Debugging Intuition

- How to calculate the probability of at least *k* successes in *n* trials?
  - X is number of successes in *n* trials each with probability *p*
  - $P(X \geq k) =$

First clue that something is wrong. Think about *p* = 1

$$\binom{n}{k} p^k$$

Don't care about the rest

ways to choose slots for success

Probability that each is success

Not mutually exclusive…

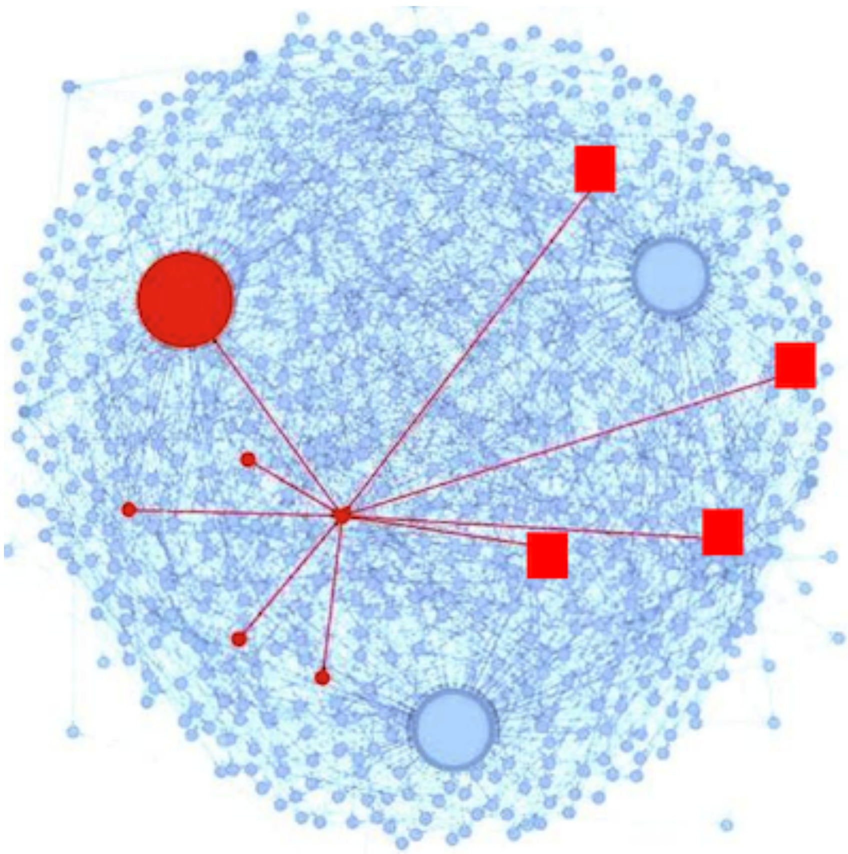
Correct: $P(X \geq k) = \sum_{i=k}^{n} \binom{n}{i} p^k (1-p)^{n-k}$

Variance

Lagunita

CS 109
Lecture 7
April 11th, 2016

Is Peer Grading Accurate Enough?



Peer Grading on Coursera HCI.

31,067 peer grades for 3,607 students.

Tuned Models of Peer Assessment. C Piech, J Huang, A Ng, D Koller

Today's Topics

Last time:
Random Variables
Expectation + PDF

Today:
Variance
Bernoulli + Binominal RVs

Next time:
All the other discrete RVs

St Petersburg

- Game set-up
  - We have a fair coin (come up "heads" with p = 0.5)
  - Let n = number of coin flips ("heads") before first "tails"
  - You win $\$2^n$
- How much would you pay to play?
- Solution
  - Let X = your winnings
  - E[X] = $\left(\frac{1}{2}\right)^1 2^0 + \left(\frac{1}{2}\right)^2 2^1 + \left(\frac{1}{2}\right)^3 2^2 + \left(\frac{1}{2}\right)^4 2^3 + \ldots = \sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^{i+1} 2^i$

    $= \sum_{i=0}^{\infty} \frac{1}{2} = \infty$
  - I'll let you play for $1 thousand... but just once! Takers?

St Petersburg + Reality

- What if Chris has only $65,536?
  - Same game
  - If you win over $65,536 I leave the country.
- Solution
  - Let X = your winnings
  - E[X] $= \left(\frac{1}{2}\right)^1 2^0 + \left(\frac{1}{2}\right)^2 2^1 + \left(\frac{1}{2}\right)^3 2^2 + \left(\frac{1}{2}\right)^4 2^3 + \ldots$

$$= \sum_{i=0}^{k} \left(\frac{1}{2}\right)^{i+1} 2^i \text{ s.t. } k = \log_2(65,536)$$

$$= \sum_{i=0}^{16} \frac{1}{2} = 8$$

Utility

- Utility is value of some choice
  - 2 options, each with n consequences: $c_1, c_2, \ldots, c_n$
  - One of c_i will occur with probability p_i
  - Each consequence has some value (utility): $U(c_i)$
  - Which choice do you make?
- Example: Buy a \$1 lottery ticket (for \$1M prize)?
  - Probability of winning is $1/10^7$
  - **Buy**: c_1 = win, c_2 = lose, $U(c_1) = 10^6 – 1$, $U(c_2) = -1$
  - **Don't Buy**: c_1 = lose, $U(c_1) = 0$
  - $E(\text{buy}) = 1/10^7 (10^6 – 1) + (1 – 1/10^7) (-1) \approx -0.9$
  - $E(\text{don't buy}) = 1 (0) = 0$
  - *"You can't lose if you don't play!"*

And Then There's This

Lottery: A tax on people who are bad at math.

– Ambrose Bierce

Recall, Geometric Series

$$a^0 + a^1 + a^2 + \ldots$$

$$= \sum_{i=0}^{\infty} a^i$$

$$= \frac{1}{1-a}$$

$$\text{where } 0 < a < 1$$

Breaking Vegas

- Consider even money bet (e.g., bet "Red" in roulette)
  - p = 18/38 you win \$Y, otherwise (1 – p) you lose \$Y
  - Consider this algorithm for one series of bets:
    1. Y = \$1
    2. Bet Y
    3. If Win then stop
    4. If Loss then Y = 2 * Y, goto 2
  - Let Z = winnings upon stopping
  - E[Z] $= \left(\frac{18}{38}\right)1 + \left(\frac{20}{38}\right)\left(\frac{18}{38}\right)(2-1) + \left(\frac{20}{38}\right)^2\left(\frac{18}{38}\right)(4-2-1) + \ldots$

$$= \sum_{i=0}^{\infty}\left(\frac{20}{38}\right)^i\left(\frac{18}{38}\right)\left(2^i - \sum_{j=0}^{i-1} 2^j\right) = \left(\frac{18}{38}\right)\sum_{i=0}^{\infty}\left(\frac{20}{38}\right)^i = \left(\frac{18}{38}\right)\frac{1}{1-\frac{20}{38}} = 1$$

  - Expected winnings ≥ 0. Use algorithm infinitely often!

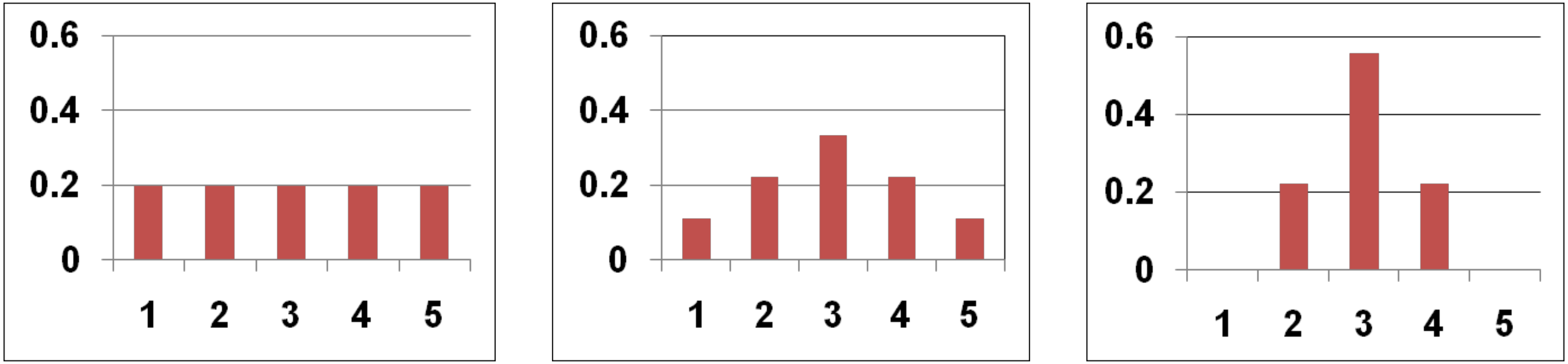
Vegas Breaks You

- Why doesn't everyone do this?
  - Real games have maximum bet amounts
  - You have finite money
    - Not able to keep doubling bet beyond certain point
  - Casinos can kick you out
- But, if you had:
  - No betting limits, and
  - Infinite money, and
  - Could play as often as you want...
- Then, go for it!
  - And tell me which planet you are living on

Is $E[X]$ enough?

Variance

- Consider the following 3 distributions (PMFs)



- All have the same expected value, E[X] = 3
- But “spread” in distributions is different
- Variance = a formal quantification of “spread”

Variance

- If X is a random variable with mean μ then the **variance** of X, denoted Var(X), is:

$$\text{Var}(X) = E[(X - \mu)^2]$$

- Note: $\text{Var}(X) \geq 0$

- Also known as the 2nd **Central** Moment, or square of the Standard Deviation

Computing Variance

$$\text{Var}(X) = E[(X-\mu)^2]$$
$$= \sum_x (x-\mu)^2 p(x)$$
$$= \sum_x (x^2 - 2\mu x + \mu^2) p(x)$$
$$= \sum_x x^2 p(x) - 2\mu \sum_x x p(x) + \mu^2 \sum_x p(x)$$
$$= \boxed{E[X^2]} - 2\mu E[X] + \mu^2$$
$$= E[X^2] - 2\mu^2 + \mu^2$$
$$= E[X^2] - \mu^2$$
$$\boxed{= E[X^2] - (E[X])^2}$$

Ladies and gentlemen, please welcome the 2nd moment!

