

Why We Use Recursion

If you haven't played Towers of Hanoi yet (or even if you have), do that now!

mathsisfun.com/games/towerofhanoi.html



Roadmap

C++ basics

User/client

vectors + grids

stacks + queues

sets + maps

Core
Tools

testing

algorithmic
analysis

recursive
problem-solving

Object-Oriented
Programming

Implementation

arrays

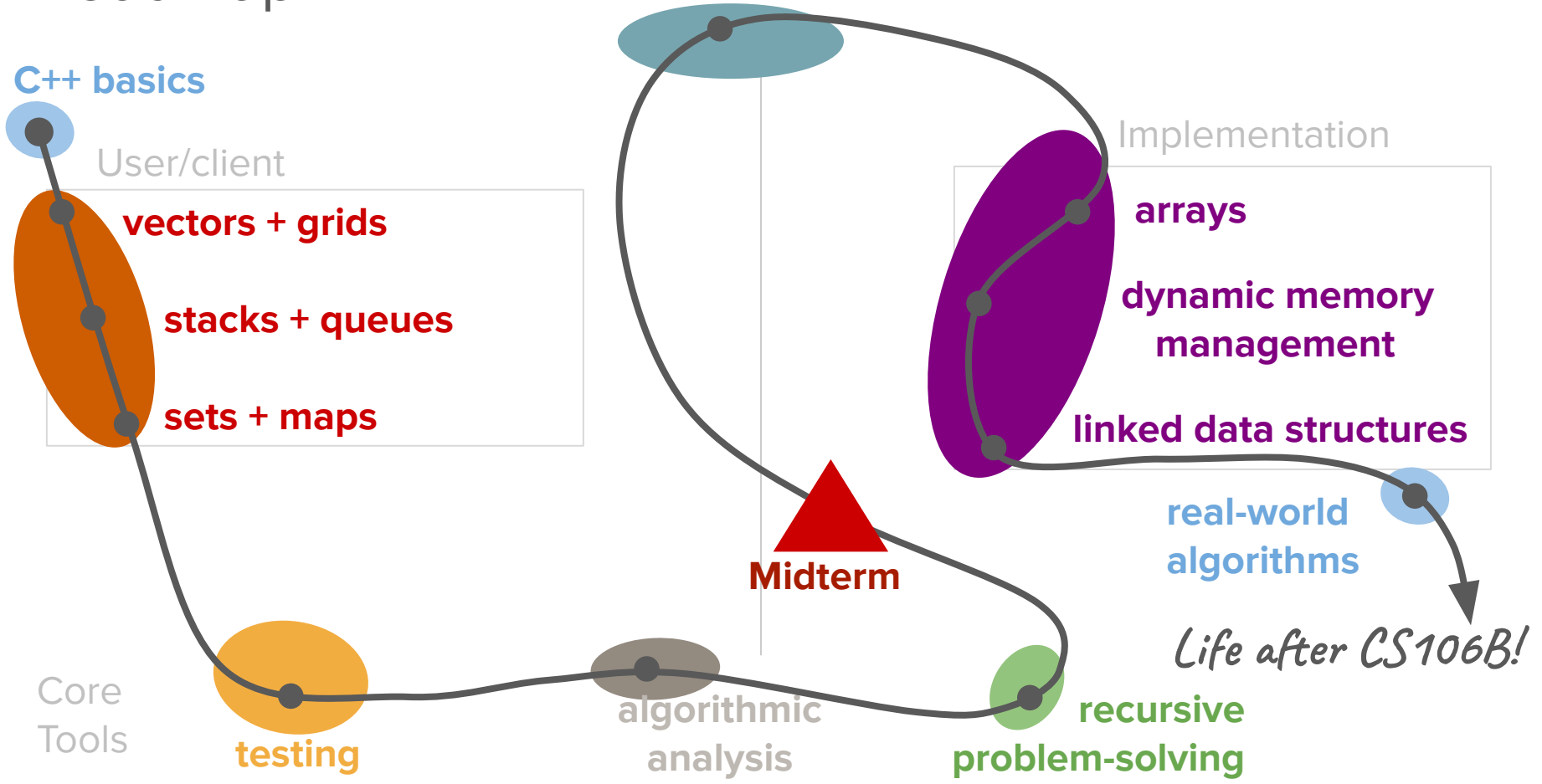
dynamic memory
management

linked data structures

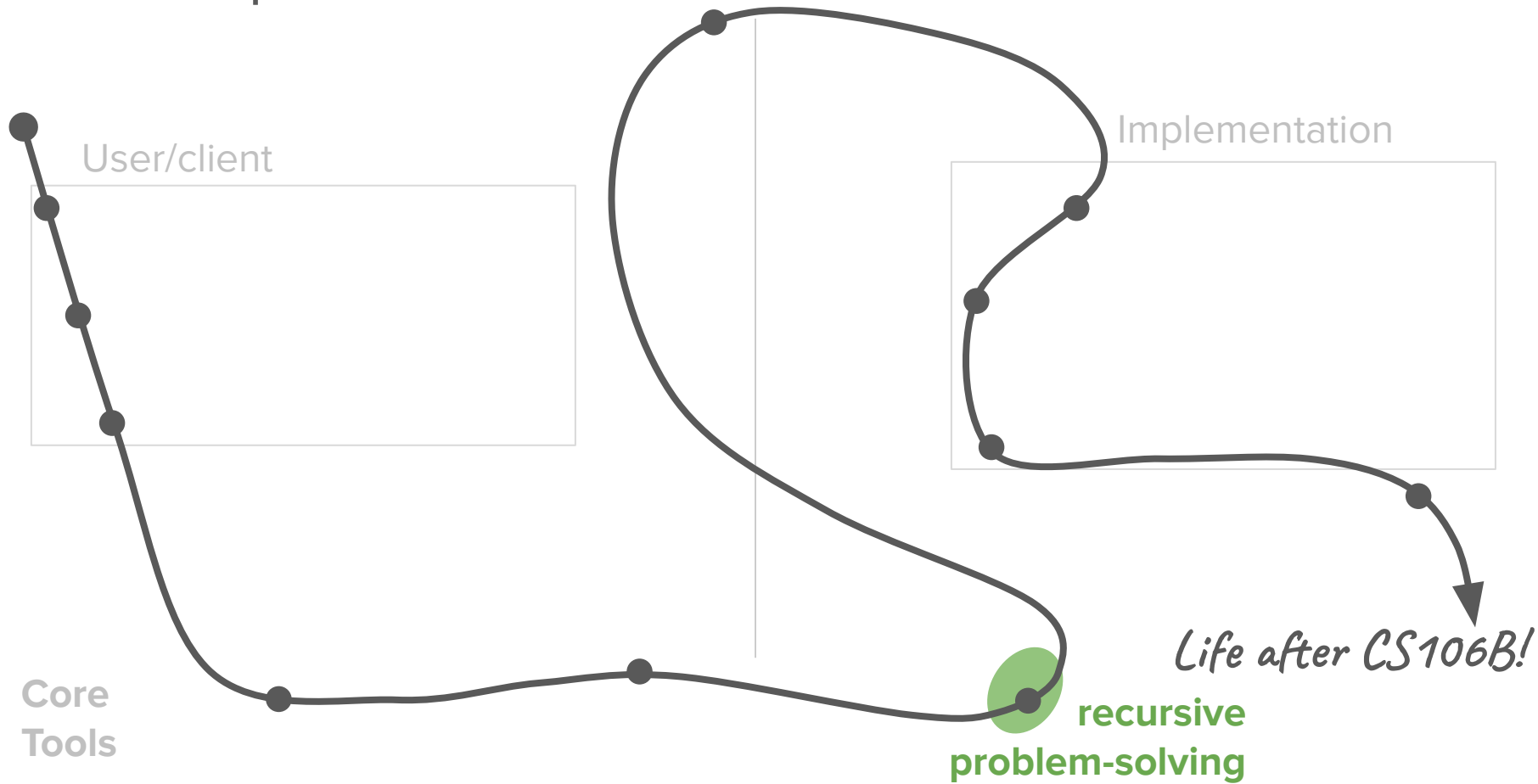
real-world
algorithms

Life after CS106B!

Midterm



Roadmap



Today's question

Why is recursion such a powerful problem-solving tool?

Today's topics

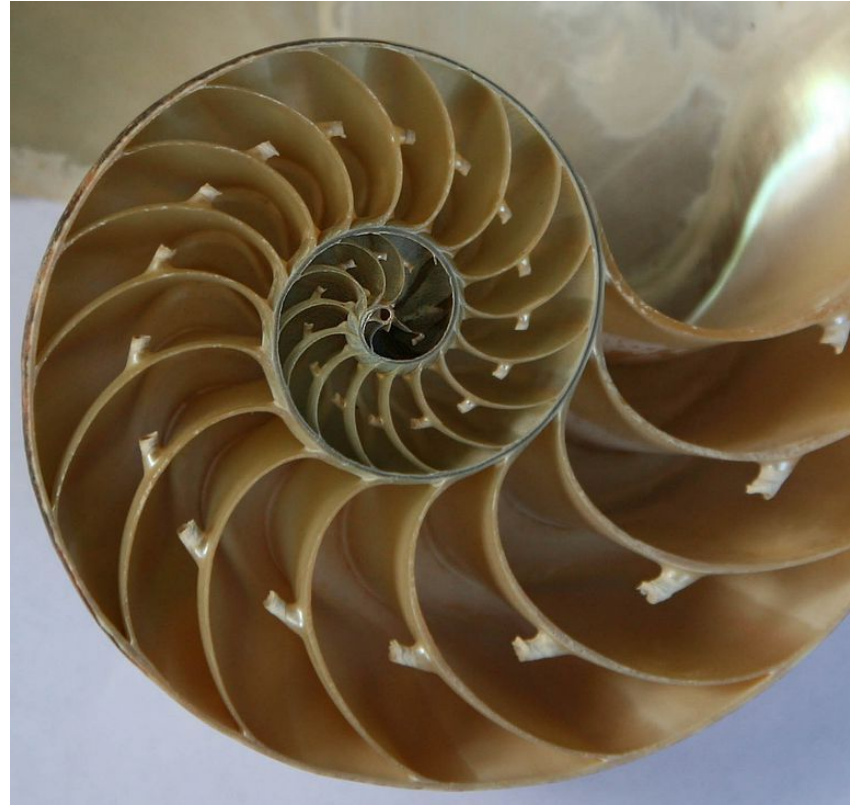
1. Review
2. Elegance
3. Efficiency
(the return of Big O)
4. Recursive Backtracking

Review

(fractals)

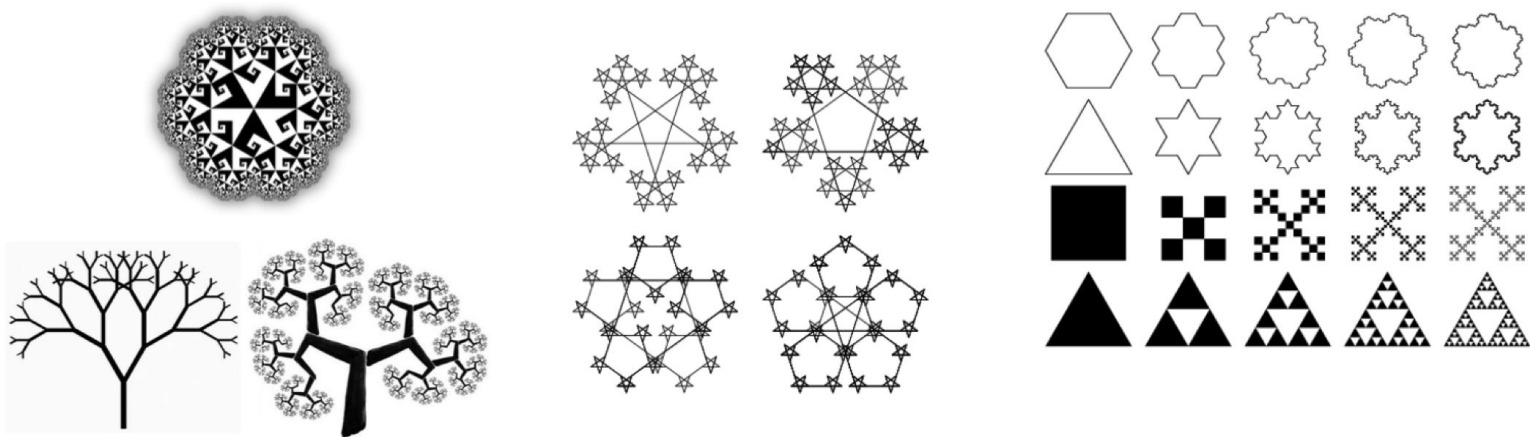
Self-Similarity

- Solving problems recursively and analyzing recursive phenomena involves identifying **self-similarity**
- An object is **self-similar** if it contains a smaller copy of itself.



Fractals

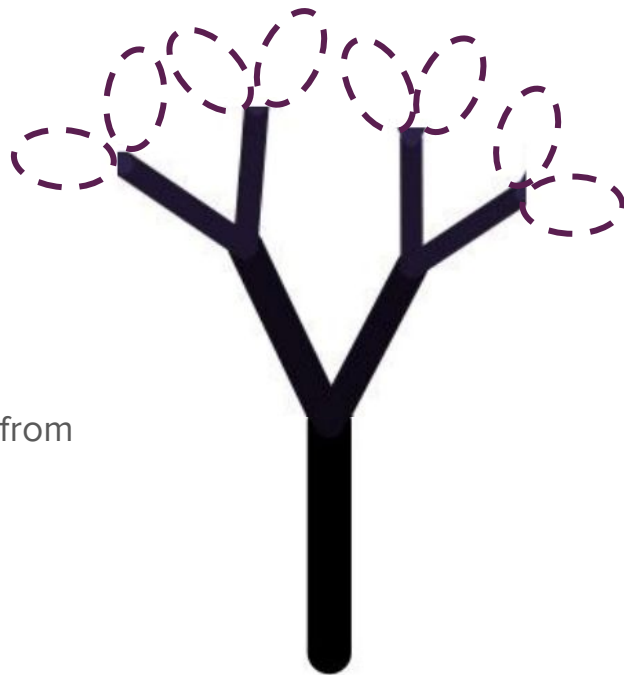
- A **fractal** is any repeated, graphical pattern.
- A fractal is composed of **repeated instances of the same shape or pattern**, arranged in a structured way.



An order-3 tree

An order-0 tree is nothing at all.

