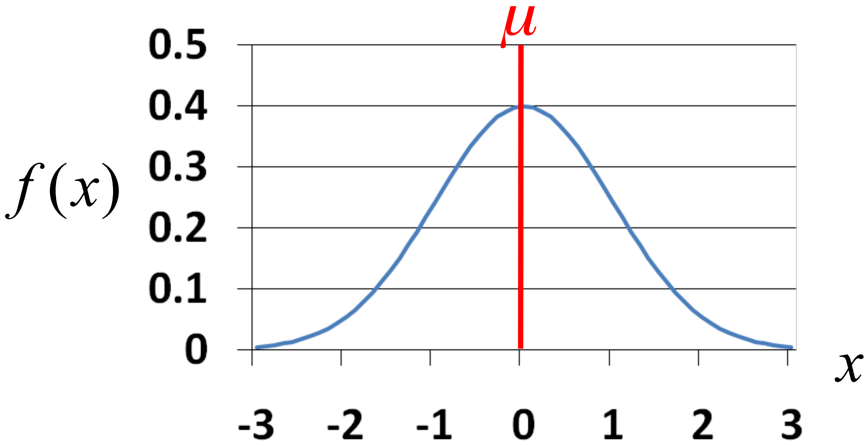
Gaussian

CS 109
Lecture 10
April 18th, 2016

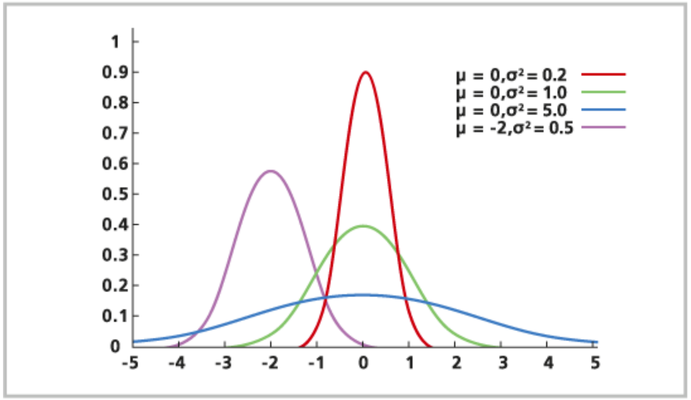
The Normal Distribution

- X is a **Normal Random Variable**: $X \sim N(\mu, \sigma^2)$
  - Probability Density Function (PDF):

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} \quad \text{where} \quad -\infty < x < \infty$$



  - $E[X] = \mu$
  - $Var(X) = \sigma^2$
  - Also called "Gaussian"
  - Note: $f(x)$ is symmetric about μ



Why Use the Normal?

- Common for natural phenomena: heights, weights, etc.
- Often results from the sum of multiple variables
- Most noise is Normal

Or that is what they want you to believe

Except Not

These are log-normal

- Common for natural phenomena: heights, weights, etc.

