# CS107 Lecture 10
## Floating Point

reading:

*B&O 2.4*

Slides by Jerry Cain and Lisa Yan, who leveraged prior work by Nick Troccoli, Julie Zelenski, Marty Stepp, Cynthia Lee, Chris Gregg, and others.

# CS107 Topic 5: How can a computer represent real numbers in addition to integer numbers?

# Learning Goals

Understand the design and compromises of the floating point representation, including:

- Fixed point vs. floating point
- How a floating point number is represented in binary
- Issues with floating point imprecision
- Other potential pitfalls using floating point numbers in programs

# Plan For Today

- Representing real numbers and (thought experiment) fixed point
- Floating Point: Normalized values
- **Break**: Announcements
- Floating Point: Special/denormalized values
- Floating Point Arithmetic

```
cp -r /afs/ir/class/cs107/samples/lectures/lect10 .
```

# Review: Function pointers

`fn_ptr_review.c`

# Warm-up (?): Function pointers

Review

```
1  int compare_strings(const void *p,
2                      const void *q) {
3    char *str1 = *(const char **) p;
4    char *str2 = *(const char **) q;
5    return strcmp(str1, str2);
6  }
7
8  void test_bsearch() {
9    char *words[] = {"aardvark", "beaver", "capybara"}; // sorted
10   int n = sizeof(words) / sizeof(*words);
11   char *searchkey = strdup("beaver");
12   char **found = bsearch(&searchkey, words, n, sizeof(words[0]),
13                          compare_strings);
14   printf("found %s\n", found ? *found : "none");
15   free(searchkey);
16 }
```

1. What are types are the parameters passed into `bsearch`?
2. Let's change line 3 to:
   `char *str1 = (const char *) p;`
   What happens?

fn_ptr_review.c

Tips: (1) draw pictures, and (2) read man pages carefully.

# Course Topic Overview



1. **Bits and Bytes -** *How can a computer represent integer numbers?*
2. **Chars and C-Strings -** *How can a computer represent and manipulate more complex data like text?*
3. **Pointers, Stack and Heap –** *How can we effectively manage all types of memory in our programs?*
4. **Generics -** *How can we use our knowledge of memory and data representation to write code that works with any data type?*

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5. **Floats -** *How can a computer represent floating point numbers in addition to integer numbers?*
6. **Assembly -** *How does a computer interpret and execute C programs?*
7. **Heap Allocators -** *How do core memory-allocation operations like malloc and free work?*

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- Floating point arithmetic

# Base 2 conversion

Convert the following number/fractions to base 10 (decimal) and base 2.

| | Number/fraction | Base 10 | Base 2 |
|---|---|---|---|
| 1. | `1/2` | `0.5` | `0.1` |
| 2. | `5` | | |
| 3. | `9/8` | | |
| 4. | `1/3` | | |
| 5. | `1/10` (bonus) | | |

# Base 2 conversion

Convert the following number/fractions to base 10 (decimal) and base 2.

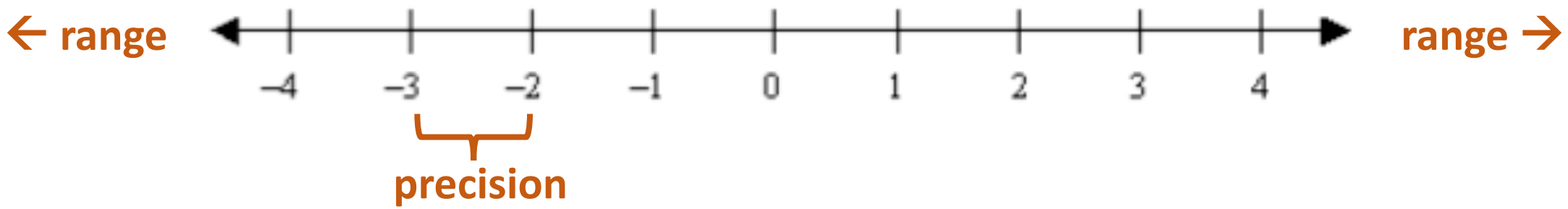
| | Number/fraction | Base 10 | Base 2 |
|---|---|---|---|
| 1. | 1/2 | 0.5 | 0.1 |
| 2. | 5 | 5.0 | 101.0 |
| 3. | 9/8 | 1.125 | 1.001 |
| 4. | 1/3 | 0.3333… | 0.01010101… |
| 5. | 1/10 (bonus) | 0.1 | 0.00011001100… |

Conceptual goal: How can we represent real numbers with a ***fixed*** number of bits?

