CS 106B, Lecture 11
Exhaustive Search and Backtracking

Plan for Today

- New recursive problem-solving techniques
  - Exhaustive Search
  - Backtracking

Exhaustive search

- **exhaustive search**: Exploring every possible combination from a set of choices or values.
  - often implemented recursively
  - Sometimes called *recursive enumeration*

  Applications:
  - producing all permutations of a set of values
  - enumerating all possible names, passwords, etc.
  - combinatorics and logic programming

- Often the search space consists of many ***decisions***, each of which has several available ***choices***.
  - Example: When enumerating all 5-letter strings, each of the 5 letters is a *decision*, and each of those decisions has 26 possible *choices*.

Exhaustive search

A general pseudo-code algorithm for exhaustive search:

Explore(*decisions*):

- if there are no more decisions to make: Stop.
- else, let's handle one decision ourselves, and the rest by recursion.
  for each available choice *C* for this decision:
  - **Choose** *C* by modifying parameters.
  - **Explore** the remaining decisions that could follow *C*.
  - **Un-choose** *C* by returning parameters to original state (if necessary).

Exhaustive Search Model

Choosing

1. We generally iterate over **decisions**. What are we iterating over here? The iteration will be done by recursion.
2. What are the **choices** for each decision? Do we need a for loop?

Exploring

3. How can we *represent* that choice? How should we **modify the parameters**?
    a) Do we need to use a **wrapper** due to extra parameters?

Un-Choosing

4. How do we **un-modify** the parameters from step 3? Do we need to explicitly un-modify, or are they copied? Are they un-modified at the same level as they were modified?

Base Case

5. What should we do in the **base case** when we're out of decisions?

Exercise: printAllBinary

printBinary

- Write a recursive function **`printAllBinary`** that accepts an integer number of digits and prints all binary numbers that have exactly that many digits, in ascending order, one per line.
  - `printAllBinary(2);` `printAllBinary(3);`

```
00          000
01          001
10          010
11          011
            100
            101
            110
            111
```

printAllBinary

Choosing

1. We generally iterate over **decisions**. What are we iterating over here? The iteration will be done by recursion.
2. What are the **choices** for each decision? Do we need a for loop?

Exploring

3. How can we *represent* that choice? How should we **modify the parameters** and **store our previous choices**?
   a) Do we need to use a **wrapper** due to extra parameters?

Un-Choosing

4. How do we **un-modify** the parameters from step 3? Do we need to explicitly un-modify, or are they copied? Are they un-modified at the same level as they were modified?

Base Case

5. What should we do in the **base case** when we're out of decisions?

printAllBinary

Choosing

1. We generally iterate over **decisions**. What are we iterating over here? ***We are iterating over characters in the binary string***
2. What are the **choices** for each decision? Do we need a for loop? ***Choose 0 or 1***

Exploring

3. How can we *represent* that choice? How should we **modify the parameters** and **store our previous choices**? ***Build up a string that we will eventually print. Add the 0 or 1 to it. String tracks our previous choices***
   a) Do we need to use a **wrapper** due to extra parameters? ***Yes***

Un-Choosing

4. How do we **un-modify** the parameters from step 3? Do we need to explicitly un-modify, or are they copied? Are they un-modified at the same level as they were modified? ***If new strings for each call, we don't need to un-choose***