Only if they are equally weighted

- Often results from the sum of multiple variables

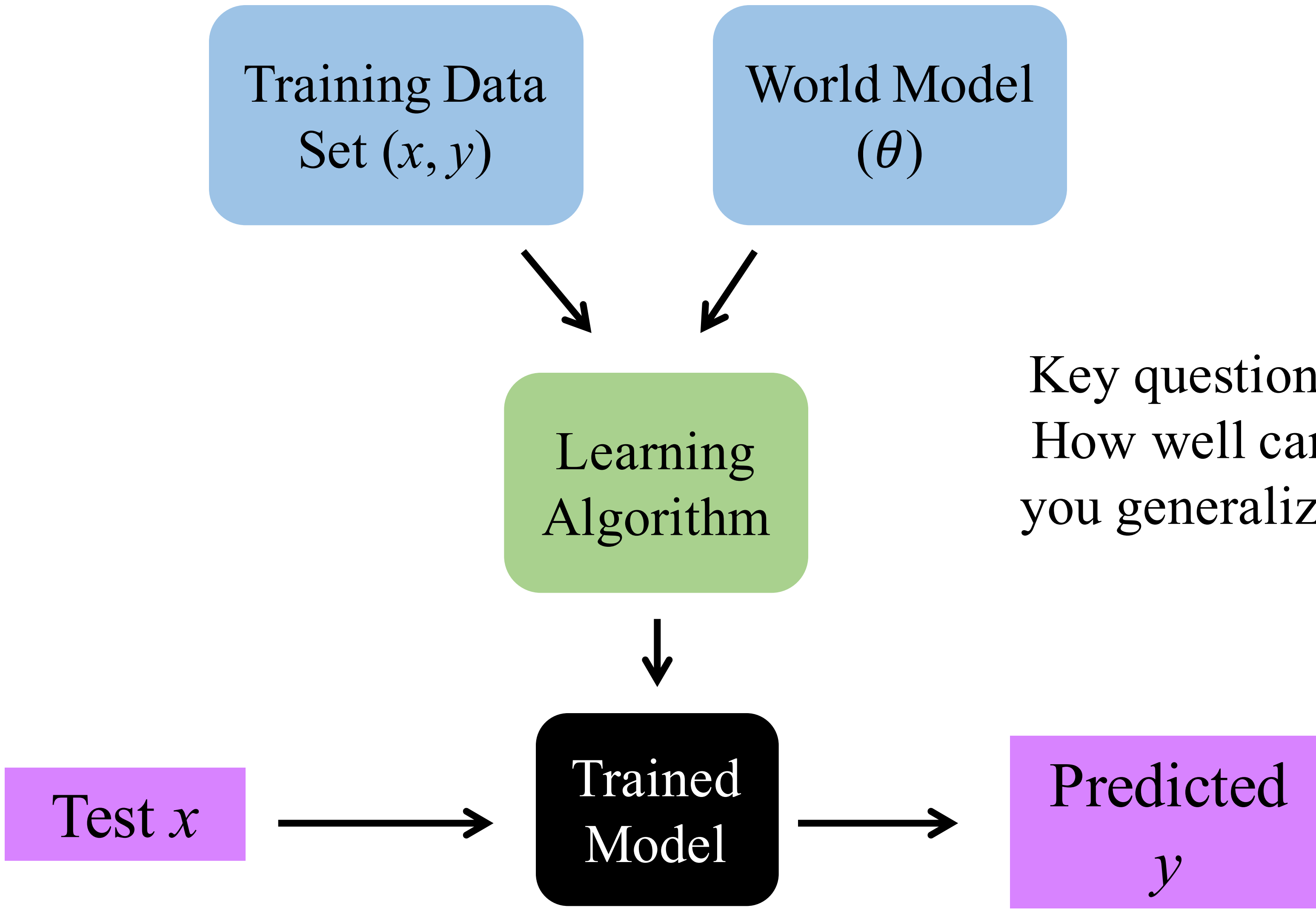
I wish it were that easy

- Most noise is Normal

It is the most important distribution

Because of a deeper truth…

Supervised Learning

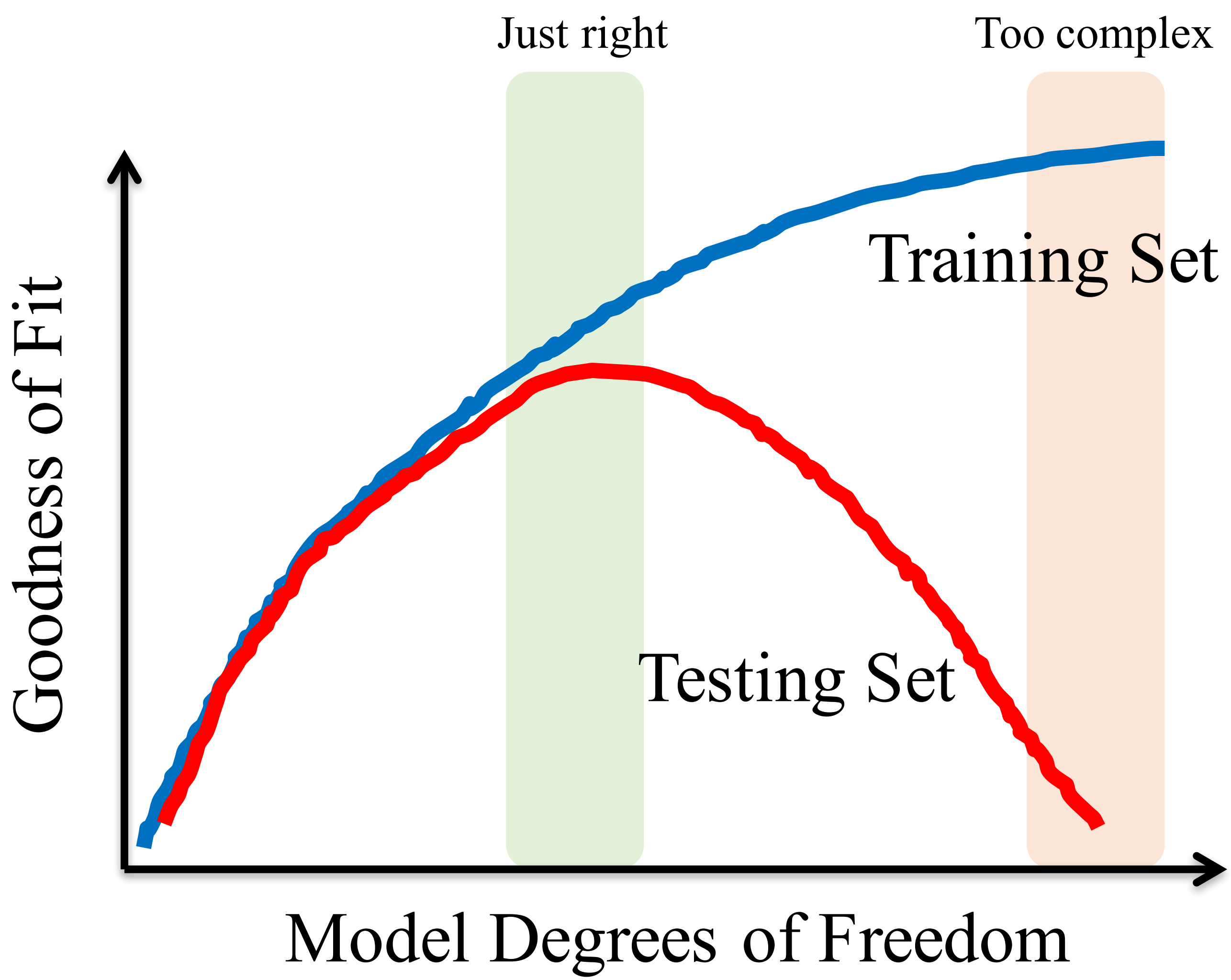


Key question: How well can you generalize

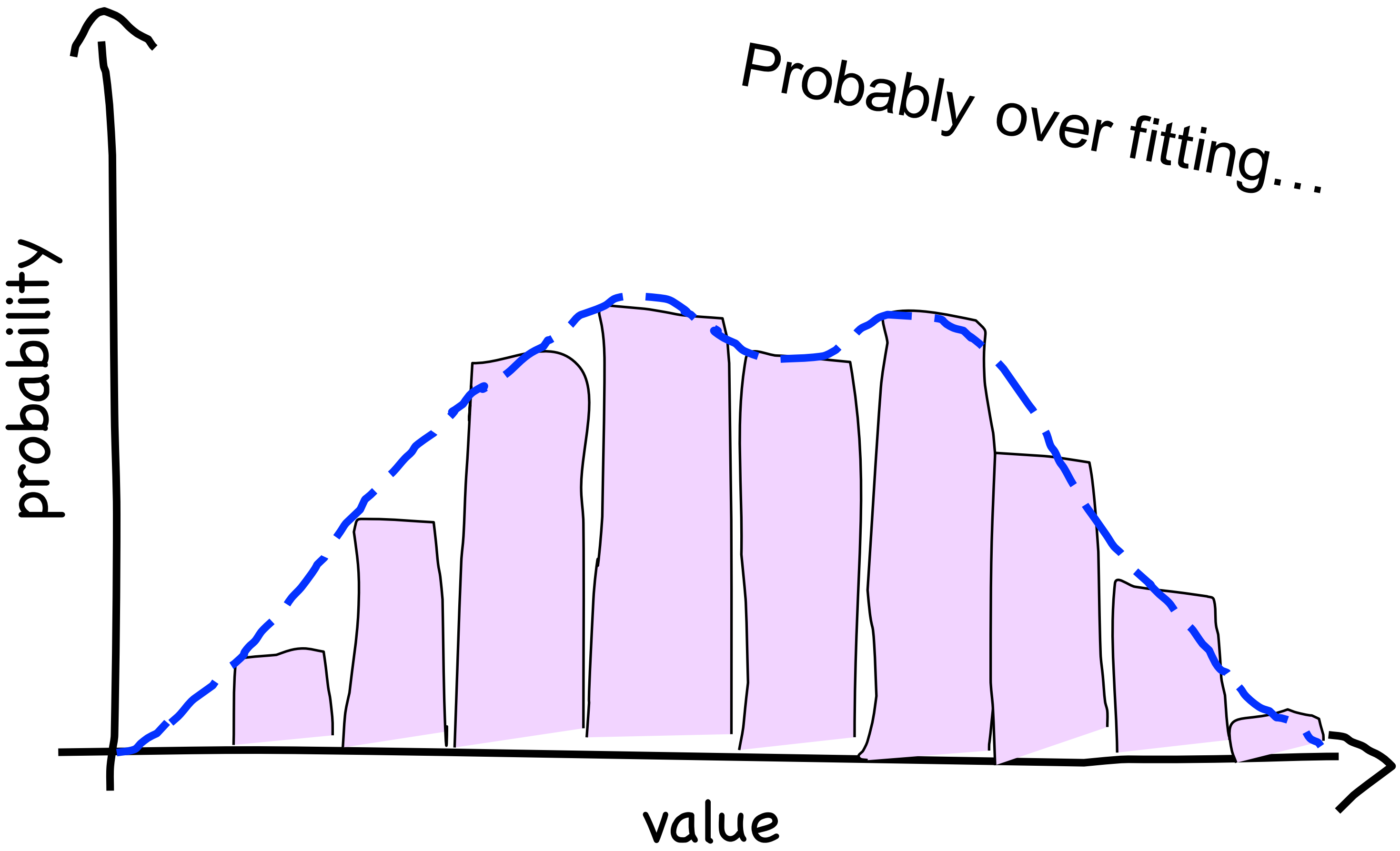
Occam's Razor

"The simplest explanation is usually the best one"