**Learning goal**: Appreciate the IEEE floating point format!

# Approximating real numbers

How can we represent real numbers with a ***fixed*** number of bits?




In the world of real numbers:

- The real number line extends forever (infinite **range**).
- Real numbers have infinite resolution (infinite **precision**).

In the base-2 world of computers, we must **approximate**:

- Each variable type is fixed width (`float`: 32 bits, `double`: 64 bits).
- Compromises are inevitable (**range** and **precision**). Like with `int`, we need to make choices about which numbers make the cut and which don't.

# Thought experiment: Fixed point

Base 10, decimal case:

$5934.216123121\ldots_{10}$

5 9 3 4 . 2 1 6 7

$10^3$ $10^2$ $10^1$ $10^0$ $10^{-1}$ $10^{-2}$ $10^{-3}$ $10^{-4}$

Base 2, binary case:

$1011.0101010101\ldots_2$

1 0 1 1 . 0 1 0 1

$2^3$ $2^2$ $2^1$ $2^0$ $2^{-1}$ $2^{-2}$ $2^{-3}$ $2^{-4}$

- Decide on fixed granularity, e.g., 1/16
- Assign bits to represent powers from $2^3$ to $2^{-4}$

# Thought experiment: Fixed point

Strategy evaluation

**What values can be represented?**

- Largest magnitude? Smallest? To what precision?

**How hard to implement?**

- How to convert `int` into 32-bit fixed point? 32-bit fixed point to `int`?
- Can existing integer ops (add, multiply, shift) be repurposed?

**How well does this meet our needs?**

Base 2, binary case:

$1011.0101010101\ldots_2$

1 0 1 1 . 0 1 0 1

$2^3$ $2^2$ $2^1$ $2^0$ $2^{-1}$ $2^{-2}$ $2^{-3}$ $2^{-4}$

- Decide on fixed granularity, e.g., 1/16
- Assign bits to represent powers from $2^3$ to $2^{-4}$

# The problem with fixed point

**Problem:** We must fix where the decimal point is in our representation. This fixes our **precision**.

$$6.022e23 = \underbrace{11\ldots0}_{79\text{ bits}}.0$$

(base 10) (base 2)

$$6.626e\text{-}34 = 0.\underbrace{0\ldots01}_{113\text{ bits}}$$

To store both these numbers in the same fixed-point representation, the bit width of the type would need to be at least 192 bits wide!

# Scientific notation to the rescue (1/2)

We have a need for relative rather than absolute precision.

- How much error/approximation is tolerable? Radius of atom, bowling ball, planet?

Consider for decimal values:

$3{,}650{,}123 \rightarrow 3.65 \times 10^{6}$

$0.0000072491 \rightarrow 7.25 \times 10^{-6}$

1 digit for exponent

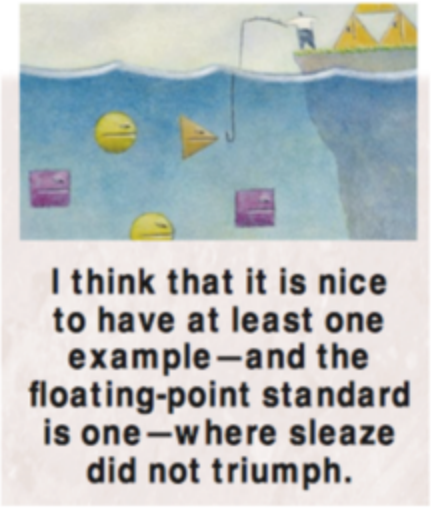
3 digits for mantissa (and round)

- As a datatype, store mantissa and exponent separately
- Division of digits into exponent/mantissa determines range and precision

