	Is defined as...	To ***prove*** that this is true...	If you ***assume*** this is true...
$x \in S \cap T$	$x \in S$ and $x \in T$	Prove $x \in A$. Then prove $x \in B$.	Assume $x \in A$. Then assume $x \in B$.
$x \in S \cup T$	$x \in S$ or $x \in T$	Prove either $x \in S$ or that $x \in T$.	Consider two cases: Case 1: $x \in S$. Case 2: $x \in T$.
$S \subseteq T$	For every $x \in S$, we have $x \in T$	Pick an arbitrary $x \in S$. Prove $x \in T$.	Initially, ***do nothing***. Once you find some $x \in S$, conclude $x \in T$.
$S = T$	$S \subseteq T$ and $T \subseteq S$	Prove $S \subseteq T$. Then prove $T \subseteq S$.	Assume $S \subseteq T$ and $T \subseteq S$.
$X \in \wp(A)$	$X \subseteq A$.	Prove $X \subseteq A$.	Assume $X \subseteq A$.

	To ***prove*** that this is true…	If you ***assume*** this is true…
$\forall x.\ A$	Have the reader pick an arbitrary x. We then prove A is true for that choice of x.	Initially, ***do nothing***. Once you find a z through other means, you can state it has property A.
$\exists x.\ A$	Find an x where A is true. Then prove that A is true for that specific choice of x.	Introduce a variable x into your proof that has property A.
$A \to B$	Assume A is true, then prove B is true.	Initially, ***do nothing***. Once you know A is true, you can conclude B is also true.
$A \land B$	Prove A. Then prove B.	Assume A. Then assume B.
$A \lor B$	Either prove $\neg A \to B$ or prove $\neg B \to A$. *(Why does this work?)*	Consider two cases. Case 1: A is true. Case 2: B is true.
$A \leftrightarrow B$	Prove $A \to B$ and $B \to A$.	Assume $A \to B$ and $B \to A$.
$\neg A$	Simplify the negation, then consult this table on the result.	Simplify the negation, then consult this table on the result.

Lecture 0: Set Theory Symbols

- $\in$ & $\notin$ — Element of / not an element of
- $\subseteq$ — Subset of

- $\emptyset$ — Empty set
- $\{ n \mid n \in \mathbb{N} \text{ and } n \text{ is even} \}$ — Example of set-builder notation

- $A \cup B$ — Set union
- $A \cap B$ — Set intersection
- $A - B$ — Set difference
- $A \Delta B$ — Set symmetric difference

- $|S|$ — Cardinality
- $\wp(S)$ — Power set of S
- $\aleph_0$ — Aleph-zero (the cardinality of $\mathbb{N}$)

Lecture 1 / Problem Set 1 Definitions

Sets of numbers

- $\mathbb{N}$ is the set of **natural numbers**: {0, 1, 2, 3, …}
- $\mathbb{Z}$ is the set of **integers**: {…, -3, -2, -1, 0, 1, 2, 3, …}
- $\mathbb{R}$ is the set of **real numbers**

Parity

- An integer n is called **even** if there is an integer k where n = 2k.
- An integer n is called **odd** if there is an integer k where n = 2k + 1.

Modular congruence

- For integers a and b, we say that $a \equiv_k b$ if there is an integer q such that a = b + kq.

Lecture 1: Proof Strategies

- A **universally-quantified statement** is a statement of the form

For all x, [some-property] holds for x.

- An **existentially-quantified statement** is a statement of the form

There is some x where [some-property] holds for x.

- To **directly prove a universally quantified statement**, allow the reader to pick an x and show that it has *[some-property]*.
- To **directly prove an existentially quantified statement**, specifically say what to pick for x and show that it has *[some-property]*.

See also the Guide to Proofs

Lecture 2: Definitions

- A **proposition** is a statement that is either true or false.
- The **negation** of a proposition X is a proposition that is true when X is false and is false when X is true
    - The negation of a universal is an existential.
    - The negation of an existential is a universal.
    - The negation of an implication “if P, then Q” is “P and not Q.”

