1 Proof Writing

1.1 Consolidated Proof Writing Checklist

- Clearly articulate your assumptoins and want-to-shows
- Make each sentence “load-bearing"
- Scope and properly introduce variables
- Make specific claims about specific variables
- Don’t repeat definitions – use them instead

Assume → Want-To-Show → Reasoning → Return to Want-To-Show

1.2 Proof Templates

Direct (Universally Quantified):

> ***Theorem:*** For all [subject], [fact].
> ***Proof:*** Pick and arbitrary [subject]. We want to show [fact]. (Optional sentence) Since we know [some useful information about [subject]], [we can represent [subject] like this]. Then we see that
>
> [turn what we know about [subject] into [fact]]
>
> Therefore, [fact] is true, as required. ■

Direct (Existentially Quantified):

> ***Theorem:*** For some [subject], [fact].
> ***Proof:*** Pick [[subject] = a value that works]. We want to show [fact]. (Optional sentence) Since we know [some useful information about [subject]], [we can represent [subject] like this]. Then we see that
>
> [turn [subject] into [fact] by plugging in the value that works]
>
> Therefore, [fact] is true, as required. ■

Contrapositive:

> ***Theorem:*** [fact] (usually an implication [if this, then that]).
> ***Proof:*** We will prove the contrapositive of this statement, namely, [contrapositive of [fact]] ([if not that, then not this] for implication). [copy relevant direct proof template]. ■

Contradiction

> ***Theorem:*** [fact]
> ***Proof:*** Assume for the sake of contradiction that [negation of [fact]]. Since we know [something useful from our assumption], [we can represent our statement some useful way]. Then we see that
>
> [simplify expression to something that breaks our assumption]
>
> [what we found], but this is impossible because [the opposite of what we found is true] by assumption.
> We have reached a contradiction, so our assumption must have been wrong. Therefore, [fact].■

Biconditional: P if and only if Q. Prove two separate statements → If P is true, then Q is true and If Q is true, then P is true.

2 Set Theory

- ***Set:*** an unordered collection of distinct objects, which may be anything, including other sets.
- ***Elements:*** the objects that make up a set. $S \in T$
- ***Subset:*** A set S is called a subset of a set T when all elements of S are also elements of T. $S \subseteq T$
- ***Power Set:*** the set of all subsets of S. $\wp(S) = \{T | T \subseteq S\}$
- ***Cardinality:*** number of elements elements a set contains. $|S|$
- ***Cantor's Theorem:*** If S is a set, then $|S| < |\wp(S)|$

	Is defined as…	If you ***assume*** this is true…	To ***prove*** that this is true…
$S \subseteq T$	$\forall x \in S.\ x \in T$	Initially, ***do nothing***. Once you find some $z \in S$, conclude $z \in T$.	Ask the reader to pick an $x \in S$. Then prove $x \in T$
$S = T$	$S \subseteq T \land T \subseteq S$	Assume $S \subseteq T$ and $T \subseteq S$.	Prove $S \subseteq T$. Also prove $T \subseteq S$.
$x \in S \cap T$	$x \in S \land x \in T$	Assume $x \in S$. Then assume $x \in T$.	Prove $x \in S$. Also prove $x \in T$.
$x \in S \cup T$	$x \in S \lor x \in T$	Consider two cases: Case 1: $x \in S$. Case 2: $x \in T$.	Either prove $x \in S$ or prove $x \in T$.
$X \in \wp(S)$	$X \subseteq S$.	Assume $X \subseteq S$.	Prove $X \subseteq S$.
$x \in \{ y \mid P(y) \}$	$P(x)$	Assume $P(x)$.	Prove $P(x)$.

Figure 1: Proofs with sets

3 Propositional Logic

Expression	English Translation	Negation of Expression
A ∧ B	A and B	2 possibilities: ¬A ∨ ¬B, A → ¬B
A ∨ B	A or B	¬A ∧ ¬B
¬A	Not A	A
A → B	A implies B	A ∧ ¬B
A ↔ B	A if and only if B	2 possibilities: A ↔ ¬B, ¬A ↔ B
⊤	True	⊥
⊥	False	⊤

Figure 2: Propositional Logic to English and Negations

4 First Order Logic

Expression	English Translation	Negation of Expression
$\forall x.(P(x))$	Everything is a P.	$\exists x.(\neg P(x))$
$\exists x.(P(x))$	Something is a P.	$\forall x.(\neg P(x))$
$\forall x.(\neg P(x))$	Nothing is a P.	$\exists x.(P(x))$
$\exists x.(\neg P(x))$	Something isn't a P.	$\forall x.(P(x))$
$\forall x.(A(x) \to B(x))$	All A's are B's.	$\exists x.(A(x) \land \neg B(x))$
$\exists x.(A(x) \land B(x))$	Some A is a B.	$\forall x.(A(x) \to \neg B(x))$
$\forall x.(A(x) \to \neg B(x))$	No A is a B.	$\exists x.(A(x) \land B(x))$
$\exists x.(A(x) \land \neg B(x))$	Some A isn't a B.	$\forall x.(A(x) \to B(x))$

Figure 3: First Order English Translations and Negations

	If you ***assume*** this is true...	To ***prove*** that this is true...
$\forall x.\ A$	Initially, ***do nothing***. Once you find a z through other means, you can state it has property A.	Have the reader pick an arbitrary x. We then prove A is true for that choice of x.
$\exists x.\ A$	Introduce a variable x into your proof that has property A.	Find an x where A is true. Then prove that A is true for that specific choice of x.
$A \to B$	Initially, ***do nothing***. Once you know A is true, you can conclude B is also true.	Assume A is true, then prove B is true.
$A \land B$	Assume A. Also assume B.	Prove A. Also prove B.
$A \lor B$	Consider two cases. Case 1: A is true. Case 2: B is true.	Either prove $\neg A \to B$ or prove $\neg B \to A$. *(Why does this work?)*
$A \leftrightarrow B$	Assume $A \to B$ and $B \to A$.	Prove $A \to B$ and $B \to A$.
$\neg A$	Simplify the negation, then consult this table on the result.	Simplify the negation, then consult this table on the result.

Figure 4: Proofs with first order logic

Useful Facts:

- An implication with a false antecedent is vacuously true
- Existentially-quantified statements are false in an empty world, since nothing exists
- Universally-quantified statements are said to be vacuously true in empty worlds
- Universally quantified statements are true unless there's a counterexample