Basic set-builder notation

Set-builder notation allows us to specify a set by describing its elements. A set written in set-builder notation has three parts: an expression, a vertical bar, and a property. Here's an example:

$$\{n \mid n \text{ is odd}\}$$
means "the set of all n where n is odd"

The vertical bar represents "where" or "with the property that".

Here, n **is a placeholder variable name**. The variable name on the left-hand side of the bar doesn't matter, as long as we refer to the same variable on the right-hand side: $\{n \mid n \text{ is odd}\}$ is the same set as $\{m \mid m \text{ is odd}\}$.

Crucially, **the set includes every possible thing that meets the condition** on the right-hand side of the bar. $\{n \mid n \text{ is odd}\}$ isn't a set with one number or just some of the odd numbers; it's the set of every single n that has the property "n is odd". As a result, set-builder notation unambiguously refers to one specific set.

Limiting our set to elements from another set

We can put $\in$ and the name of a set on the left-hand side of the bar in order to indicate that our set will only contain elements from the named set. Those elements still need to have the property on the right-hand side of the bar. Here's an example:

$$\{n \in \mathbb{N} \mid n \text{ is odd}\}$$
means "the set of all n where n is an element in $\mathbb{N}$ and n is odd"

If it helps you to understand, you can mentally shift the element-of symbol to the right-hand side of the bar: $\{n \in \mathbb{N} \mid n \text{ is odd}\}$ is the same set as $\{n \mid n \in \mathbb{N} \text{ and } n \text{ is odd}\}$.

Other expressions on the left-hand side of the bar

Here's a guiding example for this section:

$$\{103n + 1 \mid n \in \mathbb{N}\}$$
means "the set of all things that can be expressed as $103n + 1$, where n is an element in $\mathbb{N}$"

Unlike the previous examples, n does not represent the elements in the set. Instead, the elements of the set look like $103n + 1$, since that expression is on the left-hand side of the bar. The set will contain all elements that can be expressed in that way where n is a natural number.

Here's another way to think about this type of set: we'll find all things that meet the condition on the right-hand side of the bar, then transform them based on the left-hand side of the bar to assemble our set. In this case, each building block n needs to have the property that $n \in \mathbb{N}$. For each building block n, we'll add $103n + 1$ to the final set.

First-order logic on the right-hand side of the bar

Since we can use first-order logic to write statements that are true or false, we can express properties for set-builder notation sets in first-order logic. Here's an example:

$$\{n \mid n \in \mathbb{N} \land \exists k.\ n = 2k + 1\}$$

means "the set of all n where n is a natural number and there exists a k such that $n = 2k + 1$"

When dealing with these sets in proofs, remember the basic rules of set-builder notation: the left-hand side of the bar represents elements in the set, and the right-hand side of the bar represents a property of elements in the set. Every element in the set must have the property, and every possible thing with the given property must be an element in the set.

For example, let's say we have assumed that x is in the above set. What would we know about x? Since n is on the left-hand side of the bar, we know that n is the placeholder variable representing an element in the set. Then, when interpreting the right-hand side of the bar in relation to x, we can replace all of the n's in the property with x's. So overall, we'd know that $x \in \mathbb{N}$ and that there is some k where $x = 2k + 1$.

On the other hand, let's say we want to prove that the variable y is in the above set. What would we have to show? Since the set contains all possible elements with the property on the right-hand side of the bar, all we need to do is show that y has that property. Similarly to what we did with x, because we know n is a placeholder variable, we can put y's in the place of n's. We'd need to prove that $y \in \mathbb{N}$ and that there is some k where $y = 2k + 1$.

Miscellaneous simplifications

When nothing has the property on the right-hand side of the bar, the set is empty. For example, $\{n \mid n \text{ is both even and odd}\}$ is the empty set. The same results when it's impossible for anything to satisfy both the left- and right-hand conditions: $\{n \in \mathbb{N} \mid n < 0\}$ is the empty set, too.

The name of a set refers to all of its elements, so we don't need to use set-builder notation for a set that already has a name or symbol: $\{n \mid n \in \mathbb{N}\}$ is the same set as $\mathbb{N}$.