Overfiting

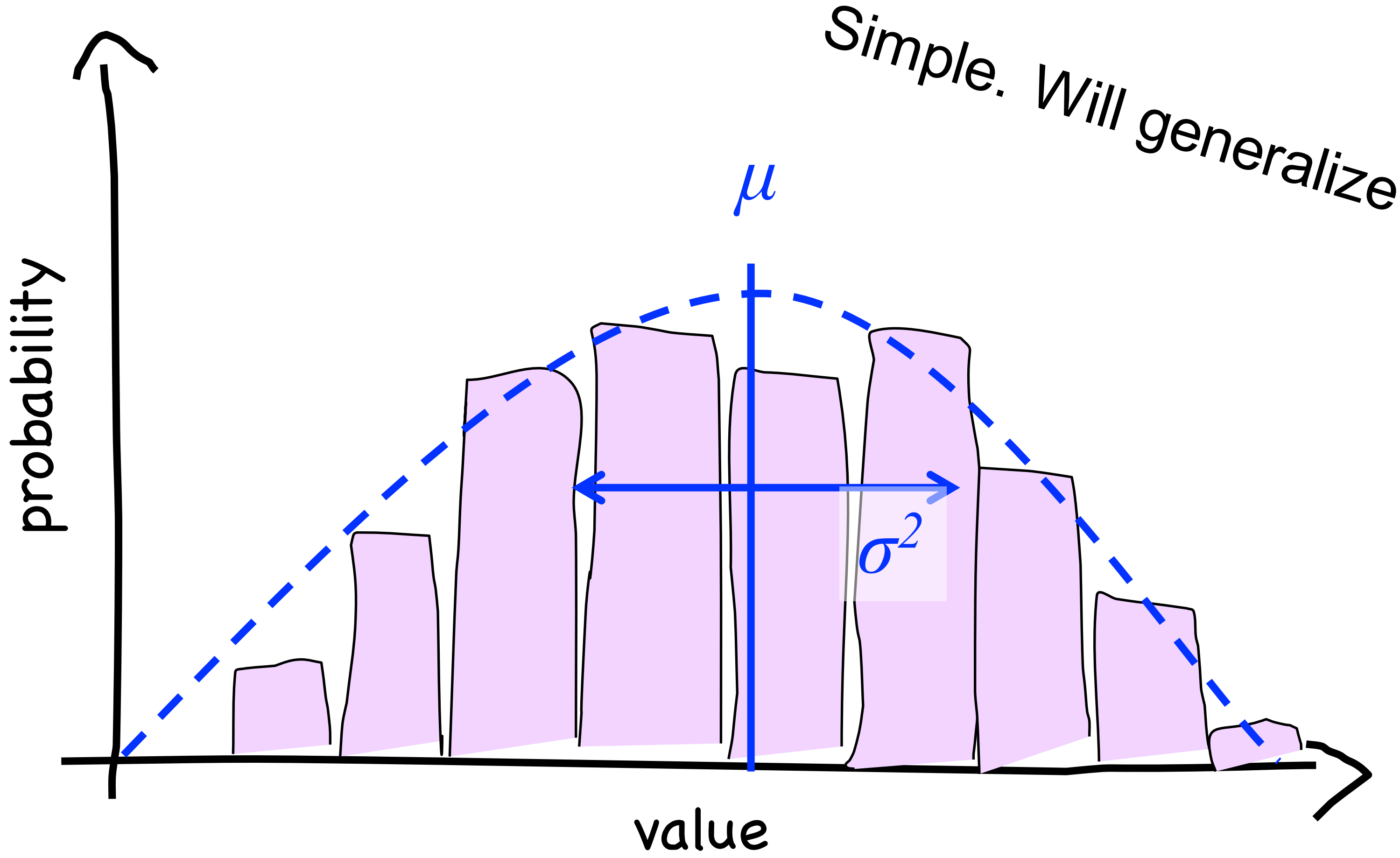

Just right
Too complex
Training Set
Testing Set
Goodness of Fit
Model Degrees of Freedom

Complexity is Tempting



* That describes the training data, but will it generalize?

Simplicity is Humble



* A Gaussian maximizes entropy for a given mean and variance

Carl Friedrich Gauss

- Carl Friedrich Gauss (1777-1855) was a remarkably influential German mathematician

- Started doing groundbreaking math as teenager
  - Did not invent Normal distribution, but popularized it
- He looked more like Martin Sheen
  - Who is, of course, Charlie Sheen's father

Anatomy of a beautiful equation

$$\mathcal{N}(\mu, \sigma^2)$$

"exponential"

the distance to the mean

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

probability density at x

a constant

sigma shows up twice

Let's try and integrate it!

$$P(a \leq X \leq b) =$$

$$\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx$$

* Call me if you find an equation for this

No closed form for the integral

No closed form for $F(x)$

Spoiler Alert

$\mathcal{N}(\mu, \sigma^2)$

A function that has been solved for numerically

$$F(x) = \Phi\left(\frac{x - \mu}{\sigma}\right)$$

The cumulative density function of any normal

* We are going to spend the next few slides getting here

Linear Transform of Normal is Normal

$$\text{Let } X \sim \mathcal{N}(\mu, \sigma^2)$$

If $Y = aX + b$ then Y is also Normal

$$\begin{aligned} E[Y] &= E[aX + b] \\ &= aE[X] + b \\ &= a\mu + b \end{aligned}$$

$$\begin{aligned} Var(Y) &= Var(aX + b) \\ &= a^2 Var(X) \\ &= a^2\sigma^2 \end{aligned}$$

$$Y \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$$

Special Linear Transform

If $Y = aX + b$ then Y is also Normal

$$Y \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$$

There is a special case of linear transform for any X:

$$Z = \frac{X - \mu}{\sigma} = \frac{1}{\sigma}X - \frac{\mu}{\sigma} \qquad a = \frac{1}{\sigma} \quad b = -\frac{\mu}{\sigma}$$

$$\begin{aligned} Z &\sim \mathcal{N}(a\mu + b, a^2\sigma^2) \\ &\sim \mathcal{N}(\frac{\mu}{\sigma} - \frac{\mu}{\sigma}, \frac{\sigma^2}{\sigma^2}) \\ &\sim \mathcal{N}(0, 1) \end{aligned}$$

