CS106B
Winter 2019

Handout 13S
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Section Solutions 4

Problem One: Weights and Balances

Imagine that we start off by putting the amount to be measured (call it *n*) on the left side of the balance. This makes the imbalance on the scale equal to *n*. Imagine that there is some way to measure *n*. If we put the weights on the scale one at a time, we can look at where we put that first weight (let's suppose it weighs *w*). It must either

- go on the left side, making the net imbalance on the scale $n + w$,
- go on the right side, making the net imbalance on the scale $n – w$, or
- not get used at all, leaving the net imbalance *n*.

If it is indeed truly possible to measure *n*, then one of these three options has to be the way to do it, even if we don't know which one it is. The question we then have to ask is whether it's then possible to measure the new net imbalance using the weights that remain – which we can determine recursively! On the other hand, if it's not possible to measure *n*, then no matter which option we choose, we'll find that there's no way to use the remaining weights to make everything balanced!

If we're proceeding recursively, which we are here, we need to think about our base case. There are many options we can choose from. One simple one is the following: imagine that we don't have any weights at all, that we're asked to see whether some weight is measurable using no weights. In what circumstances can we do that? Well, if what we're weighing has a nonzero weight, we can't possibly measure it – placing it on the scale will tip it to some side, but that doesn't tell us how much it weighs. On the other hand, if what we're weighing is completely weightless, then putting it on the scale won't cause it to tip, convincing us that, indeed, it is weightless! So as our base case, we'll say that when we're down to no remaining weights, we can measure *n* precisely if $n = 0$. With that in mind, here's our code:

```
/**
 * Given an amount, a list of weights, and an index, determines whether it's
 * possible to measure n using the weights at or after the index given by
 * startPoint.
 *
 * @param amount The amount to measure, which can be positive, negative or 0.
 * @param weights The weights available to us.
 * @param index The starting index into the weights Vector.
 * @return Whether the amount can be measured using the weights from the specified
 *         index and forward.
 */
bool isMeasurableRec(int amount, const Vector<int>& weights, int index) {
  if (index == weights.size()) {
      return amount == 0;
  } else {
      return isMeasurableRec(amount,                  weights, index + 1) ||
             isMeasurableRec(amount + weights[index], weights, index + 1) ||
             isMeasurableRec(amount - weights[index], weights, index + 1);
  }
}

bool isMeasurable(int amount, const Vector<int>& weights) {
    return isMeasurableRec(n, weights, 0);
}
```

Problem Two: CHeMoWIZrDy

Here's one possible implementation of the `isElementSpellable` function:

```
/**
 * Given a word and an element symbol, returns whether the word starts with that
 * particular element symbol.
 *
 * @param word The word in question
 * @param symbol The symbol in question.
 * @return Whether the word starts with that element symbol.
 */
bool startsWithElement(const string& word, const string& symbol) {
    return startsWith(toLowerCase(word), toLowerCase(symbol));
}

/**
 * Given a word and a set containing all the element symbols in the Periodic
 * Table, returns whether it's possible to spell that word using just element
 * symbols.
 *
 * @param text The word
 * @param symbols The element symbols in the Periodic Table.
 * @return Whether that text can be spelled out.
 */
bool isElementSpellable(const string& word, const Set<string>& symbols) {
    /* Base case: the empty string can be spelled out by simply using no strings
     * from the list of symbols.
     */
    if (word == "") {
        return true;
    }
    /* Recursive case: try each element symbol to see whether any of them match
     * the first characters of the input string. We could alternatively rely on
     * the fact that all element symbols are between 1 and 3 characters long, but
     * just in case that changes we won't assume that here. :-)
     */
    else {
        for (string symbol: symbols) {
            if (startsWithElement(word, symbol) &&
                isElementSpellable(word.substr(symbol.length()), symbols)) {
                return true;
            }
        }
        /* If none of those options work, there is no way to spell this word using
         * element symbols.
         */
        return false;
    }
}
```