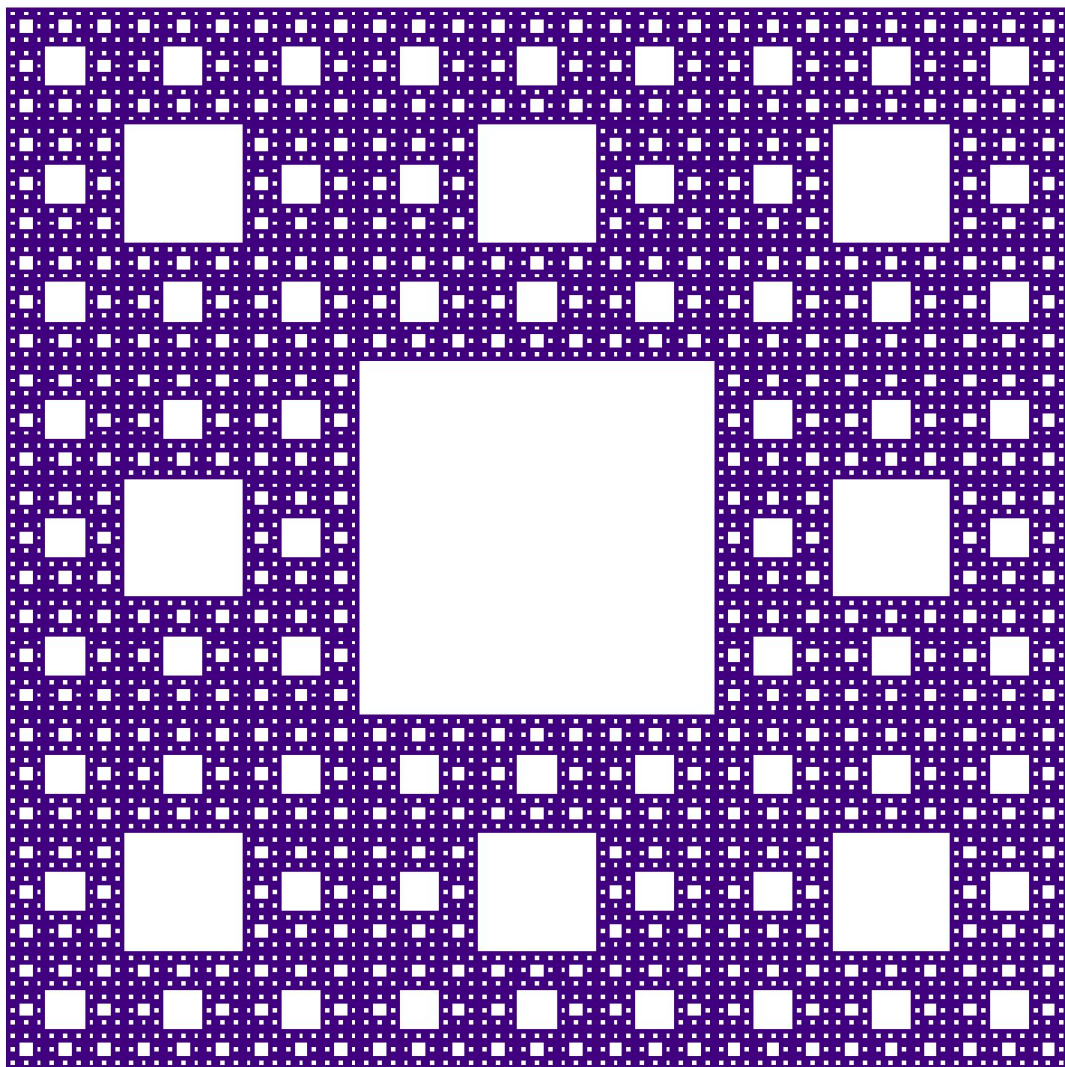
An order- n tree is a line with two smaller order- $(n-1)$ trees starting at the end of that line.



What differentiates the smaller tree from the bigger one?

1. It's at a different **position**.
2. It has a different **size**.
3. It has a different **orientation**.
4. It has a different **order**.

Fractals and self-similar structures are often defined in terms of some parameter called the **order**, which indicates the complexity of the overall structure.



Sierpinski Carpet Pseudocode (Take 2)

```
drawSierpinskiCarpet (x, y, order):  
    if (order == 0)  
        drawFilledSquare(x, y, BASE_SIZE)  
    else  
        for row = 0 to row = 2:  
            for col = 0 to col = 2:  
                if (col != 1 || row != 1):  
                    x_i = newX(x, y, row, col)  
                    y_i = newY(x, y, row, col)  
                    drawSierpinskiCarpet(x_i, y_i, order - 1)
```



Sierpinski Carpet

```
void drawSierpinskiCarpet(GWindow& window, double x, double y, double size, int order) {  
    // Base case: A carpet of order 0 is a filled square.  
    if (order == 0) {  
        drawSquare(window, x, y, size);  
    } else {  
        for (int row = 0; row < 3; row++) {  
            for (int col = 0; col < 3; col++) {  
                // The only square to skip is the very center one.  
                if (row != 1 || col != 1) {  
                    double newX = x + col * size / 3;  
                    double newY = y + row * size / 3;  
                    drawSierpinskiCarpet(window, newX, newY, size / 3, order - 1);  
                }  
            }  
        }  
    }  
}
```

Iteration + Recursion

- It's completely reasonable to mix iteration and recursion in the same function.
- Here, we're firing off eight recursive calls, and the easiest way to do that is with a double for loop.
- Recursion doesn't mean "the absence of iteration." It just means "solving a problem by solving smaller copies of that same problem."
- Iteration and recursion can be very powerful in combination!

Homework from yesterday

- Play Towers of Hanoi:

<https://www.mathsisfun.com/games/towerofhanoi.html>

- Look for and write down patterns in how to solve the problem as you increase the number of disks. Try to get to at least 5 disks!
- **Extra challenge** (optional): How would you define this problem recursively?
 - Don't worry about data structures here. Assume we have a function `moveDisk(X, Y)` that will handle moving a disk from the top of post `X` to the top of post `Y`.

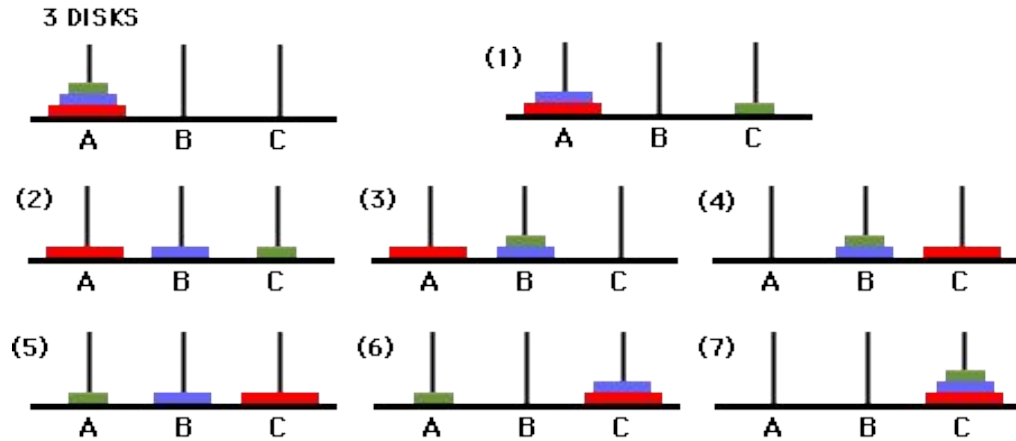
Why do we use recursion?

Why do we use recursion?

- Elegance
 - Allows us to solve problems with very clean and concise code
- Efficiency
 - Allows us to accomplish better runtimes when solving problems
- Dynamic
 - Allows us to solve problems that are hard to solve iteratively

An **elegant** example:
Towers of Hanoi

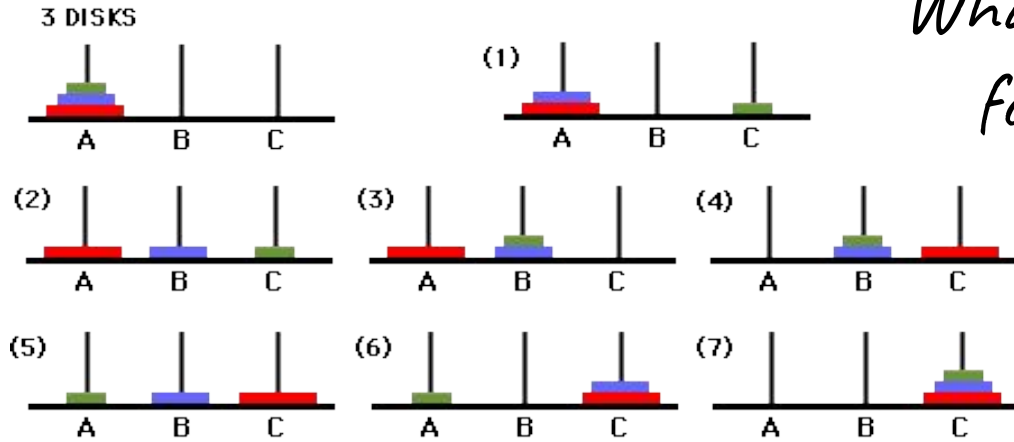
Pseudocode for 3 disks



- (1) Move disk 1 to destination
- (2) Move disk 2 to auxiliary
- (3) Move disk 1 to auxiliary
- (4) Move disk 3 to destination

- (5) Move disk 1 to source
- (6) Move disk 2 to destination
- (7) Move disk 1 to destination

Pseudocode for 3 disks



What if we add a fourth disk?

- (1) Move disk 1 to destination
- (2) Move disk 2 to auxiliary
- (3) Move disk 1 to auxiliary
- (4) Move disk 3 to destination

- (5) Move disk 1 to source
- (6) Move disk 2 to destination
- (7) Move disk 1 to destination

Towers of Hanoi with 4 disks



source



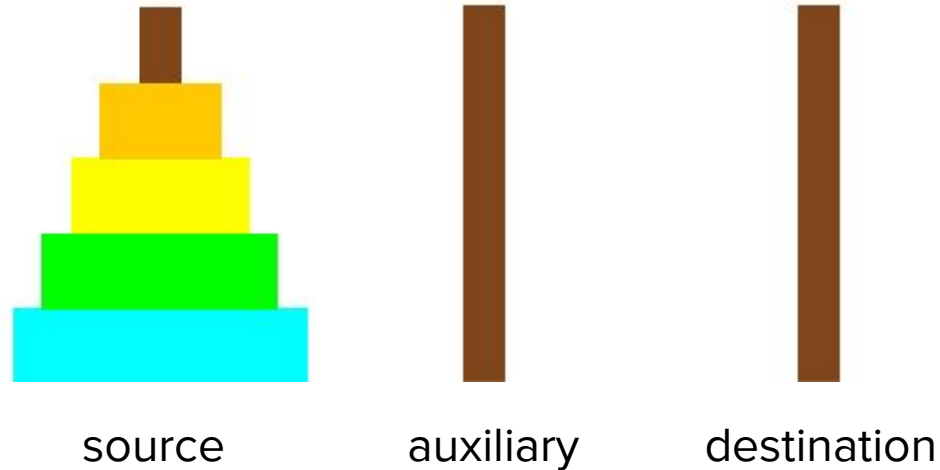
auxiliary



destination

Towers of Hanoi with 4 disks

- We want to first move the biggest disk over to the destination peg.



Towers of Hanoi with 4 disks

- We want to first move the biggest disk over to the destination peg.
 - We need to get the top three disks out of the way.



source



auxiliary



destination

Towers of Hanoi with 4 disks

- We want to first move the biggest disk over to the destination peg.
 - We need to get the top three disks out of the way.
 - We already have an algorithm for moving three disks from a source peg to a destination peg!



source



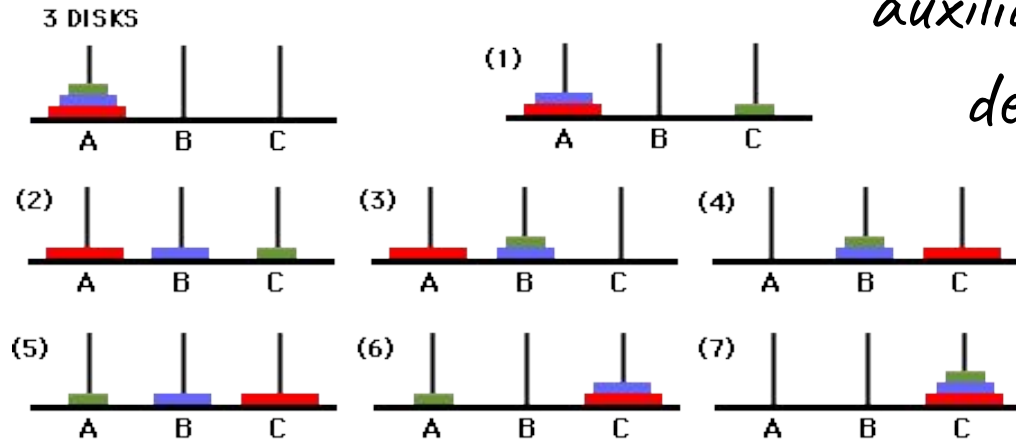
auxiliary



destination

Pseudocode for 3 disks

Idea: Move disks to auxiliary instead of destination!

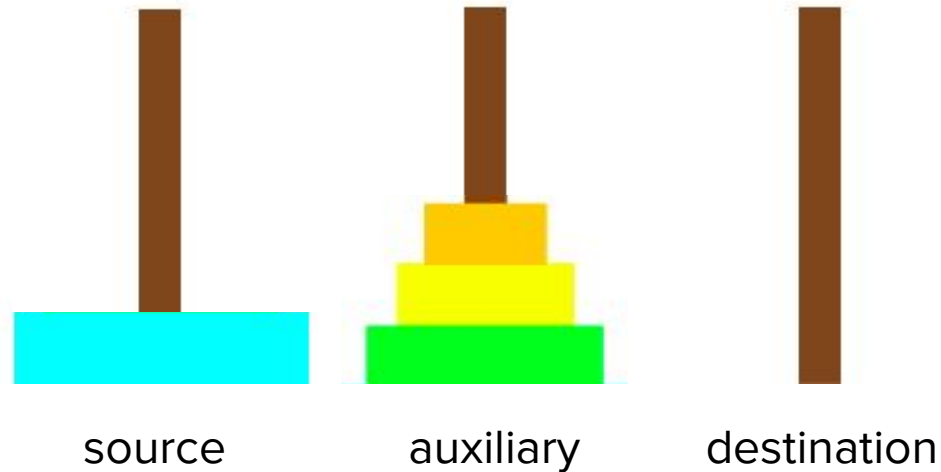


- (1) Move disk 1 to destination
- (2) Move disk 2 to auxiliary
- (3) Move disk 1 to auxiliary
- (4) Move disk 3 to destination

- (5) Move disk 1 to source
- (6) Move disk 2 to destination
- (7) Move disk 1 to destination

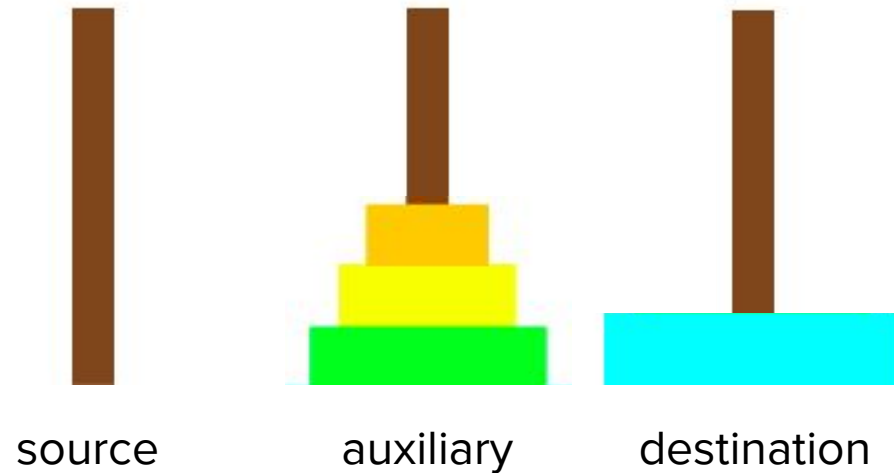
Towers of Hanoi with 4 disks

- We want to first move the biggest disk over to the destination peg.