Base Case

5. What should we do in the **base case** when we're out of decisions? ***Print the string***

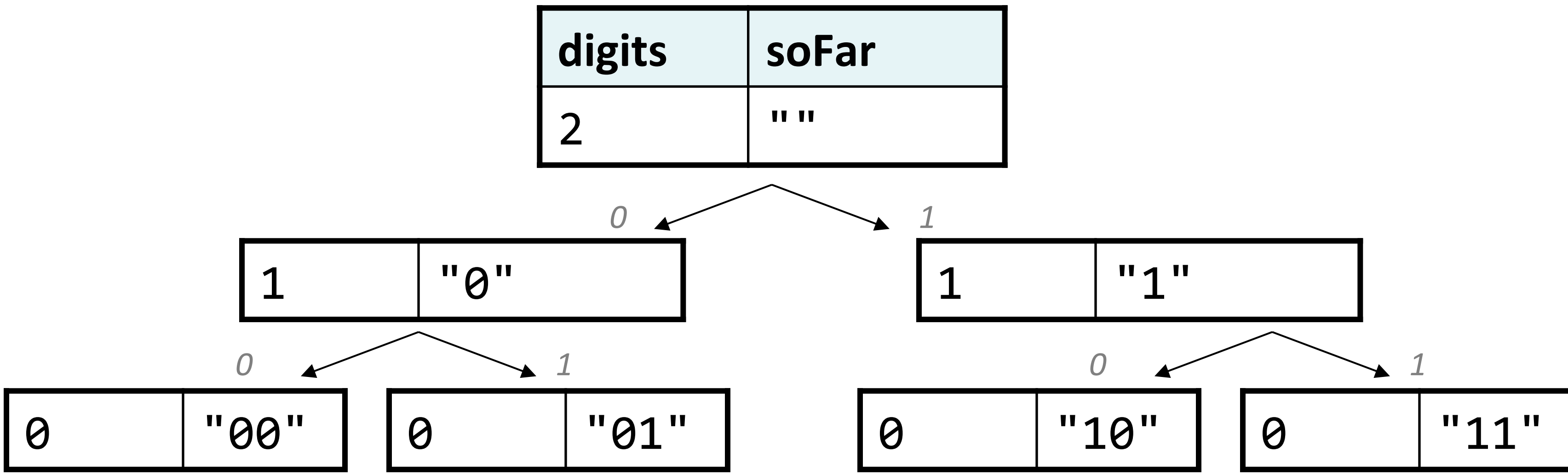
printAllBinary solution

```
void printAllBinary(int numDigits) {
    printAllBinaryHelper(numDigits, "");
}

void printAllBinaryHelper(int digits, string soFar) {
    if (digits == 0) {
        cout << soFar << endl;
    } else {
        printAllBinaryHelper(digits - 1, soFar + "0");
        printAllBinaryHelper(digits - 1, soFar + "1");
    }
}
```

A tree of calls

- `printAllBinary(2);`



  - This kind of diagram is called a *call tree* or *decision tree*.
  - Think of each call as a choice or decision made by the algorithm:
    - Should I choose 0 as the next digit?
    - Should I choose 1 as the next digit?

The base case

```
void printAllBinaryHelper(int digits, string soFar) {
    if (digits == 0) {
        cout << soFar << endl;
    } else {
        printAllBinaryHelper(digits - 1, soFar + "0");
        printAllBinaryHelper(digits - 1, soFar + "1");
    }
}
```

- The **base case** is where the code stops after doing its work.
  - `pAB(3) -> pAB(2) -> pAB(1) -> pAB(0)`
- Each call should keep track of the work it has done.
- Base case should print the result of the work done by prior calls.
  - Work is kept track of in some variable(s) - in this case, `string soFar`.

Exercise: printDecimal

- Write a recursive function **`printDecimal`** that accepts an integer number of digits and prints all base-10 numbers that have exactly that many digits, in ascending order, one per line.
  - `printDecimal(2);` `printDecimal(3);`

```
00          000
01          001
02          002
..          ...
98          997
99          998
            999
```

  - Use recursion. *

printDecimal

Choosing

1. We generally iterate over **decisions**. What are we iterating over here? The iteration will be done by recursion.
2. What are the **choices** for each decision? **<u>Do we need a for loop?</u>**

Exploring

3. How can we *represent* that choice? How should we **modify the parameters store our previous choices**?
   a) Do we need to use a **wrapper** due to extra parameters?

Un-Choosing

4. How do we **un-modify** the parameters from step 3? Do we need to explicitly un-modify, or are they copied? Are they un-modified at the same level as they were modified?