Standard (Unit) Normal Variable

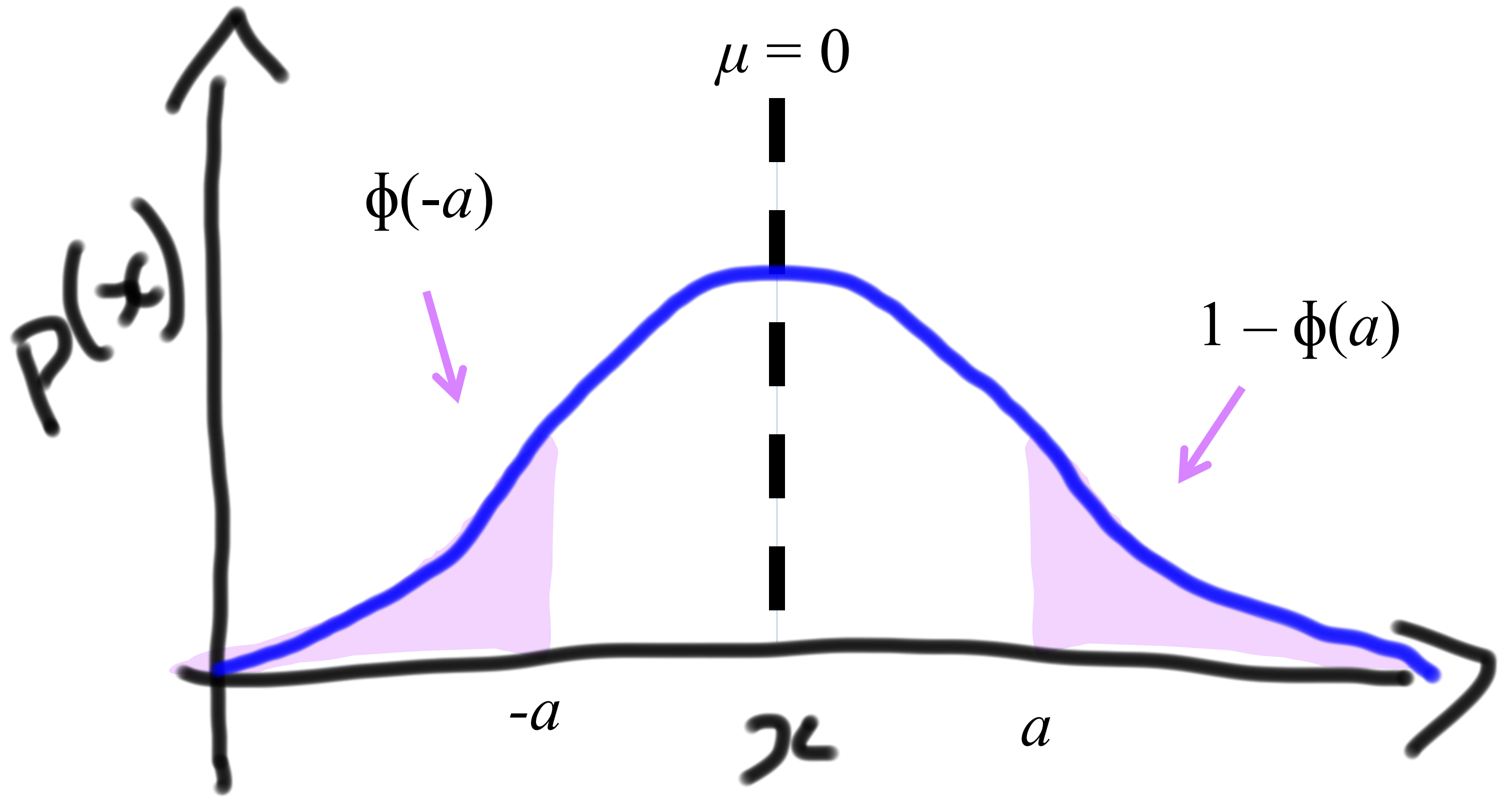
- Z is a **<u>Standard (or Unit) Normal RV</u>**: $Z \sim \mathrm{N}(0, 1)$
  - $\mathrm{E}[\mathrm{Z}] = m = 0$ $\mathrm{Var}(\mathrm{Z}) = s^2 = 1$ $\mathrm{SD}(\mathrm{Z}) = s = 1$
  - CDF of Z, $F_Z(z)$ does not have closed form
  - We denote $F_Z(z)$ as $\Phi(z)$: "phi of z"

$$\Phi(z) = P(Z \le z) = \int_{-\infty}^{z} \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} dx = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

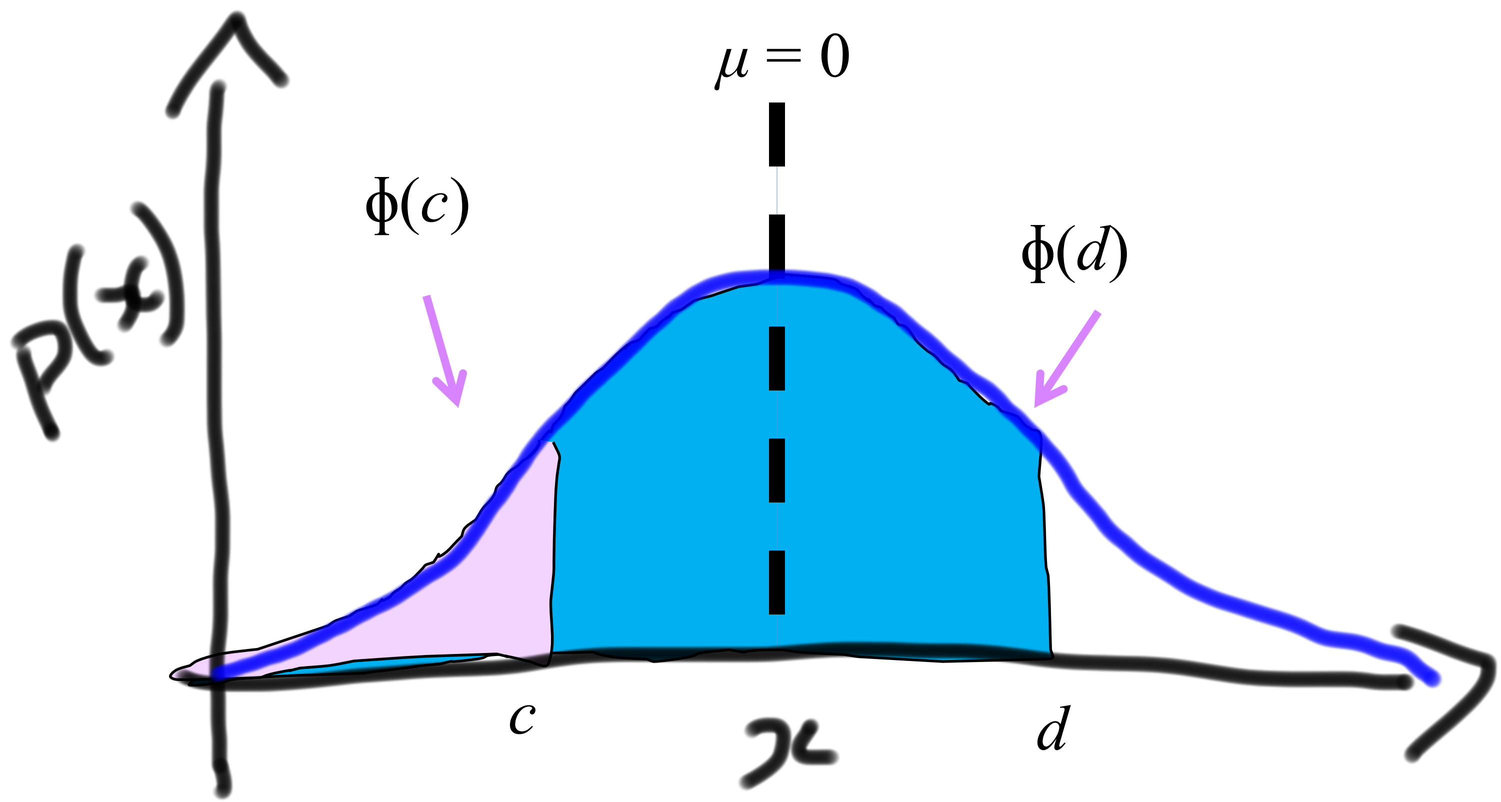
  - By symmetry: $\Phi(-z) = \mathrm{P}(Z \le -z) = \mathrm{P}(Z \ge z) = 1 - \Phi(z)$