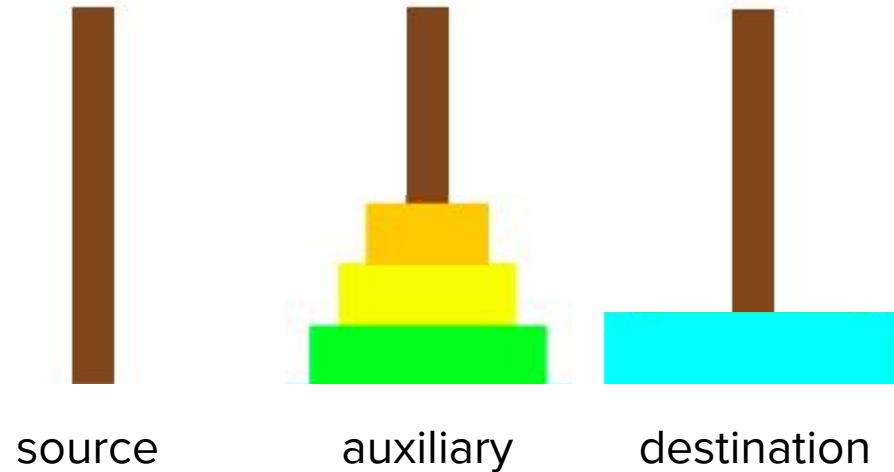
Towers of Hanoi with 4 disks

- We want to first move the biggest disk over to the destination peg.



Towers of Hanoi with 4 disks

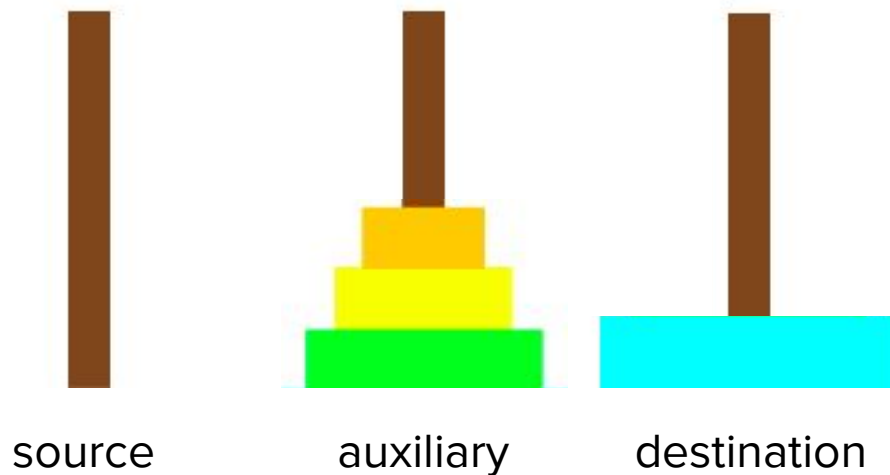
- We want to first move the biggest disk over to the destination peg.
- Now we need to move the stack of three from auxiliary to destination.



Towers of Hanoi with 4 disks

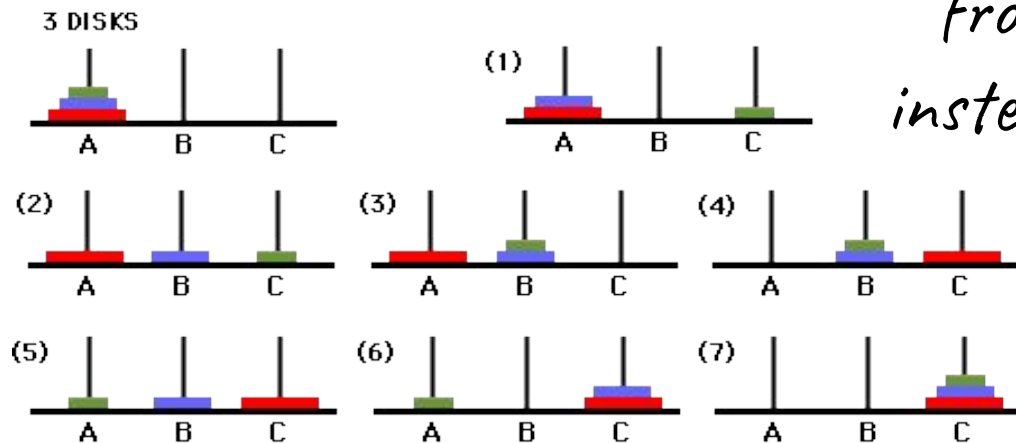
- We want to first move the biggest disk over to the destination peg.
- Now we need to move the stack of three from auxiliary to destination.

*Use our
existing 3-disk
algorithm!*



Pseudocode for 3 disks

*Idea: Move disks
from auxiliary
instead of source!*



- (1) Move disk 1 to destination
- (2) Move disk 2 to auxiliary
- (3) Move disk 1 to auxiliary
- (4) Move disk 3 to destination

- (5) Move disk 1 to source
- (6) Move disk 2 to destination
- (7) Move disk 1 to destination

Discuss: How could we solve Towers of Hanoi recursively?

- Assume we have a function called `moveSingleDisk(start, end)` that moves a single disk from a **start** peg to a **end** peg.
- Our function prototype that we're trying to fill in looks like this:
`findSolution(n, source, dest, aux)`

Where **n** is the number of disks we have starting on the **source** peg. The goal is to have all disks end up on the **dest** peg, and you can use the **aux** peg to help.

- What are our base and recursive cases for this problem?

Towers of Hanoi solution

[live coding]

An **efficient** example: Binary Search

Finding a number in a sorted list

-1	2	5	18	37	59	77	82	89	101
0	1	2	3	4	5	6	7	8	9

Finding a number in a sorted list

-1	2	5	18	37	59	77	82	89	101
0	1	2	3	4	5	6	7	8	9

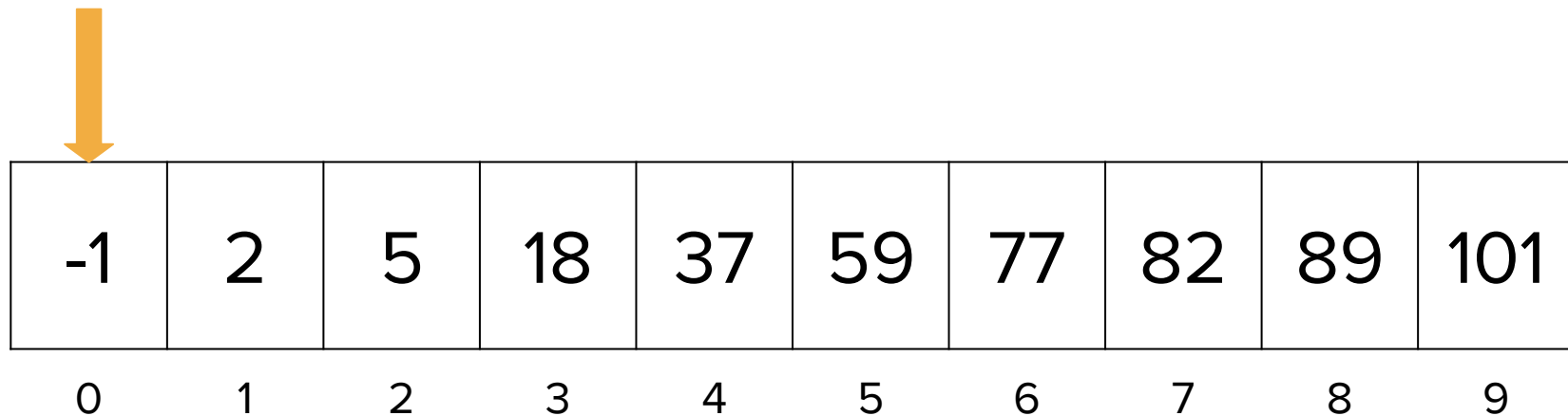
Where is 89?

Finding a number in a sorted list

-1	2	5	18	37	59	77	82	89	101
0	1	2	3	4	5	6	7	8	9

Idea #1: We could just go through each element in order and do a linear search.

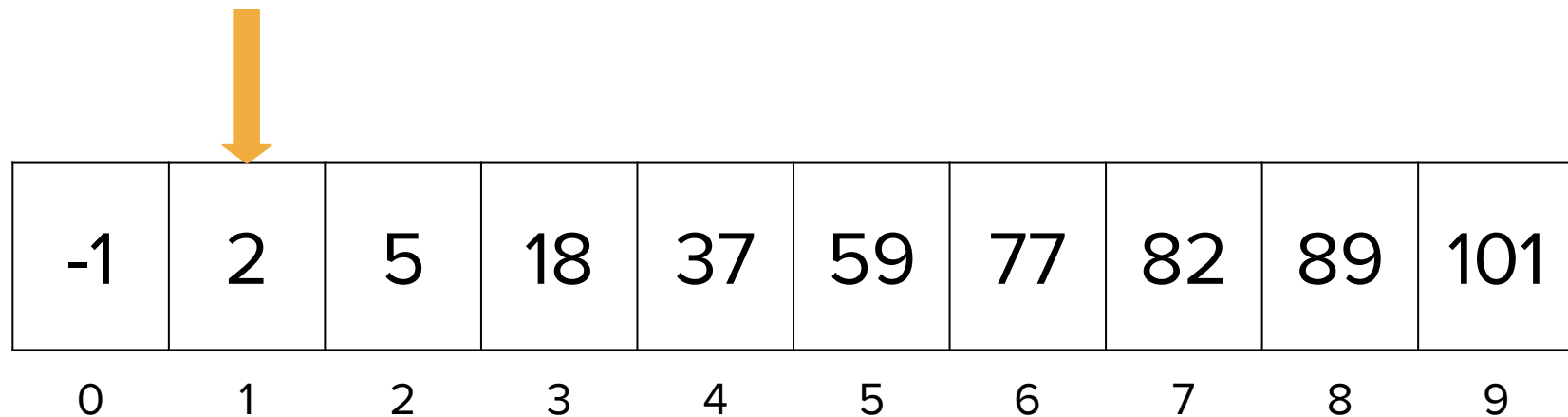
Finding a number in a sorted list



-1	2	5	18	37	59	77	82	89	101
0	1	2	3	4	5	6	7	8	9

*We could just go through each element in order
and do a linear search.*

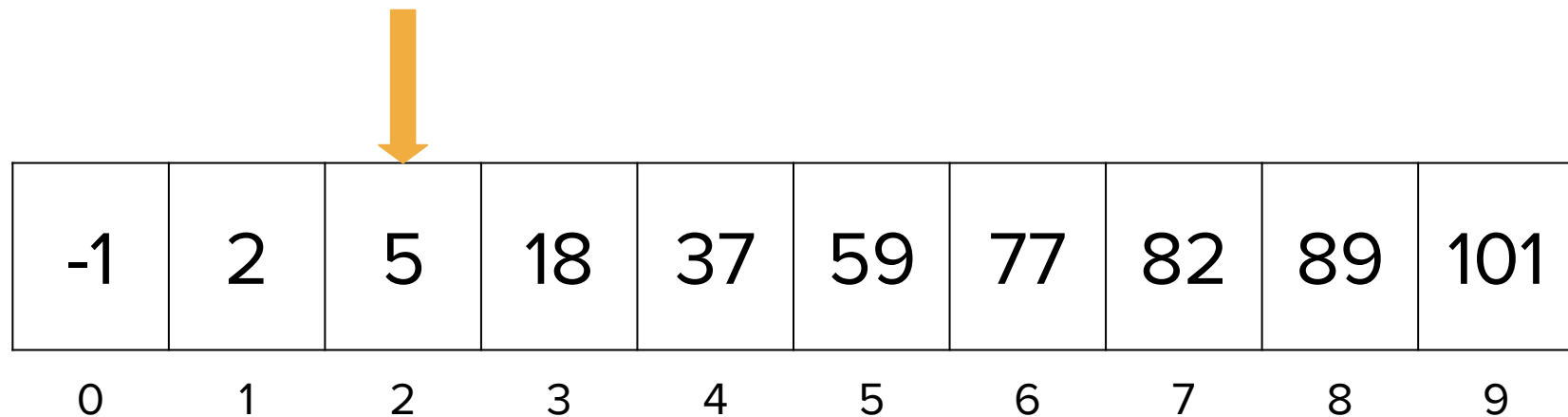
Finding a number in a sorted list



-1	2	5	18	37	59	77	82	89	101
0	1	2	3	4	5	6	7	8	9

*We could just go through each element in order
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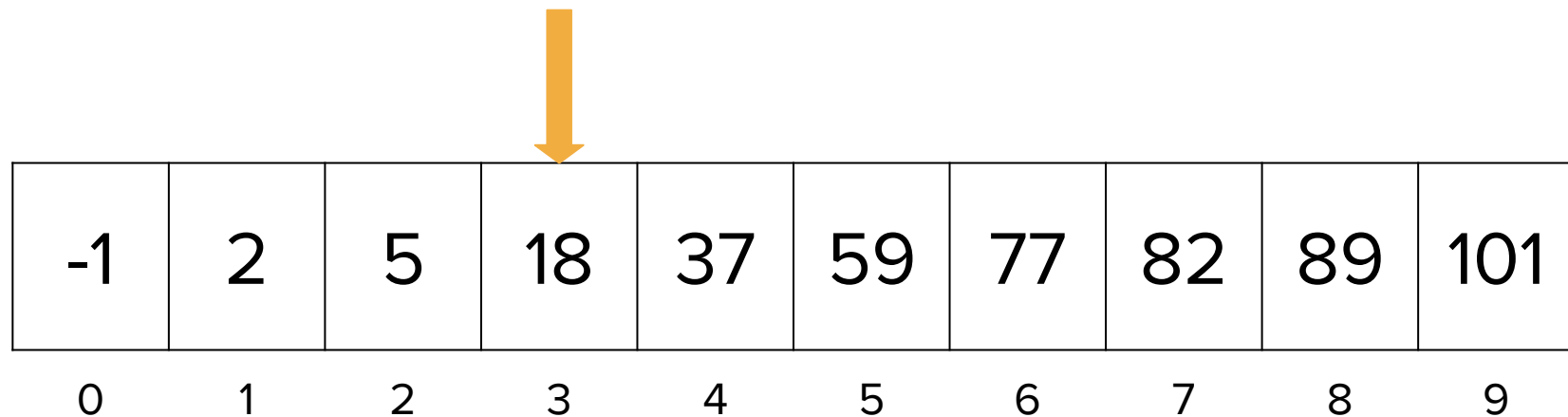
Finding a number in a sorted list



