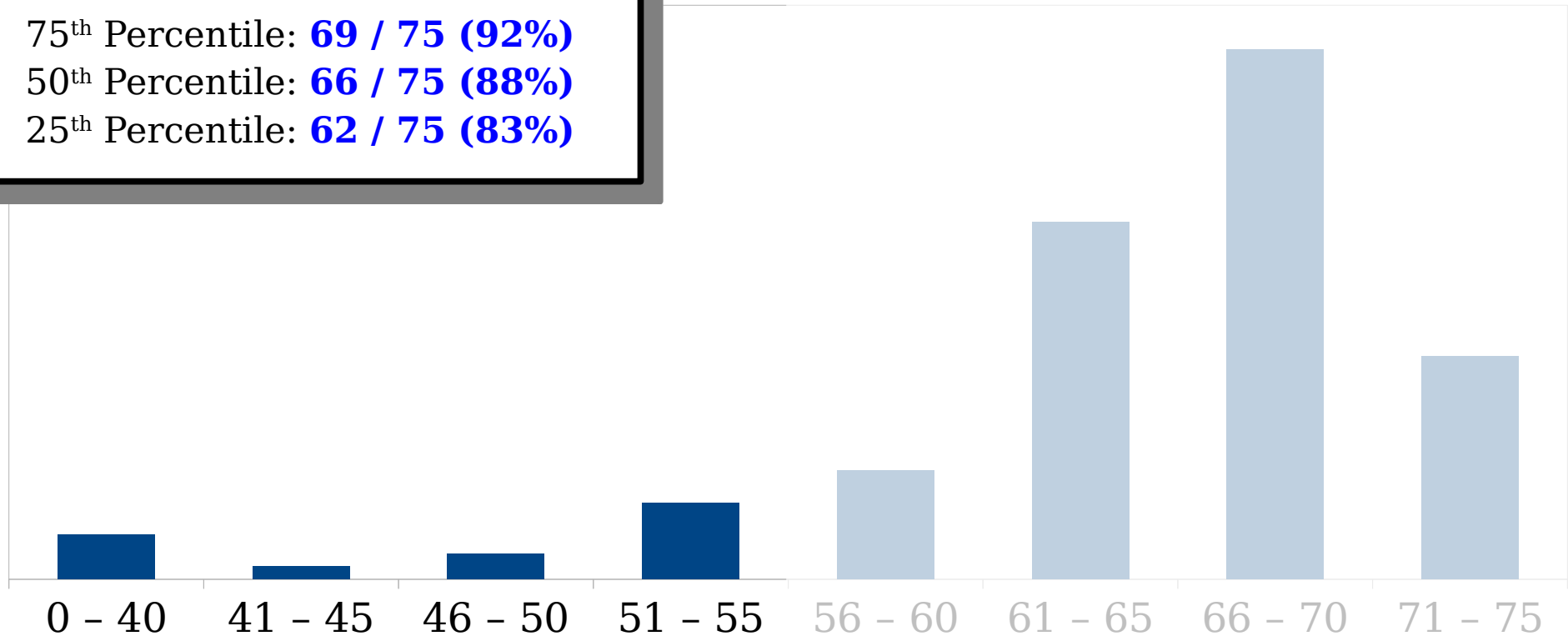
"You're on the right track, but there are some areas where you need to improve. Review your feedback and ask us questions when you have them."

Problem Set One Graded

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"Looks like something hasn't quite clicked yet. Get in touch with us and stop by office hours to get some extra feedback and advice. Don't get discouraged - you can do this!"