We can modify this code to report the way in which the string would match by making a slight modification to the recursive step to accumulate element symbols together as we unwind back up.

```
bool isElementSpellable(const string& word, const Set<string>& symbols,
                        string& result) {
    /* Base case: the empty string can be spelled out by simply using no strings
     * from the list of symbols.
     */
    if (word == "") {
        result = "";  // This is the proper way to spell this word.
        return true;
    }
    /* Recursive case: try each element symbol to see whether any of them match
     * the first characters of the input string. We could alternatively rely on
     * the fact that all element symbols are between 1 and 3 characters long, but
     * just in case that changes we won't assume that here. :-)
     */
    else {
        for (string symbol: symbols) {
            if (startsWithElement(word, symbol)) {
                /* See if we can spell what's left. */
                if (isElementSpellable(word.substr(symbol.length()),
                    symbols, result)) {
                    /* Because we could, we know that result is now filled in with
                     * how to spell the rest of the word (that's what the function
                     * says it will do!). We just need to prepend the element
                     * symbol we used.
                     */
                    result = symbol + result;
                    return true;
                }
            }
        }
        /* If none of those options work, there is no way to spell this word using
         * element symbols.
         */
        return false;
    }
}
```

Now, to the challenge problem of getting the best optimization. This one is a lot harder because we might not end up using all the letters in the original string – in fact, we might delete large chunks of the string in order to make more things fit.

We could do this by going one character at a time, seeing what to do with that character, but that turns out to be fairly tricky. Instead, we'll opt for another approach. We'll ask the question: which element symbol should go at the start of the approximation? For each possible element symbol, we need to check that the characters within that symbol actually appear somewhere in the input string. But the tricky bit is that they don't have to be consecutive. For example, in converting "`chemowizardry`" to "`CHeMoWIZrDy`," we deleted the a between the `z` and the `r` to make zirconium (`Zr`) fit, and we deleted the `r` between the `d` and `y` to get dysprosium (`Dy`) to fit. So when we try using an element symbol, we need to find all the characters that make it up, in sequence, possibly with spaces in them. That's tricky but doable.

And what happens if no element symbol fits? Then we just end up approximating things with the empty string.

Here's what this might look like:

```
string closestApproximationTo(const string& word, const Set<string>& symbols) {
    /* Base case: If the string is empty, the best approximation is to use no
     * element symbols.
     */
    if (word == "") return "";

    /* Recursive case: Try all possible elements to see which one goes first. */
    string best = ""; // In case nothing matches, we return the empty string.

    for (string element: symbols) {
        /* See where this element fits. We need to find each character in sequence
         * but possibly with gaps between them.
         */
        int index = word.find(element[0]);
        for (int i = 1; i < element.length() && index != string::npos; ) {
            index = word.find(element[i], index + 1);
        }

        /* If we found everything, this is a possible match. */
        if (index != string::npos) {
            auto with = element + closestApproximationTo(word.substr(index + 1),
                                                         symbols);
            if (best.length() < with.length()) best = with;
        }
    }
    return best;
}
```

Problem Three: Barnstorming Brainstorming

This problem essentially boils down to generating all permutations of the sites and seeing whether any of them fit in the specified timeframe. The intuition we'll use in writing up this solution is similar to the one we used to generate permutations in class – we'll look at all options for the next place to go, consider what would happen if we visited any of them, and see if any of those options lead to success.

One catch here is that in order to measure distances we need to remember where we just were, since we have to measure distances based on where we used to be. That in itself is somewhat interesting because the very first place we visit isn't proceeded by anything, so we'll separate that from the rest of the recursion logic.