-1	2	5	18	37	59	77	82	89	101
0	1	2	3	4	5	6	7	8	9

*We could just go through each element in order
and do a linear search.*

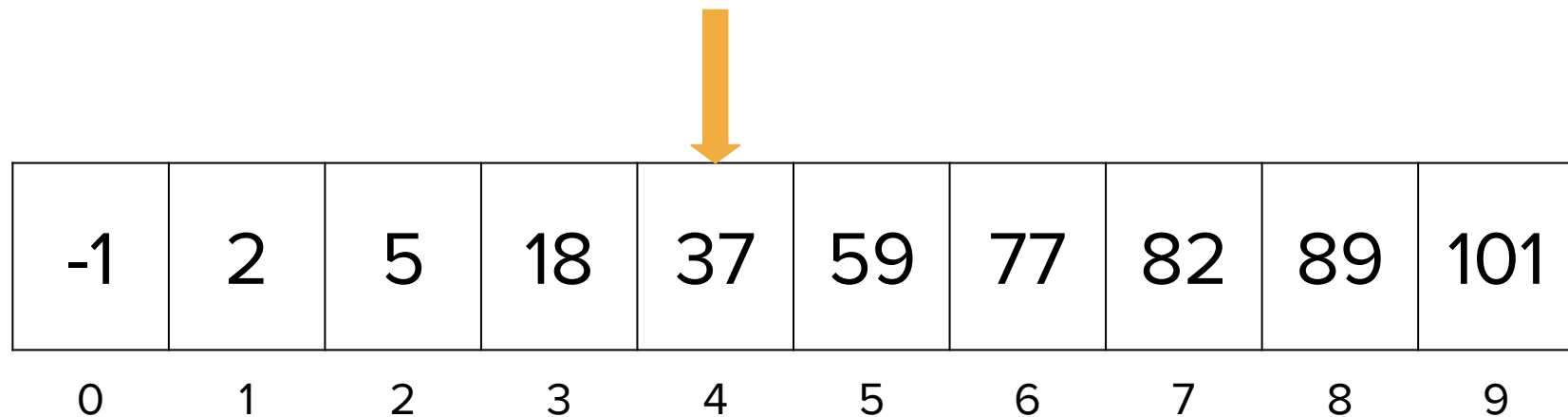
Finding a number in a sorted list



-1	2	5	18	37	59	77	82	89	101
0	1	2	3	4	5	6	7	8	9

*We could just go through each element in order
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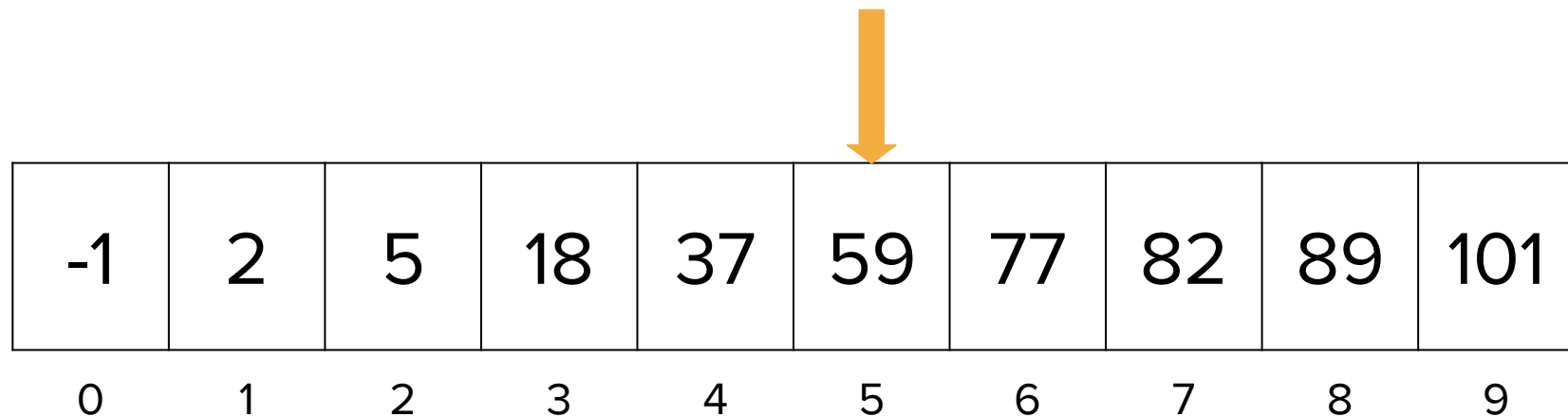
Finding a number in a sorted list



-1	2	5	18	37	59	77	82	89	101
0	1	2	3	4	5	6	7	8	9

*We could just go through each element in order
and do a linear search.*

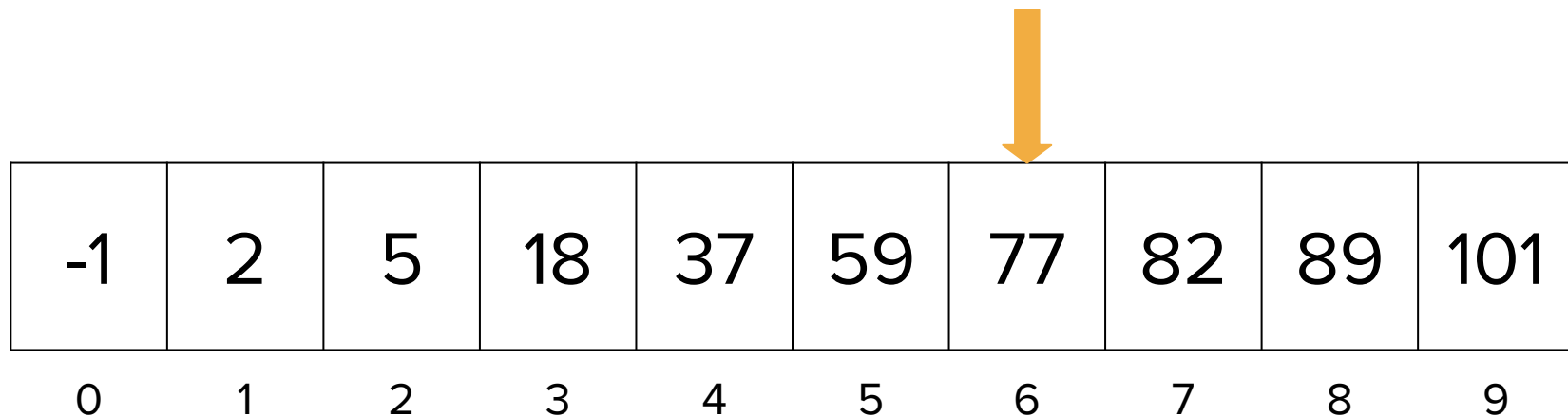
Finding a number in a sorted list



-1	2	5	18	37	59	77	82	89	101
0	1	2	3	4	5	6	7	8	9

*We could just go through each element in order
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Finding a number in a sorted list

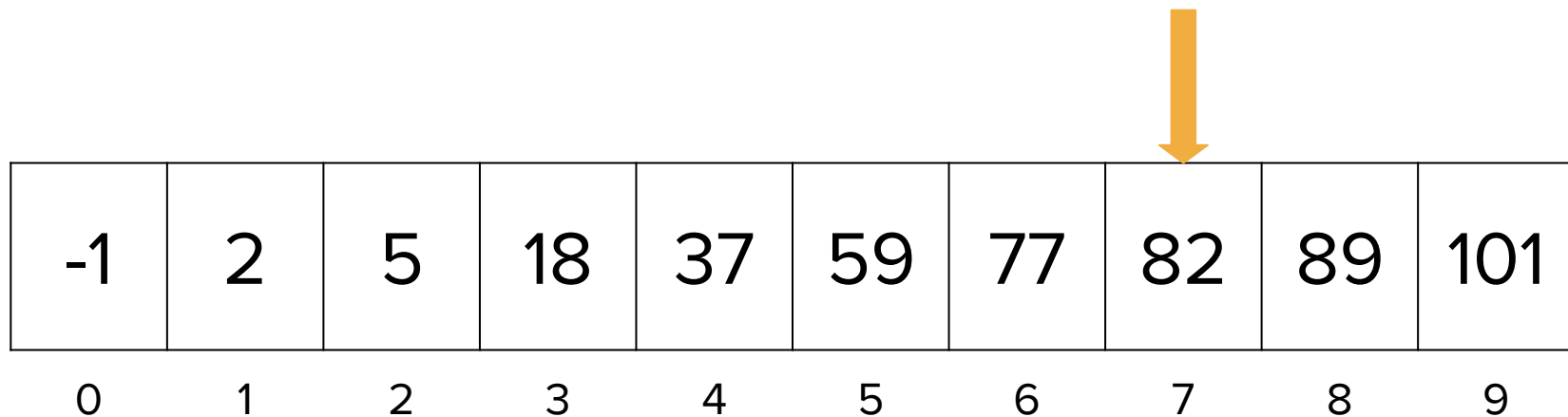


A horizontal array of 10 cells, each containing a number. The numbers are sorted in ascending order: -1, 2, 5, 18, 37, 59, 77, 82, 89, 101. Below each cell is its corresponding index from 0 to 9. An orange arrow points down to the cell containing the number 77 at index 6.

-1	2	5	18	37	59	77	82	89	101
0	1	2	3	4	5	6	7	8	9

*We could just go through each element in order
and do a linear search.*

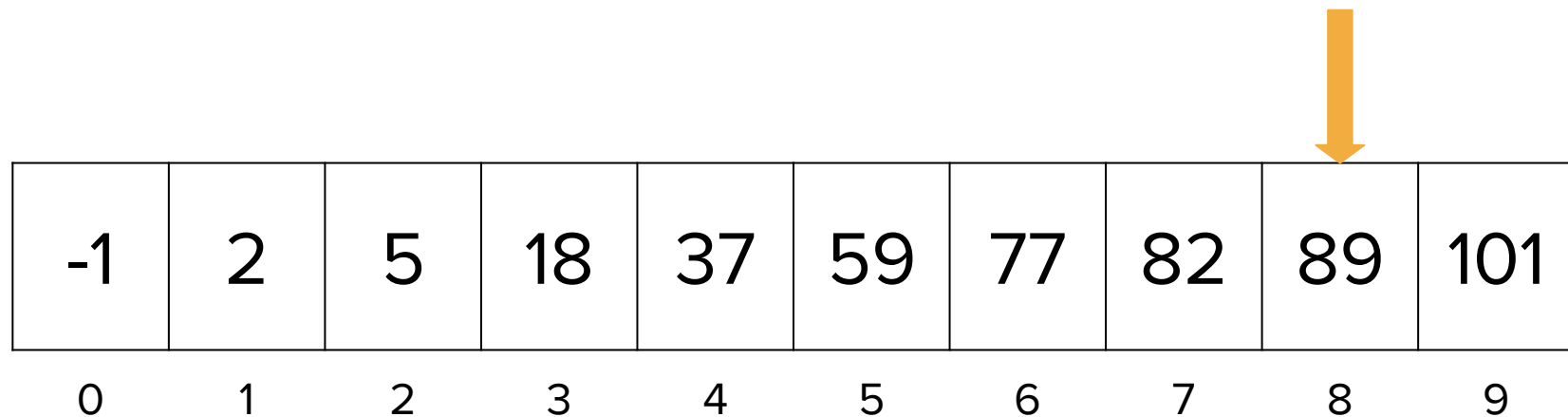
Finding a number in a sorted list



-1	2	5	18	37	59	77	82	89	101
0	1	2	3	4	5	6	7	8	9

*We could just go through each element in order
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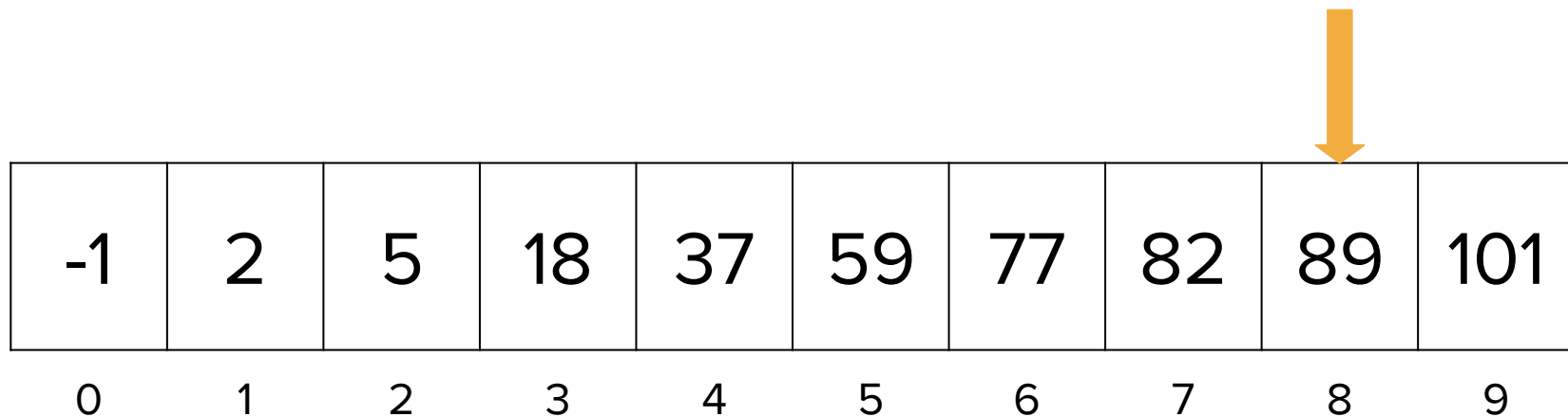
Finding a number in a sorted list



-1	2	5	18	37	59	77	82	89	101
0	1	2	3	4	5	6	7	8	9

*We could just go through each element in order
and do a linear search.*

Finding a number in a sorted list



-1	2	5	18	37	59	77	82	89	101
0	1	2	3	4	5	6	7	8	9

*Linear search is **$O(n)$***

Finding a number in a sorted list

-1	2	5	18	37	59	77	82	89	101
0	1	2	3	4	5	6	7	8	9

Can we do better? Can we take advantage of the structure of the data?

