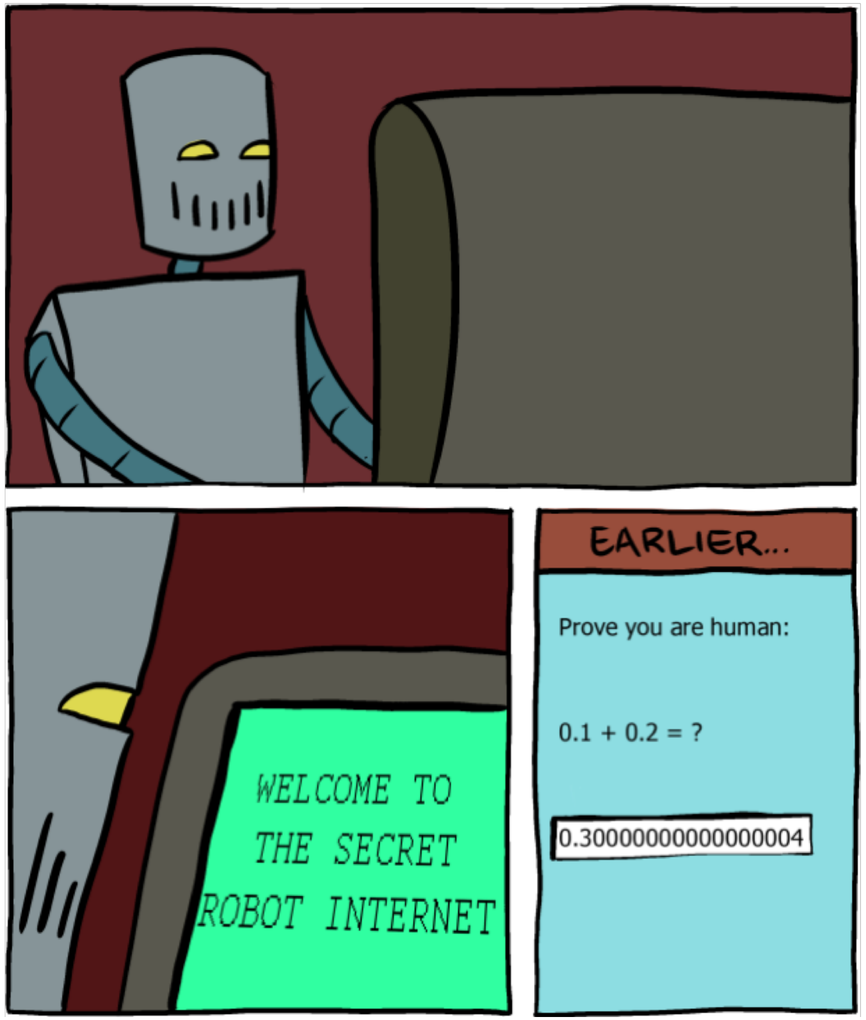
CS107, Lecture 10
Floating Point

Reading: B&O 2.4



Learning Goals

Understand the design and compromises of the floating point representation, including:

- Fixed point vs. floating point
- How a floating point number is represented in binary
- Issues with floating point imprecision
- Other potential pitfalls using floating point numbers in programs

Plan For Today

- **Recap:** Generics with Function Pointers
- Representing real numbers
- Fixed Point
- **Break:** Announcements
- Floating Point

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Function Pointers

- In C, there is a variable type for functions!
- We can pass functions as parameters, store functions in variables, etc.
- Why is this useful?

