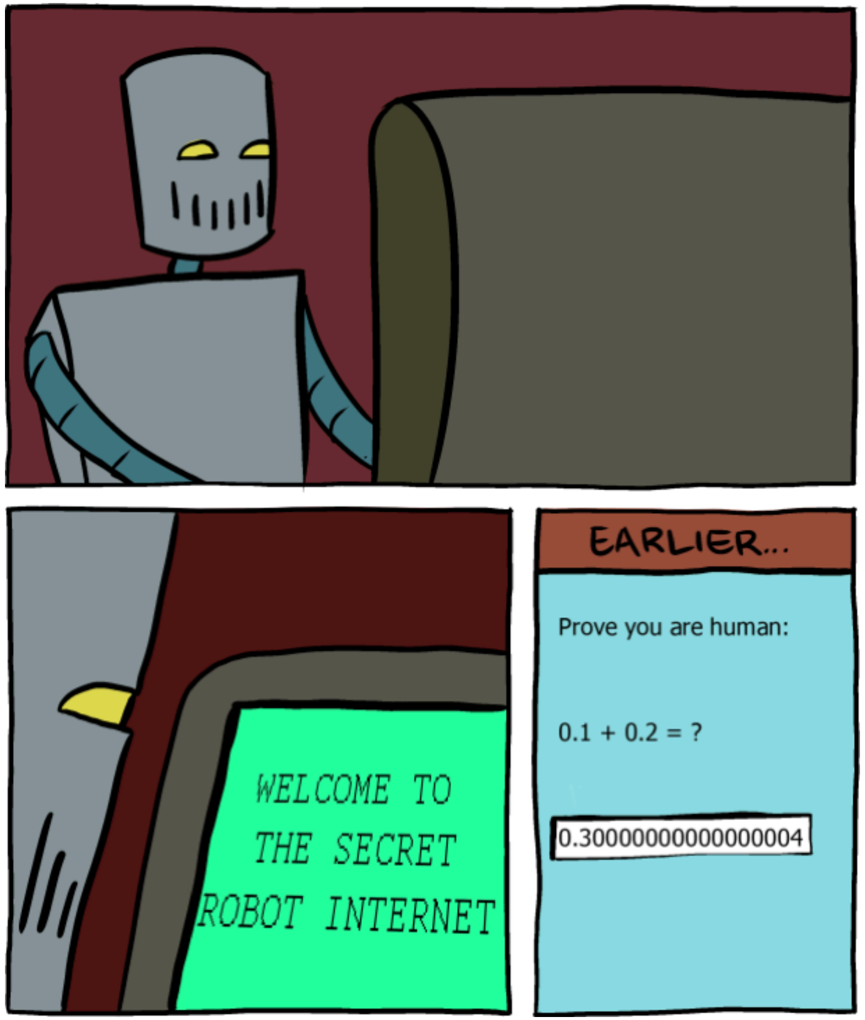
CS107, Lecture 10
Floating Point

Reading: B&O 2.4



<u>CS107 Topic 5</u>: How can a computer represent real numbers in addition to integer numbers?

Learning Goals

Understand the design and compromises of the floating point representation, including:

- Fixed point vs. floating point
- How a floating point number is represented in binary
- Issues with floating point imprecision
- Other potential pitfalls using floating point numbers in programs

Plan For Today

- **Recap:** Generics with Function Pointers
- Representing real numbers
- Fixed Point
- **Break:** Announcements
- Floating Point
- Floating Point Arithmetic

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Function Pointers

- In C, there is a variable type for functions!
- We can pass functions as parameters, store functions in variables, etc.
- Why is this useful?