ADT Big-O Matrix

- Vectors

- `.size()` - $O(1)$
- `.add()` - $O(1)$
- `v[i]` - $O(1)$
- `.insert()` - $O(n)$
- `.remove()` - $O(n)$
- `.clear()` - $O(n)$
- `traversal` - $O(n)$

- Grids

- `.numRows()` / `.numCols()`
- $O(1)$
- `g[i][j]` - $O(1)$
- `.inBounds()` - $O(1)$
- `traversal` - $O(n^2)$

- Queues

- `.size()` - $O(1)$
- `.peek()` - $O(1)$
- `.enqueue()` - $O(1)$
- `.dequeue()` - $O(1)$
- `.isEmpty()` - $O(1)$
- `traversal` - $O(n)$

- Stacks

- `.size()` - $O(1)$
- `.peek()` - $O(1)$
- `.push()` - $O(1)$
- `.pop()` - $O(1)$
- `.isEmpty()` - $O(1)$
- `traversal` - $O(n)$

- Sets

- `.size()` - $O(1)$
- `.isEmpty()` - $O(1)$
- `.add()` - ???
- `.remove()` - ???
- `.contains()` - ???
- `traversal` - $O(n)$

- Maps

- `.size()` - $O(1)$
- `.isEmpty()` - $O(1)$
- `m[key]` - ???
- `.contains()` - ???
- `traversal` - $O(n)$

ADT Big-O Matrix

- Vectors

- `.size()` - $O(1)$
- `.add()` - $O(1)$
- `v[i]` - $O(1)$
- `.insert()` - $O(n)$
- `.remove()` - $O(n)$
- `.clear()` - $O(n)$
- `traversal` - $O(n)$

- Grids

- `.numRows()` / `.numCols()`
- $O(1)$
- `g[i][j]` - $O(1)$
- `.inBounds()` - $O(1)$
- `traversal` - $O(n^2)$

- Queues

- `.size()` - $O(1)$
- `.peek()` - $O(1)$
- `.enqueue()` - $O(1)$
- `.dequeue()` - $O(1)$
- `.isEmpty()` - $O(1)$
- `traversal` - $O(n)$

- Stacks

- `.size()` - $O(1)$
- `.peek()` - $O(1)$
- `.push()` - $O(1)$
- `.pop()` - $O(1)$
- `.isEmpty()` - $O(1)$
- `traversal` - $O(n)$

- Sets

- `.size()` - $O(1)$
- `.isEmpty()` - $O(1)$
- `.add()` - ???
- `.remove()` - ???
- `.contains()` - ???
- `traversal` - $O(n)$

- Maps

- `.size()` - $O(1)$
- `.isEmpty()` - $O(1)$
- `m[key]` - ???
- `.contains()` - ???
- `traversal` - $O(n)$

Note: Sets and Maps don't actually use a sorted list to store information, but the general idea of searching sorted data is similar.

Remember how their elements/keys always printed out in alphabetical order?



Note: Sets and Maps don't actually use a sorted list to store information, but the general idea of searching sorted data is similar.

Finding a number in a sorted list

-1	2	5	18	37	59	77	82	89	101
0	1	2	3	4	5	6	7	8	9

Where is 89?

Idea #2: Binary search

- Eliminate half of the data at each step.
- **Algorithm:** Check the middle element at $(\text{startIndex} + \text{endIndex}) / 2$
 - If the middle element is bigger than your desired value, eliminate the right half of the data and repeat.
 - If the middle element is smaller than your desired value, eliminate the left half of the data and repeat.
 - Otherwise, you've found your element!

Finding a number in a sorted list

-1	2	5	18	37	59	77	82	89	101
0	1	2	3	4	5	6	7	8	9

Where is 89?

Finding a number in a sorted list

-1	2	5	18	37	59	77	82	89	101
0	1	2	3	4	5	6	7	8	9

Start by looking at index:
 $(\text{startIndex} + \text{endIndex}) / 2$

Finding a number in a sorted list

-1	2	5	18	37	59	77	82	89	101
0	1	2	3	4	5	6	7	8	9

Start by looking at index:

$$(0 + 9) / 2$$

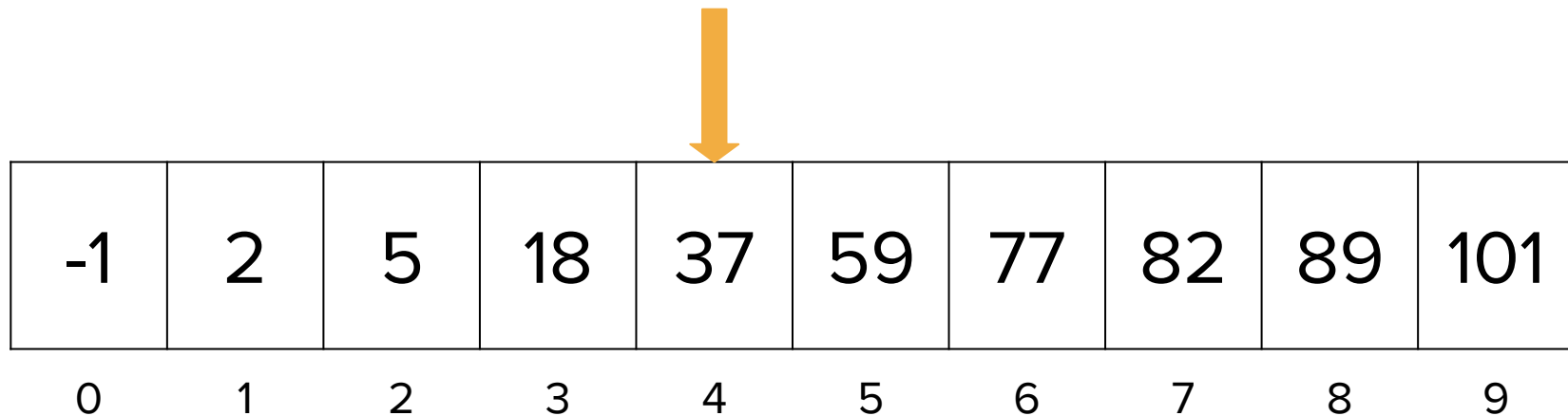
Finding a number in a sorted list

-1	2	5	18	37	59	77	82	89	101
0	1	2	3	4	5	6	7	8	9

Start by looking at index:

4

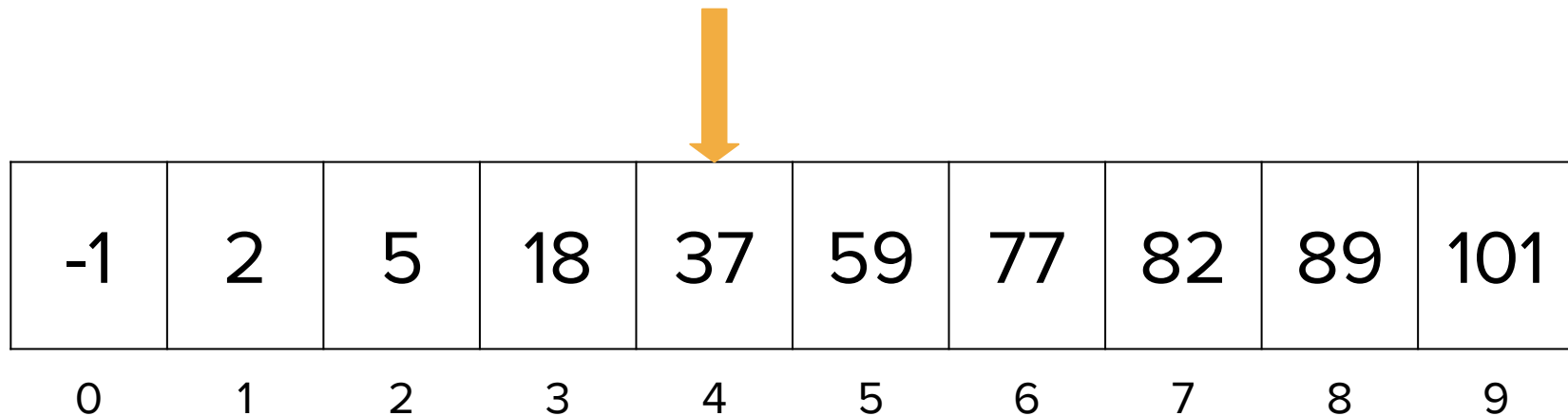
Finding a number in a sorted list



A diagram illustrating a sorted list of numbers. The list is represented as a horizontal row of ten cells, each containing a number. Below each cell is its corresponding index, ranging from 0 to 9. An orange arrow points down to the cell containing the number 37, which is at index 4.

-1	2	5	18	37	59	77	82	89	101
0	1	2	3	4	5	6	7	8	9

Finding a number in a sorted list



-1	2	5	18	37	59	77	82	89	101
0	1	2	3	4	5	6	7	8	9

Too small

Finding a number in a sorted list

-1	2	5	18	37	59	77	82	89	101
0	1	2	3	4	5	6	7	8	9

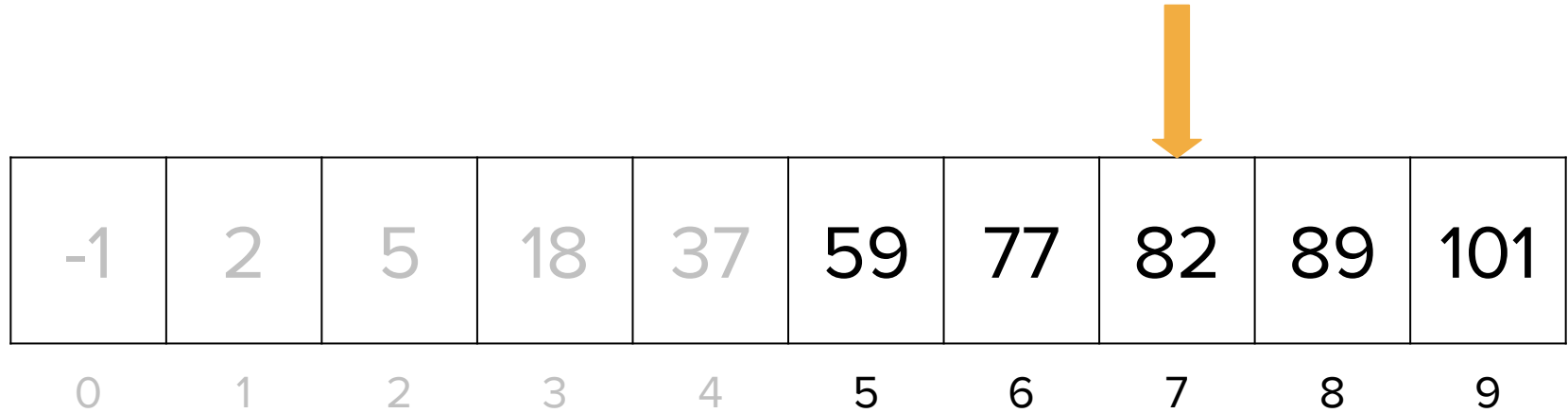
Eliminate left half

Finding a number in a sorted list

-1	2	5	18	37	59	77	82	89	101
0	1	2	3	4	5	6	7	8	9

$$\begin{aligned} &(\text{startIndex} + \text{endIndex}) / 2 = \\ & (5 + 9) / 2 = \\ & 7 \end{aligned}$$

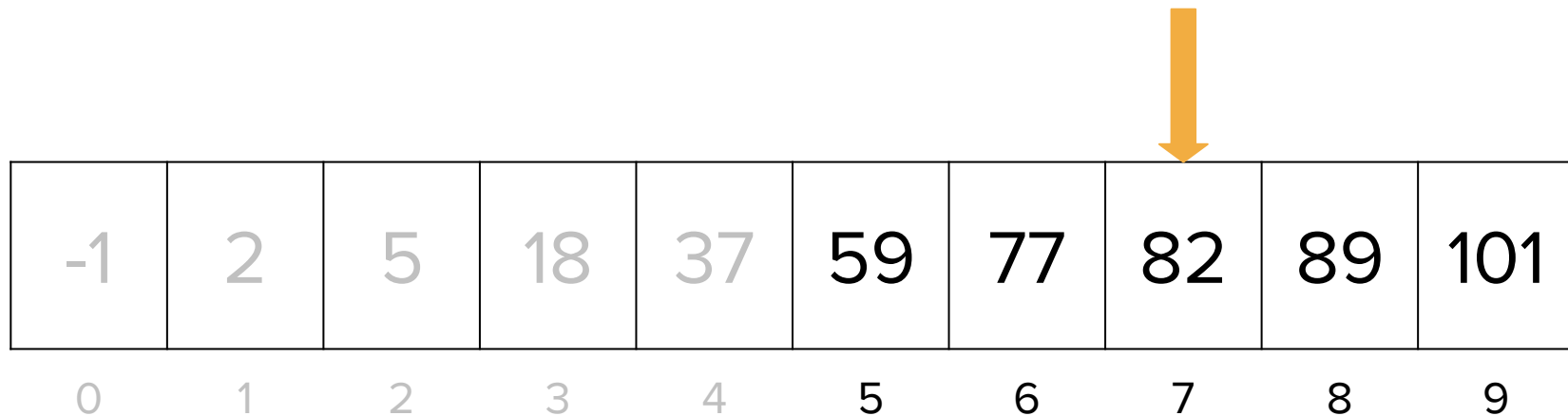
Finding a number in a sorted list



-1	2	5	18	37	59	77	82	89	101
0	1	2	3	4	5	6	7	8	9

$$\begin{aligned} &(\text{startIndex} + \text{endIndex}) / 2 = \\ & (5 + 9) / 2 = \\ & 7 \end{aligned}$$

Finding a number in a sorted list



-1	2	5	18	37	59	77	82	89	101
0	1	2	3	4	5	6	7	8	9

The image shows a horizontal array of ten cells, each containing a number. The numbers are sorted in ascending order: -1, 2, 5, 18, 37, 59, 77, 82, 89, 101. Below each cell is its corresponding index from 0 to 9. An orange arrow points down to the cell containing the number 82 at index 7.