Variance of a 6 sided dice

- Let X = value on roll of 6 sided die
- Recall that E[X] = 7/2
- Compute $E[X^2]$

$$E[X^2] = (1^2)\frac{1}{6} + (2^2)\frac{1}{6} + (3^2)\frac{1}{6} + (4^2)\frac{1}{6} + (5^2)\frac{1}{6} + (6^2)\frac{1}{6} = \frac{91}{6}$$

$$\text{Var}(X) = E[X^2] - (E[X])^2$$

$$= \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{35}{12}$$

Properties of Variance

- $\text{Var}(aX + b) = a^2\text{Var}(X)$
  - Proof:

    $\text{Var}(aX + b) = E[(aX + b)^2] - (E[aX + b])^2$

    $= E[a^2X^2 + 2abX + b^2] - (aE[X] + b)^2$

    $= a^2E[X^2] + 2abE[X] + b^2 - (a^2(E[X])^2 + 2abE[X] + b^2)$

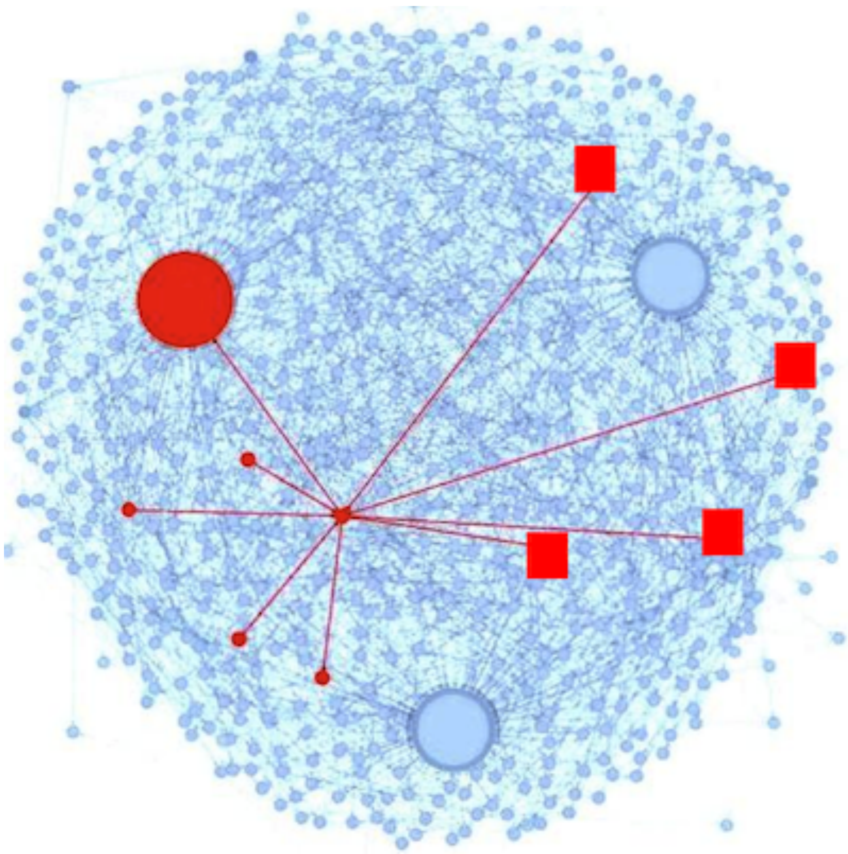
    $= a^2E[X^2] - a^2(E[X])^2 = a^2(E[X^2] - (E[X])^2)$

    $= a^2\text{Var}(X)$
- Standard Deviation of X, denoted SD(X), is:

$$\text{SD}(X) = \sqrt{\text{Var}(X)}$$

  - Var(X) is in units of X^2
  - SD(X) is in same units as X

Intuition

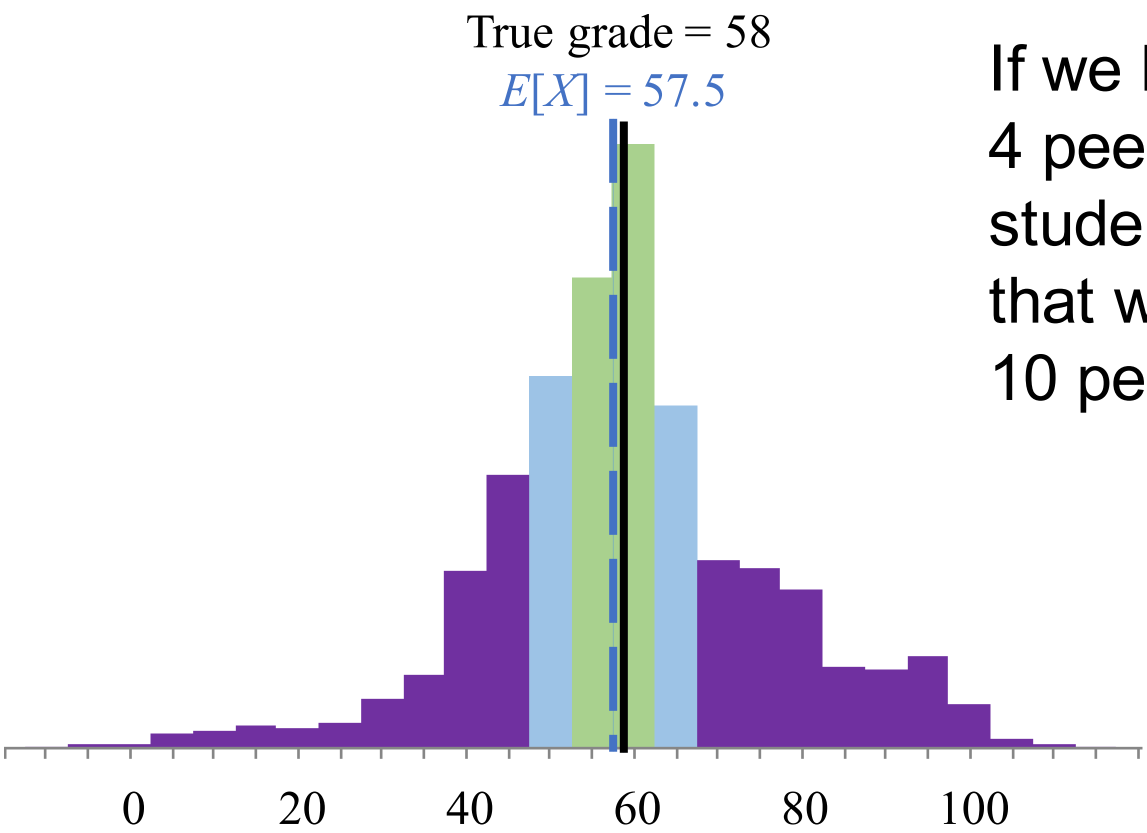


Peer Grading on Coursera HCI.

31,067 peer grades for 3,607 students.

Peer Grades in Coursera HCI

Let X be a random variable that represents a peer grade

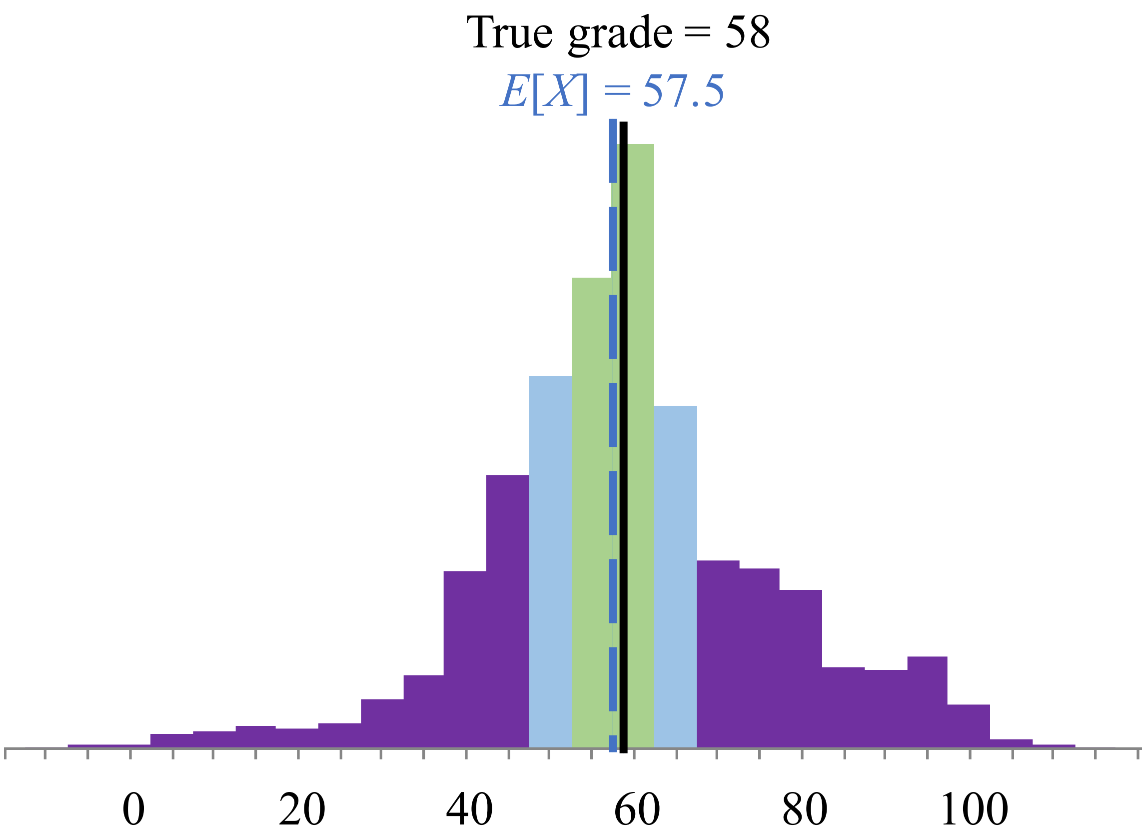


If we base a final score off 4 peer-grades, 19% of students would get a score that was off by more than 10 percentage points