Here's what that looks like:

```
/**
 * Given a list of sites to visit and a total travel time, plus the location of
 * the last city visited, returns whether it's possible to visit all of those
 * locations in the specified amount of time.
 *
 * @param sites The list of sites left to visit.
 * @param timeAvailable How much time is left.
 * @param last The last place we visited.
 * @return Whether we can visit those sites starting at the given location.
 */
bool canVisitAllSitesRec(const Vector<GPoint>& sites, double timeAvailable,
                         const GPoint& last);

/**
 * Given a Vector, returns a new Vector formed by removing the element at the
 * specified index.
 *
 * @param sites The list of sites.
 * @param index The index in question.
 * @return That vector with that index removed.
 */
Vector<GPoint> removeAt(Vector<GPoint> sites, int index);

bool canVisitAllSites(const Vector<GPoint>& sites, double timeAvailable) {
    /* If there aren't any sites, we can always visit them all! */
    if (sites.isEmpty()) return true;

    /* Try all possible starting points and see if any of them work. */
    for (int i = 0; i < sites.size(); i++) {
        if (canVisitAllSitesRec(removeAt(sites, i), timeAvailable, sites[i])) {
            return true;
        }
    }
    return false;
}

                          /* continued on the next page */
```

```
/**
 * Returns the Euclidean distance between two points.
 *
 * @param one The first point.
 * @param two The second point.
 * @return The distance between them.
 */
double distanceBetween(const GPoint& one, const GPoint& two) {
    double dx = one.getX() - two.getX();
    double dy = one.getY() - two.getY();
    return sqrt(dx * dx + dy * dy);
}

bool canVisitAllSitesRec(const Vector<GPoint>& sites, double timeAvailable,
                         const GPoint& last) {
    /* Base case: If no sites remain, we're done! */
    if (sites.isEmpty()) {
        return true;
    }
    /* Recursive case: see where we go next. */
    else {
        for (int i = 0; i < sites.size(); i++) {
            /* See how long this is going to take. If it's too far, then we
             * can't go there next.
             *
             * We can actually be way more aggressive here due to the triangle
             * inequality: the fastest way to a point is to go straight there.
             * If we can't make it there from here in time, there's no alternate
             * route we could take that would be any better. The only reason we
             * didn't optimize the code this way was because in general you can't
             * make assumptions like this.
             */
            double distance = distanceBetween(last, sites[i]);
            if (distance <= timeAvailable &&
                canVisitAllSitesRec(removeAt(sites, i), timeAvailable - dist,
                                    sites[i])) {
                return true;
            }
        }

        /* Looks like no options worked. Oh well! */
        return false;
    }
}
```

To update this code to not just tell us whether there is a route, but to also say what the route is, we can update the function so that, when it finds a route that works, it adds in the city that we considered at the current level of the recursion. Here's what that looks like, with the helper functions and documentation removed:

```
bool canVisitAllSites(const Vector<GPoint>& sites, double timeAvailable,
                      Vector<GPoint>& result) {
    /* If there aren't any sites, we can always visit them all! */
    if (sites.isEmpty()) {
        result.clear(); // Best option is the empty list.
        return true;
    }

    /* Try all possible starting points and see if any of them work. */
    for (int i = 0; i < sites.size(); i++) {
        if (canVisitAllSitesRec(removeAt(sites, i), timeAvailable, sites[i],
                                result) {
            /* Prepend the starting city. */
            result.insert(0, sites[i]);
            return true;
        }
    }
    return false;
}

bool canVisitAllSitesRec(const Vector<GPoint>& sites, double timeAvailable,
                         const GPoint& last, Vector<GPoint>& result) {
    /* Base case: If no sites remain, we're done! */
    if (sites.isEmpty()) {
        result.clear(); // Empty list is the correct visit order here.
        return true;
    }
    /* Recursive case: see where we go next. */
    else {
        for (int i = 0; i < sites.size(); i++) {
            double distance = distanceBetween(last, sites[i]);
            if (distance <= timeAvailable &&
                canVisitAllSitesRec(removeAt(sites, i), timeAvailable - dist,
                                    sites[i], result)) {
                /* Result will have been filled in with the best sequence to use
                 * given the remaining cities, so we just need to fill in this
                 * particular city.
                 */
                result.insert(0, sites[i]);
                return true;
            }
        }
        /* Looks like no options worked. Oh well! */
        return false;
    }
}
```

This function would ***not*** be a good candidate for memoization. It's extremely unlikely that we'd arrive at the same recursive call in two different ways, since that would mean that somehow we visited the same set of cities in two different ways and ended up using exactly the same amount of time to do so.

