CS 106B, Lecture 8
Recursion

Plan for Today

- Learn a powerful algorithmic technique called *recursion*
  - Exploit self-similarity in problems

- We will spend several days on recursion – don't worry if it doesn't make sense today
  - Goal: do as many examples as we can
  - You should **practice**: CodeStepByStep, section problems, or examples from the textbook
  - Highly encourage the reading for this week!

Recursion

- **recursion**: Function that calls itself
  - Solving a problem using recursion depends on solving smaller (simpler) occurrences of the same problem until the problem is simple enough that you can solve it directly
  - Key question: *"How is this problem self-similar?"* – what are the smaller subproblems that make up the bigger problem?

- Occurs in many places in code and in real world:
  - Looking up a word in dictionary may involve looking up other words
  - Nested structures (trees, file folders, collections) can be self-similar

Recursive Programming

- **recursive programming**: Writing functions that call themselves to solve problems that are recursive in nature.
  - An equally powerful substitute for *iteration* (loops)
  - Particularly well-suited to solving certain types of problems
  - Leads to **elegant**, simplistic, short code (when used well)
  - A key component of many of our assignments in this course

Non-recursive factorial

```
// Returns n!, or 1 * 2 * 3 * 4 * ... * n.
// Assumes n >= 1.
int factorial(int n) {
    int total = 1;
    for (int i = 1; i <= n; i++) {
        total *= i;
    }
    return total;
}
```

- Important observations:

  0! = 1! = 1

  4! = $\underline{4 * 3 * 2 * 1}$

  5! = 5 * $\underline{4 * 3 * 2 * 1}$

  = 5 * 4!

Recursive factorial

```
// Returns n!, or 1 * 2 * 3 * 4 * ... * n.
// Assumes n >= 0.
int factorial(int n) {
    if (n <= 1) {                      // base case
        return 1;
    } else {
        return n * factorial(n - 1);   // recursive case
    }
}
```

- The recursive code handles a small part of the overall task (multiplying by *n*), then makes a recursive call to handle the rest.
  - The recursive version is written <u>without using any loops</u>.
    - Recursion *replaces* the loop
  - We separate the code into a *base case* (a simple case that does not make any recursive calls), and a *recursive case*.

Recursive stack trace

```
int factorial(int n) { // 4
    if (n <= 1) {                      // base case
        return 1;
    } else {
        return n * factorial(n - 1);   // recursive case
    }
}
```

```
int factorial(int n) { // 3
    if (n <= 1) {                      // base case
        return 1;
    } else {
        return n * factorial(n - 1);   // recursive case
    }
}
```

```
int factorial(int n) { // 2
    if (n <= 1) {                      // base case
        return 1;
    } else {
        return n * factorial(n - 1);   // recursive case
    }
}
```

```
int factorial(int n) { // 1
    if (n <= 1) {                      // base case
        return 1;
    } else {
        return n * factorial(n - 1);   // recursive case
    }
}
```

Recursion and cases

- Every recursive algorithm involves at least 2 cases:
  - **base case**: A simple occurrence that can be answered directly
  - **recursive case**: A more complex occurrence of the problem that cannot be directly answered, but can instead be described in terms of smaller occurrences of the same problem
  - *Key idea:* In a recursive piece of code, you handle a small part of the overall task yourself (usually the work involves modifying the results of the smaller problems), then make a recursive call to handle the rest.
  - Ask yourself, "How is this task **self-similar**?"
    - "How can I describe this algorithm in terms of a smaller or simpler version of itself?"

Three Rules of Recursion

- Every (valid) input must have a case (either recursive or base)
- There **must** be a base case that makes no recursive calls
- The recursive case must make the problem simpler and make forward progress to the base case

Recursive tracing

- Consider the following recursive function:

```
int mystery(int n) {
    if (n < 10) {
        return n;
    } else {
        int a = n / 10;
        int b = n % 10;
        return mystery(a + b);
    }
}
```

Q: What is the result of: `mystery(648)` ?

A. 8 **B.** 9 **C.** 54 **D.** 72 **E.** 648

Recursive stack trace

```
int mystery(int n) {                    // n = 648
}
```

```
int mystery(int n) {                    // n = 72
}
```

```
int mystery(int n) {                    // n = 9
    if (n < 10) {
        return n;                       // return 9
    } else {
        int a = n / 10;
        int b = n % 10;
        return mystery(a + b);
    }
}
```

isPalindrome exercise

- Write a recursive function `isPalindrome` that accepts a `string` and returns `true` if it reads the same forwards as backwards.

```
isPalindrome("madam")                        → true
isPalindrome("racecar")                      → true
isPalindrome("step on no pets")              → true
isPalindrome("able was I ere I saw elba")    → true
isPalindrome("Q")                            → true
isPalindrome("Java")                         → false
isPalindrome("rotater")                      → false
isPalindrome("byebye")                       → false
isPalindrome("notion")                       → false
```