Peer Grades in Coursera HCI

Let X be a random variable that represents a peer grade

$$\mathrm{Var}(X) = E[(X - \mu)^2]$$



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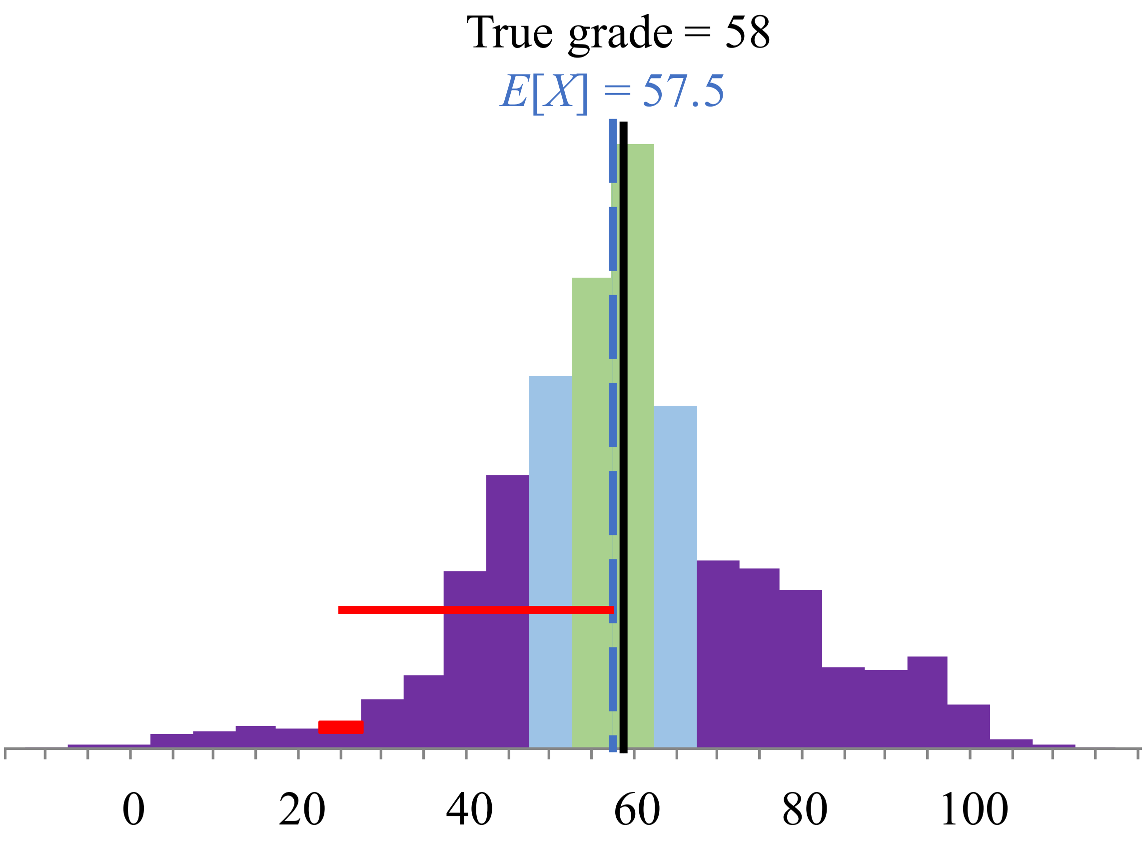
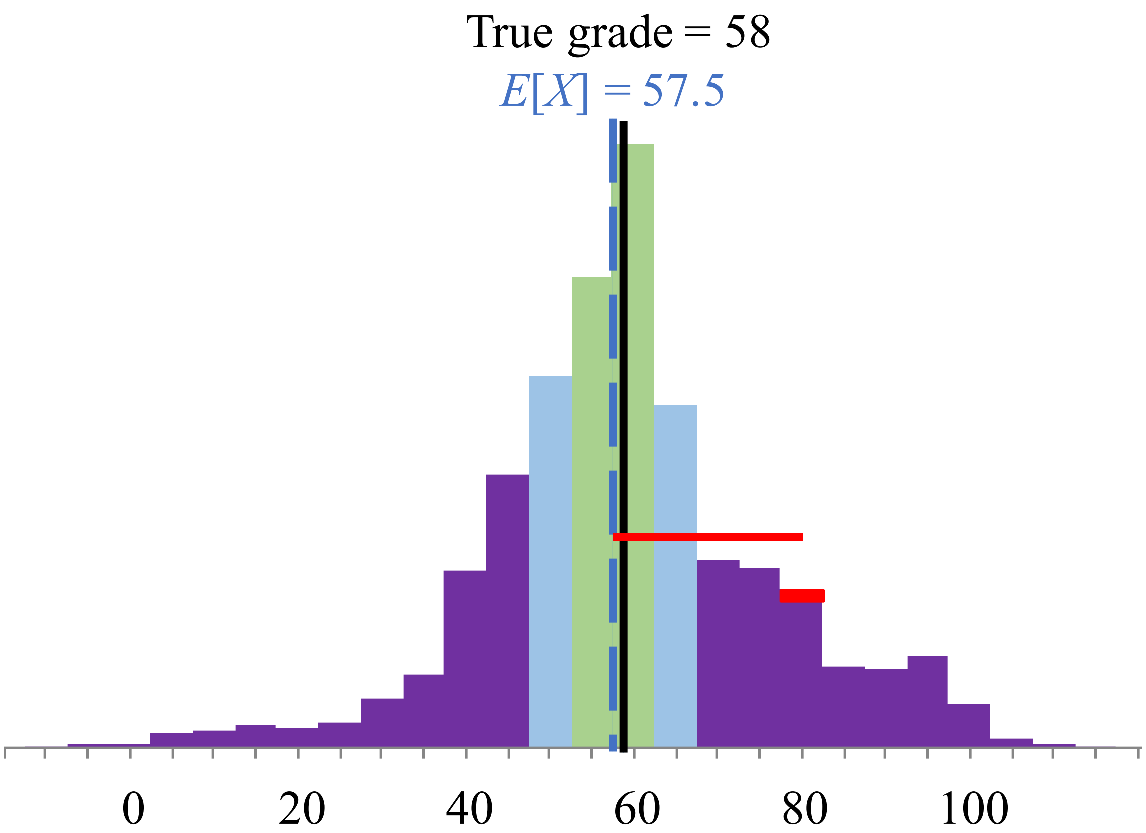