Problem Four: Pattern Matching

The recursion here works by recursively consuming both the pattern and the text, but its base case is only for the case where the *pattern* is empty, since an empty pattern only matches the empty string while an empty string can match a nonempty pattern. (Do you see why?) The solution we've introduced here uses a quick optimization that's worth keeping in your back pocket. Because we always munch from the front of the pattern and text strings, any text or pattern string we encounter later on is going to be a suffix of the original text or pattern. Therefore, rather than making lots of copies of strings by using `string::substr`, we'll just keep track of the index of the next character in each string that we need to process.

```
bool matchesRec(const string& text, int textIndex,
                const string& pattern, int patternIndex) {
    /* Base case: If we've consumed the pattern, confirm we consumed the text. */
    if (patternIndex == pattern.length()) {
        return textIndex == text.length();
    }
    /* Recursive step: there's more pattern to match. See what to do here. */
    /* Case 1: The next pattern character is a letter. */
    else if (isalpha(pattern[patternIndex])) {
        return textIndex != text.length() &&               // Text isn't empty
               text[textIndex] == pattern[patternIndex] && // That char matches
               matchesRec(text, textIndex + 1, pattern, patternIndex + 1);
    }
    /* Case 2: The next pattern character is a dot. */
    else if (pattern[patternIndex] == '.') {
        return textIndex != text.length() &&
               matchesRec(text, textIndex + 1, pattern, patternIndex + 1);
    }
    /* Case 3: The next pattern character is a ?. */
    else if (pattern[patternIndex] == '?') {
        return matchesRec(text, textIndex, pattern, patternIndex + 1) ||
            (textIndex != text.length() &&
             matchesRec(text, textIndex + 1, pattern, patternIndex + 1));
    }
    /* Case 4: The next pattern character is a star. */
    else if (pattern[patternIndex] == '*') {
        return matchesRec(text, textIndex, pattern, patternIndex + 1) ||
           (textIndex != text.length() &&
            matchesRec(text, textIndex + 1, pattern, patternIndex));
    } else {
        error("Unknown pattern character.");
    }
}

bool matches(const string& text, const string& pattern) {
    return matchesRec(text, 0, pattern, 0);
}
```

This function is very amenable to memoization, especially given that texts with multiple stars or question marks in them can possibly match the same text in several different ways. Fun fact: many years ago, Keith got this exact question as a job interview question for a technical internship at Facebook and didn't think to use memoization after writing a recursive solution. ☺

We're going to use the handy `SparseGrid` type for our memoization. It's essentially a 2D grid that may have missing entries, which is perfect for memoization where our table is initially empty and then has entries filled in as the recursion progresses.

```
bool matchesRec(const string& text, int textIndex,
                const string& pattern, int patternIndex,
                SparseGrid<bool>& memo) {

    /* Base case: If we've consumed the pattern, confirm we consumed the text. */
    if (patternIndex == pattern.length()) {
        return textIndex == text.length();
    }
    /* Base case: If we've memoized the result, return it. */
    else if (memo.isSet(textIndex, patternIndex)) {
        return memo[textIndex][patternIndex];
    }
    /* Recursive step always has to write the answer down. We'll store that value
     * in a variable that we write at the very end of the function.
     */
    bool answer;

    /* Case 1: The next pattern character is a letter. */
    else if (isalpha(pattern[patternIndex])) {
        answer = textIndex != text.length() &&
                 text[textIndex] == pattern[patternIndex] &&
                 matchesRec(text, textIndex + 1, pattern, patternIndex + 1, memo);
    }
    /* Case 2: The next pattern character is a dot. */
    else if (pattern[patternIndex] == '.') {
        answer = textIndex != text.length() &&
                 matchesRec(text, textIndex + 1, pattern, patternIndex + 1, memo);
    }
    /* Case 3: The next pattern character is a ?. */
    else if (pattern[patternIndex] == '?') {
        answer = matchesRec(text, textIndex, pattern, patternIndex + 1, memo) ||
               (textIndex != text.length() &&
               matchesRec(text, textIndex + 1, pattern, patternIndex + 1, memo));
    }
    /* Case 4: The next pattern character is a star. */
    else if (pattern[patternIndex] == '*') {
        answer = matchesRec(text, textIndex, pattern, patternIndex + 1, memo) ||
                 (textIndex != text.length() &&
                 matchesRec(text, textIndex + 1, pattern, patternIndex, memo));
    } else {
        error("Unknown pattern character.");
    }
    memo[textIndex][patternIndex] = answer;
    return answer;
}

bool matches(const string& text, const string& pattern) {
    SparseGrid<bool> memo(text.length() + 1, pattern.length() + 1);
    return matchesRec(text, 0, pattern, 0, memo);
}
```