Base Case

5. What should we do in the **base case** when we're out of decisions?

printDecimal solution

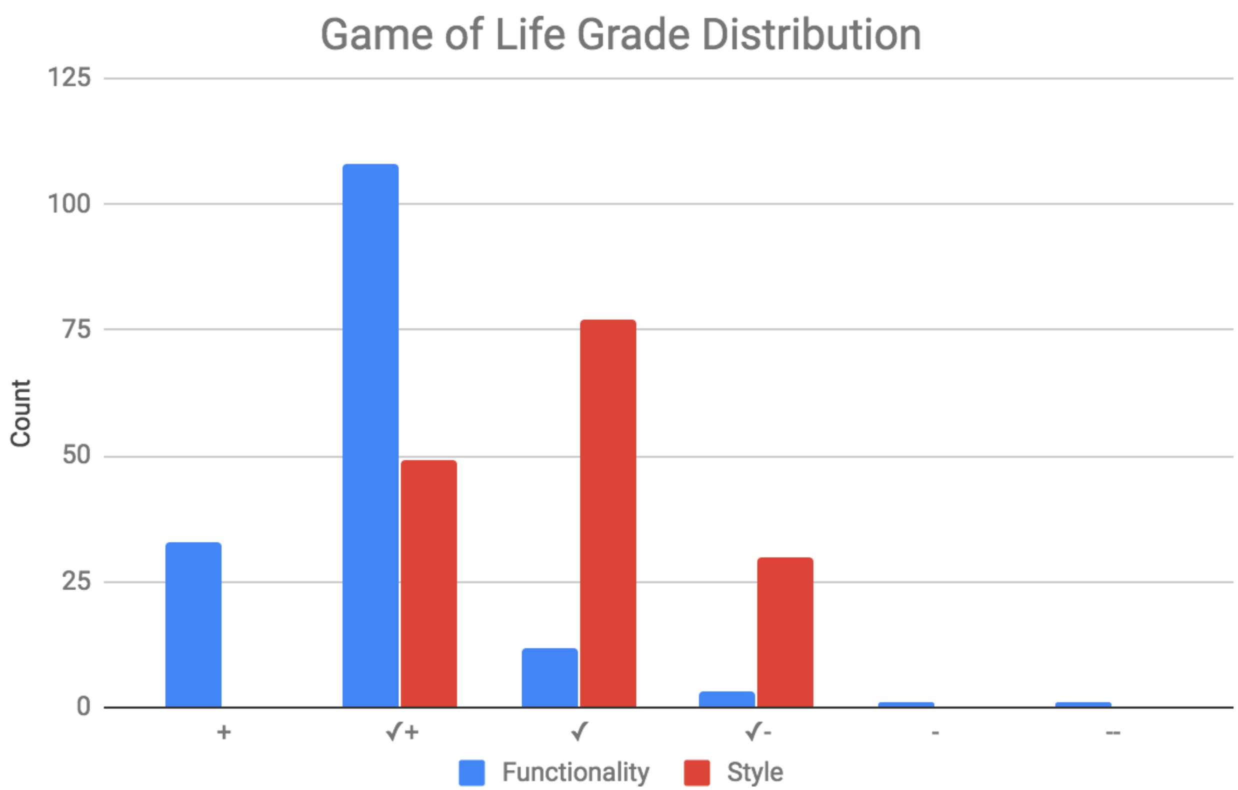
```
void printDecimal(int digits) {
    printDecimalHelper(digits, "");
}

void printDecimalHelper(int digits, string soFar) {
    if (digits == 0) {
        cout << soFar << endl;
    } else {
        for (int i = 0; i < 10; i++) {
            printDecimalHelper(digits - 1, soFar +
                                       integerToString(i));
        }
    }
}
```

- *Observation:* When the set of digit choices available is large, using a loop to enumerate them avoids redundancy. *(This is okay!)*
- *Note:* Loop over **choices**, not decisions

Announcements

- Homework 3 due on Wednesday at **5PM**
- Shreya will be guest-lecturing on Monday
  - My office hours will be cancelled that day (still available via email)
- **Midterm Review Session** on Tuesday, July 24, from 7-9PM in Gates B01