Symmetry of Phi

$$\phi(-a) = 1 - \phi(a)$$

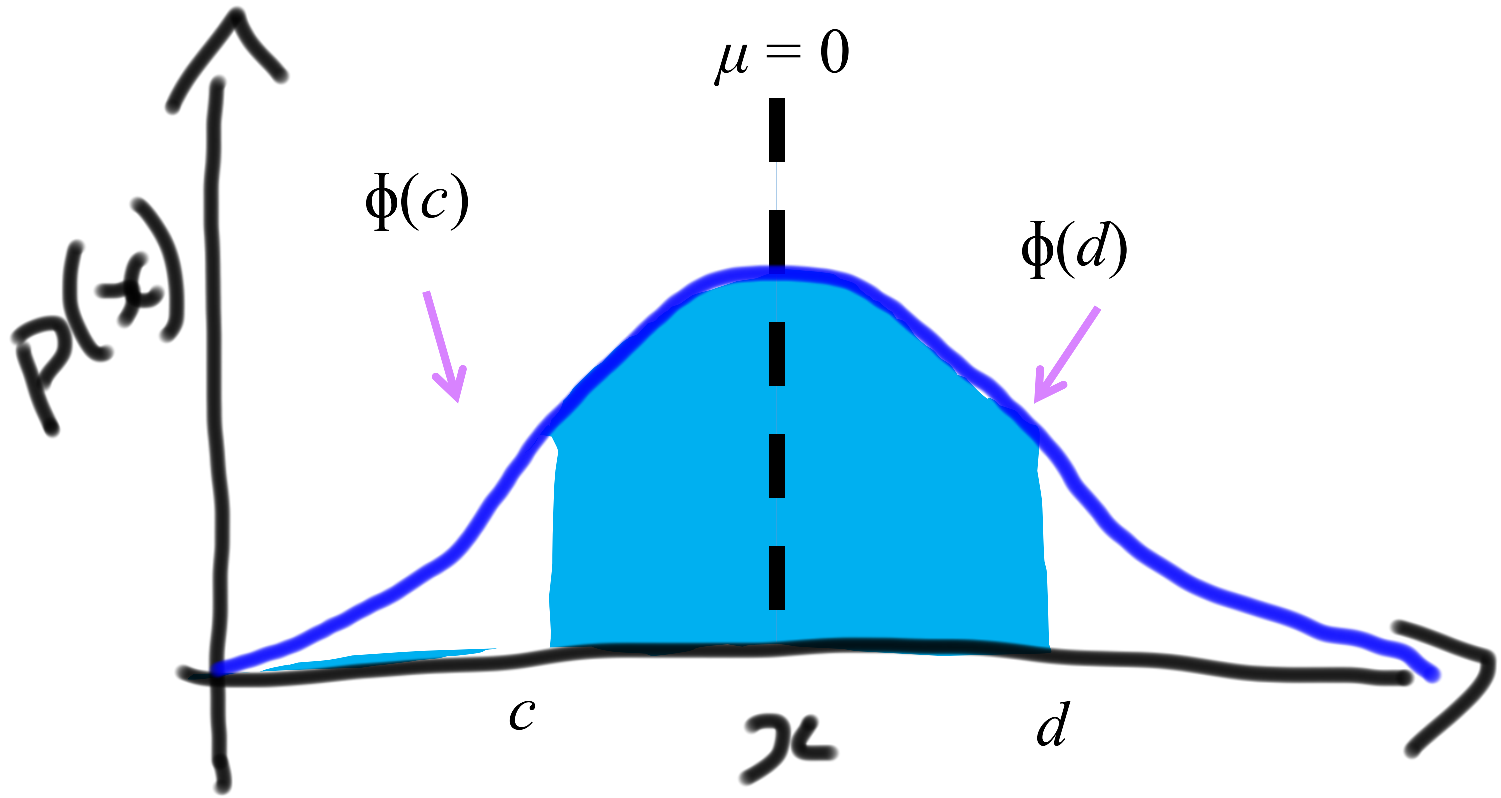


Interval of Phi



Interval of Phi

$$\phi(d) - \phi(c)$$



Compute F(x) via Transform

Let $X \sim \mathcal{N}(\mu, \sigma^2)$ $\qquad Z = \dfrac{X - \mu}{\sigma}$

Use Z to compute $F(x)$

$$
\begin{aligned}
F_X(x) &= P(X \leq x) \\
&= P(X - \mu \leq x - \mu) \\
&= P\left(\frac{X - \mu}{\sigma} \leq \frac{x - \mu}{\sigma}\right) \\
&= P\left(Z \leq \frac{x - \mu}{\sigma}\right) \\
&= \Phi\left(\frac{x - \mu}{\sigma}\right)
\end{aligned}
$$

And here we are

$\mathcal{N}(\mu, \sigma^2)$

CDF of Standard Normal: A function that has been solved for numerically

$$F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

The cumulative density function (CDF) of any normal

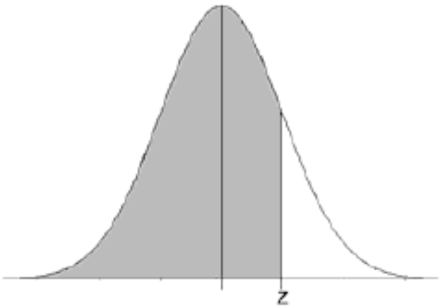
Table of $\Phi(z)$ values in textbook, p. 201 and handout

Using Table of ϕ

Standard Normal Cumulative Probability Table

$\Phi(0.54) = 0.7054$

Cumulative probabilities for **POSITIVE** z-values are shown in the following table:



z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319

Kinda old school

Using Programming Library

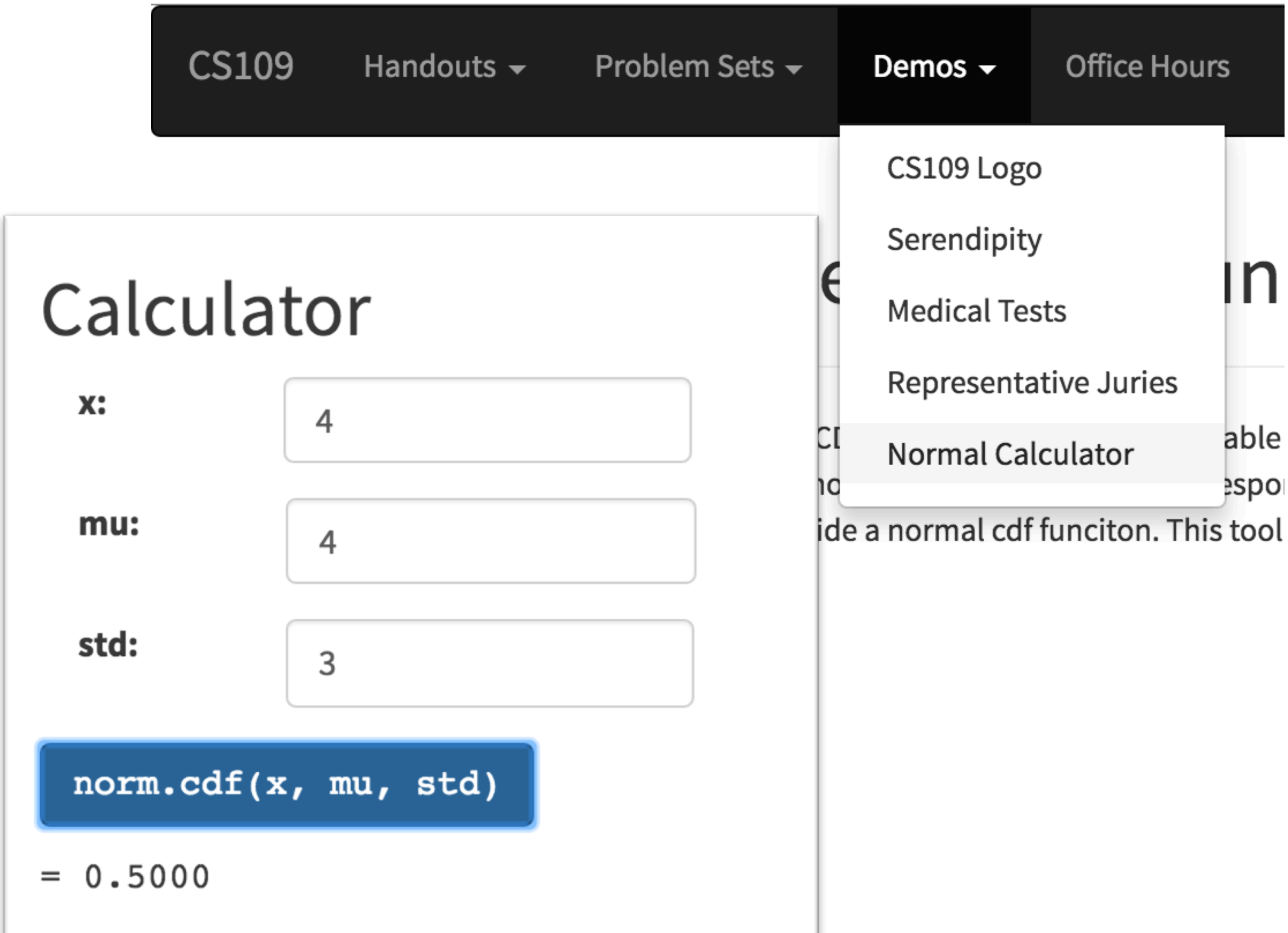
```
norm.cdf(x, mean, std)
```

$$= P(X < x) \text{ where } X \sim \mathcal{N}(\mu, \sigma^2)$$

$$= \Phi\left(\frac{x - \mu}{\sigma}\right)$$

* Every modern programming language has a normal library

I made one for you



Get Your Gaussian On

- X ~ N(3, 16) $\mu = 3$ $\sigma^2 = 16$ $\sigma = 4$
  - What is P(X > 0)?