Too small

Finding a number in a sorted list

-1	2	5	18	37	59	77	82	89	101
0	1	2	3	4	5	6	7	8	9

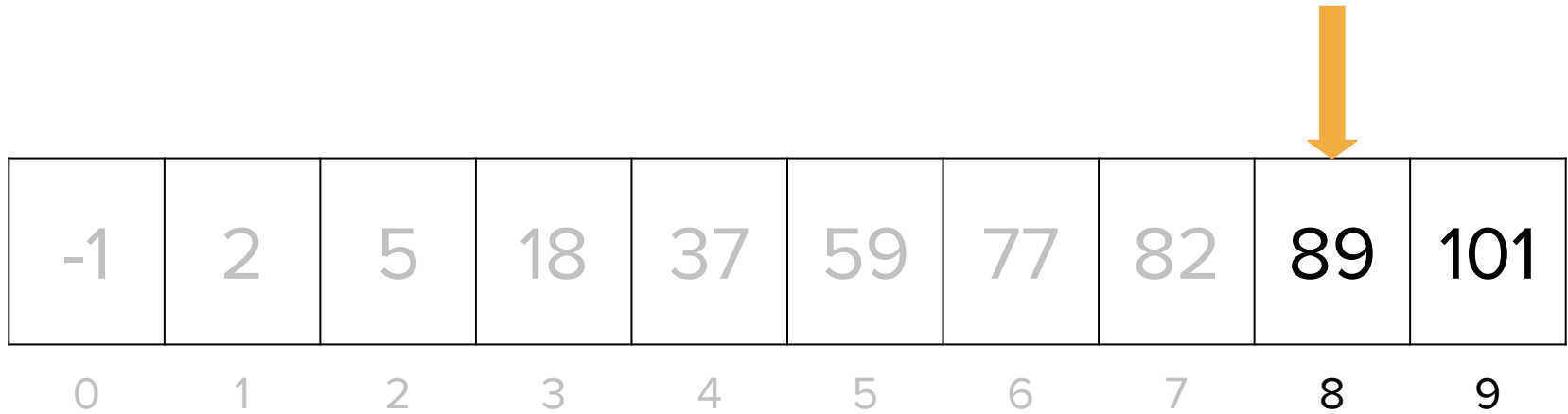
Eliminate left half

Finding a number in a sorted list

-1	2	5	18	37	59	77	82	89	101
0	1	2	3	4	5	6	7	8	9

$$\begin{aligned} &(\text{startIndex} + \text{endIndex}) / 2 = \\ & (8 + 9) / 2 = \\ & 8 \end{aligned}$$

Finding a number in a sorted list

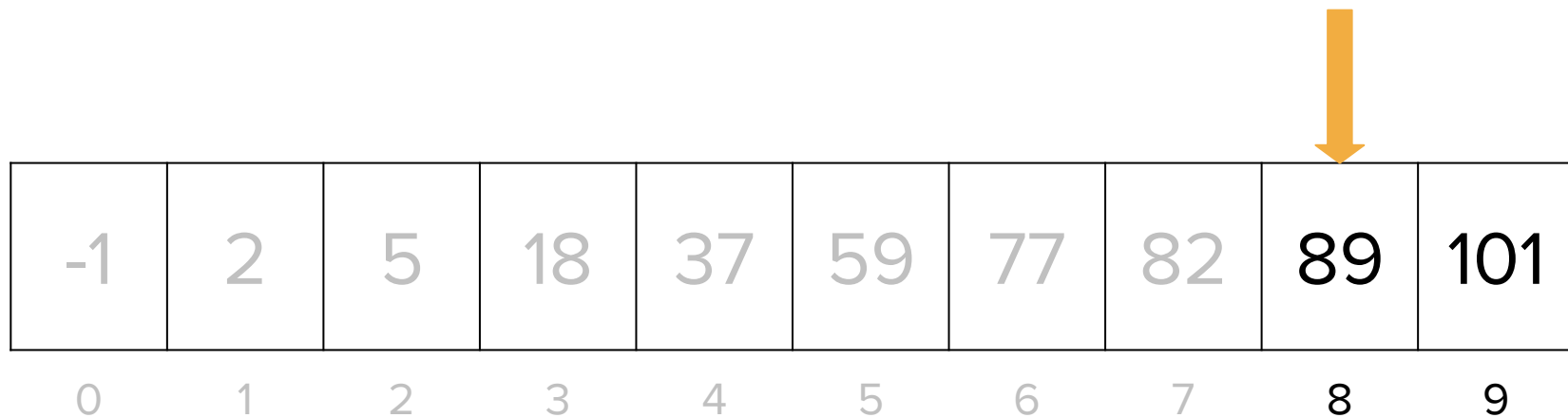


-1	2	5	18	37	59	77	82	89	101
0	1	2	3	4	5	6	7	8	9

$$\begin{aligned} &(\text{startIndex} + \text{endIndex}) / 2 = \\ & (8 + 9) / 2 = \\ & 8 \end{aligned}$$

Finding a number in a sorted list

-1	2	5	18	37	59	77	82	89	101
0	1	2	3	4	5	6	7	8	9



Success!

Defining binary search recursively

- **Algorithm:** Check the middle element at $(\text{startIndex} + \text{endIndex}) / 2$
 - If the middle element is bigger than your desired value, eliminate the right half of the data and repeat.
 - If the middle element is smaller than your desired value, eliminate the left half of the data and repeat.
 - Otherwise, you've found your element!

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- Recursive cases
 - Element at middle is too small → `binarySearch(right half of data)`
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- Recursive cases
 - Element at middle is too small → `binarySearch(right half of data)`
 - Element at middle is too large → `binarySearch(left half of data)`
- Base cases
 - Element at middle == desired element
 - Desired element is not in your data

Discuss:

Read the code for

binarySearch() and identify
the base/recursive cases.

Attendance ticket:

<https://tinyurl.com/binarySearchCases>

Please don't send this link to students who are not here. It's on your honor!

Binary search code

```
int binarySearch(Vector<int>& v, int targetVal, int startIndex, int endIndex) {
    if (startIndex > endIndex) {
        return -1;
    }

    int middleIndex = (startIndex + endIndex) / 2;
    int currentVal = v[middleIndex];
    if (targetVal == currentVal) {
        return middleIndex;
    } else if (targetVal < currentVal) {
        return binarySearch(v, targetVal, startIndex, middleIndex - 1);
    } else {
        return binarySearch(v, targetVal, middleIndex + 1, endIndex);
    }
}
```


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```

Base cases

Binary search code

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}
```

Recursive cases

Binary search code

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    } else if (targetVal < currentVal) {  
        return binarySearch(v, targetVal, startIndex, middleIndex - 1);  
    } else {  
        return binarySearch(v, targetVal, middleIndex + 1, endIndex);  
    }  
}
```

We don't want the user to have to pass these in, but we need them to update our search range

Binary search code

```
int binarySearch(Vector<int>& v, int targetVal) {  
    return binarySearchHelper(v, targetVal, 0, v.size() - 1);  
}
```

```
int binarySearchHelper(Vector<int>& v, int targetVal, int startIndex, int endIndex) {  
    ...  
}
```

Use a recursive helper function for the extra parameters!

(binarySearchHelper would have the same code as the previous slide)

Finding a number in a sorted list

-1	2	5	18	37	59	77	82	89	101
0	1	2	3	4	5	6	7	8	9

What's the runtime?

Binary search runtime

- For data of size N , it eliminates half until 1 element remains:

$N, N/2, N/4, N/8, \dots, 4, 2, 1$

- How many divisions does it take?

Binary search runtime

- For data of size **N**, it eliminates half until 1 element remains.
- Think of it from the other direction:
 - How many times do I have to multiply by 2 to reach **N**?

1, 2, 4, 8, ..., N/4, N/2, N

- Call this number of multiplications **x**:

$$2^x = N$$
$$x = \log_2 N$$

Binary search runtime

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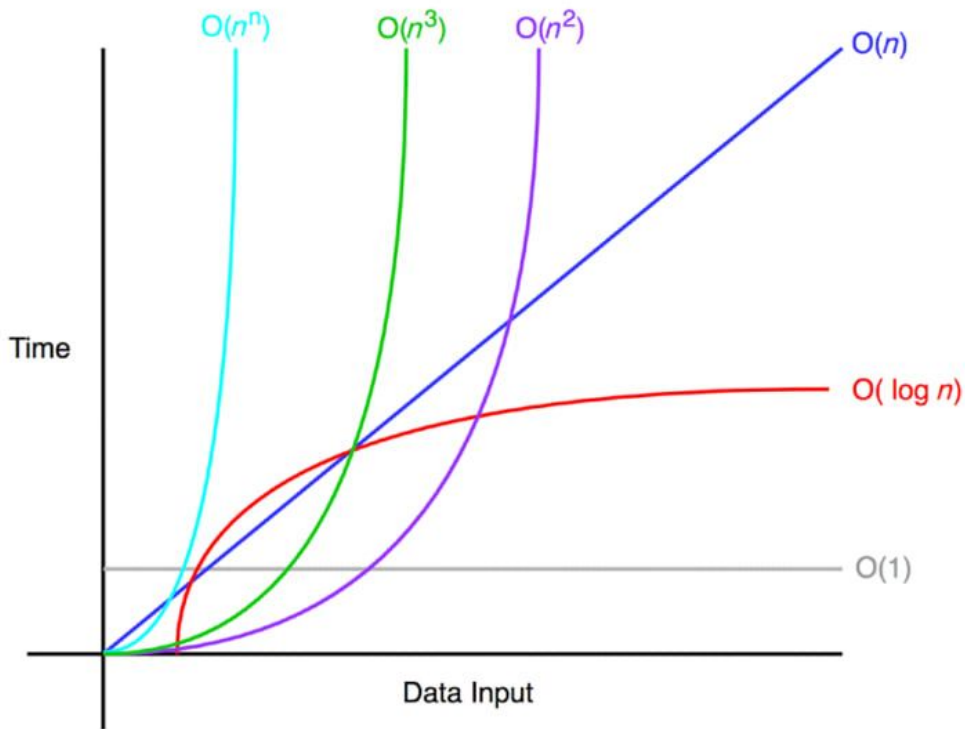
- Binary search has logarithmic Big-O: **$O(\log N)$**

binarysearch.cpp

[demo]

Logarithmic runtime

- Better than linear
- A common runtime when you're able to "divide and conquer" in your algorithm, like with binary search



ADT Big-O Matrix

- Vectors

- `.size()` - $O(1)$
- `.add()` - $O(1)$
- `v[i]` - $O(1)$
- `.insert()` - $O(n)$
- `.remove()` - $O(n)$
- `.clear()` - $O(n)$
- `traversal` - $O(n)$

- Grids

- `.numRows()` / `.numCols()` - $O(1)$
- `g[i][j]` - $O(1)$
- `.inBounds()` - $O(1)$
- `traversal` - $O(n^2)$

- Queues

- `.size()` - $O(1)$
- `.peek()` - $O(1)$
- `.enqueue()` - $O(1)$
- `.dequeue()` - $O(1)$
- `.isEmpty()` - $O(1)$
- `traversal` - $O(n)$

- Stacks

- `.size()` - $O(1)$
- `.peek()` - $O(1)$
- `.push()` - $O(1)$
- `.pop()` - $O(1)$
- `.isEmpty()` - $O(1)$
- `traversal` - $O(n)$

- Sets

- `.size()` - $O(1)$
- `.isEmpty()` - $O(1)$
- `.add()` - ???
- `.remove()` - ???
- `.contains()` - ???
- `traversal` - $O(n)$

- Maps

- `.size()` - $O(1)$
- `.isEmpty()` - $O(1)$
- `m[key]` - ???
- `.contains()` - ???
- `traversal` - $O(n)$

ADT Big-O Matrix

● Vectors

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- `.numRows()` / `.numCols()`
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- `.size()` - $O(1)$
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- `.push()` - $O(1)$
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● Sets

- `.size()` - $O(1)$
- `.isEmpty()` - $O(1)$
- `.add()` - $O(\log(n))$
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- `.contains()` - $O(\log(n))$
- `traversal` - $O(n)$

● Maps

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- `.contains()` - $O(\log(n))$
- `traversal` - $O(n)$

Announcements

Announcements

- Assignment 2 is due **tonight at 11:59pm**, and Assignment 1 revisions are due **Sunday, July 10, at 11:59pm PDT**.
- The midterm will be **next Monday, July 11** during lecture.
 - There will be a midterm review session **tomorrow around noon**. Watch Ed for a finalized location and time!
 - Today's content will only show up in the form of extra credit.
- Assignment 3, which is all about recursion, will be released after the midterm.
- Jenny will be rescheduling her individual OH next week.

A **dynamic** example:
Exploring many
possibilities

Limits of iteration

- So far, we've seen problems that could be solved iteratively or recursively.
 - Depending on the problem, you could make the argument that one of the approaches was stylistically preferable or easier to understand.
 - But both got the job done!

Limits of iteration

- So far, we've seen problems that could be solved iteratively or recursively.
- However, there is a whole class of problems that are very difficult, or nearly impossible, to solve with an iterative approach.
 - These problems have the goal of exploring **many different possibilities or solutions**.
 - Because iteration is inherently linear (and not dynamic), it is usually used to build up a single solution without exploring many possible alternatives.
 - Recursion allows us to explore many potential possibilities at once via **the power of branching** that comes when we have multiple recursive calls.

Limits of iteration

- So far, we've seen problems that could be solved iteratively or recursively.
- However, there is a whole class of problems that are very difficult, or nearly impossible, to solve with an iterative approach.
- To solve these problems and generate many possible solutions, we will have to learn a new problem-solving technique called **recursive backtracking**.
 - The key steps in recursive backtracking are that you make a choice about how to generate a solution, you use recursion to explore that choice, and then you might make a different choice and repeat the process.
 - This paradigm is called “**choose-explore-unchoose**.”

Limits of iteration

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Let's do an example!

Generating coin sequences



- Let's say that you're playing a game that involves flipping a coin a certain number of times in a row. Your success in the game depends on the exact sequence of "heads" and "tails" that you get.

Generating coin sequences



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HH

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- Let's say that you're playing a game that involves flipping a coin a certain number of times in a row. Your success in the game depends on the exact sequence of "heads" and "tails" that you get.
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HHH HHT HTH HTT THH THT TTH TTT

How do we know that we got all the possibilities? How do we avoid repeats?

Generating coin sequences

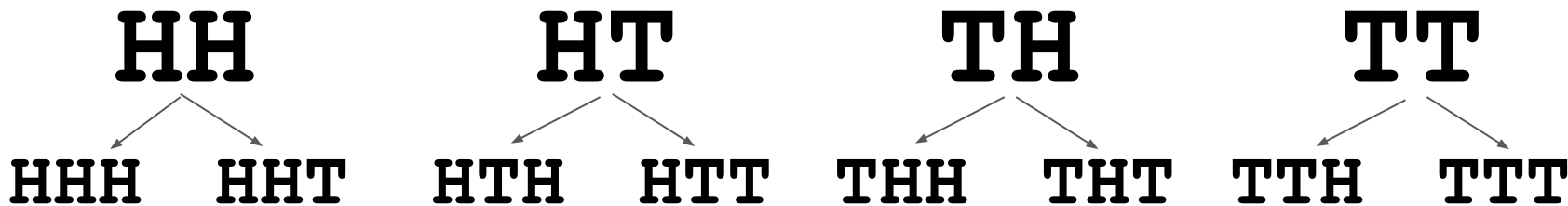


- Let's say that you're playing a game that involves flipping a coin a certain number of times in a row. Your success in the game depends on the exact sequence of "heads" and "tails" that you get.
- Can we observe any patterns between the outcomes in the game with 2 flips and the outcomes in the game with 3 flips?