Backtracking

- **backtracking**: Finding solution(s) by trying all possible paths and then abandoning them if they are not suitable.
  - a "brute force" algorithmic technique (tries all paths)
  - often implemented recursively
  - Could involve looking for **one** solution
    - If any of the paths found a solution, a solution exists! If none find a solution, no solution exists
  - Could involve finding **all solutions**
  - Idea: it's exhaustive search **with conditions**

  Applications:
  - parsing languages
  - games: anagrams, crosswords, word jumbles, 8 queens, sudoku
  - combinatorics and logic programming
  - escaping from a maze

Backtracking: One Solution

A general pseudo-code algorithm for backtracking problems searching for one solution

Backtrack(*decisions*):

- if there are no more decisions to make:
  - if our current solution is valid, return **true**
  - else, return **false**
- else, let's handle one decision ourselves, and the rest by recursion. for each available **valid** choice *C* for this decision:
  - **Choose** *C* by modifying parameters.
  - **Explore** the remaining decisions that could follow *C*. If any of them find a solution, return **true**
  - **Un-choose** *C* by returning parameters to original state (if necessary).
- If no solutions were found, **return false**

Backtracking: All Solutions

A general pseudo-code algorithm for backtracking problems searching for all solutions

Backtrack(*decisions*):

– if there are no more decisions to make:
  - if our current solution is valid, add it to our list of found solutions
  - else, do nothing or return

– else, let's handle one decision ourselves, and the rest by recursion. for each available **valid** choice *C* for this decision:
  - **Choose** *C* by modifying parameters.
  - **Explore** the remaining decisions that could follow *C*. Keep track of which solutions the recursive calls find.
  - **Un-choose** *C* by returning parameters to original state (if necessary).

– Return the list of solutions found by all the helper recursive calls.

Backtracking Model

Choosing

1. We generally iterate over **decisions**. What are we iterating over here? What are the **choices** for each decision? Do we need a for loop?

Exploring

3. How can we *represent* that choice? How should we **modify the parameters** and **store our previous choices** (avoiding *arms-length* recursion)?
   a) Do we need to use a **wrapper** due to extra parameters?
4. <u>How should we **restrict** our choices to be valid?</u>
5. <u>How should we use the **return value** of the recursive calls? Are we looking for all solutions or just one?</u>

Un-choosing

6. How do we **un-modify** the parameters from step 3? Do we need to explicitly un-modify, or are they copied? Are they un-modified at the same level as they were modified?

Base Case

7. What should we do in the base case when we're **out of decisions** <u>(usually return true)</u>?
8. <u>Is there a case for when there **aren't any valid choices left** or a "bad" state is reached (usually return false)?</u>
9. <u>Are the base cases ordered properly? Are we avoiding **arms-length** recursion?</u>

Exercise: Dice roll sum

diceSum

- Write a function **`diceSum`** that accepts two integer parameters: a number of dice to roll, and a desired sum of all die values. Output all combinations of die values that add up to exactly that sum.

```
diceSum(2, 7);

{1, 6}
{2, 5}
{3, 4}
{4, 3}
{5, 2}
{6, 1}
```

```
diceSum(3, 7);

{1, 1, 5}
{1, 2, 4}
{1, 3, 3}
{1, 4, 2}
{1, 5, 1}
{2, 1, 4}
{2, 2, 3}
{2, 3, 2}
{2, 4, 1}
{3, 1, 3}
{3, 2, 2}
{3, 3, 1}
{4, 1, 2}
{4, 2, 1}
{5, 1, 1}
```

Dice Roll Sum

Choosing

1. We generally iterate over **decisions**. What are we iterating over here? What are the **choices** for each decision? Do we need a for loop?

Exploring

3. How can we *represent* that choice? How should we **modify the parameters** and **store our previous choices** (avoiding *arms-length* recursion)?
   a) Do we need to use a **wrapper** due to extra parameters?
4. How should we **restrict** our choices to be valid?
5. How should we use the **return value** of the recursive calls? Are we looking for all solutions or just one?