Generics Limitations

Sometimes, there is functionality that *cannot* be made generic.

```
void bubble_sort(void *arr, int n, int elem_size_bytes) {
    while (true) {
        bool swapped = false;
        for (int i = 1; i < n; i++) {
            void *prev_elem = (char *)arr + (i-1)*elem_size_bytes;
            void *curr_elem = (char *)arr + i*elem_size_bytes;
            if (curr_elem should come before prev_elem) {
                swapped = true;
                swap(prev_elem, curr_elem, elem_size_bytes);
            }
        }

        if (!swapped) {
            return;
        }
    }
}
```

Generics Limitations

Sometimes, there is functionality that *cannot* be made generic. The caller can pass in a function to perform that functionality for the data they are providing.

```
void bubble_sort(void *arr, int n, int elem_size_bytes,
        bool (*cmp_fn)(const void *, const void *)) {
    while (true) {
        bool swapped = false;
        for (int i = 1; i < n; i++) {
            void *prev_elem = (char *)arr + (i-1)*elem_size_bytes;
            void *curr_elem = (char *)arr + i*elem_size_bytes;
            if (cmp_fn(prev_elem, curr_elem) > 0)) {
                swapped = true;
                swap(prev_elem, curr_elem, elem_size_bytes);
            }
        }

        if (!swapped) {
            return;
        }
    }
```

Generic C Standard Library Functions

- **qsort** – I can sort an array of any type! To do that, I need you to provide me a function that can compare two elements of the kind you are asking me to sort.
- **bsearch** – I can use binary search to search for a key in an array of any type! To do that, I need you to provide me a function that can compare two elements of the kind you are asking me to search.
- **lfind** – I can use linear search to search for a key in an array of any type! To do that, I need you to provide me a function that can compare two elements of the kind you are asking me to search.
- **lsearch** - I can use linear search to search for a key in an array of any type! I will also add the key for you if I can't find it. In order to do that, I need you to provide me a function that can compare two elements of the kind you are asking me to search.

Generic C Standard Library Functions

- **scandir** – I can create a directory listing with any order and contents! To do that, I need you to provide me a function that tells me whether or not you want me to include a given directory entry in the listing. I also need you to provide me a function that tells me the correct ordering of two given directory entries.

Function Pointers

Here's the variable type syntax for a function:

```
[return type] (*[name])([parameters])
```

Function Pointers

```
int do_something(char *str) {
    …
}

int main(int argc, char *argv[]) {
    …
    int (*func_var)(char *) = do_something;
    …
    func_var("testing");
    return 0;
}
```

Function Pointers

```
void bubble_sort(void *arr, int n, int elem_size_bytes,
        int (*cmp_fn)(const void *, const void *)) {

…
}

int cmp_double(const void *, const void *) {…}

int main(int argc, char *argv[]) {
    …
    double values[] = {1.2, 3.5, 12.2};
    int n = sizeof(values) / sizeof(values[0]);
    bubble_sort(values, n, sizeof(*values), cmp_double);
    …
}
```

Comparison Functions

- Comparison functions are a common use of function parameters, because many generic functions must know how to compare elements of your type.
- Comparison functions always take *pointers to the data they care about,* since the data could be any size!

When writing a comparison function callback, use the following pattern:

1) Cast the void *argument(s) and set typed pointers equal to them.
2) Dereference the typed pointer(s) to access the values.
3) Perform the necessary operation.

(steps 1 and 2 can often be combined into a single step)

Comparison Functions

- It should return:
  - < 0 if first value should come before second value
  - > 0 if first value should come after second value
  - 0 if first value and second value are equivalent
- This is the same return value format as **strcmp**!

```
int (*compare_fn)(const void *a, const void *b)
```

Function Pointers

```
int integer_compare(void *ptr1, void *ptr2) {
    // cast arguments to int *s and dereference
    int num1 = *(int *)ptr1;
    int num2 = *(int *)ptr2;

    // perform operation
    return num1 – num2;
}

…
qsort(mynums, count, sizeof(*mynums), integer_compare);
```

String Comparison Function

```
int string_compare(void *ptr1, void *ptr2) {
    // cast arguments and dereference
    char *str1 = *(char **)ptr1;
    char *str2 = *(char **)ptr2;

    // perform operation
    return strcmp(str1, str2);
}

…
qsort(mystrs, count, sizeof(*mystrs), string_compare);
```

Generics Wrap-Up

- We use **void *** pointers and memory operations like **memcpy** and **memmove** to make data operations generic.
- We use **function pointers** to make logic/functionality operations generic.