Problem Five: Cell Towers Redux

Here’s what this code would look like if we chose to rewrite it using the strategy suggested:

```
int bestCoverageRec(const Vector<int>& populations, int index) {
    /* Base case 1: If we are out of cities, we can't cover anyone. */
    if (index == populations.size()) {
        return 0;
    }
    /* Base case 2: If there's just one city, cover it. */
    else if (index == populations.size() - 1) {
        return populations[index];
    }
    /* Recursive step: consider our options when we do or do not cover the first
     * tower. If we cover the first tower, we have to give up the second.
     */
    else {
        return max(bestCoverageRec(populations, index + 2) + populations[0],
                   bestCoverageRec(populations, index + 1));
    }
}

int bestCoverageFor(const Vector<int>& populations) {
    return bestCoverageRec(populations, 0);
}
```

Wow, that’s a lot cleaner!

This code is just screaming for memoization, since we can get to the same subproblems in all sorts of different ways. For example, the subproblem consisting of all but the first two cities can be reached either by choosing to include the first city or choosing to exclude the first two cities. Here's what that might look like:

```
int bestCoverageRec(const Vector<int>& populations, int index,
                    Map<int, int>& memo) {
    /* Base case 1: If we are out of cities, we can't cover anyone. */
    if (index == populations.size()) {
        return 0;
    }
    /* Base case 2: If there's just one city, cover it. */
    else if (index == populations.size() - 1) {
        return populations[index];
    }
    /* Base case 3: If we memoized the result, just return it. */
    else if (memo.containsKey(index)) {
        return memo[index];
    }
    /* Recursive step: consider our options when we do or do not cover the first
     * tower. If we cover the first tower, we have to give up the second.
     */
    else {
        int with = bestCoverageRec(populations, index + 2, memo) +
                       populations[index];
        int without = bestCoverageRec(populations, index + 1, memo);

        int result = max(with, without);
        memo[index] = result;
        return result;
    }
}

int bestCoverageFor(const Vector<int>& populations) {
    Map<int, int> memo;
    return bestCoverageRec(populations, 0, memo);
}
```

So what's the efficiency of this code? Notice that at each recursive call we do O(1) work doing a bunch of multiplications, additions, function calls, etc., plus some amount of table reading. We'll end up having recursive calls made for the index ranging from 0 to the number of cities, so there's going to be O(n) work done, plus the work done to read and write from the `Map`.

We haven't talked about this yet, but the map has lookup time O(log n) for fun reasons we'll discuss in a few weeks. That means that this code runs in time O(n log n). If you used a different structure, like a `Vector`, though, you can drop this down to O(n) runtime, a ***massive*** improvement over the original!

Problem Six: Advocating for Exponents

```
int raiseToPower(int m, int n) {
    int result = 1;
    for (int i = 0; i < n; i++) {
        result *= m;
    }
    return result;
}
```

i. What is the big-O complexity of the above function, written in terms of m and n? You can assume that it takes time O(1) to multiply two numbers.