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Generating coin sequences



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- Can we observe any patterns between the outcomes in the game with 2 flips and the outcomes in the game with 3 flips?
 - There is a self-similar tree-like relationship between the possible outcomes of 2 flips and the possible outcomes of 3 flips.
 - The branching in the tree comes from deciding whether or not to add an H or a T to the existing sequence.
 - Together these branching sequences of decisions define a **decision tree**.

Why decision trees?

- We've seen trees in the context of fractals (drawing pretty shapes), but now we're going to apply meaningful context to these trees.

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- In problems where we care about many possible outcomes, decision trees can help illustrate the recursive backtracking strategy for generating outcomes. They model the **options we can choose from** and the **“decisions” we make along the way**.

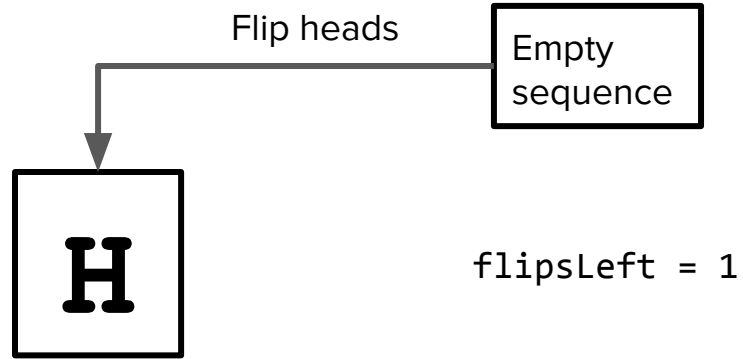
Why decision trees?

- We've seen trees in the context of fractals (drawing pretty shapes), but now we're going to apply meaningful context to these trees.
- In problems where we care about many possible outcomes, decision trees can help illustrate the recursive backtracking strategy for generating outcomes. They model the options we can choose from and the “decisions” we make along the way.
- Let's create a visualization of the possible space of outcomes that could result from N coin flips. Each decision is one flip, and the options for a single flip are either heads or tails.

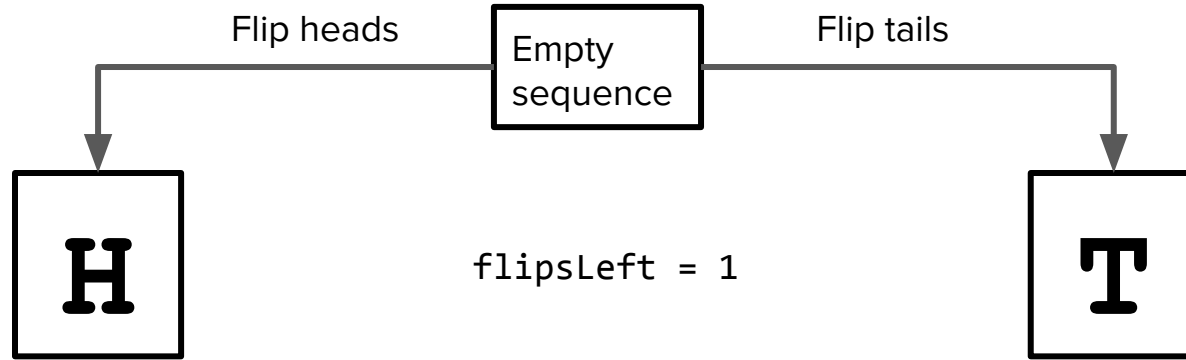
Example decision tree for $N=2$

Empty
sequence

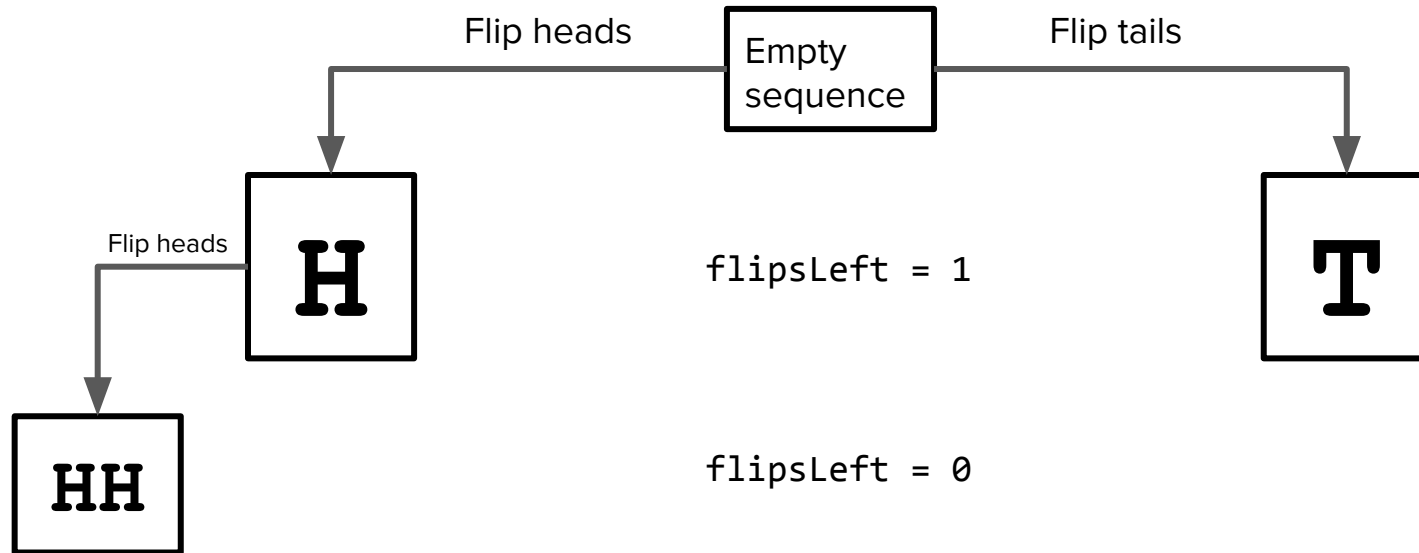
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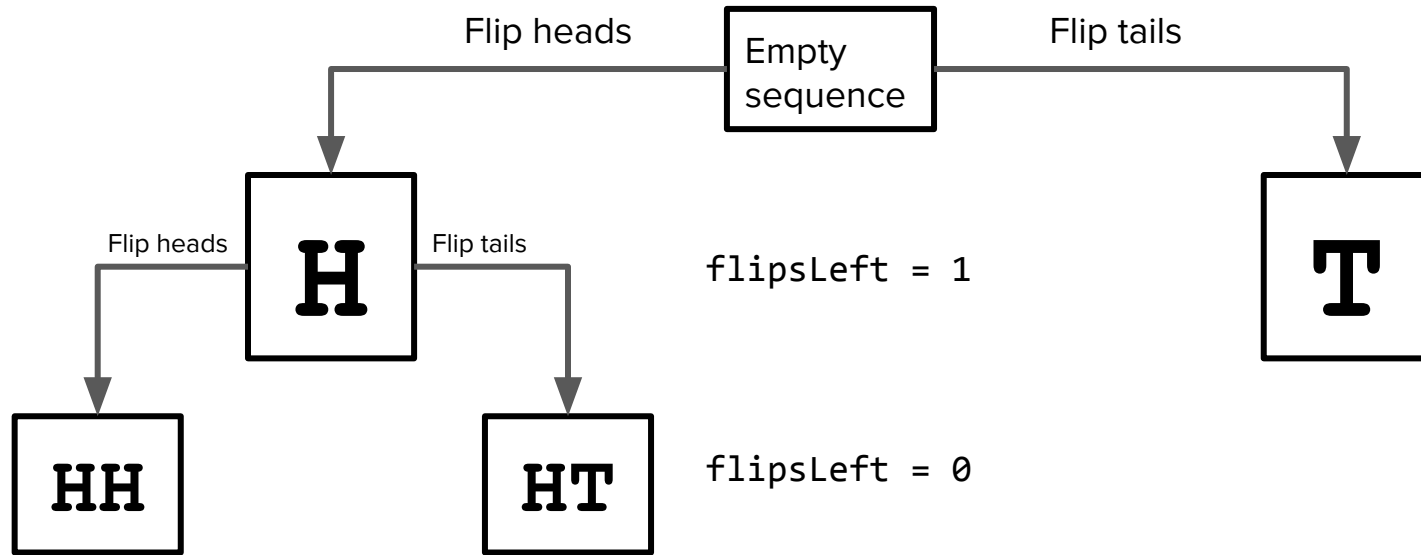
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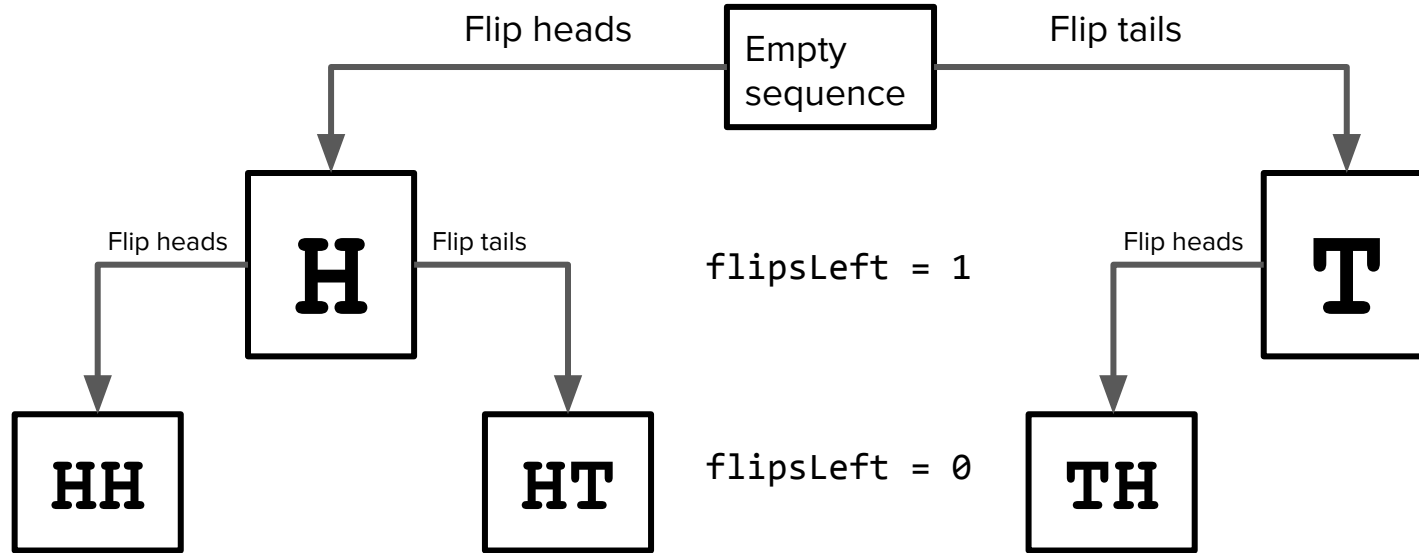
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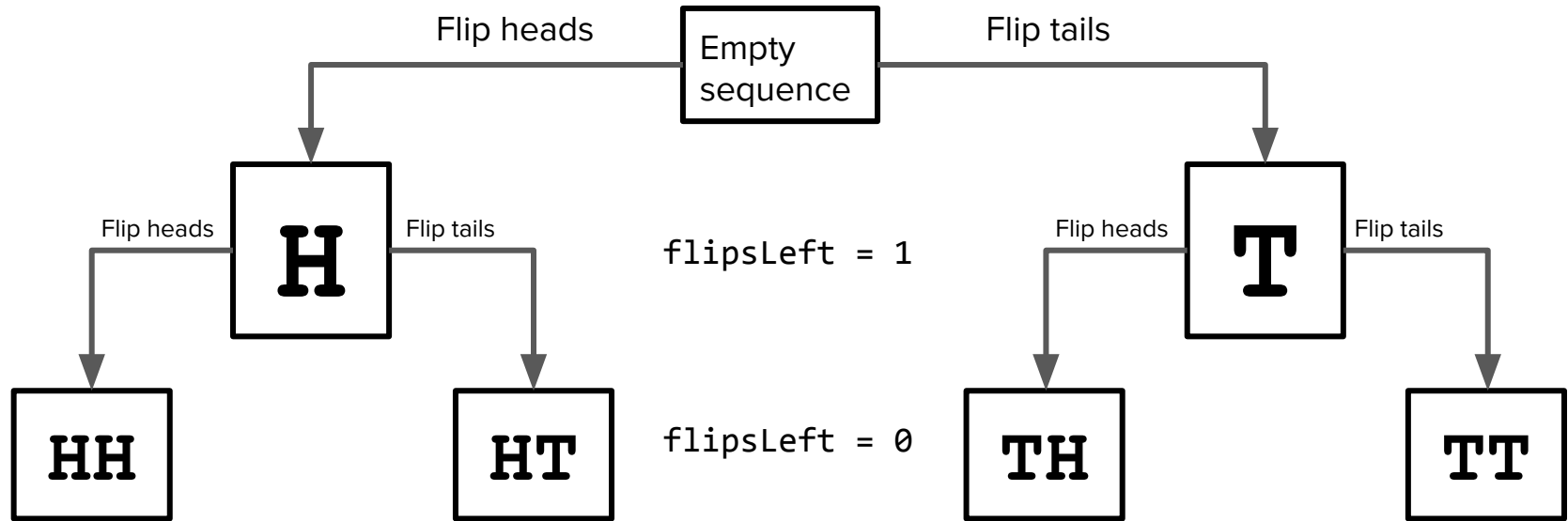
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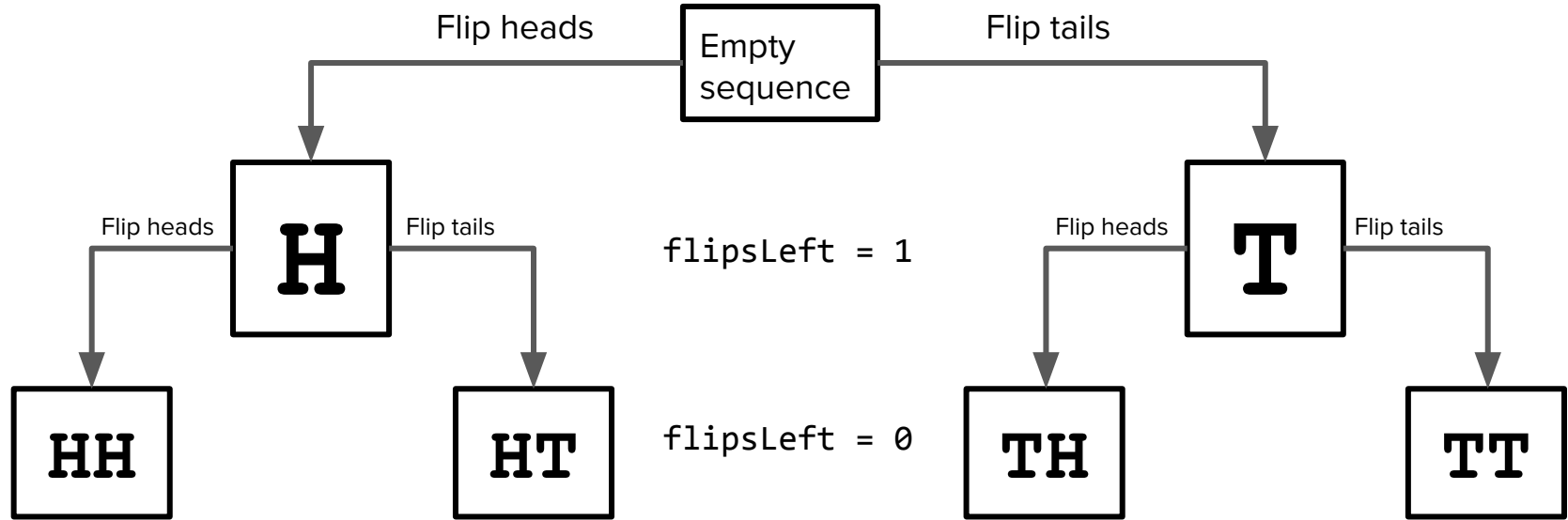
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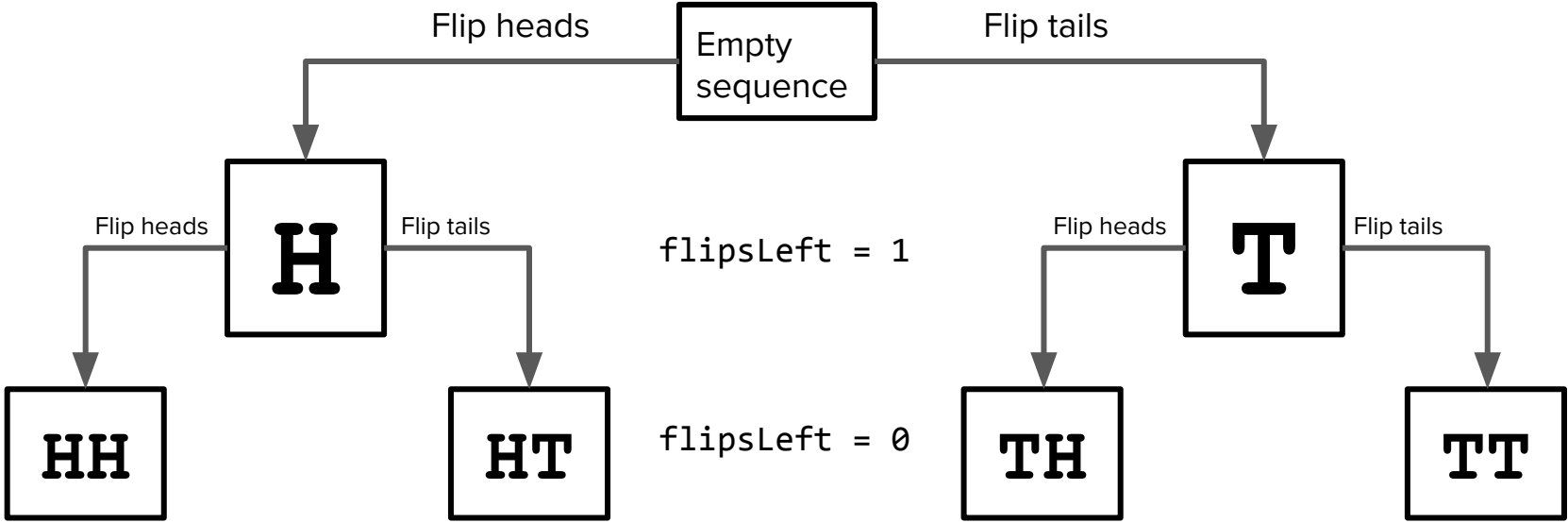