memset

memset is a function that sets a specified amount of bytes at one address to a certain value.

```
void *memset(void *s, int c, size_t n);
```

It fills n bytes starting at memory location **s** with the byte **c**. (It also returns **s**).

```
int counts[5];
memset(counts, 0, 3);   // zero out first 3 bytes at counts
memset(counts + 3, 0xff, 4) // set 3rd entry’s bytes to 1s
```

Plan For Today

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- Fixed Point
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- Floating Point Arithmetic

Real Numbers

- We previously discussed representing integer numbers using two's complement.
- However, this system does not represent real numbers such as 3/5 or 0.25.
- How can we design a representation for real numbers?

Real Numbers

Problem: unlike with the integer number line, where there are a finite number of values between two numbers, there are an *infinite* number of real number values between two numbers!

Integers between 0 and 2: 1

Real Numbers Between 0 and 2: 0.1, 0.01, 0.001, 0.0001, 0.00001,…

We need a fixed-width representation for real numbers. Therefore, by definition, *we will not be able to represent all numbers*.

Real Numbers

Problem: every number base has un-representable real numbers.

Base 10: $1/6_{10} = 0.16666666\ldots\ldots_{10}$

Base 2: $1/10_{10} = 0.0001100110011001100110011\ldots_{2}$

Therefore, by representing in base 2, *we will not be able to represent all numbers*, even those we can exactly represent in base 10.

Fixed Point

- **Idea:** Like in base 10, let's add binary decimal places to our existing number representation.

$$5\ 9\ 3\ 4\ .\ 2\ 1\ 6$$

$$10^3 \quad 10^2 \quad 10^1 \quad 10^0 \qquad 10^{-1} \quad 10^{-2} \quad 10^{-3}$$

$$1\ 0\ 1\ 1\ .\ 0\ 1\ 1$$

$$2^3 \quad 2^2 \quad 2^1 \quad 2^0 \qquad 2^{-1} \quad 2^{-2} \quad 2^{-3}$$

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Fixed Point

- **Idea:** Like in base 10, let's add binary decimal places to our existing number representation.

1	0	1	1	.	0	1	1
8s	4s	2s	1s		1/2s	1/4s	1/8s

- **Pros:** arithmetic is easy! And we know exactly how much precision we have.

Fixed Point

- **Problem:** we have to fix where the decimal point is in our representation. What should we pick? This also fixes us to 1 place per bit.

.	0	1	1	0	0	1	1
	1/2s	1/4s	1/8s	…			

1	0	1	1	0	.	1	1
16s	8s	4s	2s	1s		1/2s	1/4s

Fixed Point

- **Problem:** we have to fix where the decimal point is in our representation. What should we pick? This also fixes us to 1 place per bit.

.	0	1	1	0	0	1	1
	1/2s	1/4s	1/8s	…			

1	0	1	1	0	.	1	1
16s	8s	4s	2s	1s		1/2s	1/4s

Fixed Point

- **Problem:** we have to fix where the decimal point is in our representation. What should we pick? This also fixes us to 1 place per bit.

Base 10 | Base 2

$$5.07E30 = 10\underbrace{\dots\dots}_{\text{100 zeros}}0.1$$

$$9.86E\text{-}32 = 0.0\underbrace{\dots\dots}_{\text{100 zeros}}01$$

To be able to store both these numbers using the same fixed point representation, the bitwidth of the type would need to be at least 207 bits wide!

Let’s Get Real

What would be nice to have in a real number representation?

- Represent widest range of numbers possible
- Flexible “floating” decimal point
- Represent scientific notation numbers, e.g. 1.2×10^6
- Still be able to compare quickly
- Have more predictable over/under-flow behavior

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Midterm Exam

- The midterm exam is **Fri. 5/10 12:30-2:20PM in Nvidia Aud. and 420-041**
  - **Last names A-R: Nvidia Auditorium**
  - **Last Names S-Z: 420-041**
- Covers material through **lab4/assign4** (no floats or assembly language)
- Closed-book, 1 2-sided page of notes permitted, C reference sheet provided
- Administered via BlueBook software (on your laptop)
- Practice materials and BlueBook download available on course website
- If you have academic (e.g. OAE) or athletics accommodations, please let us know by **Sunday 5/5** if possible.
- If you do not have a workable laptop for the exam, you **<u>must</u>** let us know by **Sunday 5/5**. Limited charging outlets will be available for those who need them.