Generics Limitations

Sometimes, there is functionality that *cannot* be made generic.

```
void bubble_sort(void *arr, int n, int elem_size_bytes) {
    while (true) {
        bool swapped = false;
        for (int i = 1; i < n; i++) {
            void *prev_elem = (char *)arr + (i-1)*elem_size_bytes;
            void *curr_elem = (char *)arr + i*elem_size_bytes;
            if (curr_elem should come before prev_elem) {
                swapped = true;
                swap(prev_elem, curr_elem, elem_size_bytes);
            }
        }

        if (!swapped) {
            return;
        }
    }
}
```

Generics Limitations

Sometimes, there is functionality that *cannot* be made generic. The caller can pass in a function to perform that functionality for the data they are providing.

```
void bubble_sort(void *arr, int n, int elem_size_bytes,
        bool (*cmp_fn)(const void *, const void *)) {
    while (true) {
        bool swapped = false;
        for (int i = 1; i < n; i++) {
            void *prev_elem = (char *)arr + (i-1)*elem_size_bytes;
            void *curr_elem = (char *)arr + i*elem_size_bytes;
            if (cmp_fn(prev_elem, curr_elem) > 0)) {
                swapped = true;
                swap(prev_elem, curr_elem, elem_size_bytes);
            }
        }

        if (!swapped) {
            return;
        }
    }
```

Generic C Standard Library Functions

- **qsort** – I can sort an array of any type! To do that, I need you to provide me a function that can compare two elements of the kind you are asking me to sort.
- **bsearch** – I can use binary search to search for a key in an array of any type! To do that, I need you to provide me a function that can compare two elements of the kind you are asking me to search.
- **lfind** – I can use linear search to search for a key in an array of any type! To do that, I need you to provide me a function that can compare two elements of the kind you are asking me to search.
- **lsearch** - I can use linear search to search for a key in an array of any type! I will also add the key for you if I can't find it. In order to do that, I need you to provide me a function that can compare two elements of the kind you are asking me to search.

Generic C Standard Library Functions

- **scandir** – I can create a directory listing with any order and contents! To do that, I need you to provide me a function that tells me whether or not you want me to include a given directory entry in the listing. I also need you to provide me a function that tells me the correct ordering of two given directory entries.

Function Pointers

Here's the variable type syntax for a function:

```
[return type] (*[name])([parameters])
```

Function Pointers

```
int do_something(char *str) {
    …
}

int main(int argc, char *argv[]) {
    …
    int (*func_var)(char *) = do_something;
    …
    func_var("testing");
    return 0;
}
```

Function Pointers

```
void bubble_sort(void *arr, int n, int elem_size_bytes,
        int (*cmp_fn)(const void *, const void *)) {

…
}

int cmp_double(const void *, const void *) {…}

int main(int argc, char *argv[]) {
    …
    double values[] = {1.2, 3.5, 12.2};
    int n = sizeof(values) / sizeof(values[0]);
    bubble_sort(values, n, sizeof(*values), cmp_double);
    …
}
```

Comparison Functions

- Comparison functions are a common use of function parameters, because many generic functions must know how to compare elements of your type.
- Comparison functions always take *pointers to the data they care about,* since the data could be any size!

When writing a comparison function callback, use the following pattern:

1) Cast the void *argument(s) and set typed pointers equal to them.
2) Dereference the typed pointer(s) to access the values.
3) Perform the necessary operation.

(steps 1 and 2 can often be combined into a single step)

Comparison Functions

- It should return:
  - < 0 if first value should come before second value
  - > 0 if first value should come after second value
  - 0 if first value and second value are equivalent
- This is the same return value format as **strcmp**!

```
int (*compare_fn)(const void *a, const void *b)
```

Function Pointers

```
int integer_compare(void *ptr1, void *ptr2) {
     // cast arguments to int *s and dereference
     int num1 = *(int *)ptr1;
     int num2 = *(int *)ptr2;

     // perform operation
     return num1 – num2;
}

…
qsort(mynums, count, sizeof(*mynums), integer_compare);
```

String Comparison Function

```
int string_compare(void *ptr1, void *ptr2) {
    // cast arguments and dereference
    char *str1 = *(char **)ptr1;
    char *str2 = *(char **)ptr2;

    // perform operation
    return strcmp(str1, str2);
}

…
qsort(mystrs, count, sizeof(*mystrs), string_compare);
```