$$P(X > 0) = P\left(\frac{X-3}{4} > \frac{0-3}{4}\right) = P\left(Z > -\frac{3}{4}\right) = 1 - P\left(Z \leq -\frac{3}{4}\right)$$

$$1 - \Phi\left(-\tfrac{3}{4}\right) = 1 - \left(1 - \Phi\left(\tfrac{3}{4}\right)\right) = \Phi\left(\tfrac{3}{4}\right) = 0.7734$$

  - What is P(2 < X < 5)?

$$P(2 < X < 5) = P\left(\frac{2-3}{4} < \frac{X-3}{4} < \frac{5-3}{4}\right) = P\left(-\frac{1}{4} < Z < \frac{2}{4}\right)$$

$$\Phi\left(\tfrac{2}{4}\right) - \Phi\left(-\tfrac{1}{4}\right) = \Phi\left(\tfrac{1}{2}\right) - \left(1 - \Phi\left(\tfrac{1}{4}\right)\right) = 0.6915 - (1 - 0.5987) = 0.2902$$

  - What is P(|X – 3| > 6)?

$$P(X < -3) + P(X > 9) = P\left(Z < \frac{-3-3}{4}\right) + P\left(Z > \frac{9-3}{4}\right)$$

$$\Phi\left(-\tfrac{3}{2}\right) + \left(1 - \Phi\left(\tfrac{3}{2}\right)\right) = 2\left(1 - \Phi\left(\tfrac{3}{2}\right)\right) = 2(1 - 0.9332) = 0.1337$$

Noisy Wires

- Send voltage of 2 or -2 on wire (to denote 1 or 0)
  - X = voltage sent
  - R = voltage received = X + Y, where noise Y ~ N(0, 1)
  - Decode R: if (R ≥ 0.5) then 1, else 0
  - What is P(error after decoding | original bit = 1)?

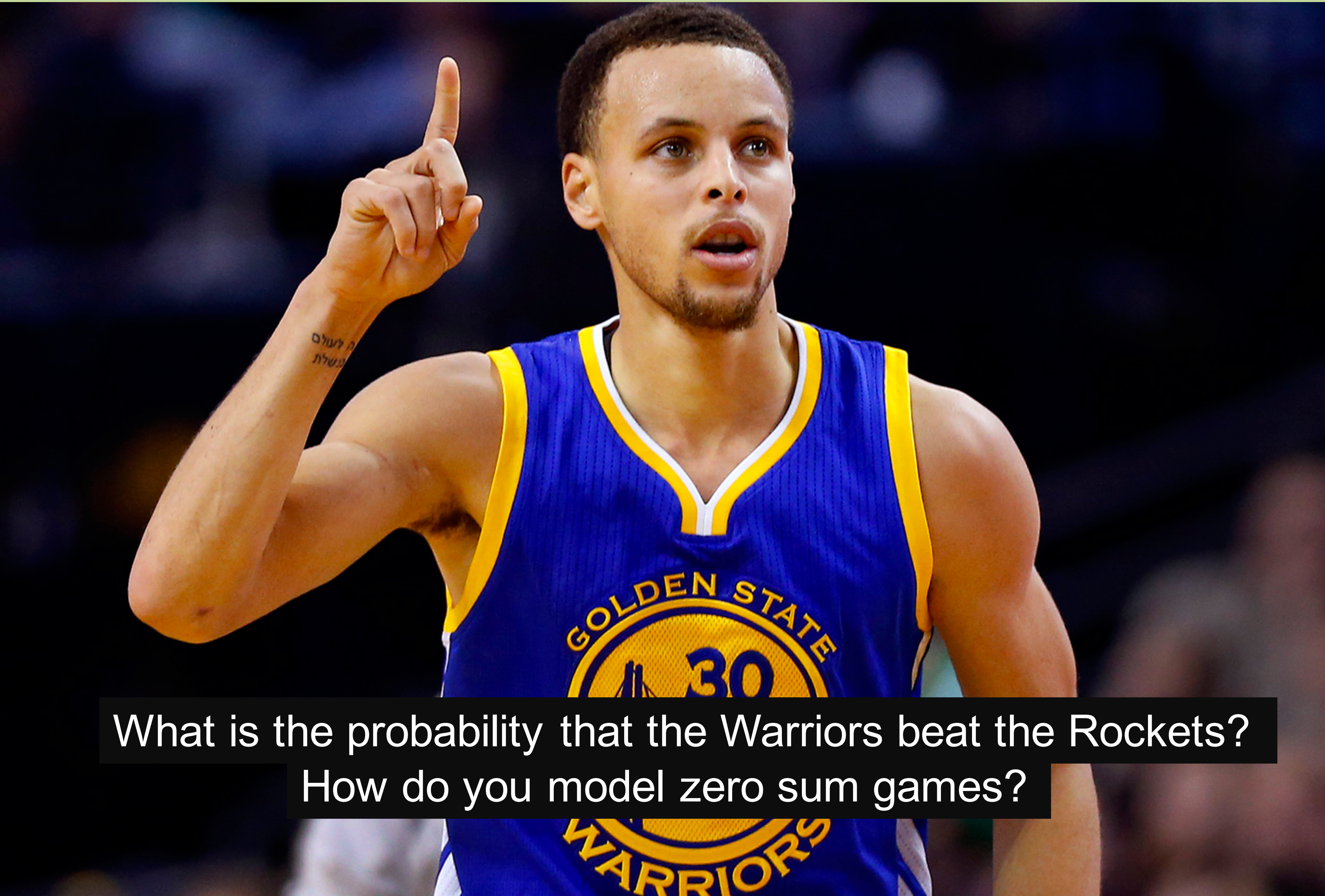
$$P(2 + Y < 0.5) = P(Y < -1.5) = \Phi(-1.5) = 1 - \Phi(1.5) \approx 0.0668$$

  - What is P(error after decoding | original bit = 0)?

$$P(-2 + Y \geq 0.5) = P(Y \geq 2.5) = 1 - \Phi(2.5) \approx 0.0062$$

Gaussian for uncertainty

ELO Ratings



What is the probability that the Warriors beat the Rockets?

How do you model zero sum games?