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- Have more predictable over/under-flow behavior

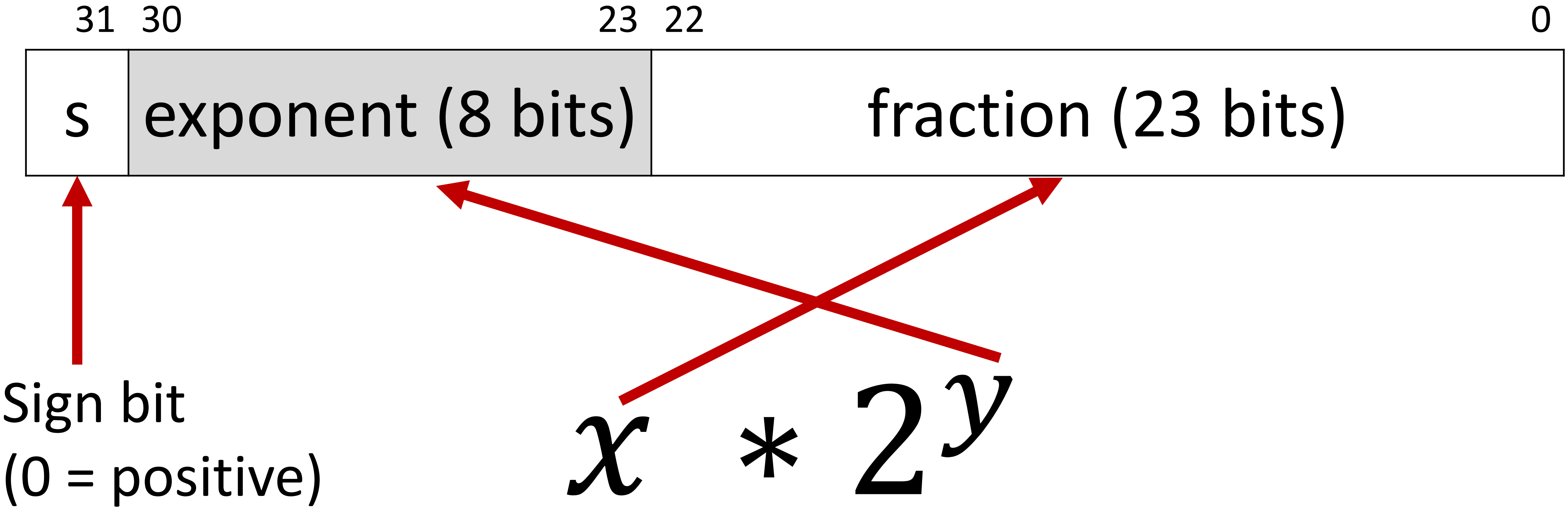
IEEE Floating Point

Let's aim to represent numbers of the following scientific-notation-like format:

$$x * 2^y$$

With this format, 32-bit floats represent numbers in the range ~1.2 $\times 10^{-38}$ to ~3.4 $\times 10^{38}$! Is every number between those representable? **No**.

IEEE Single Precision Floating Point



Exponent

s	exponent (8 bits)	fraction (23 bits)

Exponent (Binary)	Exponent (Base 10)
11111111	?
11111110	?
11111101	?
11111100	?
…	?
00000011	?
00000010	?
00000001	?
00000000	?

