1. Proving Injectivity and Surjectivity

a. What does the notation $f : A \to B$ mean? Which sets are the domain and codomain? Explain what the domain and codomain are.

b. Here are two ways to state the definition of injectivity:

$$\forall a_1 \in A.\ \forall a_2 \in A.\ (a_1 \neq a_2 \to f(a_1) \neq f(a_2))$$

$$\forall a_1 \in A.\ \forall a_2 \in A.\ (f(a_1) = f(a_2) \to a_1 = a_2)$$

(1) Explain what an injective function is in your own words.

(2) What's the difference between this definition of an injective function and the following property, which is one of the requirements for something to be called a function?

$$\forall a_1 \in A.\ \forall a_2 \in A.\ (a_1 = a_2 \to f(a_1) = f(a_2))$$

(3) Based on the structure of each formula, what are two ways to prove that f is injective?

(4) Negate either formula and simplify it. How would you prove that f is **not** injective?

c. Here's the definition of surjectivity:

$$\forall b \in B.\ \exists a \in A.\ (f(a) = b)$$

(1) Explain what a surjective function is in your own words.

(2) What's the difference between this definition of a surjective function and the following property, which is one of the requirements for something to be called a function?

$$\forall a \in A.\ \exists b \in B.\ (f(a) = b)$$

(3) Based on the structure of this formula, how would you write a proof that f is surjective?

(4) Negate the formula and simplify it. How would you write a proof that f is **not** surjective?

d. How would you write a proof that a function f is (1) bijective, (2) **not** bijective?

2. Function Composition

For these questions, use the following strategies: (1) Write out all of the definitions in the problem.
(2) Use the proof strategies table to determine what you will **assume** and **want to show**.
(3) Use the two-column proof organizer to keep track of introducing new variables, expanding definitions, and so on.

a. Let $f : A \to B$ and $g : B \to C$ be functions. Prove that if $g \circ f$ is injective, then f is injective.

b. Let $f : A \to B$ and $g : B \to C$ be functions. Prove that if $g \circ f$ is surjective, then g is surjective.

3. First-Order Definitions and Functions Proofs

Interpreting definitions given in terms of first-order logic is really important for the remainder of CS 103. In this problem, we'll practice with setting up proofs involving first-order logic.

Let $f : A \to B$ be a function. We call f **right-cancellative** if the following property holds for any functions $g : B \to C$ and $h : B \to C$:

$$\big(\forall a \in A. (g \circ f)(a) = (h \circ f)(a)\big) \to \big(\forall b \in B. g(b) = h(b)\big)$$

Prove that if f is surjective, then f is right-cancellative.

Key question: When we want to show an implication, what should we do?

4. Set Theory Proofs

If we have time, we'll prove the following result as a group: for arbitrary sets A and B, $\wp(A) \cap \wp(B) \subseteq \wp(A \cap B)$. In conjunction with the result in part b, this means that $\wp(A \cap B) = \wp(A) \cap \wp(B)$.

For the following proofs, proceed by clearly articulating what you are assuming and what you want to show, unpacking definitions, and focusing on individual elements of sets. In these statements, A, B, and C are arbitrary sets.

a. Prove that if $A \subseteq B$, then $A \cap C \subseteq B \cap C$.

b. Prove that $\wp(A \cap B) \subseteq \wp(A) \cap \wp(B)$.