Generics Wrap-Up

- We use **void *** pointers and memory operations like **memcpy** and **memmove** to make data operations generic.
- We use **function pointers** to make logic/functionality operations generic.

memset

memset is a function that sets a specified amount of bytes at one address to a certain value.

```
void *memset(void *s, int c, size_t n);
```

It fills n bytes starting at memory location **s** with the byte **c**. (It also returns **s**).

```
int counts[5];
memset(counts, 0, 3);   // zero out first 3 bytes at counts
memset(counts + 3, 0xff, 4) // set 3rd entry’s bytes to 1s
```

Plan For Today

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Real Numbers

- We previously discussed representing integer numbers using two's complement.
- However, this system does not represent real numbers such as 3/5 or 0.25.
- How can we design a representation for real numbers?

Real Numbers

Problem: unlike with the integer number line, where there are a finite number of values between two numbers, there are an *infinite* number of real number values between two numbers!

Integers between 0 and 2: 1

Real Numbers Between 0 and 2: 0.1, 0.01, 0.001, 0.0001, 0.00001,…

We need a fixed-width representation for real numbers. Therefore, by definition, *we will not be able to represent all numbers*.

Real Numbers

Problem: every number base has un-representable real numbers.

Base 10: $1/6_{10} = 0.16666666\ldots\ldots_{10}$

Base 2: $1/10_{10} = 0.000110011001100110011\ldots_2$

Therefore, by representing in base 2, *we will not be able to represent all numbers*, even those we can exactly represent in base 10.

Fixed Point

- **Idea:** Like in base 10, let's add binary decimal places to our existing number representation.

5 9 3 4 . 2 1 6

10^3 10^2 10^1 10^0 10^{-1} 10^{-2} 10^{-3}

1 0 1 1 . 0 1 1

2^3 2^2 2^1 2^0 2^{-1} 2^{-2} 2^{-3}

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Fixed Point

- **Idea:** Like in base 10, let's add binary decimal places to our existing number representation.

1	0	1	1	.	0	1	1
8s	4s	2s	1s		1/2s	1/4s	1/8s

- **Pros:** arithmetic is easy! And we know exactly how much precision we have.

Fixed Point

- **Problem:** we have to fix where the decimal point is in our representation. What should we pick? This also fixes us to 1 place per bit.

. 0 1 1 0 0 1 1

1/2s 1/4s 1/8s …

1 0 1 1 0 . 1 1

16s 8s 4s 2s 1s 1/2s 1/4s

Let's Get Real

What would be nice to have in a real number representation?

- Represent widest range of numbers possible
- Flexible "floating" decimal point
- Represent scientific notation numbers, e.g. 1.2×10^6
- Still be able to compare quickly
- Have more predictable over/under-flow behavior

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Announcements

- Functions like **versionsort** and **alphasort** prohibited on assign4
- Brown Institute "Magic Grants" application open! See brown.stanford.edu
- CURIS undergraduate summer research applications open! See curis.stanford.edu. Due 2/10.

Midterm Exam

- The midterm exam is **Fri. 2/15 12:30-2:20PM in Hewlett 200 and Hewlett 201**
  - **Last names A-G: Hewlett 201**
  - **Last Names H-Z: Hewlett 200**
- Covers material through **lab4/assign4** (no floats or assembly language)
- Closed-book, 1 2-sided page of notes permitted, C reference sheet provided
- Administered via BlueBook software (on your laptop)
- Practice materials and BlueBook download available on course website
- If you have academic (e.g. OAE) or athletics accommodations, please let us know by **Sunday 2/10** if possible.
- If you do not have a workable laptop for the exam, you <u>**must**</u> let us know by **Sunday 2/10**. Limited charging outlets will be available for those who need them.

