Computer Systems

CS107

Cynthia Lee

Today's Topics

LECTURE:

- Floating point!

Real Numbers and Approximation

MATH TIME!

Some preliminary observations on approximation

- We know that some non-integer numbers can be represented precisely by a finite-length decimal value
  - $1/5_{10} = 0.2_{10}$
- …and some can't (in base 10)
  - π
  - $1/3_{10} = .3333333\overline{3}_{10}$

- But this isn't consistent across different bases
  - $1/3_{10} = 0.1_{3}$
  - $1/5_{10} = 0.012012\overline{012}_{3}$
- And of course even with `int`, we needed to make choices about which integers make the cut and which don't, when *we have only a limited storage space*

Real number types: `float`, `double`

- `float` is 4 bytes = 32 bits
  - 2^{32} = 4,294,967,296 possible values
  - (note: obviously same number of possible values as 32-bit int)
- `double` is 8 bytes = 64 bits
  - 2^{64} = 18,446,744,073,709,551,616 possible
- This is the same number of different possible representable values as int and long, of course
  - But floating point range goes much higher than max int:
  - Max 32-bit int: $2^{31} - 1$ = 2,147,483,647
  - Max 32-bit float: $(2 - 2^{-23}) \times 2^{127} \approx 3.402823 \times 10^{38}$

Real number types: `float`, `double`

- If you're deciding how to divide up a large number of possible bit patterns into numbers they can represent, there are any number of ways you cold do this
  - What are your priorities?
  - Do you want to leave space for very big ones? Or very small ones?
  - Do you prefer positive or negative?

The “mini-float”

NOT A REAL TYPE IN C, BUT WILL SHOW US THE PRINCIPLES THAT WILL SCALE UP TO THE ACTUAL BIT SIZES FLOAT AND DOUBLE

IEEE format, squished down to example-size "mini-float"

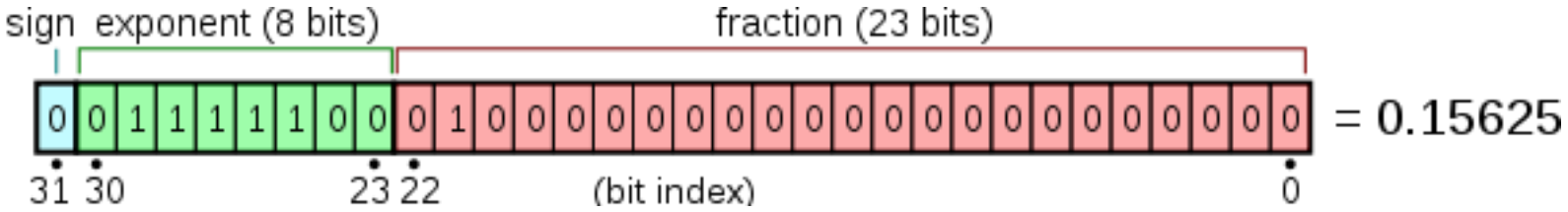
Sign	Exponent				Mantissa		

- Sign bit is 1 for negative, 0 for positive
  - Operates completely independently, *unlike* 2's complement int
- The rest of the number is in something like scientific notation:
  - (+/-) M * 2^{Exp}
    - **M**antissa provides the significant digits of the number
      - *Note: a*lso called the coefficient or the significand
    - **Exp**onent is used to scale the number up or down to very large or very small values
  - Both exponent and mantissa are binary numbers, 4 bits and 3 bits, respectively
    - *but not in normal 2's complement form!*

IEEE format, squished down to example-size “mini-float”

Sign	Exponent				Mantissa		

- Compare to the bit distribution for 32-bit `float`:



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https://en.wikipedia.org/wiki/Single-precision_floating-point_format#/media/File:Float_example.svg

“Mini-float”: the mantissa

- A 3-bit binary number representing a number after an implicit 1.
  - What??
- Yes, it looks like this: 1.*[mantissa digits]*
- Examples:
  - Mantissa = 100_2 → 1.100_2
    - = 1 + ½ = 1.5_{10}
  - Mantissa = 110_2 → 1.110_2
    - = 1 + ½ + ¼ = 1.75_{10}

“Mini-float”: the exponent

- A 4-bit binary number representing the exponent:
  - 0000 → reserved
  - 0001 → -6
  - 0010 → -5
  - 0011 → -4
  - 0100 → -3
  - 0101 → -2
  - 0110 → -1
  - 0111 → 0
  - 1000 → 1
  - 1001 → 2
  - 1010 → 3
  - 1011 → 4
  - 1100 → 5
  - 1101 → 6
  - 1110 → 7
  - 1111 → reserved

Lets do an example!