What Not to Think

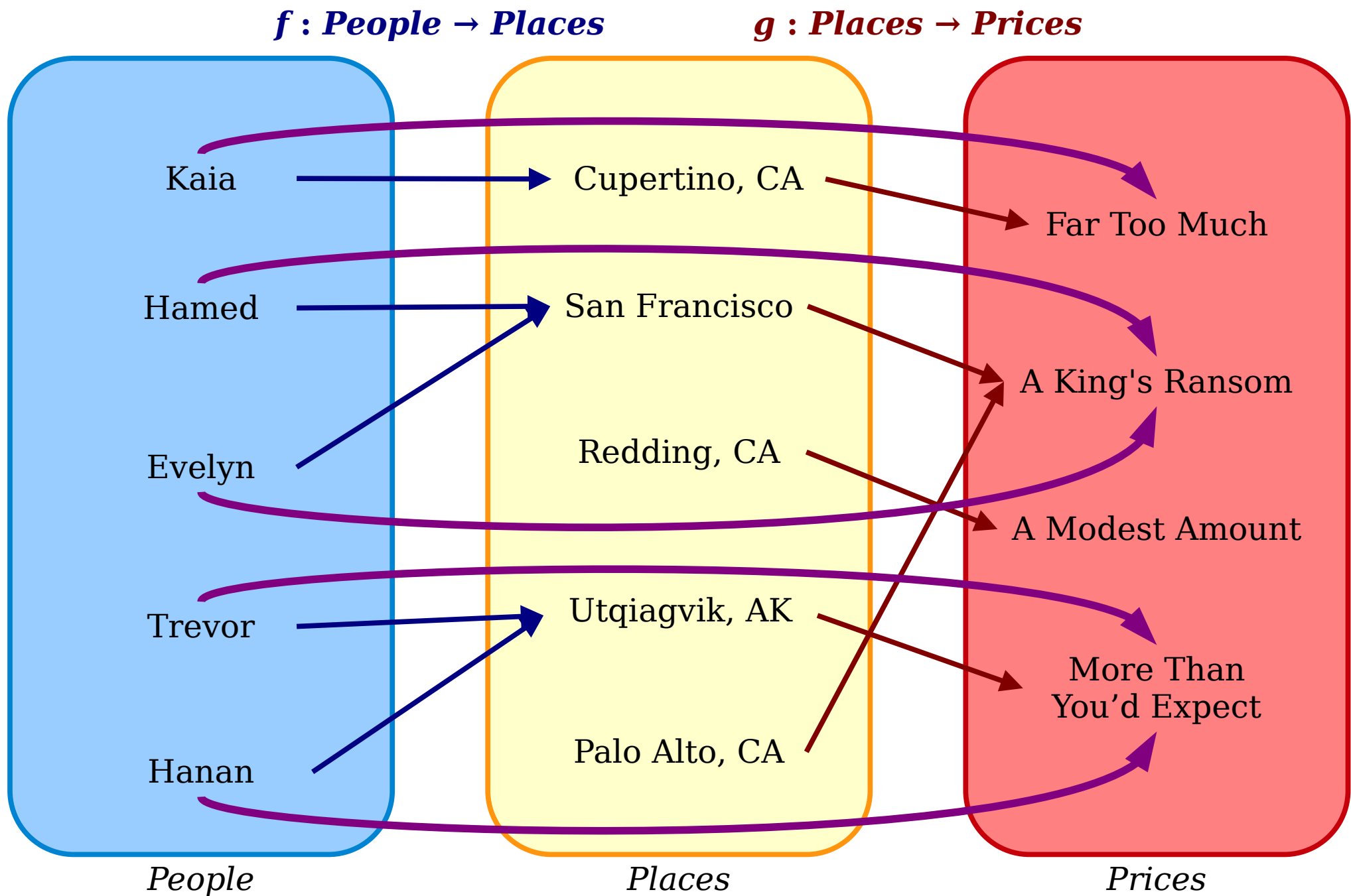
- “Well, I guess I’m just not good at math.”
 - For most of you, this is your first time doing any rigorous proof-based math.
 - Don’t judge your future performance based on a single data point.
 - Life advice: adopt a growth mindset!
- “Hey, I did above the median. That’s good enough.”
 - There’s always some area where you can improve. Take the time to see what that is.

Regrade Requests

- We're human. We make mistakes. And we're happy to correct them!
- Regrades open a few days after grades are released, generally Friday 12:01 AM through Wednesday 11:59 PM.
- Notes on regrades:
 - Please be civil. We make mistakes. We're happy to correct them.
 - We have to grade what you submitted; we can't take any clarifications into account during regrades.
 - Regrades are for where we made deductions we shouldn't have, rather than for the magnitude of deductions.