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Let’s Get Real

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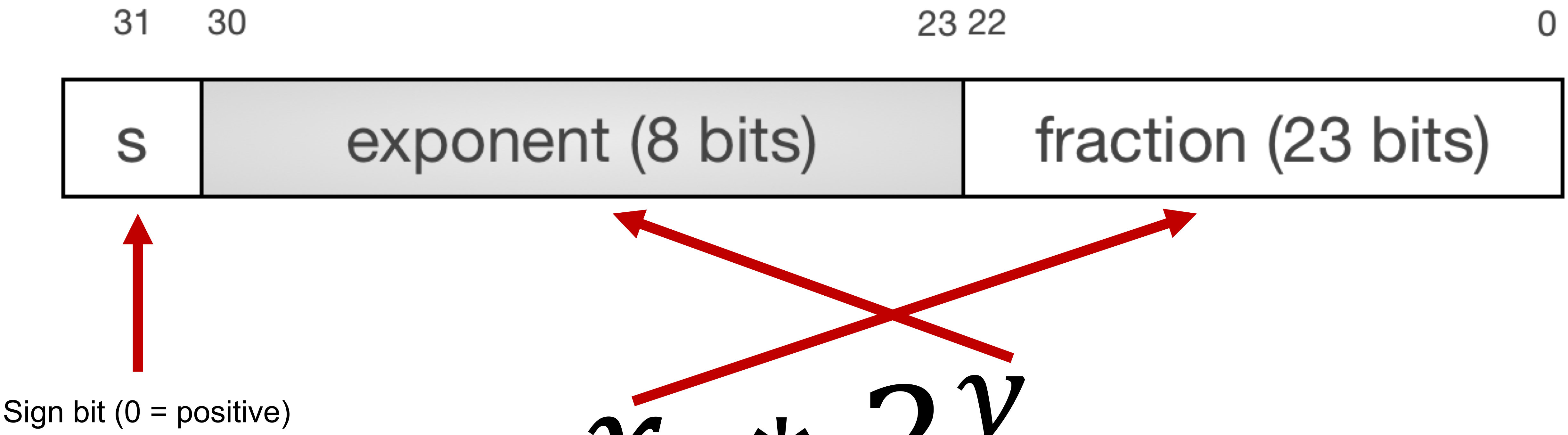
IEEE Floating Point

Let's aim to represent numbers of the following scientific-notation-like format:

$$x * 2^y$$

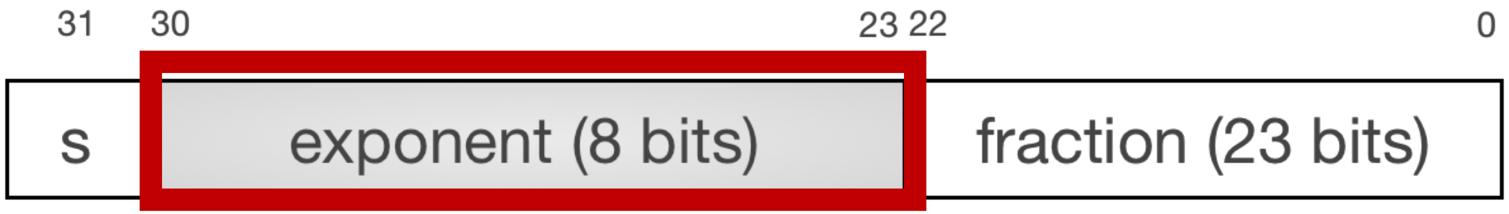
With this format, 32-bit floats represent numbers in the range -3.4E+38 to 3.4E+38! Is every number between those representable? **No**.