X	$(X - \mu)^2$
25 points	1056 points^2

Peer Grades in Coursera HCI

Let X be a random variable that represents a peer grade

$$\mathrm{Var}(X) = E[(X - \mu)^2]$$

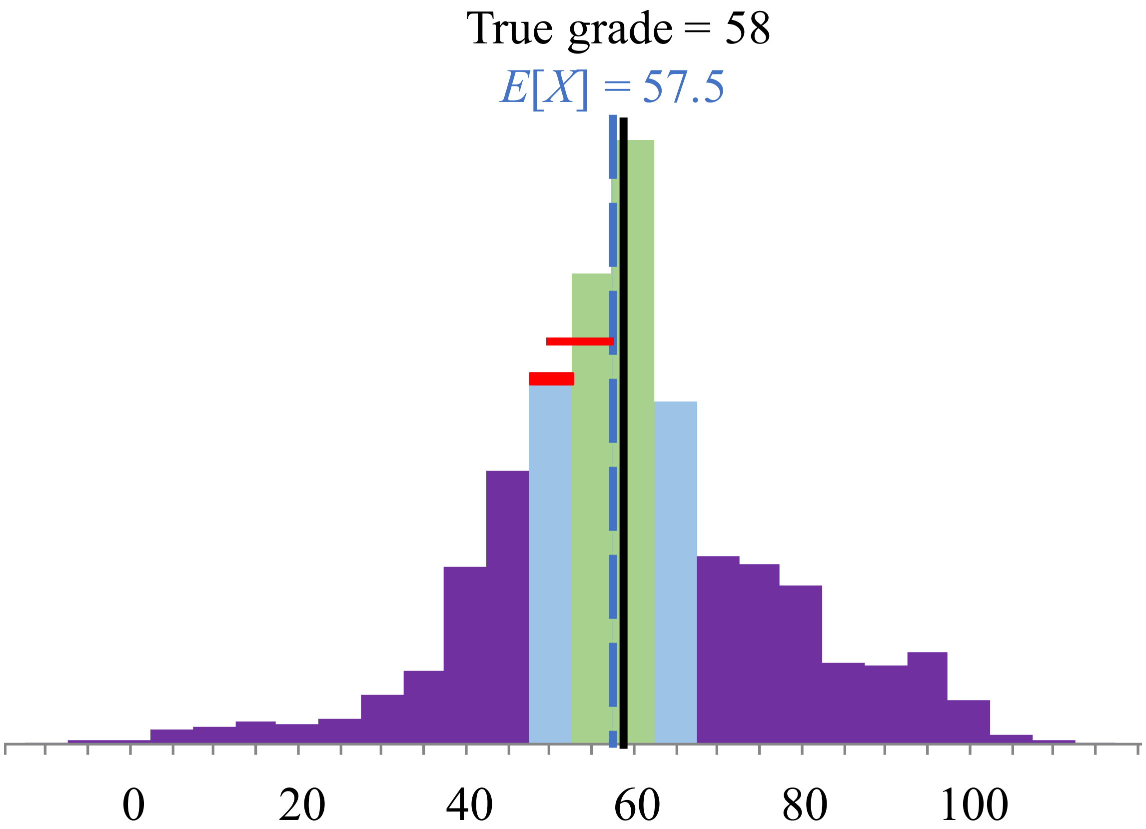


X	$(X - \mu)^2$
25 points	1056 points^2
80 points	506 points^2

Peer Grades in Coursera HCI

Let X be a random variable that represents a peer grade

$$\mathrm{Var}(X) = E[(X - \mu)^2]$$

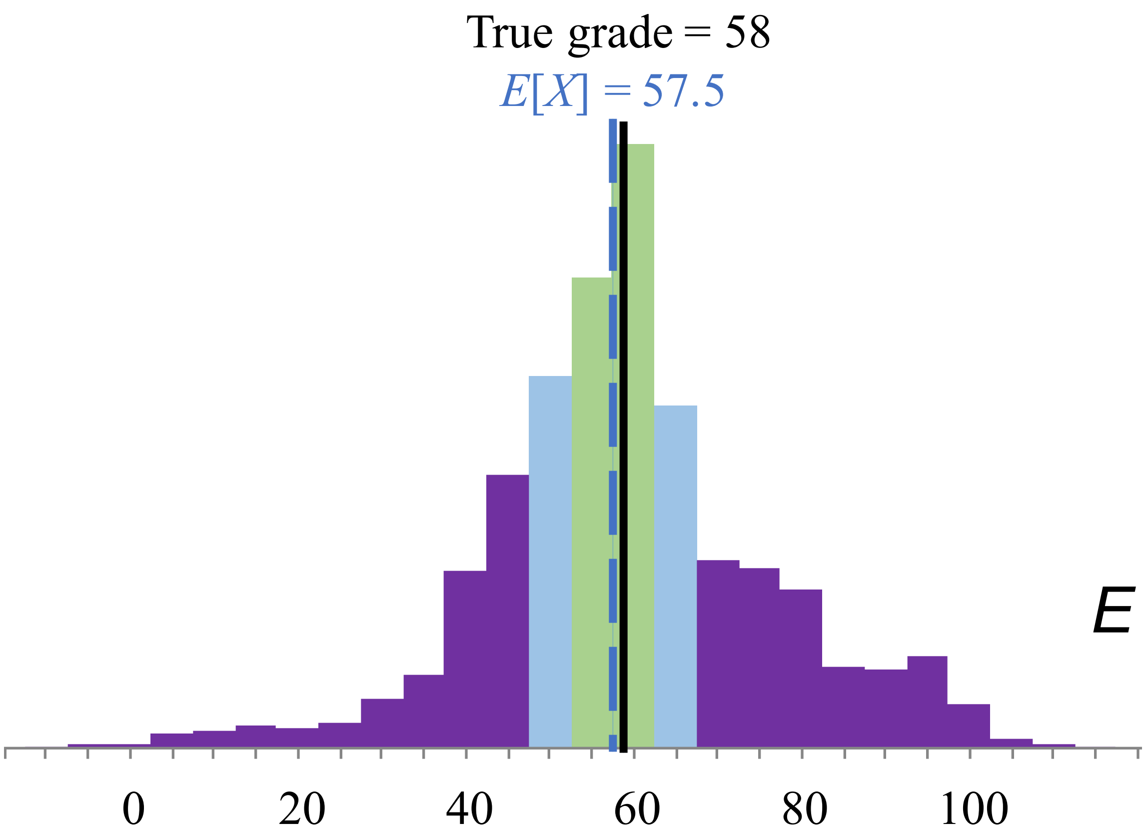


X	$(X - \mu)^2$
25 points	1056 points^2
80 points	506 points^2
50 points	56 points^2

Peer Grades in Coursera HCI

Let X be a random variable that represents a peer grade

$$\mathrm{Var}(X) = E[(X - \mu)^2]$$



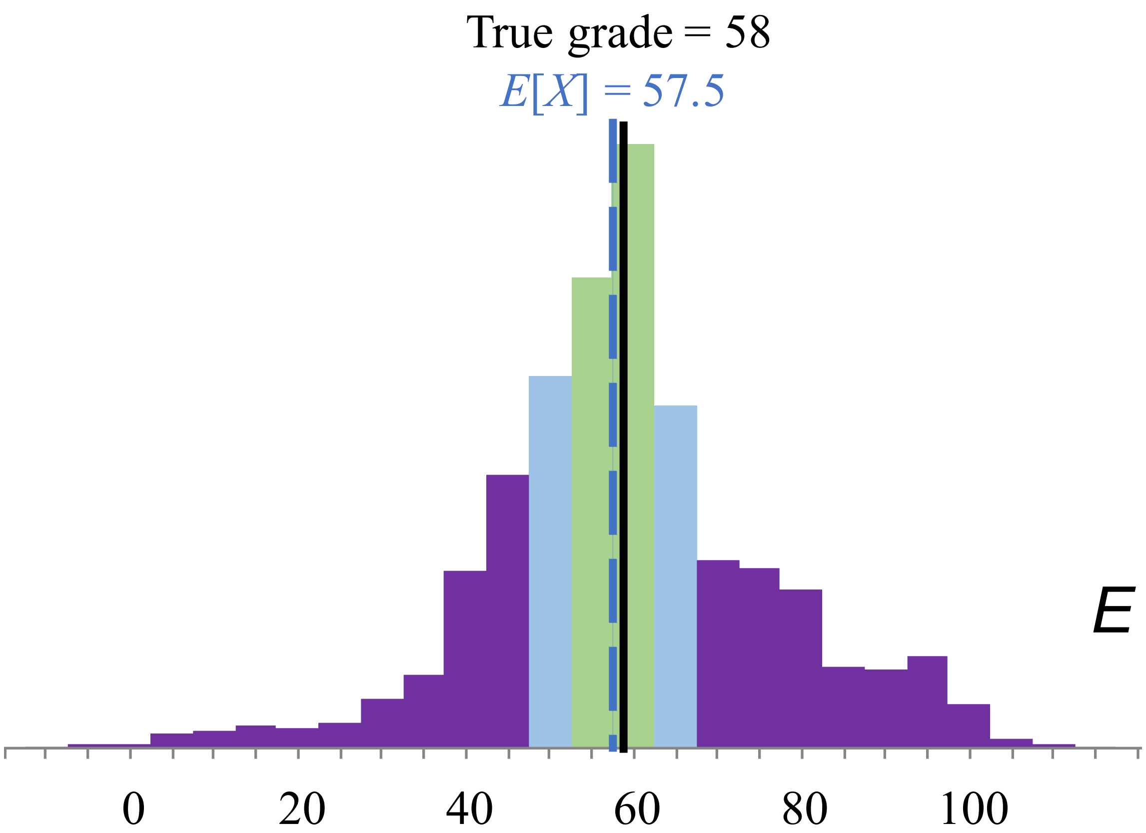
X	$(X - \mu)^2$
25 points	1056 points^2
80 points	506 points^2
50 points	56 points^2
…	

$$E[(X - \mu)^2] = 52 \text{ points}^2$$

Peer Grades in Coursera HCI

Let X be a random variable that represents a peer grade

$$\mathrm{Var}(X) = E[(X - \mu)^2]$$



X	$(X - \mu)^2$
25 points	1056 points^2
80 points	506 points^2
50 points	56 points^2

…

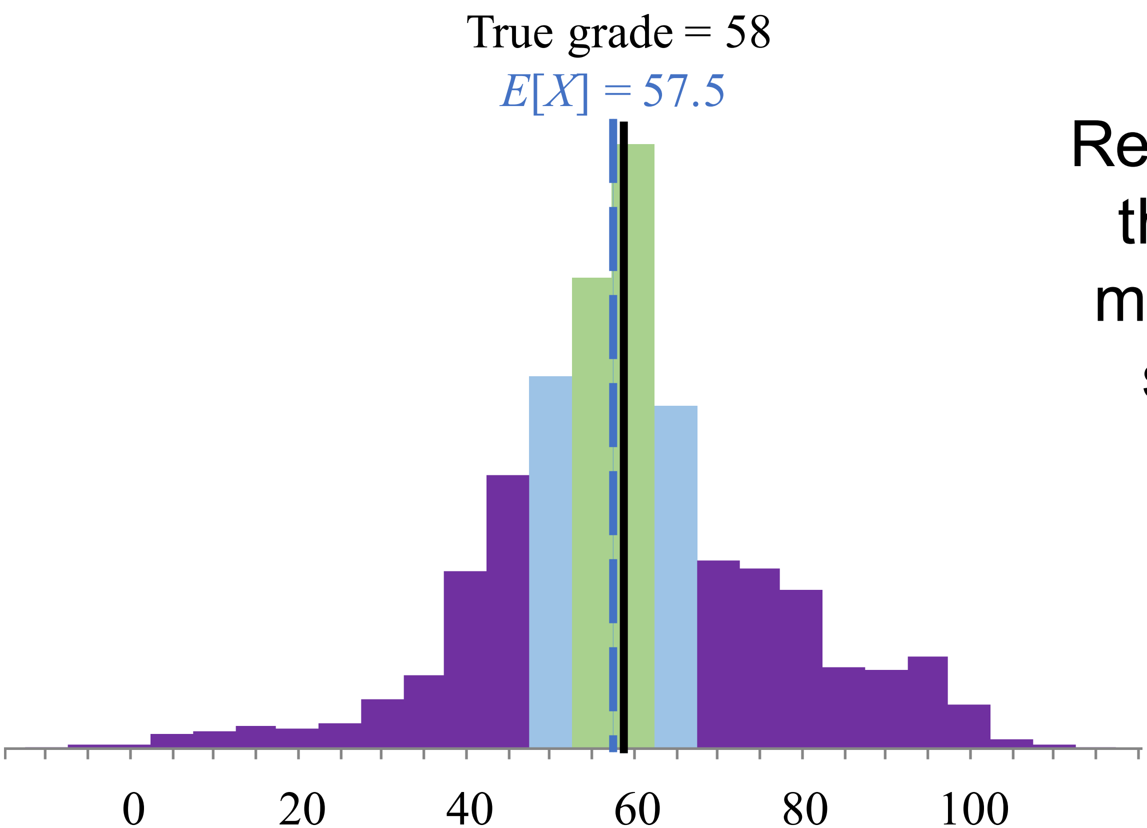
$$E\,[(X - \mu)^2] = 52 \text{ points}^2$$