- An **implication** is a statement of the form “If P is true, then Q is true.” P is called the **antecedent** and Q is called the **consequent**.
- The **contrapositive** of the implication “If P is true, then Q is true” is the implication “If Q is false, then P is false.”
- A **biconditional** is a statement of the form “P if and only if Q”

Lecture 2: Proof Strategies

See the Guide to Proofs

- **Proving an implication:** First, assume the antecedent. Then, prove the consequent.
- **Proof by contradiction:** First, assume that P is false. The goal is to show that this assumption is silly. Next, show this leads to an impossible result. Finally, conclude that since P can’t be false, we know that P must be true.
- **Proof by contrapositive:** Start by announcing that we're going to use a proof by contrapositive so that the reader knows what to expect. Then, explicitly state the contrapositive of what we want to prove. Then, go prove the contrapositive. (See “proving an implication”!)

Lecture 3: Propositional Connectives

Expression	English Translation	Negation of Expression
$A \wedge B$	A and B	2 possibilities: $\neg A \vee \neg B$, $A \rightarrow \neg B$
$A \vee B$	A or B	$\neg A \wedge \neg B$
$\neg A$	Not A	A
$A \rightarrow B$	A implies B	$A \wedge \neg B$
$A \leftrightarrow B$	A if and only if B	2 possibilities: $A \leftrightarrow \neg B$, $\neg A \leftrightarrow B$
$\top$	True	$\bot$
$\bot$	False	$\top$

Truth table tool

Lecture 3: Operators

Operator precedence:

- $\neg$ binds to whatever immediately follows it.
- $\wedge$ and $\vee$ bind more tightly than $\rightarrow$: $p \wedge q \rightarrow r$ is equivalent to $(p \wedge q) \rightarrow r$
- Operators are right-associative: $p \rightarrow q \rightarrow r$ is equivalent to $p \rightarrow (q \rightarrow r)$

De Morgan's Laws:

- $\neg(p \wedge q)$ is the same as $\neg p \vee \neg q$
- $\neg(p \vee q)$ is the same as $\neg p \wedge \neg q$

Lecture 4: Quantifiers

- $\exists$ is the **existential quantifier**. A statement of the form

  $$\exists x.\ \textit{some-formula}$$

  is true for a certain world if there exists a choice of x where *some-formula* is true when that x is plugged into it. Existentially-quantified statements are false unless there's a positive example. The $\exists$ quantifier usually is paired with $\land$.

- $\forall$ is the **universal quantifier**. A statement of the form

  $$\forall x.\ \textit{some-formula}$$

  is true for a certain world if, for every choice of x, the statement *some-formula* is true when x is plugged into it. Universally-quantified statements are true unless there's a counterexample. The $\forall$ quantifier usually is paired with $\to$.

Lecture 4: The Type-Checking Table

	… operate on …	… and produce
Connectives (↔, ∧, etc.) …	propositions	a proposition
Predicates (=, etc.) …	objects	a proposition
Functions …	objects	an object

Lecture 5: The Aristotelian Forms

"All *A*s are *B*s"

$$\forall x.\ (A(x) \to B(x))$$

"Some *A*s are *B*s"

$$\exists x.\ (A(x) \land B(x))$$

"No *A*s are *B*s"

$$\forall x.\ (A(x) \to \neg B(x))$$

"Some *A*s aren't *B*s"

$$\exists x.\ (A(x) \land \neg B(x))$$

Lecture 5: The Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	For all objects x, $P(x)$ is true.	$\exists x.\ \neg P(x)$
$\exists x.\ P(x)$	There is an x where $P(x)$ is true.	$\forall x.\ \neg P(x)$
$\forall x.\ \neg P(x)$	For all objects x, $P(x)$ is false.	$\exists x.\ P(x)$
$\exists x.\ \neg P(x)$	There is an x where $P(x)$ is false.	$\forall x.\ P(x)$

Lecture 5: Quantifiers over Sets

The notation

$$\forall x \in S.\ P(x)$$

means "for any element x of set S, $P(x)$ holds." (It's vacuously true if S is empty.)

The notation

$$\exists x \in S.\ P(x)$$

means "there is an element x of set S where $P(x)$ holds." (It's false if S is empty.)