  – What is a good **base case**?

isPalindrome

- What is our stopping point (*base case*)?
- How is this problem *self-similar*?
- What is the minimum *amount of work*?
- How can we make the problem *simpler* by doing the least amount of work?

isPalindrome

- What is our stopping point (*base case*)?
  - Empty string or string of length 1
- How is this problem *self-similar*?
  - Palindromes can be written as: x[SMALLER_PALINDROME]x, where x stands for some letter
- What is the minimum *amount of work*?
  - Testing the equality of outside characters
- How can we make the problem *simpler* by doing the least amount of work?
  - Peel off the outside characters and test if the middle is a palindrome

isPalindrome solution

```
// Returns true if the given string reads the same
// forwards as backwards.
// Trivially true for empty or 1-letter strings.
bool isPalindrome(string s) {
    if (s.length() < 2) {   // base case
        return true;
    } else {                // recursive case
        if (s[0] != s[s.length() - 1]) {
            return false;
        }
        string middle = s.substr(1, s.length() - 2);
        return isPalindrome(middle);
    }
}
```

Announcements

- Homework 2 due on Wednesday at **5PM**
- Homework 1 grades will be released by your section leader on or before Wednesday
- Alternate midterms are being scheduled this week. Keep an eye out for an email from Kate

Multiple calls tracing

```
int mystery(int n) {
    if (n < 10) {
        return (10 * n) + n;
    } else {
        int a = mystery(n / 10);
        int b = mystery(n % 10);
        return (100 * a) + b;
    }
}
```

Q: What is the result of: `mystery(348) ?`

A. 3828 **B.** 348348 **C.** 334488 **D.** 80403 **E.** none

Multiple calls tracing

```
// call 1:    348
int mystery(int n) {
  if (n < 10) {
    return (10 * n) + n;
  } else {
    int a = mystery(n / 10);
    int b = mystery(n % 10);
    return (100 * a) + b;
```

```
// call 2a:    34
int mystery(int n) {
  if (n < 10) {
    return (10 * n) + n;
  } else {
    int a = mystery(n / 10);
    int b = mystery(n % 10);
    return (100 * a) + b;
```

```
// call 2b:     8
int mystery(int n) {
  if (n < 10) {
    return (10 * n) + n;
  } else {
    int a = mystery(n / 10);
    int b = mystery(n % 10);
    return (100 * a) + b;
```

```
// call 3a:     3
int mystery(int n) {
  if (n < 10) {
    return (10 * n) + n;
  } else {
    int a = mystery(n / 10);
    int b = mystery(n % 10);
    return (100 * a) + b;
  }
}
```

```
// call 3b:     4
int mystery(int n) {
  if (n < 10) {
    return (10 * n) + n;
  } else {
    int a = mystery(n / 10);
    int b = mystery(n % 10);
    return (100 * a) + b;
  }
}
```

Recursive Big O

- Below is the "pseudocode" for finding Big O of a function
  - Note that this is not real code; this is to show the recursive nature of finding Big O
  - Self-similarity: find Big O of smaller code blocks and combine them
  - This Big O pseudocode doesn't cover function calls and some other cases (for pedagogical purposes)

```
findBigO(codeSnippet):
   if codeSnippet is a single statement:
       return O(1)
   if codeSnippet is loop:
       return number of times loop runs * findBigO(loop inside)
   for codeBlock in codeSnippet:
       return the sum of findBigO(codeBlock)
```

Finding Big O Recursively

```
findBigO(codeSnippet):
   if codeSnippet is a single statement:
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   if codeSnippet is loop:
       return number of times loop runs * findBigO(loop inside)
   for codeBlock in codeSnippet:
       return the sum of findBigO(codeBlock)

for (int i = 0; i < N * N; i += 3) {
    for (int j = 3; j <= 219; j++) {
        cout << "sum: " << i + j << endl;
    }
}

cout << "Have a nice Life!" << endl;
```

Finding Big O Recursively

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for (int i = 0; i < N * N; i += 3) {
    for (int j = 3; j <= 219; j++) {
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    }
}

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```

$\boxed{O(N^2)}$

```
for (int i = 0; i < N * N; i += 3) {
    for (int j = 3; j <= 219; j++) {
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    }
}

cout << "Have a nice Life!" << endl;
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O(1)

```
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    for (int j = 3; j <= 219; j++) {
        cout << "sum: " << i + j << endl;
    }
}

cout << "Have a nice Life!" << endl;
```

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```