IEEE Single Precision Floating Point



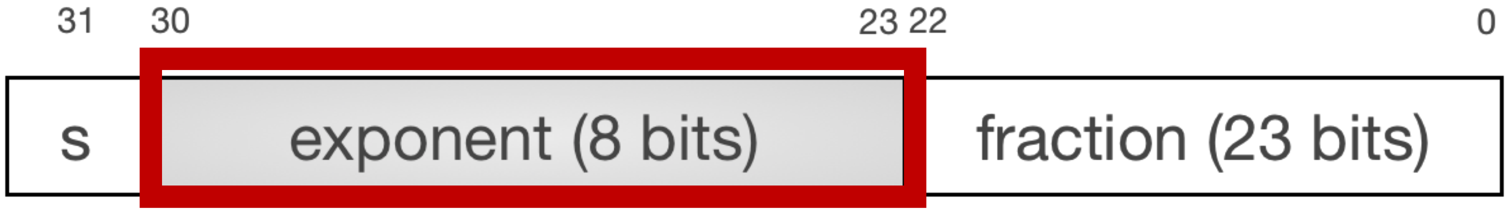
$$x * 2^y$$

Exponent



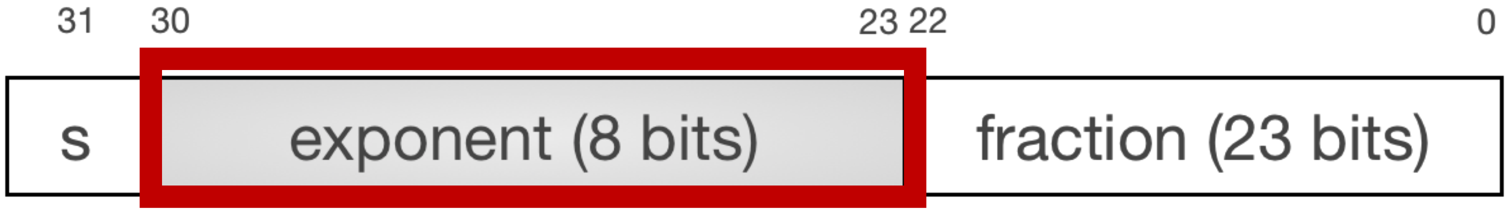
Exponent (Binary)	Exponent (Base 10)
00000000	?
00000001	?
00000010	?
00000011	?
…	?
11111100	?
11111101	?
11111110	?
11111111	?

Exponent



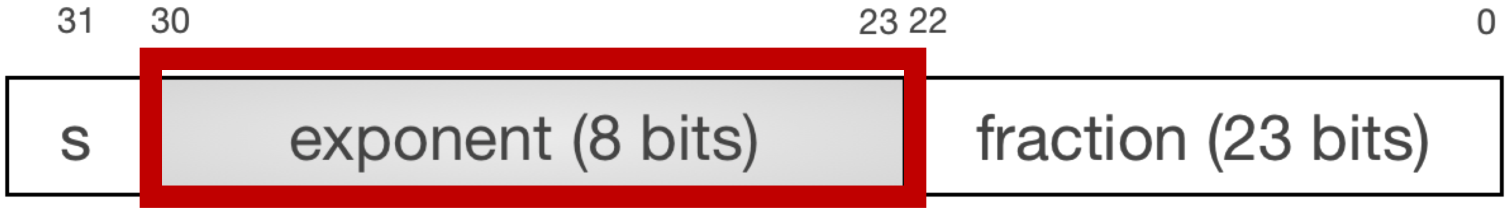
Exponent (Binary)	Exponent (Base 10)
00000000	RESERVED
00000001	?
00000010	?
00000011	?
…	?
11111100	?
11111101	?
11111110	?
11111111	RESERVED

Exponent



Exponent (Binary)	Exponent (Base 10)
00000000	RESERVED
00000001	-126
00000010	-125
00000011	-124
…	…
11111100	125
11111101	126
11111110	127
11111111	RESERVED

Exponent



- The exponent is **not** represented in two's complement.
- Instead, exponents are sequentially represented starting from 000…1 (most negative) to 111…10 (most positive).
- **Actual value = binary value – 127**

00000001	1 – 127 = -126
00000010	2 – 127 = -125
…	…
11111101	253 – 127 = 126
11111110	254 – 127 = 127

Fraction

$$x * 2^y$$

- We could just encode whatever x is in the fraction field. But there's a trick we can use to make the most out of the bits we have.