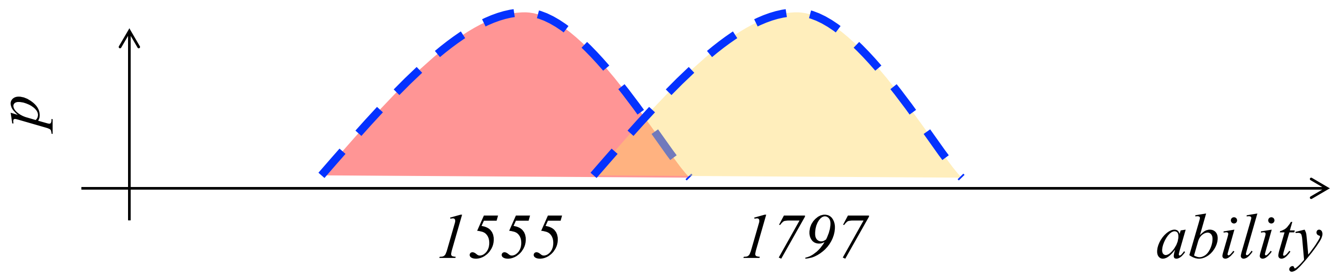
ELO Ratings

How it works:

- Each team has an "ELO" score *S*, calculated based on their past performance.
- Each game, the team has ability $A \sim N(S, 200^2)$
- The team with the higher sampled ability wins.

Arpad Elo

$A_R \sim \mathcal{N}(1555, 200^2)$ $A_W \sim \mathcal{N}(1797, 200^2)$



$$P(\text{Warriors win}) = P(A_W > A_R)$$
$$\approx 0.87$$

Calculated via sampling

Poll of polls?

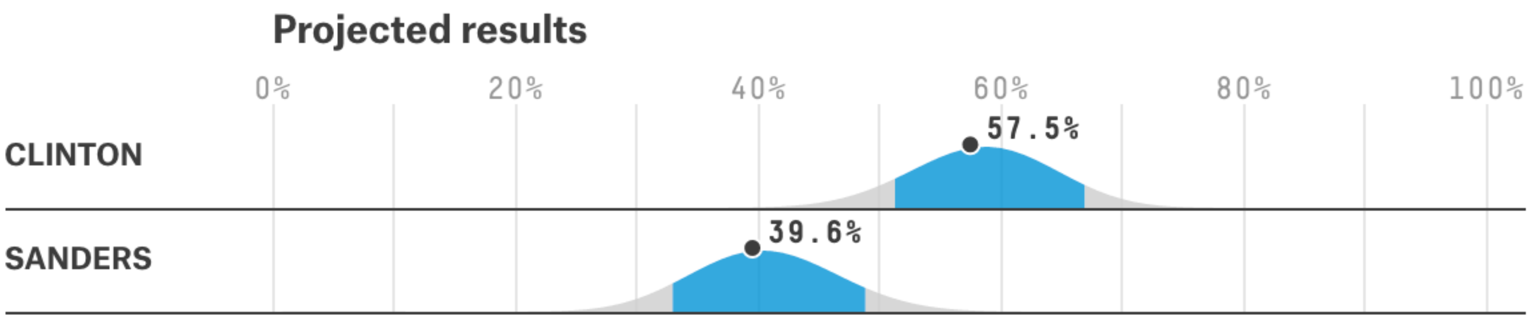
Democratic Pennsylvania Primary

	POLLSTER	SAMPLE	WEIGHT	LEADER	CLINTON	SANDERS
APR. 4-7	Fox News	805 LV	0.54	Clinton +11	49%	38%
MAR. 30-APR. 4	Quinnipiac University	514 LV	0.31	Clinton +6	50%	44%
APR. 2-3	Harper Polling	603 LV	0.25	Clinton +22	55%	33%
MAR. 14-20	Franklin & Marshall College	408 RV	0.03	Clinton +25	53%	28%
MAR. 1-2	Harper Polling	347 LV	0.00	Clinton +30	57%	27%
FEB. 13-21	Franklin & Marshall College	486 RV	0.00	Clinton +21	48%	27%
FEB. 11-16	Robert Morris University	232 LV	0.00	Clinton +7	48%	41%
JAN. 22-23	Harper Polling	640 LV	0.00	Clinton +27	55%	28%
JAN. 18-23	Franklin & Marshall College	361 RV	0.00	Clinton +17	46%	29%
OCT. 19-25	Franklin & Marshall College	303 RV	0.00	Clinton +34	52%	18%

Credit: fivethirtyeight.com

What is the probability that Hillary/Bernie wins?

Poll of polls?



Credit: fivethirtyeight.com

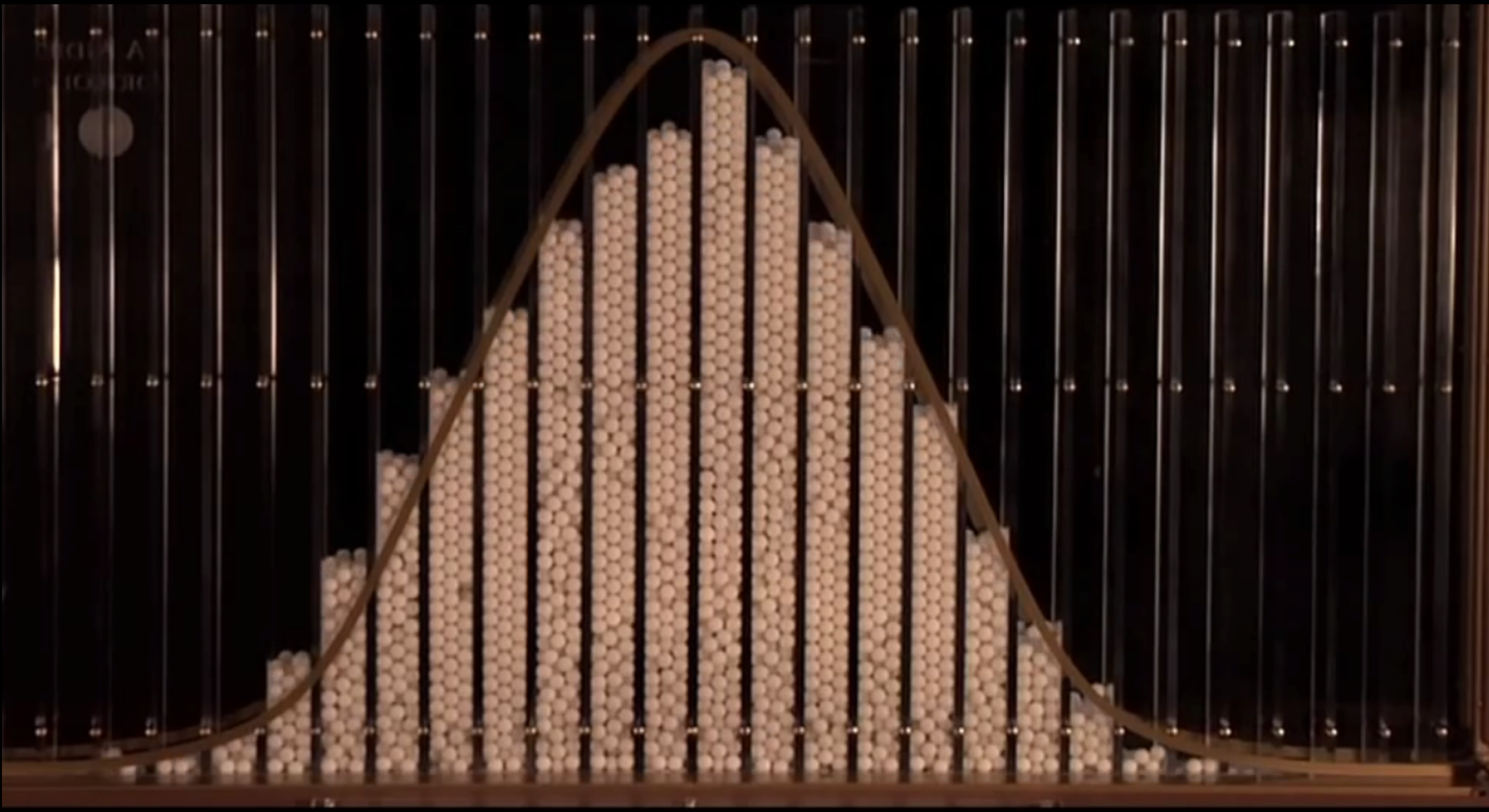
How it works:

- Estimate a Gaussian for each candidate
- Mean is a weighted average of poll means
- Variance is a weighted average of poll trust
- Sample Gaussians to find out % win
- Key: represent uncertainty

According to our latest **polls-plus** forecast, **Hillary Clinton** has a **92%** chance of winning the Pennsylvania primary.