# IEEE floating point

**IEEE Standard 754**

- Established in 1985 as uniform standard for floating point arithmetic
- Supported by all major systems today
  Hardware: specialized co-processor vs. integrated into main chip




*— Will Kahan*
*(chief architect of standard)*

**Driven by numerical concerns**

- Behavior defined in mathematical terms
- Clear standards for rounding, overflow, underflow
- Support for transcendental functions (roots, trig, exponentials, logs)
- Hard to make fast in hardware
  **Numerical analysts** predominated over hardware designers in defining standard

# Plan For Today

- Representing real numbers and fixed point
- Floating Point: Normalized values
- **Break:** Announcements
- Floating Point: Special/denormalized values
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# IEEE Floating Point

Let's aim to represent numbers of the following scientific-notation-like format:

$$V = (-1)^{s} \times M \times 2^{E}$$

Sign bit, $s$:
negative (s == 1)
positive (s == 0)

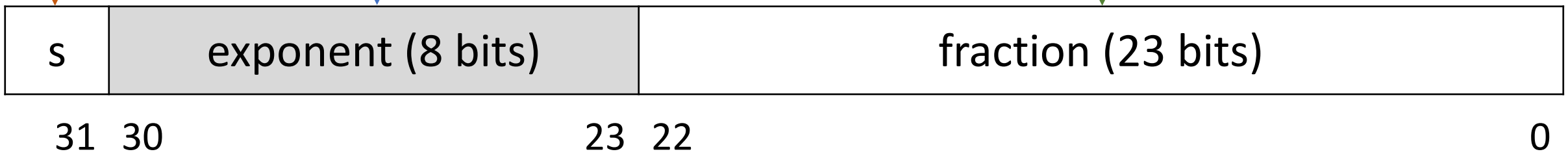
Mantissa, $M$ :
Significant digits, also called significand

Exponent, $E$:
Scales value by (possibly negative) power of 2

# IEEE Floating Point

Let's aim to represent numbers of the following scientific-notation-like format:

$$V = (-1)^{s} \times M \times 2^{E}$$




⚠ Quirky! Exponent and fraction are not encoded as $M$ and $E$ in 2's complement!

# Example for the next few slides

What is the number represented by the following 32-bit `float`?

| s | exponent (8 bits) | fraction (23 bits) |
|---|---|---|
| 0 | 1000 0000 | 0100 0000 0000 0000 0000 000 |

# Exponent


| exponent (Binary) | $E$ (Base 10) |
|---|---|
| 11111111 | RESERVED |
| 11111110 | 127 |
| 11111101 | 126 |
| 11111100 | 125 |
| … | … |
| 00000011 | -124 |
| 00000010 | -125 |
| 00000001 | -126 |
| 00000000 | RESERVED |

special

normalized

denormalized

# Exponent: Normalized values


| exponent (Binary) | $E$ (Base 10) |
|---|---|
| 11111111 | RESERVED |
| 11111110 | 127 |
| 11111101 | 126 |
| 11111100 | 125 |
| … | … |
| 00000011 | -124 |
| 00000010 | -125 |
| 00000001 | -126 |
| 00000000 | RESERVED |

- Based on this table, how do we compute an exponent from a binary value?
- Why would this be a good idea? (hint: what if we wanted to compare two floats with >, <, =?)

# Exponent: Normalized values


| exponent (Binary) | $E$ (Base 10) |
|---|---|
| 11111111 | RESERVED |
| 11111110 | 127 |
| 11111101 | 126 |
| 11111100 | 125 |
| … | … |
| 00000011 | -124 |
| 00000010 | -125 |
| 00000001 | -126 |
| 00000000 | RESERVED |

- Not 2's complement
- $E$ = exponent – bias, where `float` bias = $2^{8-1} - 1 = 127$
- Exponents are sequentially represented starting from 000…1 (most negative) to 111…10 (most positive).
- Bit-level comparison is fast!

# Fraction: Normalized values


We could just encode whatever $M$ is in the fraction field ($f$). But there's a trick we can use to make the most out of the bits we have…

$$M = 1.[\text{fraction bits}]$$

What???

# Scientific notation to the rescue (2/2)

Correct scientific notation:
In the mantissa, always keep one non-zero digit to the left of the decimal point.

