1. All or Nothing?

Cumulative Practice Problems

True or False?

a. ______ If D is a DFA over alphabet Σ and D has no accepting states, then $\mathcal{L}(D) = \emptyset$

b. ______ If D is a DFA over alphabet Σ and D has no rejecting states, then $\mathcal{L}(D) = \Sigma^*$

c. ______ If N is a NFA over alphabet Σ and N has no accepting states, then $\mathcal{L}(N) = \emptyset$

d. ______ If N is a NFA over alphabet Σ and N has no rejecting states, then $\mathcal{L}(N) = \Sigma^*$

2. Botanical Graphs

Practice Exam 7

An undirected graph $G = (V, E)$ is called ***botanical*** if

- G is connected, and
- $\exists v \in V. \forall x \in V. (v = x \leftrightarrow deg(x) \geq 3)$

where $deg(x)$ denoted the ***degree*** of x, the number of edges touching x (or the number of nodes adjacent to x). As a reminder, undirected graphs cannot have self-loops.

a. Draw the smallest possible botanical graph. By "smallest possible," we mean the botanical graph that has as few nodes as possible, and, of those graphs, the one with the fewest edges.

b. Draw an eight-node botanical graph with no cycles.

c. Draw an eight-node botanical graph with exactly one cycle. (For the purposes of this problem, we'll consider two cycles to be the same if they follow the same series of nodes and edges. So, for example, v_1, v_2, v_3, v_1 would be the same cycle as v_2, v_3, v_1, v_2 and v_3, v_1, v_2, v_3.)

d. Draw an eight-node botanical graph with exactly three simple cycles. (Use the above convention to determine if two cycles are the same.)

3. Idempotent Functions / Exploring a Function

A function $f : A \to A$ is called ***idempotent*** when the following is true:

$$\forall x \in A. f(x) = f(f(x))$$

Practice Exam 9

a. Below are four functions. Which are idempotent? No justification is required.

- ☐ $f : \mathbb{Z} \to \mathbb{Z}$ defined as $f(x) = |x|$
- ☐ $f : \mathbb{Z} \to \mathbb{Z}$ defined as $f(x) = x + |x|$
- ☐ $f : \mathbb{N} \to \mathbb{N}$ defined as $f(n) = 2n - 2\lfloor \frac{n}{2} \rfloor$
- ☐ $f : \mathbb{N} \to \mathbb{N}$ defined as $f(n) = 2n - 2\lceil \frac{n}{2} \rceil$

Practice Exam 11

Consider the function $g : \wp(\mathbb{N}) \to \wp(\mathbb{N})$ defined as follows:

$$g(S) = \{n | n \in \mathbb{N} \land \exists m \in S. n \geq m\}$$

This problem explores properties of this function.

a. Let $T = \{n | n \in \mathbb{N} \land n \text{ is odd}\}$. Select all true options from the list below. You may need to select multiple options or no options at all.

- ☐ $0 \in g(T)$
- ☐ $1 \in g(T)$
- ☐ $2 \in g(T)$
- ☐ $3 \in g(T)$

b. Select all true options from the list below. You may need to select multiple options or no options at all.

- ☐ $g(\mathbb{N}) = \emptyset$
- ☐ $g(\mathbb{N}) = \{\emptyset\}$
- ☐ $g(\mathbb{N}) = \mathbb{N}$
- ☐ $g(\mathbb{N}) = \{\mathbb{N}\}$
- ☐ $g(\mathbb{N}) = g(\{n \in \mathbb{N} | \exists k \in \mathbb{N}. n \leq k\})$
- ☐ $g(\mathbb{N}) = g(\{n \in \mathbb{N} | \forall k \in \mathbb{N}. n \leq k\})$

c. Reminder: $g : \wp(\mathbb{N}) \to \wp(\mathbb{N})$ defined as follows:

$$g(S) = \{n | n \in \mathbb{N} \land \exists m \in S. n \geq m\}$$

Reminder: A function $f : A \to A$ is called ***idempotent*** when the following is true:

$$\forall x \in A. f(x) = f(f(x))$$

Prove that g is idempotent. Something to keep in mind: how do you prove that two sets are equal?

4. Paying the Troll Toll

Practice Exam 1

Little Red Riding Hood wants to visit Grandma and bring her cookies. To get there, she has to cross a series of bridges. Beneath each bridge is a troll. Whenever Little Red Riding Hood arrives at a bridge, the following happens:

- If she's carrying an even number of cookies, the troll takes half of those cookies, then gives back one cookie.
- If she's carrying an odd number of cookies, the troll gets angry about not being able to take half of them and instead just eats Little Red Riding Hood. Yikes!

Little Red Riding Hood wants to arrive at Grandma's house with exactly three cookies. The number of cookies she has to start with will depend on how many bridges she must cross.

a. Fill in the remaining blanks below. No justification is required.

- If she begins with 3 cookies, then she can cross 0 bridges and end with exactly 3 cookies.
- If she begins with 4 cookies then she can cross 1 bridge and end with exactly 3 cookies
- If she begins with _____ cookies, then she can cross 2 bridges and end with exactly 3 cookies
- If she begins with _____ cookies, then she can cross 3 bridges and end with exactly 3 cookies
- If she begins with _____ cookies, then she can cross 4 bridges and end with exactly 3 cookies

b. Fill in the following blank with a simple mathematical expression (e.g. $b^2 + 19b + 1$, $\lfloor \frac{b}{2} \rfloor^2$, etc., but not a recurrence relation like $t_{b+1} = 3t_b^2 - 137$). No justification required.

Theorem: For all natural numbers $b \geq 0$, if she begins with $2 +$ ________ cookies, then Little Red Riding Hood can cross b bridges and end with exactly 3 cookies.

c. Prove the theorem from part (b) of this problem.
(***Theorem:*** For all natural numbers $b \geq 0$, if she begins with $2 + \_\_\_\_\_\_\_\_$ cookies, then Little Red Riding Hood can cross b bridges and end with exactly 3 cookies.)