Un-choosing

6. How do we **un-modify** the parameters from step 3? Do we need to explicitly un-modify, or are they copied? Are they un-modified at the same level as they were modified?

Base Case

7. What should we do in the base case when we're **out of decisions** (usually return true)?
8. Is there a case for when there **aren't any valid choices left** or a "bad" state is reached (usually return false)?
9. Are the base cases ordered properly? Are we avoiding **arms-length** recursion?

Easier: Dice rolls

diceRoll

- **Suggestion:** First just output <u>all</u> possible combinations of values that could appear on the dice.
- This is just exhaustive search!
- In general, starting with exhaustive search and then adding conditions is not a bad idea

```
diceSum(2, 7);
```

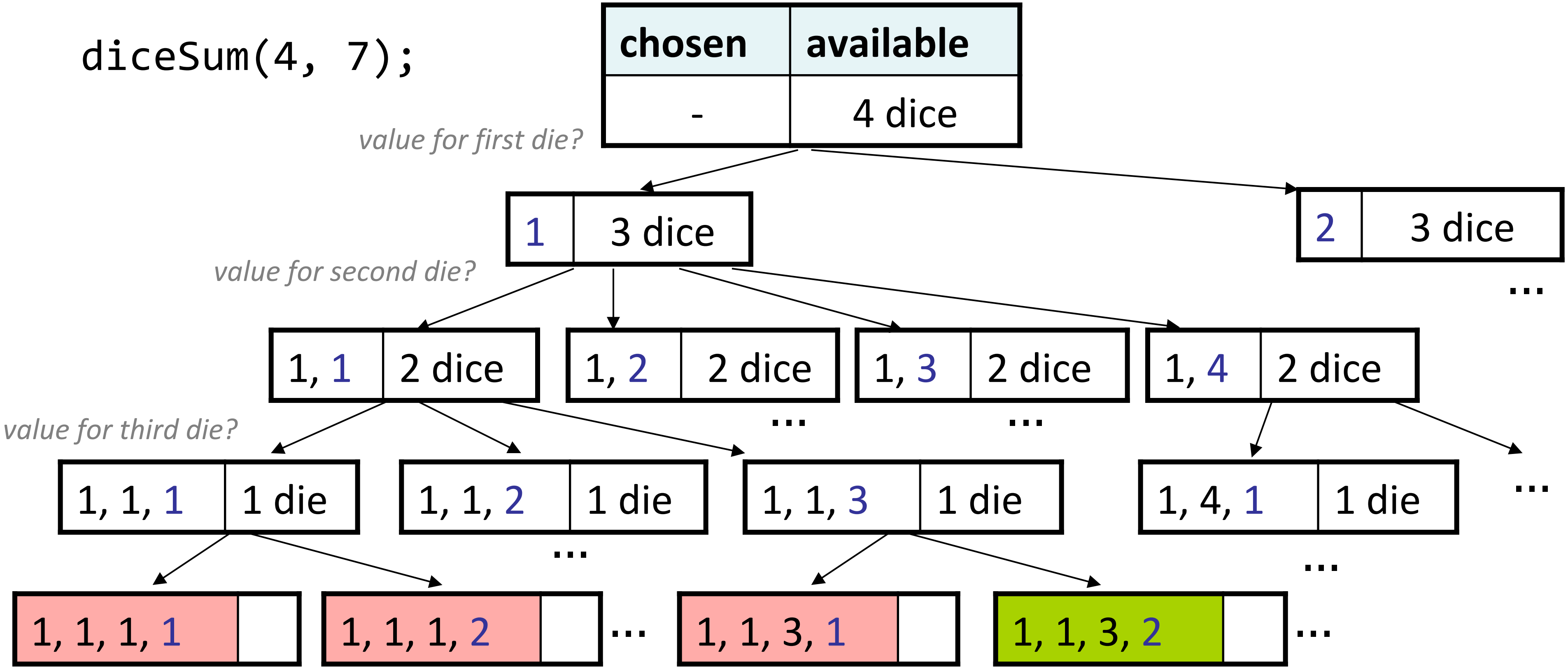
{1, 1}	{3, 1}	{5, 1}
{1, 2}	{3, 2}	{5, 2}
{1, 3}	{3, 3}	{5, 3}
{1, 4}	{3, 4}	{5, 4}
{1, 5}	{3, 5}	{5, 5}
{1, 6}	{3, 6}	{5, 6}
{2, 1}	{4, 1}	{6, 1}
{2, 2}	{4, 2}	{6, 2}
{2, 3}	{4, 3}	{6, 3}
{2, 4}	{4, 4}	{6, 4}
{2, 5}	{4, 5}	{6, 5}
{2, 6}	{4, 6}	{6, 6}

```
diceSum(3, 7);

{1, 1, 1}
{1, 1, 2}
{1, 1, 3}
{1, 1, 4}
{1, 1, 5}
{1, 1, 6}
{1, 2, 1}
{1, 2, 2}
   ...
{6, 6, 4}
{6, 6, 5}
{6, 6, 6}
```