For base 10:

$42.4 \times 10^5 \rightarrow \mathbf{4}.24 \times 10^6$

$324.5 \times 10^5 \rightarrow \mathbf{3}.245 \times 10^7$

$0.624 \times 10^5 \rightarrow \mathbf{6}.24 \times 10^4$

For base 2:

$10.1 \times 2^5 \rightarrow \mathbf{1}.01 \times 2^6$

$1011.1 \times 2^5 \rightarrow \mathbf{1}.0111 \times 2^8$

$0.110 \times 2^5 \rightarrow \mathbf{1}.10 \times 2^4$

**Observation:** in base 2, this means there is *always* a 1 to the left of the decimal point!

# Fraction: Normalized values


We could just encode whatever $M$ is in the fraction field ($f$). But there's a trick we can use to make the most out of the bits we have…

$$M = 1.[\text{fraction bits}]$$

- An "implied leading 1" representation
- Means: we get one additional bit of precision for free!

Thanks, Will!

# Example from the past few slides

What is the number represented by the following 32-bit `float`?

| s | exponent (8 bits) | fraction (23 bits) |
|---|---|---|
| 0 | 1000 0000 | 0100 0000 0000 0000 0000 000 |

Subtract bias
($2^{8-1} - 1 = 127$)

Add implicit 1

$$E = 128 - 127 = 1$$

$$M = (1.01)_2 \quad \text{(base 2)}$$
$$= 1 + 0\times2^{-1} + 1\times2^{-2} = 1.25 \quad \text{(base 10)}$$

$$V = (-1)^0 \times 1.25 \times 2^1 = 2.5$$

# Practice #1

| s | exponent (8 bits) | fraction (23 bits) |
|---|---|---|
| 0 | 0111 1110 | 0000 0000 0000 0000 0000 000 |

1. Is this number:
   A. Greater than 0?
   B. Less than 0?

2. Is this number:
   A. Less than -1?
   B. Between -1 and 1?
   C. Greater than 1?

3. Bonus: What is the number?

# Practice #1

| s | exponent (8 bits) | fraction (23 bits) |
|---|---|---|
| 0 | 0111 1110 | 0000 0000 0000 0000 0000 000 |

1. Is this number:
   A. Greater than 0?
   B. Less than 0?

2. Is this number:
   A. Less than -1?
   B. Between -1 and 1?
   C. Greater than 1?

3. Bonus: What is the number? $1.0 \times 2^{-1} = 0.5$

# Plan For Today

- Representing real numbers and fixed point
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# Midterm Exam