Gaussian for Binomial

Remember this?



There is a deep reason for the Binomial/Normal approximation…

Normal Approximation of Binomial

- X ~ Bin(*n*, *p*)
  - E[X] = *np*   Var(X) = *np*(1 – *p*)
  - Poisson approx. good: *n* large (> 20), *p* small (< 0.05)
  - For large *n:* X ≈ Y ~ N(E[X], Var(X)) = N(*np*, *np*(1 – *p*))
  - Normal approx. good : Var(X) = *np*(1 – *p*) ≥ 10

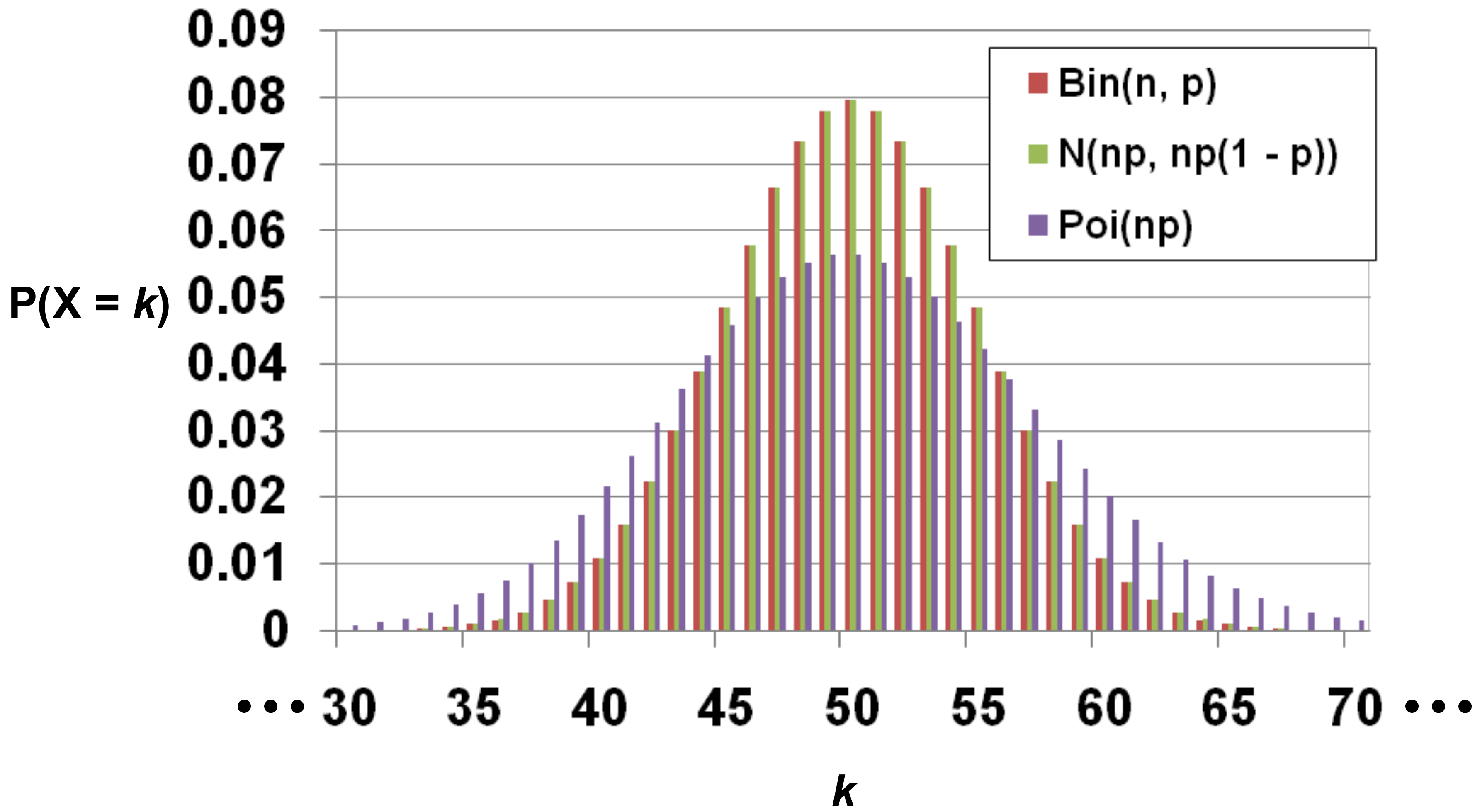
$$P(X = k) \approx P\left(k - \frac{1}{2} < Y < k + \frac{1}{2}\right) = \Phi\left(\frac{k - np + 0.5}{\sqrt{np(1-p)}}\right) - \Phi\left(\frac{k - np - 0.5}{\sqrt{np(1-p)}}\right)$$

"Continuity correction"

  - DeMoivre-Laplace Limit Theorem:
    - S_n: number of successes (with prob. *p*) in *n* independent trials

$$P\left(a \le \frac{S_n - np}{\sqrt{np(1-p)}} \le b\right) \xrightarrow{n \to \infty} \Phi(b) - \Phi(a)$$

Comparison when $n = 100$, $p = 0.5$



Website Testing

- 100 people are given a new website design
  - X = # people whose time on site increases
  - CEO will endorse new design if X ≥ 65 What is P(CEO endorses change| it has no effect)?
  - X ~ Bin(100, 0.5)

$$np = 50 \quad np(1-p) = 25 \quad \sqrt{np(1-p)} = 5$$

  - Use Normal approximation: Y ~ N(50, 25)

$$P(X \geq 65) \approx P(Y > 64.5)$$

$$P(Y > 64.5) = P\left(\frac{Y-50}{5} > \frac{64.5-50}{5}\right) = P(Z > 2.9) = 1 - \Phi(2.9) \approx 0.0019$$

  - Using Binomial:

$$P(X \geq 65) \approx 0.0018$$

Stanford Admissions

- Stanford accepts 2480 students
  - Each accepted student has 68% chance of attending
  - X = # students who will attend. X ~ Bin(2480, 0.68)
  - What is P(X > 1745)?

$$np = 1686.4 \quad np(1-p) \approx 539.65 \quad \sqrt{np(1-p)} \approx 23.23$$

  - Use Normal approximation: Y ~ N(1686.4, 539.65)

$$P(X > 1745) \approx P(Y > 1745.5)$$

$$P(Y > 1745.5) = P\left(\frac{Y-1686.4}{23.23} > \frac{1745.5-1686.4}{23.23}\right) = 1 - \Phi(2.54) \approx 0.0055$$

  - Using Binomial:

$$P(X > 1745) \approx 0.0053$$

Changes in Stanford Admissions

- Stanford Daily, March 28, 2014

  "Class of 2018 Admit Rates Lowest in University History" by Alex Zivkovic

  "Fewer students were admitted to the Class of 2018 than the Class of 2017, due to the increase in Stanford's yield rate which has increased over 5 percent in the past four years, according to Colleen Lim M.A. '80, Director of Undergraduate Admission."