A decision tree

```
diceSum(4, 7);
```



Initial solution

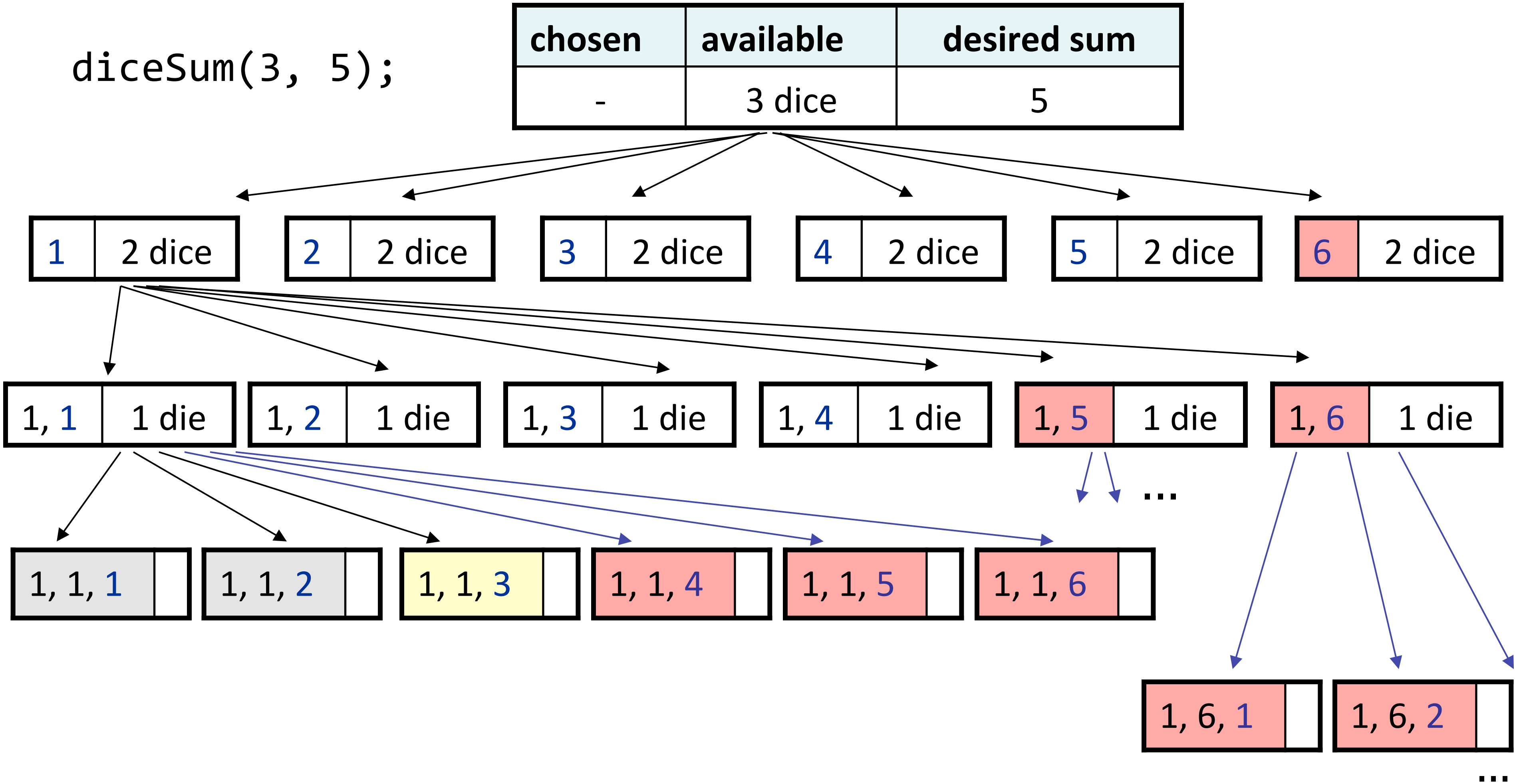
```
void diceSum(int dice, int desiredSum) {
    Vector<int> chosen;
    diceSumHelper(dice, desiredSum, chosen);
}

void diceSumHelper(int dice, int desiredSum, Vector<int>& chosen) {
    if (dice == 0) {
        if (sumAll(chosen) == desiredSum) {
            cout << chosen << endl;                      // base case
        }
    } else {
        for (int i = 1; i <= 6; i++) {
            chosen.add(i);                               // choose
            diceSumHelper(dice - 1, desiredSum, chosen); // explore
            chosen.remove(chosen.size() - 1);            // un-choose
        }
    }
}

int sumAll(const Vector<int>& v) {   // adds the values in given vector
    int sum = 0;
    for (int k : v) { sum += k; }
    return sum;
}
```