Essential Action Items

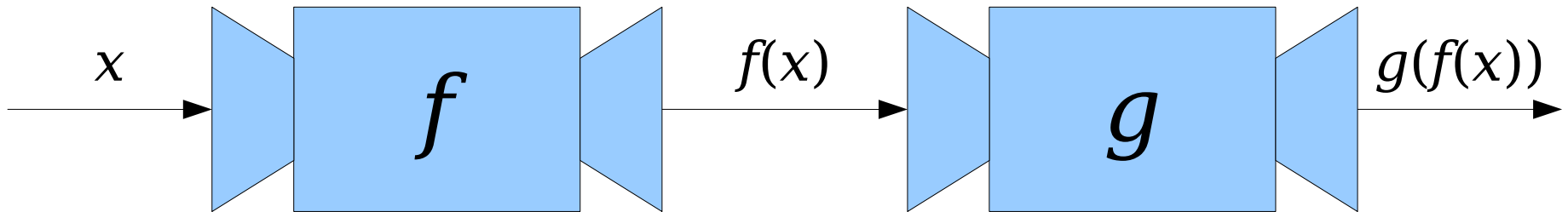
- ***Review your feedback when it comes available.***
 - Don't just look at the raw score. Make sure you really, truly understand where you need to improve.
- ***Read the solutions in depth.***
 - Make sure you understand what we were asking, why we asked it, and what we wanted you to take away.
 - (Especially for Q8, Q10) Look at our solutions and see if there's any neat lessons you can draw from them.
- ***Come to us with questions.***
 - Anything you're not sure about? That's what we're here for! Come to office hours, ask questions on EdStem, etc.



$h : \text{People} \rightarrow \text{Prices}$
 $h(x) = g(f(x))$

Function Composition

- Suppose that we have two functions $f : A \rightarrow B$ and $g : B \rightarrow C$.
- Notice that the codomain of f is the domain of g . This means that we can use outputs from f as inputs to g .



Function Composition

- Suppose that we have two functions $f : A \rightarrow B$ and $g : B \rightarrow C$.
- The **composition of f and g** , denoted **$g \circ f$** , is a function where
 - $g \circ f : A \rightarrow C$, and
 - $(g \circ f)(x) = g(f(x))$.
- A few things to notice:

The name of the function is $g \circ f$.
When we apply it to an input x , we write $(g \circ f)(x)$. I don't know why, but that's what we do.

 - The domain of $g \circ f$ is the domain of f . Its codomain is g 's codomain.
 - Even though composition is written $g \circ f$, when evaluating $(g \circ f)(x)$, the function f is evaluated first.
- Composition is **associative**: $(f \circ g) \circ h = f \circ (g \circ h)$. (Prove this!)
- Composition is not necessarily commutative: $f \circ g$ is not necessarily the same as $g \circ f$. (Prove this!)

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Theorem: If $f : A \rightarrow B$ is an injection and $g : B \rightarrow C$ is an injection, then the function $g \circ f : A \rightarrow C$ is an injection.

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What We're Assuming

$f : A \rightarrow B$ is an injection.

$\forall x \in A. \forall y \in A. (x \neq y \rightarrow$
 $f(x) \neq f(y))$

$g : B \rightarrow C$ is an injection.

$\forall x \in B. \forall y \in B. (x \neq y \rightarrow$
 $g(x) \neq g(y))$

We're *assuming* these universally-quantified statements, so we won't introduce any variables for what's here.

What We Need to Prove

$g \circ f$ is an injection.

$\forall a_1 \in A. \forall a_2 \in A. (a_1 \neq a_2 \rightarrow$
 $(g \circ f)(a_1) \neq (g \circ f)(a_2))$

We need to *prove* this universally-quantified statement. so let's introduce arbitrarily-chosen values.

Theorem: If $f : A \rightarrow B$ is an injection and $g : B \rightarrow C$ is an injection, then the function $g \circ f : A \rightarrow C$ is an injection.

What We're Assuming

$f : A \rightarrow B$ is an injection.

$$\forall x \in A. \forall y \in A. (x \neq y \rightarrow f(x) \neq f(y))$$

$g : B \rightarrow C$ is an injection.

$$\forall x \in B. \forall y \in B. (x \neq y \rightarrow g(x) \neq g(y))$$

$a_1 \in A$ is arbitrarily-chosen.

$a_2 \in A$ is arbitrarily-chosen.

What We Need to Prove

$g \circ f$ is an injection.

$$\forall a_1 \in A. \forall a_2 \in A. (a_1 \neq a_2 \rightarrow (g \circ f)(a_1) \neq (g \circ f)(a_2))$$

We need to *prove* this universally-quantified statement. so let's introduce arbitrarily-chosen values.

Theorem: If $f : A \rightarrow B$ is an injection and $g : B \rightarrow C$ is an injection, then the function $g \circ f : A \rightarrow C$ is an injection.