An Interesting Observation

In Base 10:

$42.4 \times 10^5 = 4.24 \times 10^6$

$324.5 \times 10^5 = 3.245 \times 10^7$

$0.624 \times 10^5 = 6.24 \times 10^4$

We tend to adjust the exponent until we get down to one place to the left of the decimal point.

In Base 2:

$10.1 \times 2^5 = 1.01 \times 2^6$

$1011.1 \times 2^5 = 1.0111 \times 2^8$

$0.110 \times 2^5 = 1.10 \times 2^4$

Observation: in base 2, this means there is *always* a 1 to the left of the decimal point!

Fraction

$$x * 2^y$$

- We can adjust this value to fit the format described previously. Then, x will always be in the format **1.XXXXXXXXX…**
- Therefore, in the fraction portion, we can encode just what is *to the right* of the decimal point! This means we get one more digit for precision.

Value encoded = 1._[FRACTION BINARY DIGITS]_

Practice

Sign	Exponent						Fraction			
0	0	…	0	0	0	1	0	1	0	…

Is this number:

A) **Greater than 0?**

B) **Less than 0?**

Is this number:

A) **Less than -1?**

B) **Between -1 and 1?**

C) **Greater than 1?**

Skipping Numbers

- We said that it’s not possible to represent *all* real numbers using a fixed-width representation. What does this look like?
- https://www.h-schmidt.net/FloatConverter/IEEE754.html
- https://twitter.com/D_M_Gregory/status/1044008750162604032

Let's Get Real

What would be nice to have in a real number representation?

- Represent widest range of numbers possible
- Flexible "floating" decimal point
- Represent scientific notation numbers, e.g. 1.2×10^6
- Still be able to compare quickly
- Have more predictable over/under-flow behavior

Representing Zero

- The float representation of zero is all zeros (with any value for the sign bit)

Sign	Exponent	Fraction
`any`	`All zeros`	`All zeros`

- This means there are two representations for zero! ☹

Representing Small Numbers

- If the exponent is all zeros, we switch into “denormalized” mode.

Sign	Exponent	Fraction
`any`	`All zeros`	`Any`

- We now treat the exponent as -126, and the fraction as *without* the leading 1.
- This allows us to represent the smallest numbers as precisely as possible.

Representing Exceptional Values

- If the exponent is all ones, and the fraction is all zeros, we have +- infinity.

Sign	Exponent	Fraction
`any`	`All ones`	`All zeros`

- The sign bit indicates whether it is positive or negative infinity.
- Floats have built-in handling of over/underflow!
  - Infinity + anything = infinity
  - Negative infinity + negative anything = negative infinity
  - Etc.

Representing Exceptional Values

- If the exponent is all ones, and the fraction is nonzero, we have **Not a Number.**

Sign	Exponent						Fraction
any	1	…	…	…	…	1	Any nonzero

- NaN results from computations that produce an invalid mathematical result.
  - Sqrt(negative)
  - Infinity / infinity
  - Infinity + -infinity
  - Etc.

Number Ranges

- 32-bit integer (type **int**):
  - -2,147,483,648 to 2147483647
  - Every integer in that range can be represented
- 64-bit integer (type **long**):
  - −9,223,372,036,854,775,808 to 9,223,372,036,854,775,807
- 32-bit floating point (type **float**):
  - ~1.7 $x10^{-38}$ to ~3.4 $x10^{38}$
  - Not all numbers in the range can be represented (obviously—uncountable)
  - Not even all integers in the range can be represented!
  - Gaps can get quite large! (larger the exponent, larger the gap between successive fraction values)
- 64-bit floating point (type **double**):
  - ~2 $x10^{-308}$ to ~2 $x10^{308}$

Floating Point Arithmetic

You might be thinking: oh, this is just overflowing. But it is more subtle than that.

```
float a = 3.14;
float b = 1e20;
printf("(3.14 + 1e20) - 1e20 = %f\n", (a + b) - b);
printf("3.14 + (1e20 - 1e20) = %f\n", a + (b - b));
```