```
for (int i = 0; i < N * N; i += 3) {
    for (int j = 3; j <= 219; j++) {
        cout << "sum: " << i + j << endl;
    }
}

cout << "Have a nice Life!" << endl;
```

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O(1)

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```
for (int i = 0; i < N * N; i += 3) {
    for (int j = 3; j <= 219; j++) {
        cout << "sum: " << i + j << endl;
    }
}

cout << "Have a nice Life!" << endl;
```

Finding Big O Recursively

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```

$O(N^2)$ $O(1)$

```
for (int i = 0; i < N * N; i += 3) {
    for (int j = 3; j <= 219; j++) {
        cout << "sum: " << i + j << endl;
    }
}

cout << "Have a nice Life!" << endl;
```

Finding Big O Recursively

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```

$\boxed{O(N^2)}$ +

```
for (int i = 0; i < N * N; i += 3) {
    for (int j = 3; j <= 219; j++) {
        cout << "sum: " << i + j << endl;
    }
}

cout << "Have a nice Life!" << endl;
```

Finding Big O Recursively

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```

```
for (int i = 0; i < N * N; i += 3) {
    for (int j = 3; j <= 219; j++) {
        cout << "sum: " << i + j << endl;
    }
}

cout << "Have a nice Life!" << endl;
```

Finding Big O Recursively

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   for codeBlock in codeSnippet:
       return the sum of findBigO(codeBlock)
```

$\boxed{O(N^2)} + \boxed{O(1)}$

```
for (int i = 0; i < N * N; i += 3) {
    for (int j = 3; j <= 219; j++) {
        cout << "sum: " << i + j << endl;
    }
}

cout << "Have a nice Life!" << endl;
```

Finding Big O Recursively

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findBigO(codeSnippet):
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   for codeBlock in codeSnippet:
       return the sum of findBigO(codeBlock)
```

final result: $O(N^2)$

```
for (int i = 0; i < N * N; i += 3) {
    for (int j = 3; j <= 219; j++) {
        cout << "sum: " << i + j << endl;
    }
}

cout << "Have a nice Life!" << endl;
```

power exercise

- Write a function **`power`** that accepts integer parameters for a base and exponent and computes base ^ exponent.

  - Write a recursive version of this function (one that calls itself).
  - Solve the problem without using any loops.

  - What is our stopping point (*base case*)?
  - How is this problem *self-similar*?
  - What is the minimum *amount of work*?
  - How can we make the problem *simpler* by doing the least amount of work?

Initial solution

```
// Returns base ^ exp.
// Assumes exp >= 1.
int power(int base, int exp) {
    if (exp == 1) {
        return base;
    } else {
        return base * power(base, exp - 1);
    }
}
```

The call stack

- Each previous call waits for the next call to finish.
  - `cout << power(5, 3) << endl;`

```
// first call:   5         3
int power(int base, int exp) {
    if (exp == 1) {

    }

    }
}
```

```
// second call:  5         2
int power(int base, int exp) {
    if (exp == 1) {

    }

    }
}
```

```
// third call:   5         1
int power(int base, int exp) {
    if (exp == 1) {
        return base;   // 5
    } else {
        return base * power(base, exp - 1);
    }
}
```

"Recursion Zen"

- The real, even simpler, base case is an exp of 0, not 1:

```
int power(int base, int exp) {
    if (exp == 0) {
        // base case; base^0 = 1
        return 1;
    } else {
        // recursive case:  x^y = x * x^(y-1)
        return base * power(base, exp - 1);
    }
}
```

  – **Recursion Zen**: The art of properly identifying the best set of cases for a recursive algorithm and expressing them elegantly.
    Opposite is **arms-length recursion**
    (our informal term)

Preconditions

- **precondition**: Something your code *assumes is true* when called.
  - Often documented as a comment on the function's header:

```
// Returns base ^ exp.
// Precondition: exp >= 0
int power(int base, int exp) {
```

  - Stating a precondition doesn't really "solve" the problem, but it at least documents our decision and warns the client what not to do.
  - What if the caller doesn't listen and passes a negative power anyway? What if we want to actually *enforce* the precondition?

Throwing exceptions

```
error(expression);
```

- In Stanford C++ lib's `"error.h"`
- Generates an exception that will crash the program, unless it has code to handle ("catch") the exception.
- alternative: `throw` ***`something`***
  - ***`something`*** can be an int, a string, etc.

- Why would anyone ever *want* a program to crash?

power solution 2

```
// Returns base ^ exp.
// Precondition: exp >= 0
int power(int base, int exp) {
    if (exp < 0) {
        throw "illegal negative exponent";
    } else ...
        ...
}
```