What We're Assuming

$f : A \rightarrow B$ is an injection.

$$\forall x \in A. \forall y \in A. (x \neq y \rightarrow f(x) \neq f(y))$$

$g : B \rightarrow C$ is an injection.

$$\forall x \in B. \forall y \in B. (x \neq y \rightarrow g(x) \neq g(y))$$

$a_1 \in A$ is arbitrarily-chosen.

$a_2 \in A$ is arbitrarily-chosen.

$a_1 \neq a_2$

What We Need to Prove

$g \circ f$ is an injection.

$$\forall a_1 \in A. \forall a_2 \in A. (a_1 \neq a_2 \rightarrow (g \circ f)(a_1) \neq (g \circ f)(a_2))$$

Now we're looking at an implication. Let's **assume** the antecedent and **prove** the consequent.

Theorem: If $f : A \rightarrow B$ is an injection and $g : B \rightarrow C$ is an injection, then the function $g \circ f : A \rightarrow C$ is an injection.

What We're Assuming

$f : A \rightarrow B$ is an injection.

$$\forall x \in A. \forall y \in A. (x \neq y \rightarrow f(x) \neq f(y))$$

$g : B \rightarrow C$ is an injection.

$$\forall x \in B. \forall y \in B. (x \neq y \rightarrow g(x) \neq g(y))$$

$a_1 \in A$ is arbitrarily-chosen.

$a_2 \in A$ is arbitrarily-chosen.

$$a_1 \neq a_2$$

What We Need to Prove

$g \circ f$ is an injection.

$$\forall a_1 \in A. \forall a_2 \in A. (a_1 \neq a_2 \rightarrow (g \circ f)(a_1) \neq (g \circ f)(a_2))$$

Let's write this out separately and simplify things a bit.

Theorem: If $f : A \rightarrow B$ is an injection and $g : B \rightarrow C$ is an injection, then the function $g \circ f : A \rightarrow C$ is an injection.

What We're Assuming

$f : A \rightarrow B$ is an injection.

$$\forall x \in A. \forall y \in A. (x \neq y \rightarrow f(x) \neq f(y))$$

$g : B \rightarrow C$ is an injection.

$$\forall x \in B. \forall y \in B. (x \neq y \rightarrow g(x) \neq g(y))$$

$a_1 \in A$ is arbitrarily-chosen.

$a_2 \in A$ is arbitrarily-chosen.

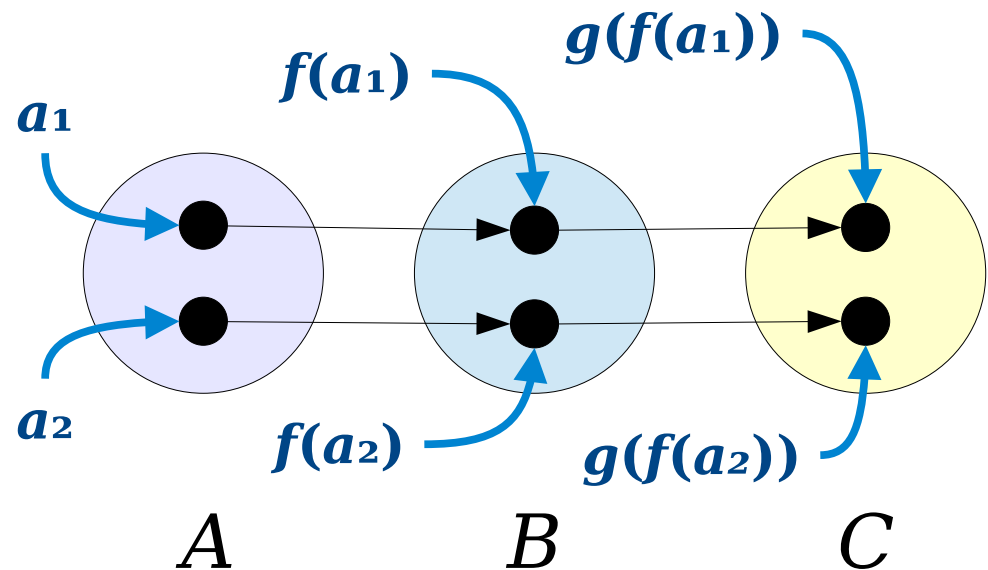
$$a_1 \neq a_2$$

What We Need to Prove

$g \circ f$ is an injection.