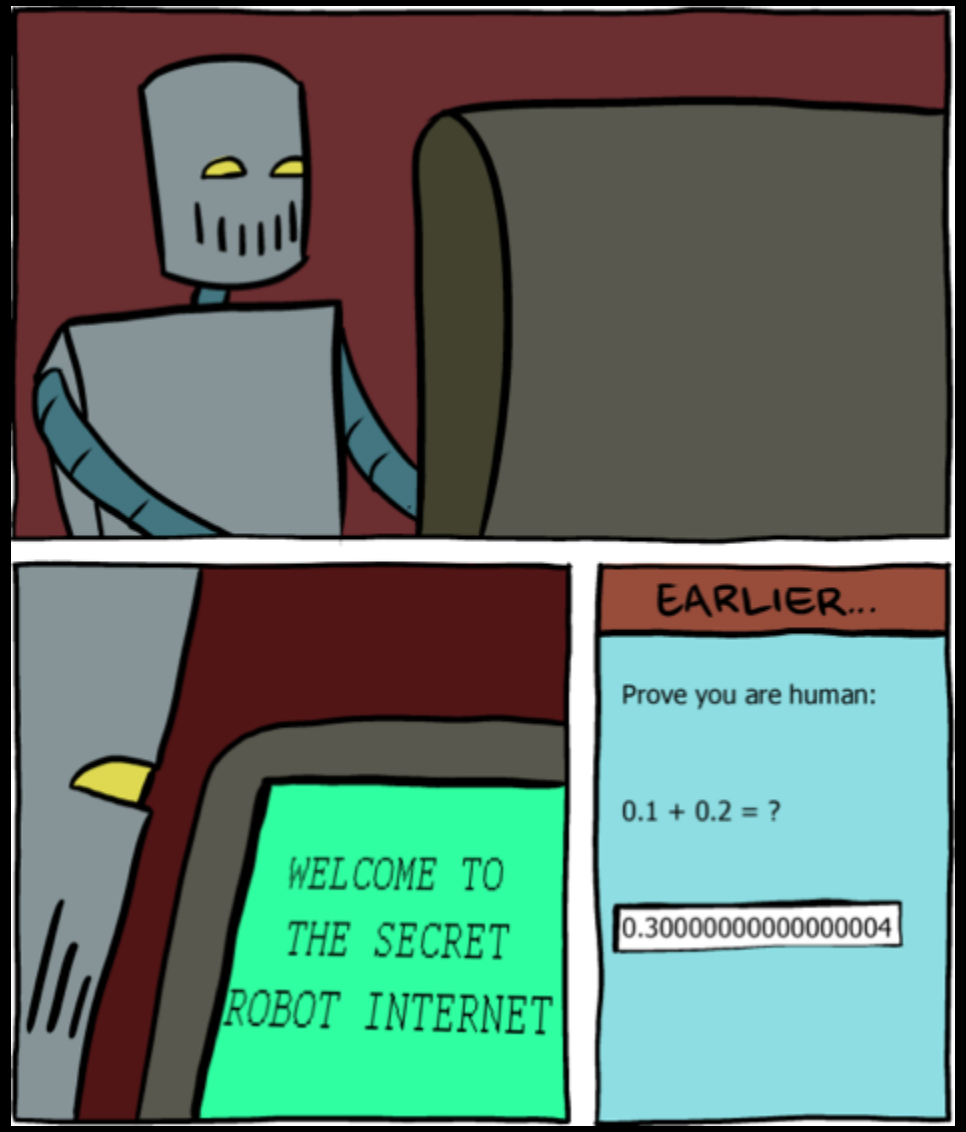
The midterm exam is **Fri. 2/14 12:30PM-2:20PM in Hewlett 200.**

- Covers material through **lab4/assign4** (no floats or assembly language)
- Closed-book, 1 2-sided page of notes permitted, C reference sheet provided

Administered via BlueBook software (on your laptop)

- Practice materials and BlueBook download available on course website
- If you have academic (e.g. OAE) or athletics accommodations, please let us know by **Sunday 2/9** if possible.
- If you do not have a workable laptop for the exam, you **must** let us know by **Sunday 2/9**. Limited charging outlets will be available for those who need them.

# Joke break




https://www.smbc-comics.com/comic/2013-06-05

Slightly off from the real float 0.3 ☺

https://www.h-schmidt.net/FloatConverter/IEEE754.html

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# Reserved exponent values


| exponent (Binary) | $E$ (Base 10) |
|---|---|
| 11111111 | RESERVED |
| 11111110 | 127 |
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| 11111100 | 125 |
| … | … |
| 00000011 | -124 |
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| 00000001 | -126 |
| 00000000 | RESERVED |

special

normalized

denormalized

# Reserved exponent values: All zeros

Zero (+0, -0)

| s | exponent (8 bits) | fraction (23 bits) |
|---|---|---|
| any | 0000 0000 | all zeros |

Why would two zeros be okay?

---

**Denormalized** floats:

| s | exponent (8 bits) | fraction (23 bits) |
|---|---|---|
| any | 0000 0000 | any nonzero |

- Smallest normalized exponent: $E = 1 - \text{bias} = -126$
- Mantissa has no leading zero: $M = 0.[\text{fraction bits}]$

$$V = (-1)^s \times M \times 2^E$$

Why would we want so much precision for tiny numbers?