$$SD(X) = 7.2 \text{ points}$$

Second Moment

Let X be a random variable that represents a peer grade

$$E[X^2]$$

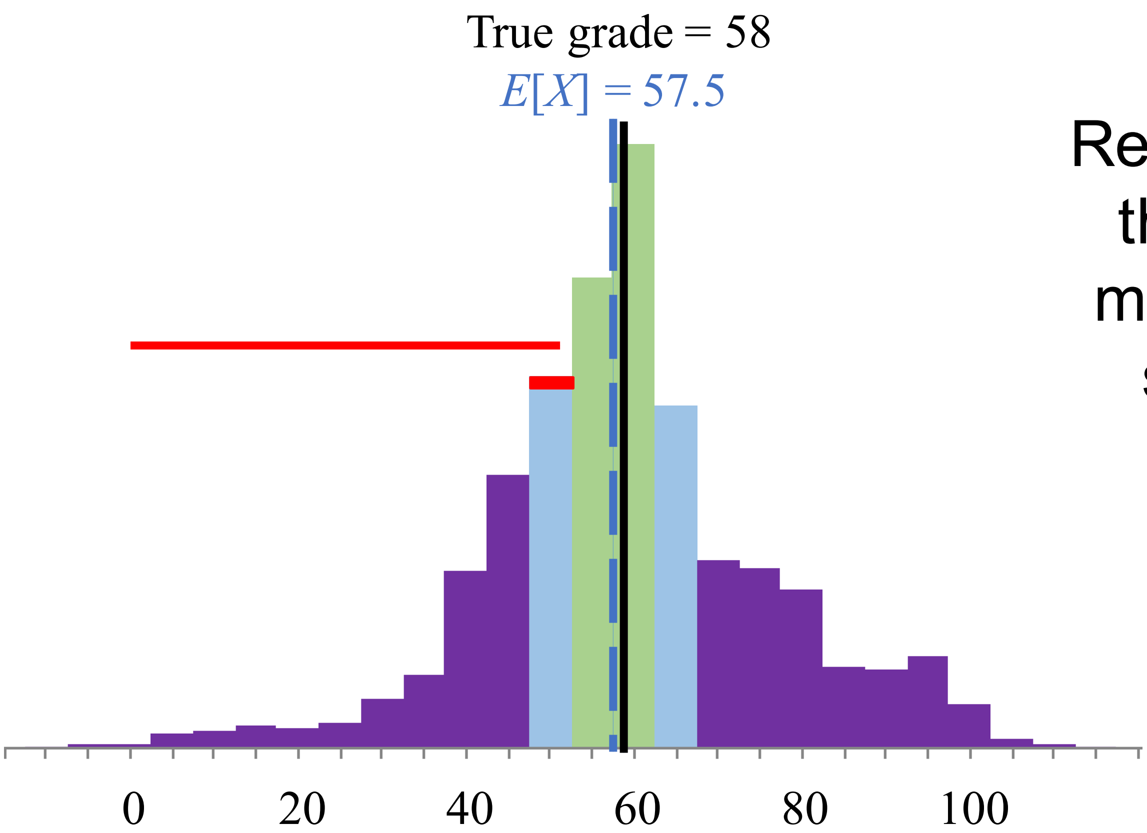


Recall that Variance is the second **central** moment. What is the second moment?

Second Moment

Let X be a random variable that represents a peer grade

$$E[X^2]$$

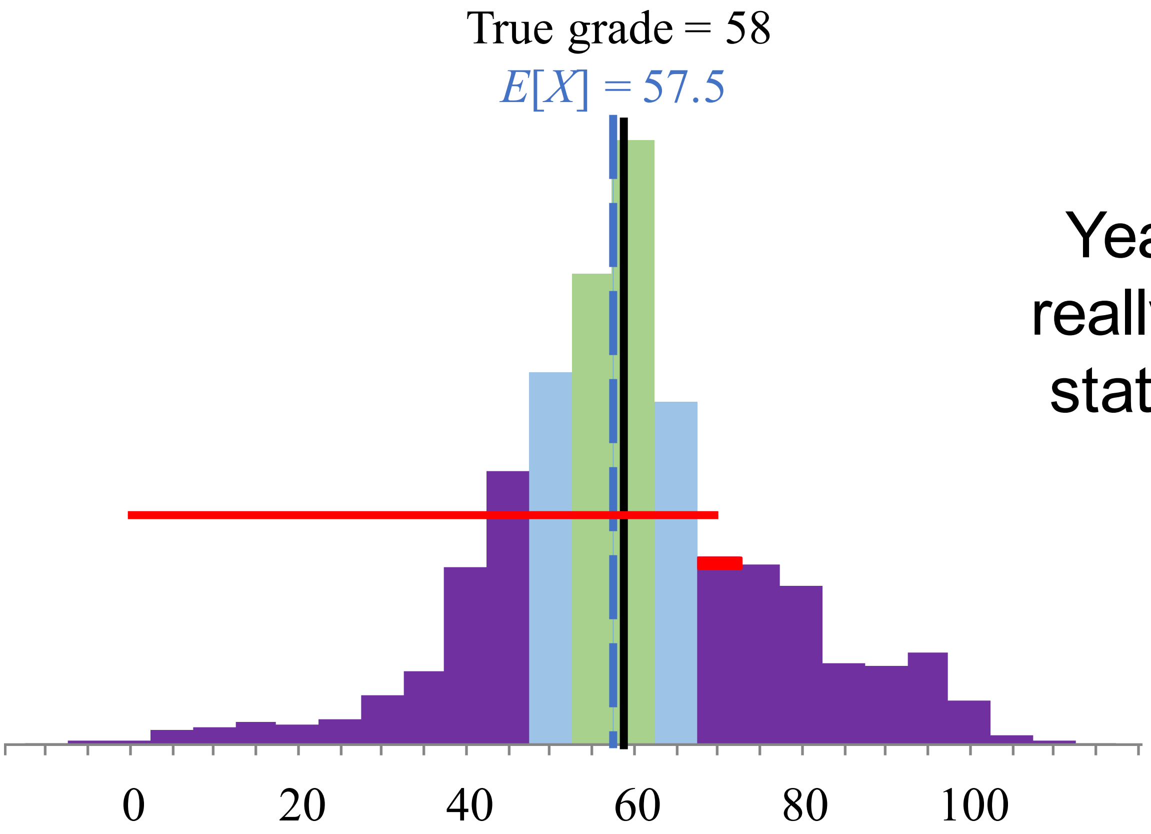


Recall that Variance is the second **central** moment. What is the second moment?

Second Moment

Let X be a random variable that represents a peer grade

$$E[X^2]$$



Yea second moment really isn't that useful a statistic in this case…

Lots of fun with Random Variables

Moaaahhhh Random Variables

Jacob Bernoulli

- Jacob Bernoulli (1654-1705), also known as "James", was a Swiss mathematician

- One of many mathematicians in Bernoulli family
- The Bernoulli Random Variable is named for him
- He is my *academic* great^{12}-grandfather
- Same eyes as Ice Cube

Bernoulli Random Variable

- Experiment results in "Success" or "Failure"
  - X is random indicator variable (1 = success, 0 = failure)
  - $P(X = 1) = p(1) = p$ $\quad$ $P(X = 0) = p(0) = 1 - p$
  - X is a **Bernoulli** Random Variable: $X \sim Ber(p)$
  - $E[X] = p$
  - $Var(X) = p(1 - p)$
- Examples
  - coin flip
  - random binary digit
  - whether a disk drive crashed
  - whether someone likes a netflix movie

Feel the Bern!

Binomial Random Variable

- Consider *n* independent trials of Ber(p) rand. var.
  - X is number of successes in *n* trials
  - X is a **<u>Binomial</u>** Random Variable: X ~ Bin(n, p)

$$P(X = i) = p(i) = \binom{n}{i} p^i (1-p)^{n-i} \quad i = 0,1,\ldots,n$$

  - By Binomial Theorem, we know that $\sum_{i=0}^{\infty} P(X = i) = 1$
- Examples
  - # of heads in *n* coin flips
  - # of 1's in randomly generated length *n* bit string
  - # of disk drives crashed in 1000 computer cluster
    - Assuming disks crash independently

Bernoulli vs Binomial

Bernoulli is a type of RV

Binomial is the sum of *n* Bernoullis

Three Coin Flips

- Three fair (“heads” with p = 0.5) coins are flipped
  - X is number of heads
  - X ~ Bin(3, 0.5)

$$P(X=0)=\binom{3}{0}p^0(1-p)^3=\frac{1}{8}$$

$$P(X=1)=\binom{3}{1}p^1(1-p)^2=\frac{3}{8}$$

$$P(X=2)=\binom{3}{2}p^2(1-p)^1=\frac{3}{8}$$

$$P(X=3)=\binom{3}{3}p^3(1-p)^0=\frac{1}{8}$$

Error Correcting Codes

- Error correcting codes
  - Have original 4 bit string to send over network
  - Add 3 “parity” bits, and send 7 bits total
  - Each bit independently corrupted (flipped) in transition with probability 0.1

Error Correcting Codes

Key for 7 bits

Send: 1110?

Receive: 1110000?

Receive: 1010100?