$$\forall a_1 \in A. \forall a_2 \in A. (a_1 \neq a_2 \rightarrow (g \circ f)(a_1) \neq (g \circ f)(a_2))$$

$$g(f(a_1)) \neq g(f(a_2))$$

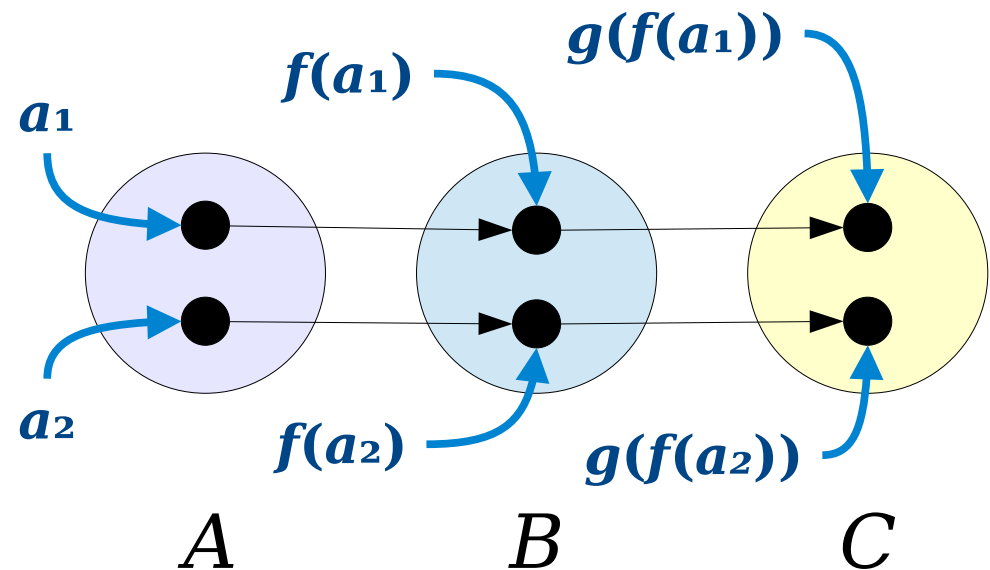


Theorem: If $f : A \rightarrow B$ is an injection and $g : B \rightarrow C$ is an injection, then the function $g \circ f : A \rightarrow C$ is also an injection.

Proof: Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be arbitrary injections. We will prove that the function $g \circ f : A \rightarrow C$ is also injective. To do so, consider any $a_1, a_2 \in A$ where $a_1 \neq a_2$. We will prove that $(g \circ f)(a_1) \neq (g \circ f)(a_2)$. Equivalently, we need to show that $g(f(a_1)) \neq g(f(a_2))$.

Since f is injective and $a_1 \neq a_2$, we see that $f(a_1) \neq f(a_2)$. Then, since g is injective and $f(a_1) \neq f(a_2)$, we see that $g(f(a_1)) \neq g(f(a_2))$, as required. ■

Great exercise: Repeat this proof using the other definition of injectivity.

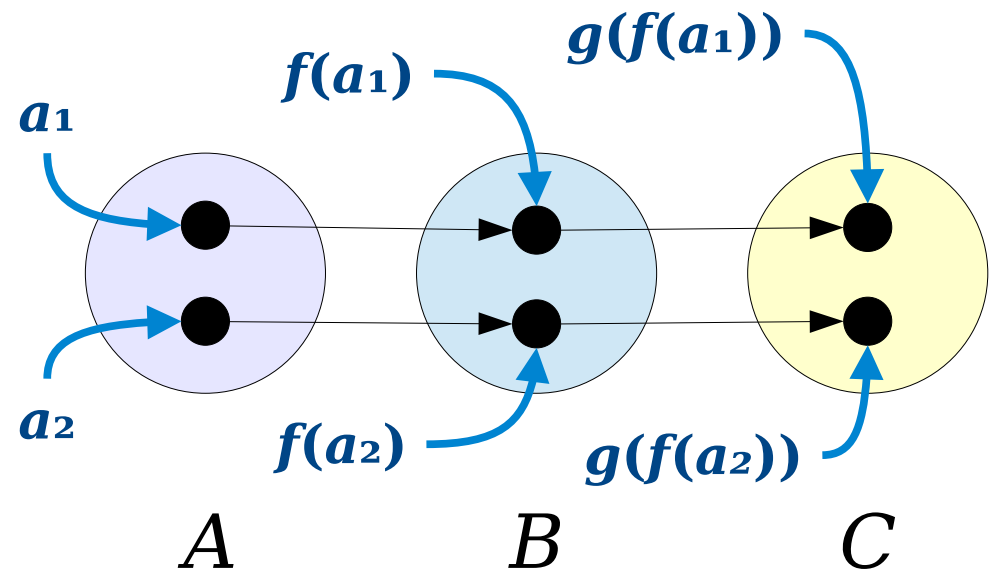


Theorem: If $f : A \rightarrow B$ is an injection and $g : B \rightarrow C$ is an injection, then the function $g \circ f : A \rightarrow C$ is also an injection.