# Reserved exponent values: All zeros

Zero (+0, -0)

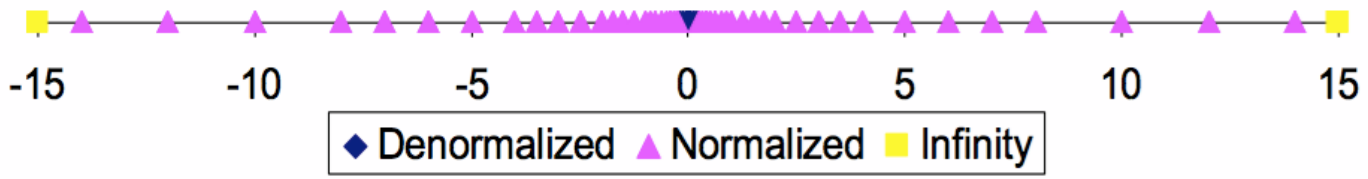
| s | exponent (8 bits) | fraction (23 bits) |
|---|---|---|
| any | 0000 0000 | all zeros |

**Denormalized** floats:

| s | exponent (8 bits) | fraction (23 bits) |
|---|---|---|
| any | 0000 0000 | any nonzero |

- Smallest normalized exponent: $E = 1 - \text{bias} = -126$
- Mantissa has no leading zero: $M = 0.[\text{fraction bits}]$

$$V = (-1)^s \times M \times 2^E$$



# Reserved exponent values: All ones

Infinity (`+inf`, `-inf`)

| s | exponent (8 bits) | fraction (23 bits) |
|---|---|---|
| any | `1111 1111` | **`all zeros`** |

Why would we want to represent infinity?

---

Not a number (**`NaN`**):

| s | exponent (8 bits) | fraction (23 bits) |
|---|---|---|
| any | `1111 1111` | **`any nonzero`** |

Computation result that is an invalid mathematical real number.

What kind of mathematical computation would result in a **non-real** number? (hint: square root)

# Reserved exponent values: All ones

Infinity (`+inf`, `-inf`)

| s | exponent (8 bits) | fraction (23 bits) |
|---|---|---|
| any | `1111 1111` | **`all zeros`** |

Floats have built-in handling of overflow:
infinity + anything = infinity

---

Not a number (**`NaN`**):

| s | exponent (8 bits) | fraction (23 bits) |
|---|---|---|
| any | `1111 1111` | **`any nonzero`** |

Computation result that is an invalid mathematical real number.

Examples: `sqrt(x)` (i.e., $\sqrt[2]{x}$), where x is negative, $\frac{\infty}{\infty}$, $\infty + (-\infty)$, etc.

# Questions?

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# Skipping Numbers

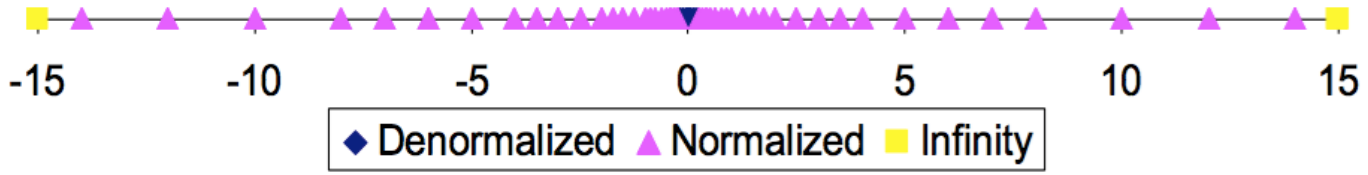
We said that it's not possible to represent *all* real numbers using a fixed-width representation.  What does this look like?

Float Converter

- https://www.h-schmidt.net/FloatConverter/IEEE754.html

Floats and Graphics

- https://www.shadertoy.com/view/4tVyDK



# `float` and `double`

`float` (32 bits)

- 8-bit exponent ranges from -126 to +127, $2^{127} = 10^{37}$


`double` (64 bits)

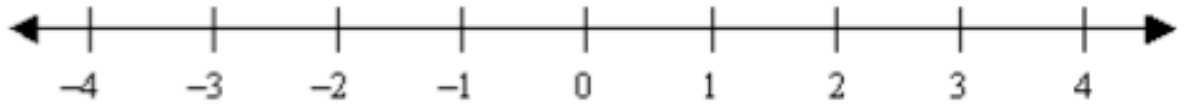
- 11-bit exponent ranges from -1022 to +1023, $2^{1023} = 10^{308}$

# `float` and `int`

32-bit integer (type **`int`**): :
–2,147,483,648 to 2147483647

64-bit integer (type **`long`**):
–9,223,372,036,854,775,80
to 9,223,372,036,854,775,807

All integers in these ranges can be represented.