Flip set: $O_i E_j^C$
for all odd sets i
and even sets j

Error Correcting Codes

- Error correcting codes
  - Have original 4 bit string to send over network
  - Add 3 “parity” bits, and send 7 bits total
  - Each bit independently corrupted (flipped) in transition with probability 0.1
  - X = number of bits corrupted: $X \sim Bin(7, 0.1)$
  - But, parity bits allow us to correct at most 1 bit error
- P(a correctable message is received)?
  - $P(X = 0) + P(X = 1)$

Error Correcting Codes

- Using error correcting codes: X ~ Bin(7, 0.1)

$$P(X=0)=\binom{7}{0}(0.1)^0(0.9)^7 \approx 0.4783$$

$$P(X=1)=\binom{7}{1}(0.1)^1(0.9)^6 \approx 0.3720$$

  - P(X = 0) + P(X = 1) = 0.8503
- What if we didn't use error correcting codes?
  - X ~ Bin(4, 0.1)
  - P(correct message received) = P(X = 0)

$$P(X=0)=\binom{4}{0}(0.1)^0(0.9)^4 = 0.6561$$

- Using error correction improves reliability ~30%!

Genetic Inheritance

- Person has 2 genes for trait (eye color)
  - Child receives 1 gene (equally likely) from each parent
  - Child has brown eyes if either (or both) genes brown
  - Child only has blue eyes if both genes blue
  - Brown is "dominant" (d) , Blue is "recessive" (r)
  - Parents each have 1 brown and 1 blue gene
- 4 children, what is P(3 children with brown eyes)?
  - Child has blue eyes: p = (½) (½) = ¼ (2 blue genes)
  - P(child has brown eyes) = 1 – (¼) = 0.75
  - X = # of children with brown eyes. X ~ Bin(4, 0.75)

$$P(X=3)=\binom{4}{3}(0.75)^3(0.25)^1 \approx 0.4219$$

Properties of Bin(n, p)

- Consider: $X \sim Bin(n, p)$
- $E[X] = np$
- $Var(X) = np(1 - p)$
- So, to compute $E[X^2]$, we have:

$$Var(X) = E[X^2] - (E[X])^2$$

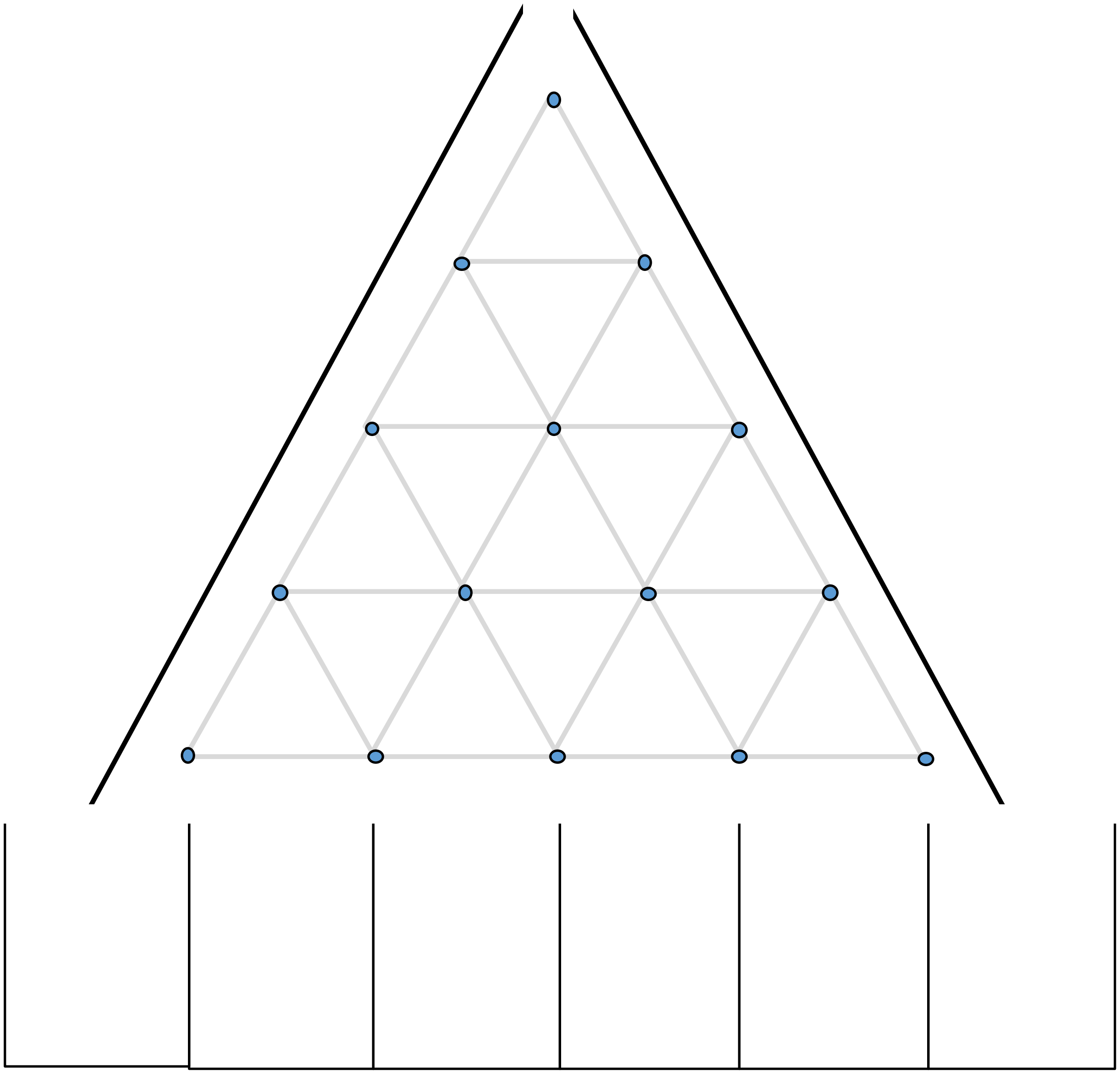
$$E[X^2] = Var(X) + (E[X])^2$$

$$= np(1 - p) + (np)^2$$

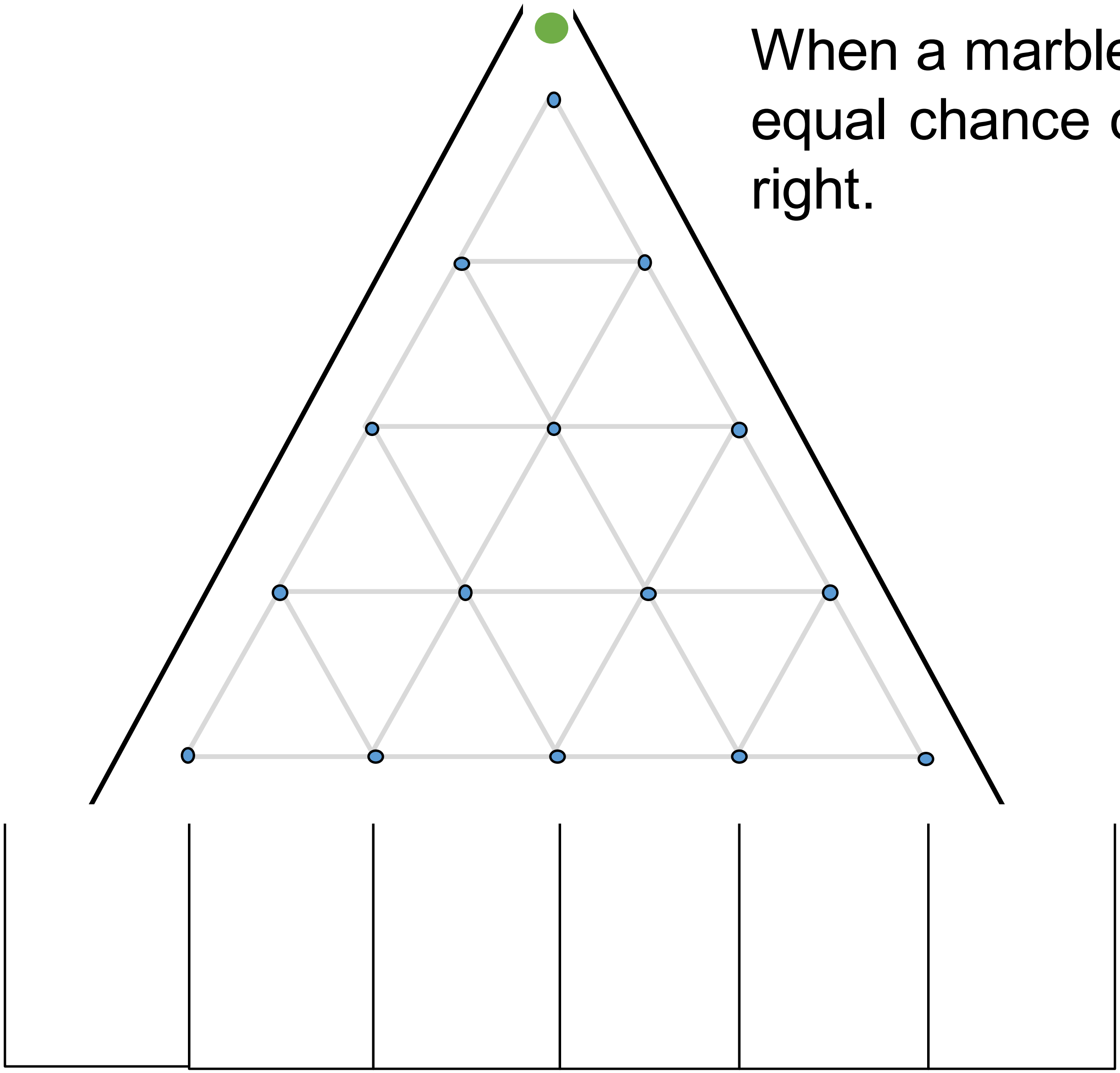
$$= n^2p^2 - np^2 + np$$

- Note: $Ber(p) = Bin(1, p)$

Galton Board

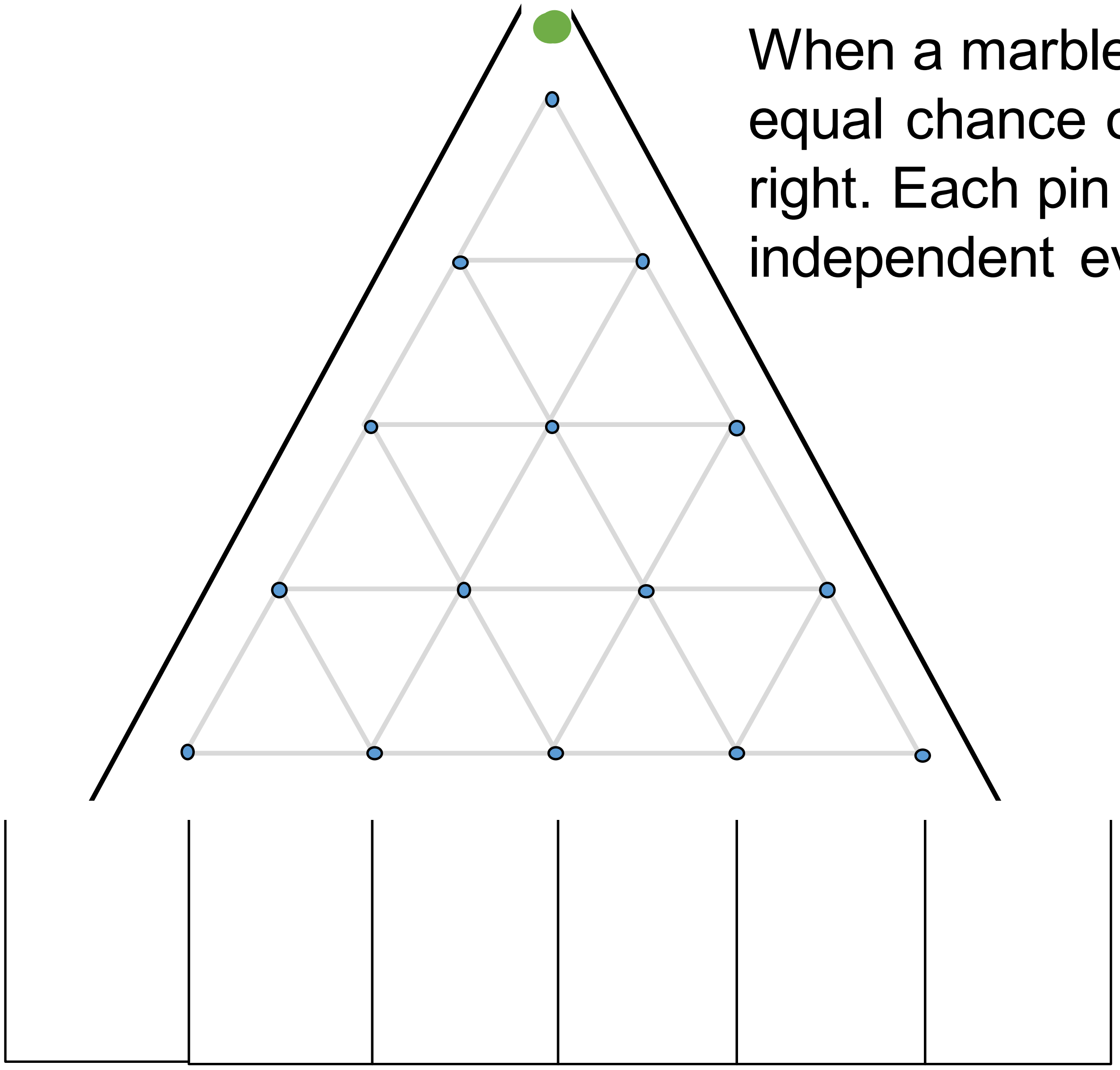


Galton Board



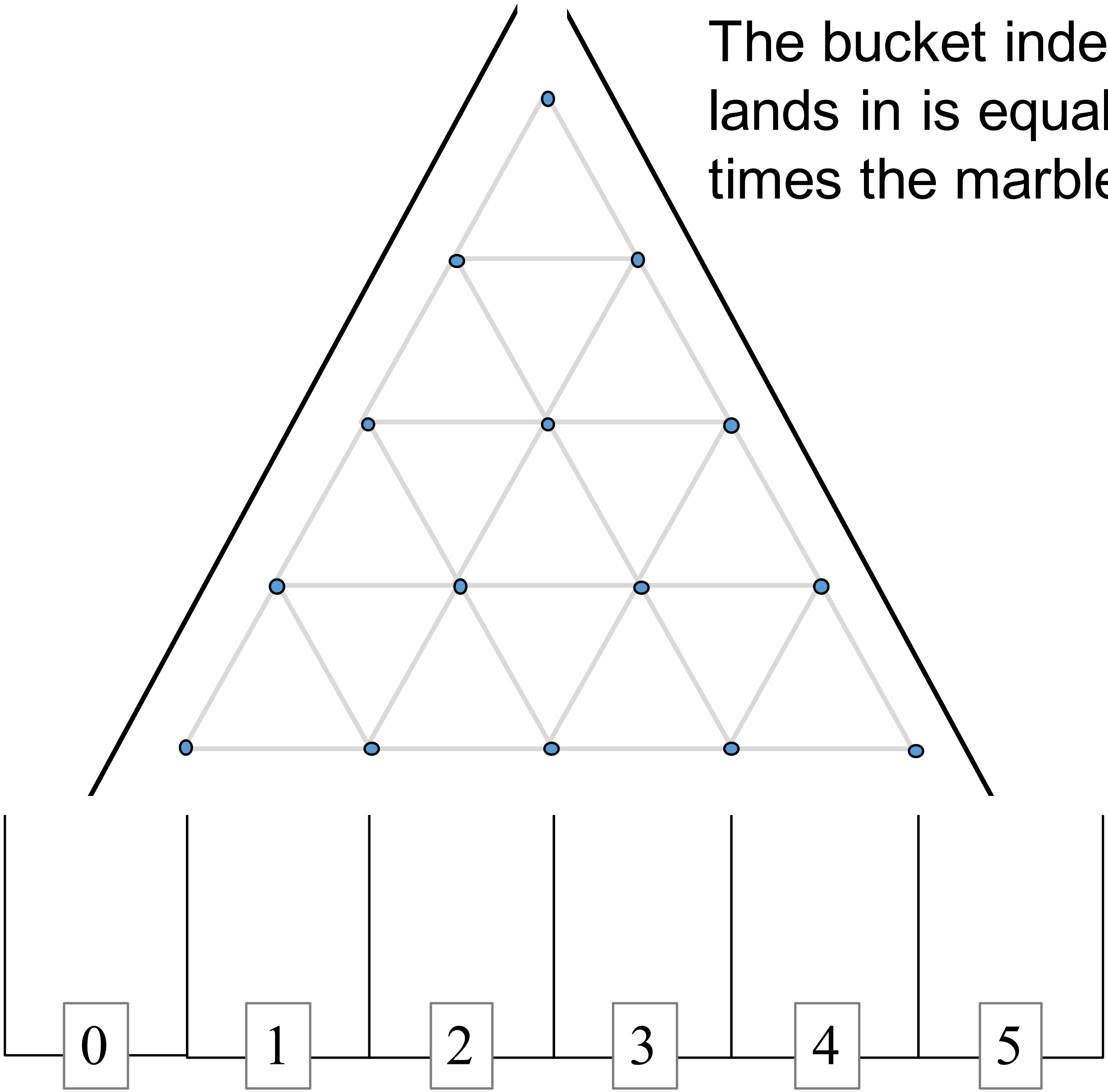
When a marble hits a pin, it has equal chance of going left or right.

Galton Board



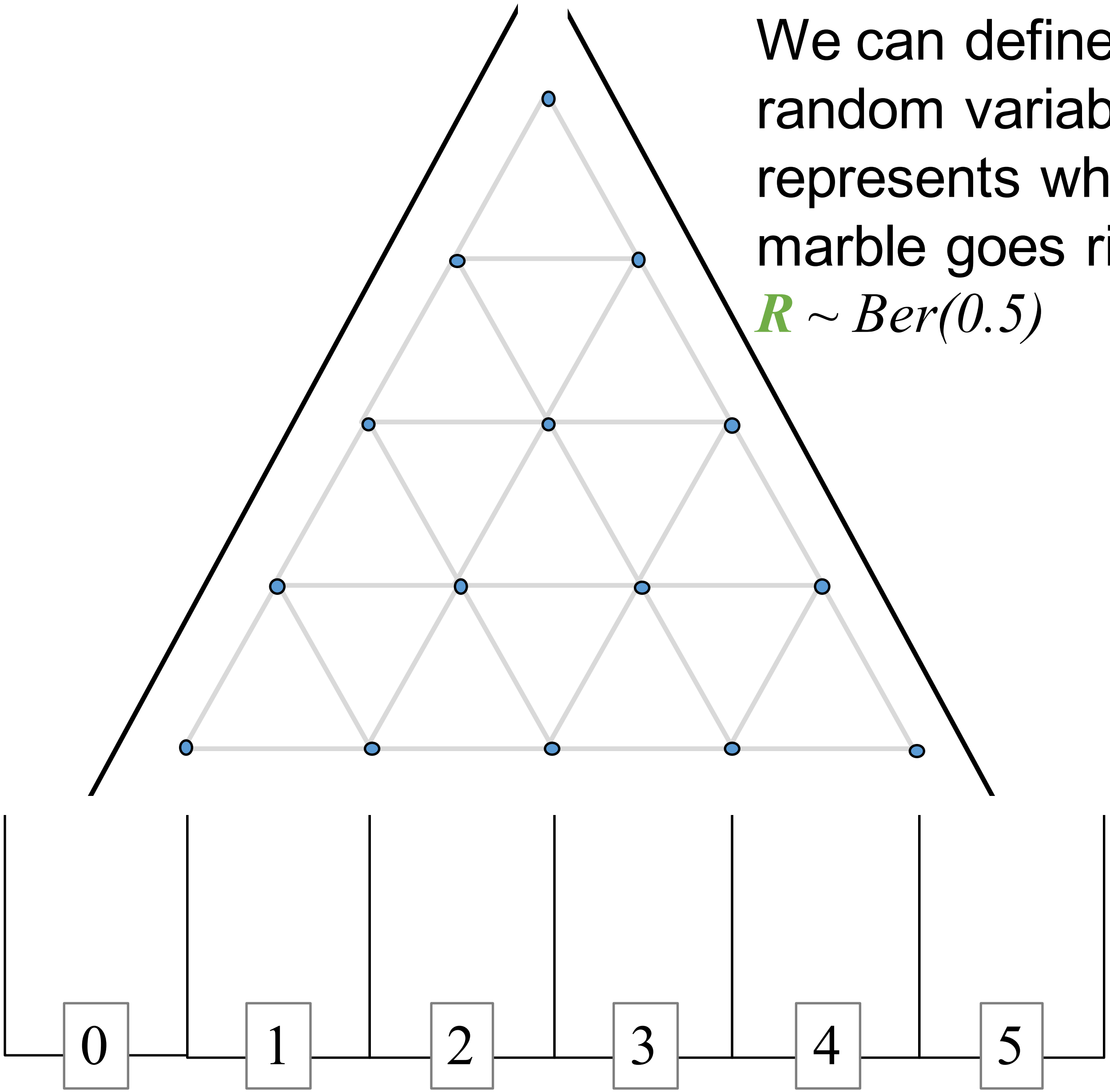
When a marble hits a pin, it has equal chance of going left or right. Each pin represents an independent event.