Wasteful decision tree

```
diceSum(3, 5);
```



Optimizations

- We need not visit every branch of the decision tree.
  - Some branches are clearly not going to lead to success.
  - We can preemptively stop, or **prune**, these branches.

- Inefficiencies in our dice sum algorithm:
  - Sometimes the current sum is already **too high**.
    - (Even rolling 1 for all remaining dice would exceed the desired sum.)
  - Sometimes the current sum is already **too low**.
    - (Even rolling 6 for all remaining dice would exceed the desired sum.)
  - The code must **re-compute** the sum many times.
    - (1+1+1 = ..., 1+1+2 = ..., 1+1+3 = ..., 1+1+4 = ..., ...)

diceSum solution

```
void diceSum(int dice, int desiredSum) {
    Vector<int> chosen;
    diceSumHelper(dice, 0, desiredSum, chosen);
}

void diceSumHelper(int dice, int sum, int desiredSum, Vector<int>& chosen) {
    if (dice == 0) {
        if (sum == desiredSum) {
            cout << chosen << endl;                 // solution found base case
        }
    } else if (sum + 1*dice > desiredSum            // invalid state base case
            || sum + 6*dice < desiredSum) {
            return;
    } else {
        for (int i = 1; i <= 6; i++) {
            chosen.add(i);                                          // choose
            diceSumHelper(dice - 1, sum + i, desiredSum, chosen); // explore
            chosen.remove(chosen.size() - 1);                    // un-choose
        }
    }
}
```

A Twist

- How would you modify **`diceSum`** so that it prints only unique combinations of dice, ignoring order?
  - (e.g. don't print both `{1, 1, 5}` and `{1, 5, 1}`)

```
diceSum2(2, 7);
```

```
{1, 6}
{2, 5}
{3, 4}
{4, 3}
{5, 2}
{6, 1}
```

```
diceSum2(3, 7);
```

```
{1, 1, 5}
{1, 2, 4}
{1, 3, 3}
{1, 4, 2}
{1, 5, 1}
{2, 1, 4}
{2, 2, 3}
{2, 3, 2}
{2, 4, 1}
{3, 1, 3}
{3, 2, 2}
{3, 3, 1}
{4, 1, 2}
{4, 2, 1}
{5, 1, 1}
```