32-bit floating point (type **`float`**):

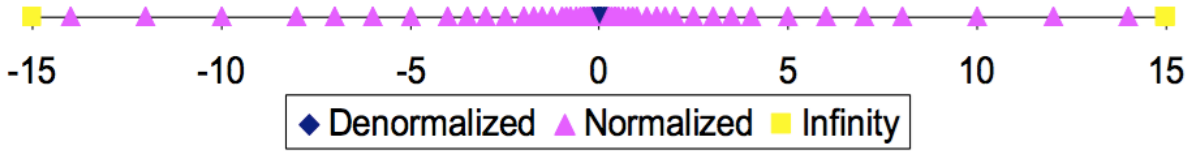
~1.7 x$10^{-38}$ to ~3.4 x$10^{38}$ (+ negative range)

64-bit floating point (type **`double`**):

~9 x$10^{-307}$ to ~1.8 x$10^{308}$ (+ negative range)

(normalized float/double ranges)

Not all numbers can be represented. **Gaps can get quite large**: larger the exponent, larger the gap between successive fraction values.



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# Key (floating) points

Approximation and rounding is inevitable.

Single operations are commutative, but sequence is not associative.

(a + b) equals (b + a)

But (a + b) + c may not equal a + (b + c)

Equality comparison operations are often unwise.

# Key (floating) points

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# Lisa's Official Guide To Making Money



# Demo: Float Arithmetic

bank.c

Try it yourself:

```
./bank 100 1          # deposit
./bank 100 -1         # withdraw
./bank 100000000 -1 # make bank
./bank 16777216 1    # lose bank
```

Why is $2^{24}$ special?

# Introducing "Minifloat"

For a more compact example representation, we will use an 8 bit "minifloat" with a 4 bit exponent, 3 bit fraction and bias of 7 (note: minifloat is just for example purposes, and is not a real datatype).

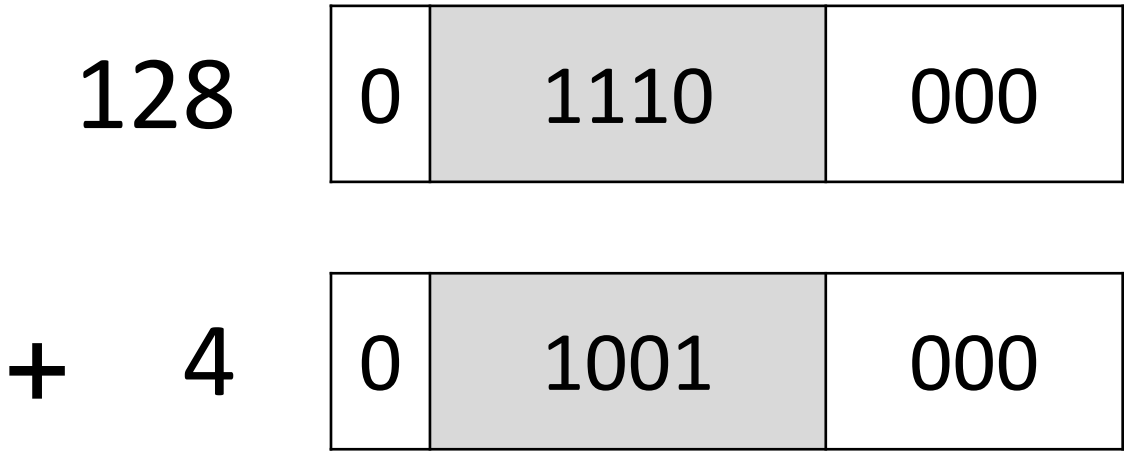
# Floating Point Arithmetic

In minifloat, with a balance of $128, a deposit of $4 **would not be recorded** at Lisa's Bank. Why not?

$$\begin{array}{r} 128 \\ +\quad 4 \\ \hline 128? \end{array}$$

Let's step through the calculations to add these two numbers (note: this is just for understanding; real float calculations are more efficient).