Proof: Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be arbitrary injections. We will prove that the function $g \circ f : A \rightarrow C$ is also injective. To do so, consider any $a_1, a_2 \in A$ where $a_1 \neq a_2$. We will prove that $(g \circ f)(a_1) \neq (g \circ f)(a_2)$. Equivalently, we need to show that $g(f(a_1)) \neq g(f(a_2))$.

Since f is injective and $a_1 \neq a_2$, we see that $f(a_1) \neq f(a_2)$. Then, since g is injective and $f(a_1) \neq f(a_2)$, we see that $g(f(a_1)) \neq g(f(a_2))$, as required. ■

This proof contains no first-order logic syntax (quantifiers, connectives, etc.). It's written in plain English, just as usual.



Theorem: If $f : A \rightarrow B$ is a surjection and $g : B \rightarrow C$ is a surjection, then the function $g \circ f : A \rightarrow C$ is a surjection.

Proof:

See appendix!

Functions

Part 2

1. Recap from Last Time
2. A Proof About Birds
3. Assuming vs. Proving
4. Leveling Up the Assume-Prove Table
5. Connecting Function Types
6. The Two-Column Proof Organizer
7. Announcements
8. Function Composition
9. Properties of Composition: A Proof
- 10. Major Ideas from Today**
11. What's Next?

Major Ideas From Today

- Proofs involving first-order definitions are heavily based on the structure of those definitions, yet FOL notation itself does *not* appear in the proof.
- Statements behave differently based on whether you're ***assuming*** or ***proving*** them.
- When you ***assume*** a universally-quantified statement, initially, do nothing. Instead, keep an eye out for a place to apply the statement more specifically.
- When you ***prove*** a universally-quantified statement, pick an arbitrary value and try to prove it has the needed property.

The Assume-Prove Table

	If you assume this is true...	To prove that this is true...
$\forall x. A$	Initially, do nothing . Once you find a z through other means, you can state it has property A .	Have the reader pick an arbitrary x . We then prove A is true for that choice of x .
$\exists x. A$	Introduce a variable x into your proof that has property A .	Find an x where A is true. Then prove A is true for that specific choice of x .
$A \rightarrow B$	Initially, do nothing . Once you know A is true, you can conclude B is also true.	Assume A is true, then prove B is true.
$A \wedge B$	Assume A . Also assume B .	Prove A . Also prove B .
$A \vee B$	Consider two cases. Case 1: A is true. Case 2: B is true.	Either prove $\neg A \rightarrow B$ or prove $\neg B \rightarrow A$. <i>(Why does this work?)</i>
$A \leftrightarrow B$	Assume $A \rightarrow B$ and $B \rightarrow A$.	Prove $A \rightarrow B$ and $B \rightarrow A$.
$\neg A$	Simplify the negation, then consult this table on the result.	Simplify the negation, then consult this table on the result.

Functions

Part 2

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What's Next?

- ***Set Theory Revisited***
 - Formalizing our definitions.
- ***Proofs on Sets***
 - How to rigorously establish set-theoretic results.

Appendix:
Additional Function Proofs!

Proof: Composing surjections
yields a surjection.

Theorem: If $f : A \rightarrow B$ is surjective and $g : B \rightarrow C$ is surjective, then $g \circ f : A \rightarrow C$ is also surjective.

Proof: Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be arbitrary surjections. We will prove that the function $g \circ f : A \rightarrow C$ is also surjective. To do so, we will prove that for any $c \in C$, there is some $a \in A$ such that $(g \circ f)(a) = c$. Equivalently, we will prove that for any $c \in C$, there is some $a \in A$ such that $g(f(a)) = c$.

Consider any $c \in C$. Since $g : B \rightarrow C$ is surjective, there is some $b \in B$ such that $g(b) = c$. Similarly, since $f : A \rightarrow B$ is surjective, there is some $a \in A$ such that $f(a) = b$. Then we see that

$$g(f(a)) = g(b) = c,$$

which is what we needed to show. ■