Exponent

s	exponent (8 bits)	fraction (23 bits)

Exponent (Binary)	Exponent (Base 10)
11111111	RESERVED
11111110	?
11111101	?
11111100	?
…	?
00000011	?
00000010	?
00000001	?
00000000	RESERVED

Exponent

s	exponent (8 bits)	fraction (23 bits)

Exponent (Binary)	Exponent (Base 10)
11111111	RESERVED
11111110	127
11111101	126
11111100	125
…	…
00000011	-124
00000010	-125
00000001	-126
00000000	RESERVED

Exponent

s	exponent (8 bits)	fraction (23 bits)

- The exponent is **not** represented in two's complement.
- Instead, exponents are sequentially represented starting from 000…1 (most negative) to 111…10 (most positive). This makes bit-level comparison fast.
- **Actual value = binary value – 127 ("bias")**

11111110	254 – 127 = 127
11111101	253 – 127 = 126
…	…
00000010	2 – 127 = -125
00000001	1 – 127 = -126

Fraction

s	exponent (8 bits)	fraction (23 bits)

$$x * 2^y$$

- We could just encode whatever x is in the fraction field. But there's a trick we can use to make the most out of the bits we have.

An Interesting Observation

In Base 10:

$42.4 \times 10^5 = 4.24 \times 10^6$

$324.5 \times 10^5 = 3.245 \times 10^7$

$0.624 \times 10^5 = 6.24 \times 10^4$

We tend to adjust the exponent until we get down to one place to the left of the decimal point.

In Base 2:

$10.1 \times 2^5 = 1.01 \times 2^6$

$1011.1 \times 2^5 = 1.0111 \times 2^8$

$0.110 \times 2^5 = 1.10 \times 2^4$

Observation: in base 2, this means there is *always* a 1 to the left of the decimal point!

Fraction

s	exponent (8 bits)	fraction (23 bits)

$$x * 2^{y}$$

- We can adjust this value to fit the format described previously. Then, x will always be in the format **1.XXXXXXXXX…**
- Therefore, in the fraction portion, we can encode just what is *to the right* of the decimal point! This means we get one more digit for precision.

Value encoded = 1._[FRACTION BINARY DIGITS]_

Practice

Sign	Exponent						Fraction			
0	0	…	0	0	0	1	0	1	0	…

Is this number:

A) **Greater than 0?**

B) **Less than 0?**

Is this number:

A) **Less than -1?**

B) **Between -1 and 1?**

C) **Greater than 1?**

Skipping Numbers

- We said that it's not possible to represent *all* real numbers using a fixed-width representation. What does this look like?

Float Converter

- https://www.h-schmidt.net/FloatConverter/IEEE754.html

Floats and Graphics

- https://www.shadertoy.com/view/4tVyDK

Let's Get Real

What would be nice to have in a real number representation?

- Represent widest range of numbers possible
- Flexible "floating" decimal point
- Represent scientific notation numbers, e.g. 1.2×10^6
- Still be able to compare quickly
- Have more predictable over/under-flow behavior

Representing Zero

The float representation of zero is all zeros (with any value for the sign bit)

Sign	Exponent	Fraction
`any`	`All zeros`	`All zeros`

- This means there are two representations for zero! ☹

Representing Small Numbers

If the exponent is all zeros, we switch into "denormalized" mode.

Sign	Exponent	Fraction
`any`	`All zeros`	`Any`

- We now treat the exponent as -126, and the fraction as *without* the leading 1.
- This allows us to represent the smallest numbers as precisely as possible.

Representing Exceptional Values

If the exponent is all ones, and the fraction is all zeros, we have +- infinity.

Sign	Exponent	Fraction
`any`	`All ones`	`All zeros`

- The sign bit indicates whether it is positive or negative infinity.
- Floats have built-in handling of over/underflow!
  - Infinity + anything = infinity
  - Negative infinity + negative anything = negative infinity
  - Etc.

Representing Exceptional Values

If the exponent is all ones, and the fraction is nonzero, we have **Not a Number.**

Sign	Exponent						Fraction
any	1	…	…	…	…	1	Any nonzero

- NaN results from computations that produce an invalid mathematical result.
  - Sqrt(negative)
  - Infinity / infinity
  - Infinity + -infinity
  - Etc.