This function runs in time O(n). It runs the loop n times, at each step doing O(1) work. There is no dependence on m in the runtime.

ii. If it takes 1μs to compute `raiseToPower(100, 100)`, approximately how long will it take to compute `raiseToPower(200, 10000)`?

We know that this code runs in time O(n), so it scales roughly linearly with the size of n. Therefore, if it took 1μs to compute a value when $n = 100$, it will take roughly 100 times longer when we plug in $n = 10000$. As a result, we'd expect this code would take about 100μs to complete.

```
int raiseToPower(int m, int n) {
    if (n == 0) return 1;
    return m * raiseToPower(m, n - 1);
}
```

iii. What is the big-O complexity of the above function, written in terms of m and n? You can assume that it takes time O(1) to multiply two numbers.

If we trace through the recursion, we'll see that we make a total of n recursive calls, each of which is only doing O(1) work. Adding up all the work done by these recursive calls gives us a total of O(n) work, as before.

iv. If it takes 1μs to compute `raiseToPower(100, 100)`, approximately how long will it take to compute `raiseToPower(200, 10000)`?

As before, this should take about 100μs.

```
int raiseToPower(int m, int n) {
    if (n == 0) {
        return 1;
    } else if (n % 2 == 0) {
        int halfPower = raiseToPower(m, n / 2);
        return halfPower * halfPower;
    } else {
        int halfPower = raiseToPower(m, n / 2);
        return m * halfPower * halfPower;
    }
}
```

v. What is the big-O complexity of the above function, written in terms of m and n? You can assume that it takes time O(1) to multiply two numbers.

Notice that each recursive call does O(1) work (there are no loops anywhere here), then calls itself on a problem that's half as big as the original one. This means that only O(log n) recursive calls will happen (remember that repeatedly dividing by two is the hallmark of a logarithm), so the total work done here is O(log n).

vi. If it takes 1µs to compute `raiseToPower(100, 100)`, approximately how long will it take to compute `raiseToPower(200, 10000)`?

We know that the runtime when $n = 100$ is roughly 1µs. Notice that $100^2 = 10{,}000$, so we're essentially asking for the runtime of this function when we square the size of the input. Also notice that via properties of logarithms that $\log n^2 = 2 \log n$. Therefore, since we know the runtime grows roughly logarithmically and we've squared the value of n, this should take about twice as long as before, roughly 2µs.

Problem Seven: Revisiting Reversals

```
string reverseOf(string str) {
    if (str == "") {
        return str;
    } else {
        return reverseOf(str.substr(1)) + str[0];
    }
}
```

Notice that when we call this function with a string of length n, we do $O(n)$ work inside the function. That comes from the cost of making the substring of length n-1, plus the work to concatenate the resulting string with `str[0]`, plus the cost of initializing the argument of the function we called, which takes its argument by value. We then make a recursive call on a problem of size n-1. The net effect is that, like with insertion and selection sort, we're roughly doing work

$$n + (n-1) + (n-2) + \ldots + 2 + 1$$

which works out to $O(n^2)$ work.

If we rewrite this code so that we have the argument passed in by `const` reference, then we still haven't changed the fact that we're doing $O(n)$ work inside the body of the function call, so the overall runtime is still going to be $O(n^2)$. However, we should expect it to run a bit faster, since we are reducing the total amount of work that we need to do. at each step.

```
string reverseOf(const string& str) {
    if (str.length() <= 1) {
        return str;
    } else {
        return reverseOf(str.substr(str.length() / 2)) +
               reverseOf(str.substr(0, str.length() / 2));
    }
}
```

Notice that any given function call to `reverseOf` will still do $O(n)$ work on a string of length n, since we have to create the two substrings (total length n) and concatenate them together to form a longer string of length n. However, this code is different from the previous part in that there are two recursive calls, not one, and each one is to a subproblem whose size is roughly $n/2$.

But we've seen this before! This is just like mergesort, which does linear work at each call and makes two subcalls on problems of size $n/2$. That means that this runs in time $O(n \log n)$.