# Floating Point Arithmetic




```
    128.00          1.00 x 2^7
+     4.00    + 1.00 x 2^2
-------------------------------
    132.00
```

aligned | not aligned

Float arithmetic (we won't go into details): x floatop y = Round(x op y)

- Manipulate significand and exponents independently
- Compute exact result (x op y) — Exact result: 132
- Round and put in floating point — (next slide)

# Practice #2

| 7 | 6 ... 3 | 2 ... 0 |
|---|---|---|
| s | exponent (4 bits) | fraction (3 bits) |
| ? | ???? | ??? |

What is 132 as a minifloat?

$$V = (-1)^s \times M \times 2^E$$

$E$ = exponent − bias

$M$ = 1.[fraction bits]

bias $= 2^{4-1} - 1 = 7$

# Practice #2

| 7 | 6 ... 3 | 2 ... 0 |
|---|---|---|
| s | exponent (4 bits) | fraction (3 bits) |
| 0 | 1110 | 000 |

$$V = (-1)^s \times M \times 2^E$$

$E$ = exponent − bias

$M$ = 1.[fraction bits]

bias = $2^{4-1} - 1 = 7$

What is 132 as a minifloat?

1. Convert to binary.
2. Convert to scientific notation $M \times 2^E$.
3. Determine exponent = $E$ + bias
4. Determine fraction, with rounding.
5. Determine sign s

$(1000\ 0100)_2$

$(1.0000100)_2 \times 2^7$

$7 + 7 = 14 = (1110)_2$

$(1.0000100)_2$

0 (positive)

# Approximation error is inevitable

| | | | |
|---|---|---|---|
| 128 | 0 | 1110 | 000 |
| + 4 | 0 | 1001 | 000 |
| 132? | 0 | 1110 | 000 |

**We didn’t have enough bits to differentiate between 128 and 132.**

# Approximation error is inevitable

| | | | |
|---|---|---|---|
| 128 | 0 | 1110 | 000 |
| + 4 | 0 | 1001 | 000 |
| 132 | 0 | 1110 | 000 |

**We didn't have enough bits to differentiate between 128 and 132.**

Another way to corroborate this: the *next-largest minifloat* that can be represented after 128 is **144**. 132 isn't representable!

| | | | |
|---|---|---|---|
| 144: | 0 | 1110 | 001 |

**Key Idea:** the smallest float hop increase we can take is to increase the fractional component by 1.

# Key (floating) points

Approximation and rounding is inevitable.

Single operations are commutative, but sequence is not associative.

(a + b) equals (b + a)

But (a + b) + c may not equal a + (b + c)

Equality comparison operations are often inaccurate.

# Floating Point Arithmetic

Is this just overflowing? It turns out it's more subtle.

```
float a = 3.14;
float b = 1e20;
printf("(3.14 + 1e20) - 1e20 = %g\n", (a + b) - b);  // prints 0
printf("3.14 + (1e20 - 1e20) = %g\n", a + (b - b)); // prints 3.14
```

**Floating point arithmetic is not associative**. The order of operations matters!

- The first line loses precision when first adding 3.14 and 1e20, as we have seen.
- The second line first evaluates 1e20 – 1e20 = 0, and then adds 3.14

# Key (floating) points

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But (a + b) + c may not equal a + (b + c)

Equality comparison operations are often inaccurate.

# Demo: Float Equality

`float_equality.c`

# Floating point in other languages

Float arithmetic is an issue with most languages, not just C!

- http://geocar.sdf1.org/numbers.html

# Key (floating) points

Approximation and rounding is inevitable.

Single operations are commutative, but sequence is not associative.

(a + b) equals (b + a)

But (a + b) + c may not equal a + (b + c)

Equality comparison operations are often ~~unwise~~ inaccurate.

# Let's Get Real (and Fixed-Width)

What would be nice to have in a real number representation?

- ✓ Represent widest range of numbers possible
- ✓ Flexible "floating" decimal point
- ✓ Still be able to compare quickly
- ✓ Represent scientific notation numbers, e.g. $1.2 \times 10^6$
- ✓ Have more predictable overflow behavior