Sign	Exponent				Mantissa		
0	0	0	0	1	0	1	0

- This number is:
  A. Greater than 0
  B. Less than 0
  C. Help!
- Extra credit: > 1 or < -1 or in between?

Lets do an example!

+/-	Exponent				Mantissa			Value
0	0	0	0	1	0	1	0	$10/8 * 2^{-6} = 10/512$
0	0	0	0	1	0	1	1	$11/8 * 2^{-6} = 11/512$
0	0	0	0	1	1	0	0	$12/8 * 2^{-6} = 12/512$
0	0	0	0	1	1	0	1	$13/8 * 2^{-6} = 13/512$
0	0	0	0	1	1	1	0	$14/8 * 2^{-6} = 14/512$
0	0	0	0	1	1	1	1	$15/8 * 2^{-6} = 15/512$
0	0	0	1	0	0	0	0	$1 * 2^{-5} = 16/512$
0	0	0	1	0	0	0	1	$9/8 * 2^{-6} = 18/512$

Lets do an example!

+/-	Exponent				Mantissa				Value
0	0	0	1	0	0	0	0		$1 * 2^{-5} = 16/512$
0	0	0	1	0	0	0	1		$9/8 * 2^{-6} = 18/512$
0	1	1	1	0	0	1	0		$10/8 * 2^{7} = 160$
0	1	1	1	0	0	1	1		$11/8 * 2^{7} = 176$
0	1	1	1	0	1	0	0		$12/8 * 2^{7} = 192$
0	1	1	1	0	1	0	1		$13/8 * 2^{7} = 208$
0	1	1	1	0	1	1	0		$14/8 * 2^{7} = 224$
0	1	1	1	0	1	1	1		$15/8 * 2^{7} = 240$

About those reserved exponents…

Denormalized and Special cases

Reserved exponent values: 0000 and 1111

0000 exponent:

- If the mantissa is all zeros: +/-0

Sign	Exponent				Mantissa		
any	0	0	0	0	0	0	0

- If the mantissa is nonzero: denormalized floats

Sign	Exponent				Mantissa
any	0	0	0	0	any nonzero

1111 exponent:

- If the mantissa is all zeros: +/-INF (infinity)

Sign	Exponent				Mantissa		
any	1	1	1	1	0	0	0

- If the mantissa is nonzero: NaN (not a number)

Sign	Exponent				Mantissa
any	1	1	1	1	any nonzero

MORE about: the mantissa

- A 3-bit binary number representing a number after an implicit 1.
  - What??
- Yes, it looks like this: 1.*[mantissa digits]*
- Examples:
  - Mantissa = 100_2 → 1.100_2
    - = 1 + ½ = 1.5_{10}
  - Mantissa = 110_2 → 1.110_2
    - = 1 + ½ + ¼ = 1.75_{10}
- **If the exponent is all zeros, then there is no implicit leading 1**
  - The exponent is implicitly smallest possible exponent (2^{-6} for mini-float)
  - This allows us to eke out a few numbers that are even (closer to zero) than would otherwise be possible
  - Called “denormalized” or “denorm” floats

Comparing float and int

Advantages and disadvantages

Comparing float and int

- 32-bit integer (type **`int`**):
  - -2,147,483,648 to 2147483647
  - Every integer in that range can be represented
- 64-bit integer (type **`long`**):
  - −9,223,372,036,854,775,808 to 9,223,372,036,854,775,807
    - (Even Psy will have a hard time overflowing this!)

- 32-bit floating point (type **`float`**):
  - ~1.7 x10^{-38} to ~3.4 x10^{38}
  - Not all numbers in the range can be represented (obviously—uncountable)
    - Not even all integers in the range can be represented!
    - Gaps can get quite large!! (larger the exponent, larger the gap between successive mantissa values)
- 64-bit floating point (type **`double`**):
  - ~9 x10^{-307} to ~2 x10^{308}

Doing arithmetic in float

FUN TIMES

Adding two mini-floats

- Your bank account balance on your 12th birthday was $128.00
- Your bank stores this amount as a mini-float:

Sign	Exponent				Mantissa		

- Each week of your childhood, starting at age 12, you deposited your weekly allowance of $8:

Sign	Exponent				Mantissa		

- **What was your account balance on the day you turned 18?**

A. About $10000
B. About $5000
C. About $2500
D. About $100
E. Other

Doing arithmetic in float

FUN TIMES

Adding two mini-floats

- Your bank account balance on your 12th birthday was $128.00
- **Your bank stores this amount as a mini-float:**

Sign	Exponent				Mantissa		

- Each week, starting at age 12, you deposited your weekly allowance of $14:

Sign	Exponent				Mantissa		

- **What was your account balance on the day you turned 18?**

A. About $10000
B. About $5000
C. About $2500
D. About $100
E. Other