Example decision tree for N=2



Base case: when `flipsLeft = 0`

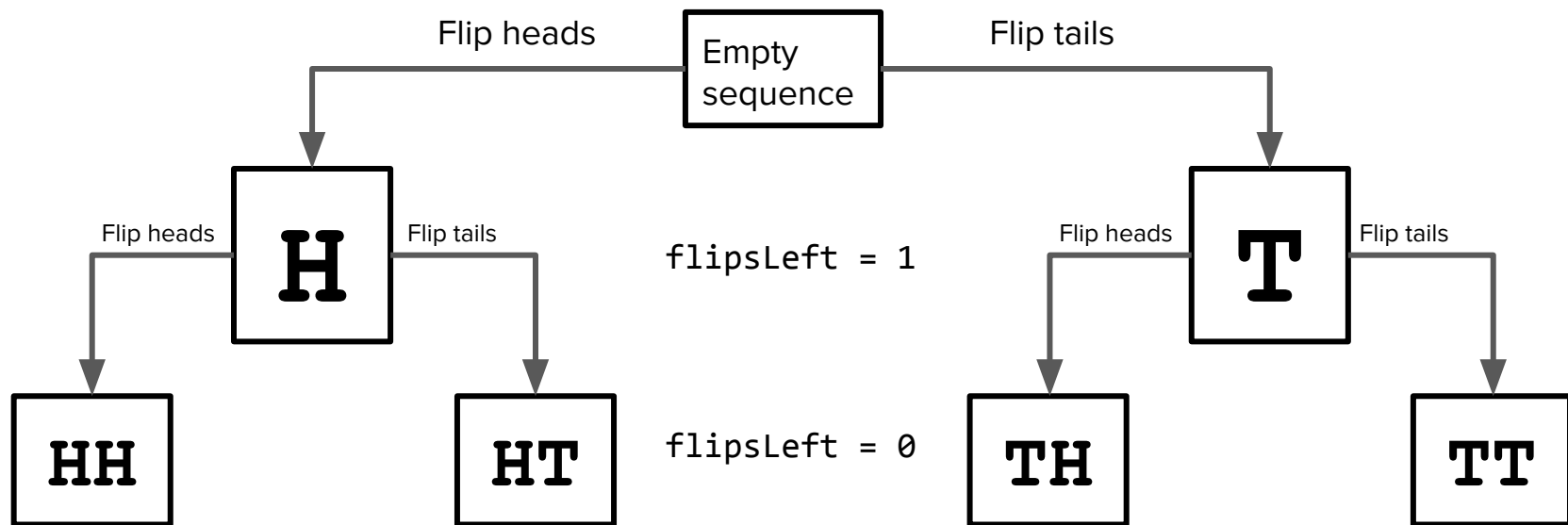
Example decision tree for N=2



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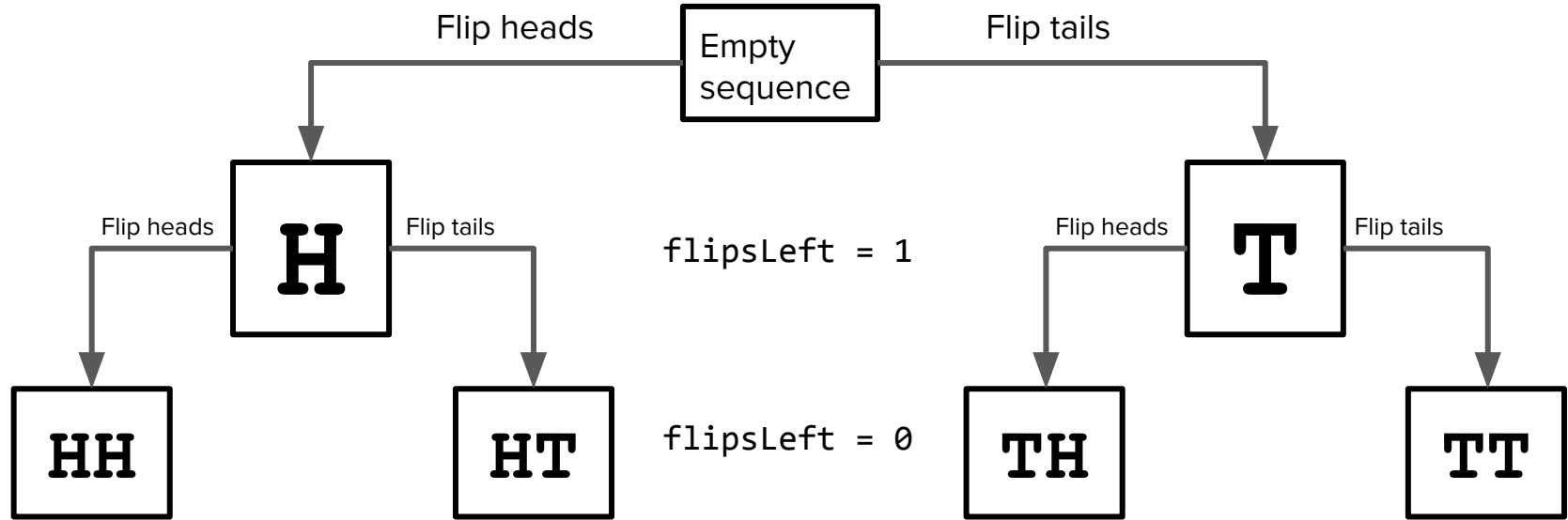
We reach the base case when we reach the leaves of our decision tree.

Example decision tree for N=2



Recursive cases: add 'H' or 'T' to the sequence

Example decision tree for N=2



Recursive cases: add 'H' or 'T' to the sequence

The branching points in our tree. We'll have a recursive call for each option.

Let's code it!

```
void generateSequences(int length);
```

Takeaways: recursive backtracking + decision trees

- Unlike our previous recursion paradigm in which a solution gets built up as recursive calls return, in backtracking our final outputs occur at our base cases (leaves) and get built up as we go down the decision tree.

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Takeaways: recursive backtracking + decision trees

- Unlike our previous recursion paradigm in which a solution gets built up as recursive calls return, in backtracking our final outputs occur at our base cases (leaves) and get built up as we go down the decision tree.
- The **height** of the tree corresponds to the **number of decisions** we have to make. The **width** at each decision point corresponds to the **number of options**.
- To exhaustively explore the entire search space, we must **try every possible option for every possible decision**. That can be a lot of paths to walk!
 - For the previous example, we have to make N decisions, with 2 choices for each decision. This means 2^N total possible outcomes!

Summary

Why do we use recursion?

- Elegance
 - Allows us to solve problems with very clean and concise code
- Efficiency
 - Allows us to accomplish better runtimes when solving problems
- Dynamic
 - Allows us to solve problems that are hard to solve iteratively

Two types of recursion

Basic recursion

- One repeated task that builds up a solution as you come back up the call stack
- The final base case defines the initial seed of the solution and each call contributes a little bit to the solution
- Initial call to recursive function produces final solution

Backtracking recursion

- Build up many possible solutions through multiple recursive calls at each step
- Seed the initial recursive call with an “empty” solution
- At each base case, you have a potential solution

What's next?

Roadmap

C++ basics

User/client

vectors + grids

stacks + queues

sets + maps

Core
Tools

testing

algorithmic
analysis

recursive
problem-solving

Object-Oriented
Programming

Implementation

arrays

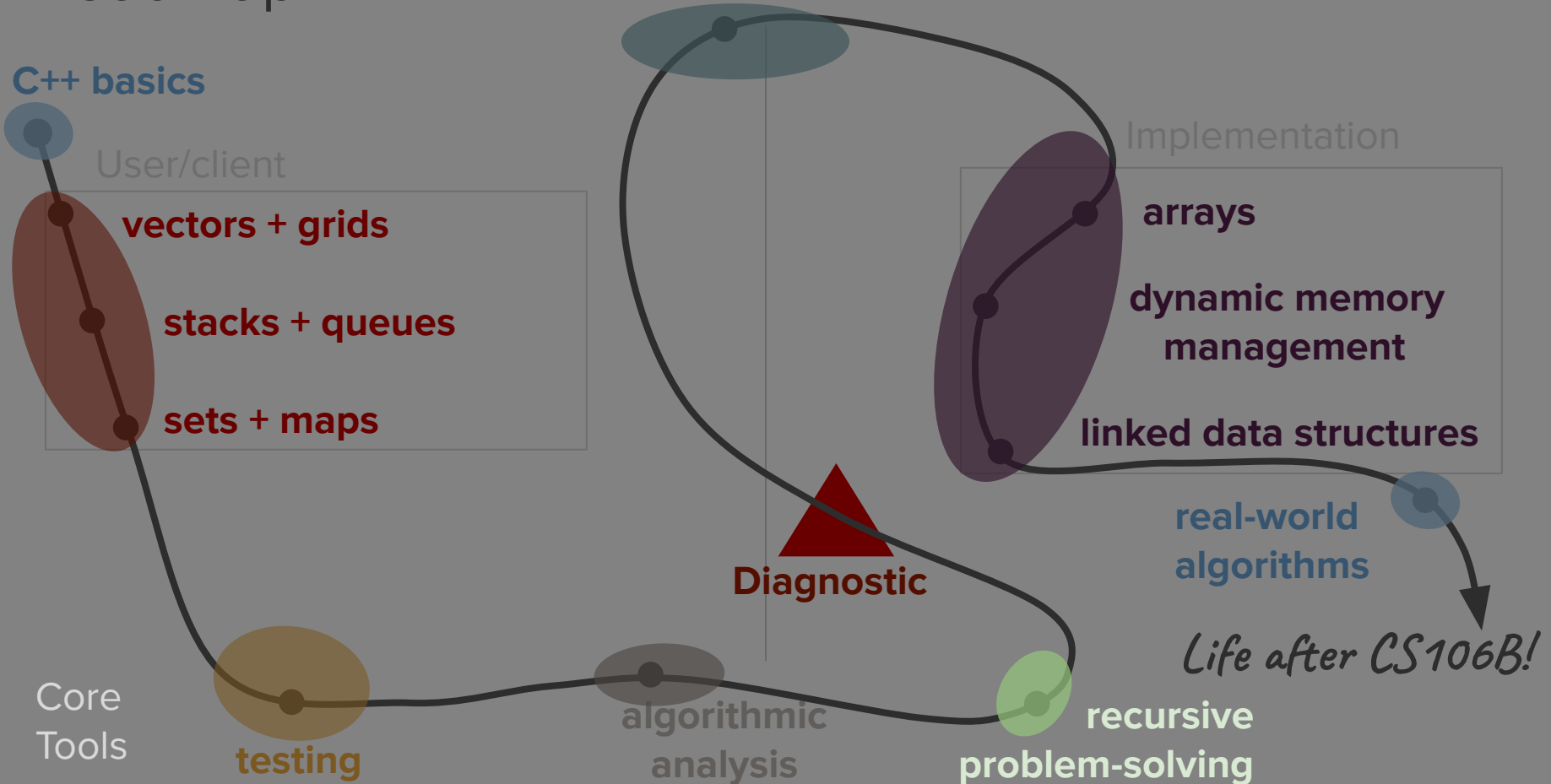
dynamic memory
management

linked data structures

real-world
algorithms

Life after CS106B!

Diagnostic



Recursive Backtracking

List all *subsets* of {A, H, I}

This is called a *decision tree*.