Let's look at the binary representations for 3.14 and 1e20:

	31	30 … 23	22 … 0
3.14:	0	10000000	10010001111010111000011

	31	30 … 23	22 … 0
1e20:	0	11000001	01011010111100011101100

Floating Point Arithmetic

	31	30 – 23	22 – 0
3.14:	0	10000000	10010001111010111000011
1e20:	0	11000001	01011010111100011101100

You cannot simply add the two significands together, you have to align their binary points. If we wanted to add the decimal values, it would look like this:

```
                           3.14
+ 100000000000000000000.00
  100000000000000000003.14
```

What does this number look like in 32-bit IEEE format?

Floating Point Arithmetic

Step 1: convert from base 10 to binary

What is 100000000000000000003.14 in binary? Let's find out!

http://web.stanford.edu/class/archive/cs/cs107/cs107.1184/float/convert.html

1010110101111000111010111100010110101100011000100000000000000000011.0010001111010111000010100011…

Floating Point Arithmetic

Step 2: find most significant 1 and take the next 23 digits for the fractional component, rounding if needed.

10101101011110001110101111000101101011000110001000000000000000000011.0010001111010111000010100011…

1 01011010111100011101100

Floating Point Arithmetic

Step 3: find how many places we need to shift **left** to put the number in 1.xxx format. This fills in the exponent component.

1010110101111000111010111100010110101100011000100000000000000000011.0010001111010111000010100011…

66 shifts -> 66 + 127 = 193

Floating Point Arithmetic

Step 4: if the sign is positive, the sign bit is 0. Otherwise, it’s 1.

1010110101111000111010111100010110101100011000100000000000000000011.0010001111010111000010100011…

Sign bit is 0.

Floating Point Arithmetic

So, we are left with the following for 100000000000000000003.14 decimal:

31	30 … 23	22 … 0
0	11000001	01011010111100011101100

Let's compare this to 1e20 that we had before:

31	30 … 23	22 … 0
0	11000001	01011010111100011101100

Identical! We didn't have enough bits to differentiate between 1e20 and 100000000000000000003.14

Floating Point Arithmetic

Back to our original example:

```
float a = 3.14;
float b = 1e20;
printf("(3.14 + 1e20) - 1e20 = %f\n", (a + b) - b);
printf("3.14 + (1e20 - 1e20) = %f\n", a + (b - b));
```

```
$ ./floatMultTest
(3.14 + 1e20) - 1e20 = 0.000000
3.14 + (1e20 - 1e20) = 3.140000
```

Clearly, `1e20 - 1e20` will produce `0` (no need to shift the binary points). What this really means is that **floating point arithmetic is not associative**. In other words, the order of operations matters.

Floating Point Arithmetic

Here is another example:

```
int main()
{
    double a = 0.1;
    double b = 0.2;
    double c = 0.3;
    double d = a + b;
    printf("0.1 + 0.2 == 0.3 ? %s\n", a + b == c ? "true" : "false");
    return 0;
}
```

```
$ ./floatEquality
0.1 + 0.2 == 0.3 ? false
```

The rounding that happens during the calculation of 0.1 + 0.2 produces a different number than 0.3!

Floating Point Arithmetic

- http://geocar.sdf1.org/numbers.html

Let’s Get Real

What would be nice to have in a real number representation?

- Represent widest range of numbers possible
- Flexible “floating” decimal point
- Represent scientific notation numbers, e.g. 1.2×10^6
- Still be able to compare quickly
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Floats Summary

- IEEE Floating Point is a carefully-thought-out standard. It's complicated, but engineered for their goals.
- Floats have an extremely wide range, but cannot represent every number in that range.
- Some approximation and rounding may occur! This means you definitely don't want to use floats e.g. for currency.
- Associativity does not hold for numbers far apart in the range
- Equality comparison operations are often unwise.

Recap

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Next time: assembly language