Number Ranges

- 32-bit integer (type **int**):
  - -2,147,483,648 to 2147483647
  - Every integer in that range can be represented
- 64-bit integer (type **long**):
  - −9,223,372,036,854,775,808 to 9,223,372,036,854,775,807
- 32-bit floating point (type **float**):
  - ~1.2 $x10^{-38}$ to ~3.4 $x10^{38}$
  - Not all numbers in the range can be represented (not even all integers in the range can be represented!)
  - Gaps can get quite large! (larger the exponent, larger the gap between successive fraction values)
- 64-bit floating point (type **double**):
  - ~2.2 $x10^{-308}$ to ~1.8 $x10^{308}$

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Demo: Float Arithmetic

Floating Point Arithmetic

Is this just overflowing? It turns out it's more subtle.

```
float a = 3.14;
float b = 1e20;
printf("(3.14 + 1e20) - 1e20 = %g\n", (a + b) - b);  // prints 0
printf("3.14 + (1e20 - 1e20) = %g\n", a + (b - b)); // prints 3.14
```

Let's look at the binary representations for 3.14 and 1e20:

	31	30 … 23	22 … 0
3.14:	0	10000000	10010001111010111000011
1e20:	0	11000001	01011010111100011101100

Floating Point Arithmetic

3.14:

31	30 … 23	22 … 0
0	10000000	10010001111010111000011

1e20:

31	30 … 23	22 … 0
0	11000001	01011010111100011101100

To add real numbers, we must align their binary points:

```
                     3.14
+ 100000000000000000000.00
--------------------------
  100000000000000000003.14
```

What does this number look like in 32-bit IEEE format?

Floating Point Arithmetic

Step 1: convert from base 10 to binary

What is 100000000000000000003.14 in binary? Let's find out!

http://web.stanford.edu/class/archive/cs/cs107/cs107.1184/float/convert.html

1010110101111000111010111100010110101100011000100000000000000000011.0010001111010111000010100011…

Floating Point Arithmetic

Step 2: find most significant 1 and take the next 23 digits for the fractional component, rounding if needed.

1010110101111000111010111100010110101100011000100000000000000000000011.0010001111010111000010100011…

1 01011010111100011101100

Floating Point Arithmetic

Step 3: find how many places we need to shift **left** to put the number in 1.xxx format. This fills in the exponent component.

1010110101111000111010111100010110101100011000100000000000000000000011.0010001111010111000010100011…

66 shifts -> 66 + 127 = 193

Step 4: if the sign is positive, the sign bit is 0. Otherwise, it’s 1.

10101101011110001110101111000101101011000110001000000000000000000011.0010001111010111000010100011…

Sign bit is 0.

Floating Point Arithmetic

The binary representation for 1e20 + 3.14 thus equals the following:

31	30 … 23	22 … 0
0	11000001	01011010111100011101100

This is the **same** as the binary representation for 1e20 that we had before!

We didn't have enough bits to differentiate between 1e20 and 1e20 + 3.14.

Floating Point Arithmetic

Is this just overflowing? It turns out it's more subtle.

```
float a = 3.14;
float b = 1e20;
printf("(3.14 + 1e20) - 1e20 = %g\n", (a + b) - b);  // prints 0
printf("3.14 + (1e20 - 1e20) = %g\n", a + (b - b)); // prints 3.14
```

Floating point arithmetic is not associative. The order of operations matters!

- The first line loses precision when first adding 3.14 and 1e20, as we have seen.
- The second line first evaluates 1e20 – 1e20 = 0, and then adds 3.14

Demo: Float Equality

Floating Point Arithmetic

Float arithmetic is an issue with most languages, not just C!

- http://geocar.sdf1.org/numbers.html

Let’s Get Real

What would be nice to have in a real number representation?

- Represent widest range of numbers possible
- Flexible “floating” decimal point
- Represent scientific notation numbers, e.g. 1.2×10^6
- Still be able to compare quickly
- Have more predictable over/under-flow behavior

Floats Summary

- IEEE Floating Point is a carefully-thought-out standard. It's complicated, but engineered for their goals.
- Floats have an extremely wide range, but cannot represent every number in that range.
- Some approximation and rounding may occur! This means you definitely don't want to use floats e.g. for currency.
- Associativity does not hold for numbers far apart in the range
- Equality comparison operations are often unwise.

Recap

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Next time: assembly language