Galton Board



The bucket index that a marble lands in is equal to the number of times the marble went right

Galton Board

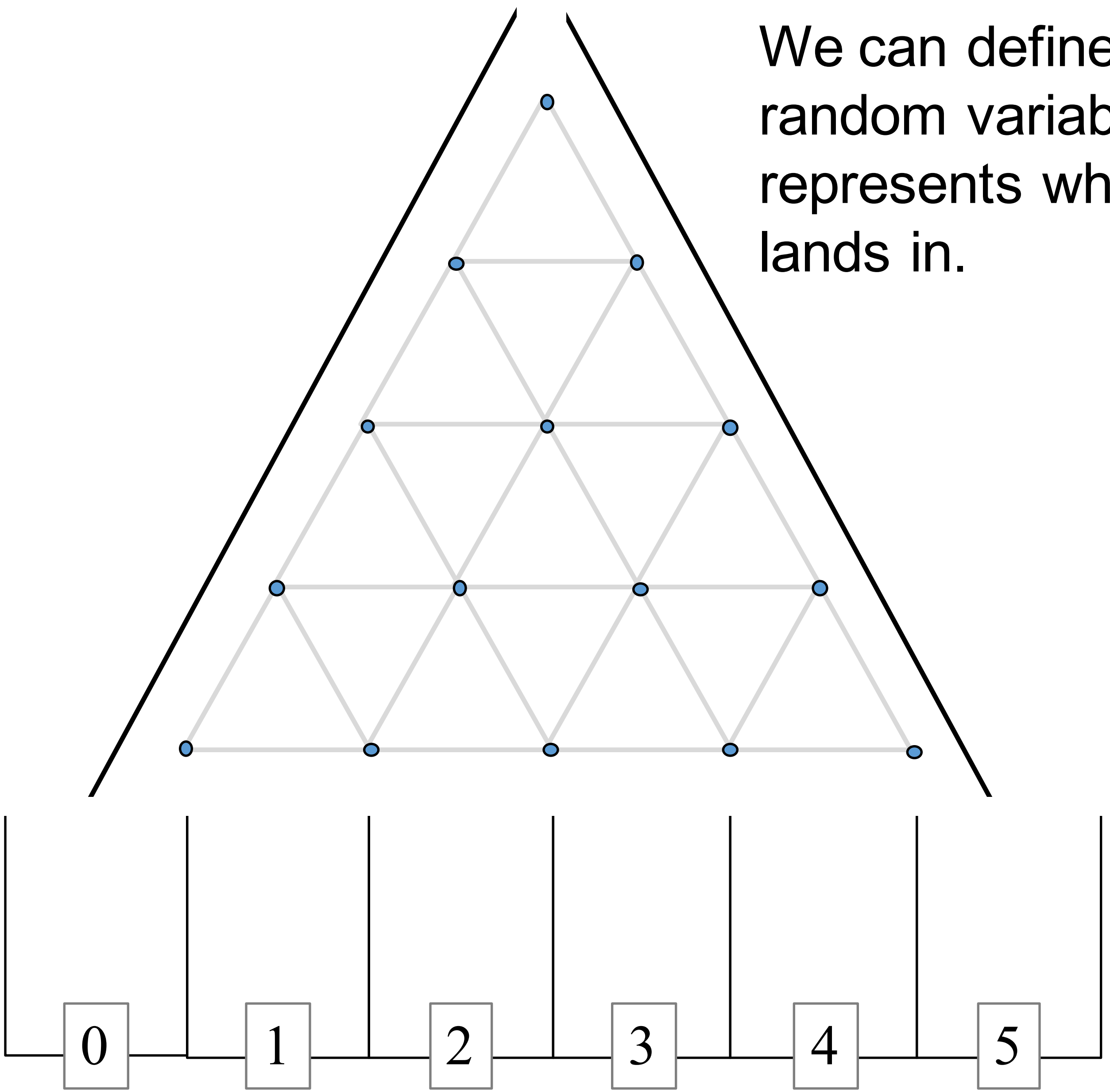


We can define an indicator random variable (R) which represents whether a particular marble goes right as a Bernoulli

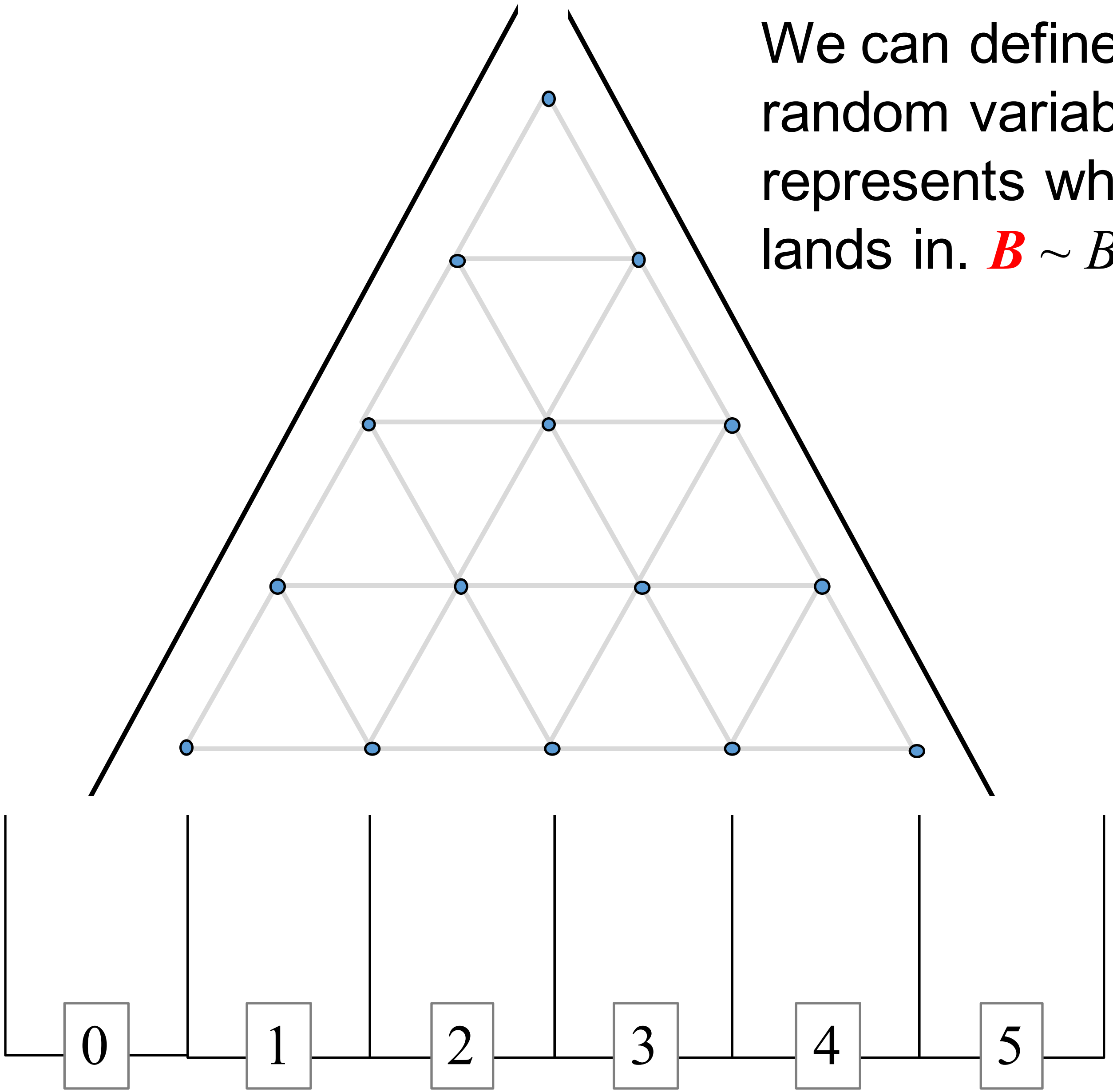
$R \sim Ber(0.5)$

Galton Board

We can define an indicator random variable (***B***) which represents what bucket a marble lands in.

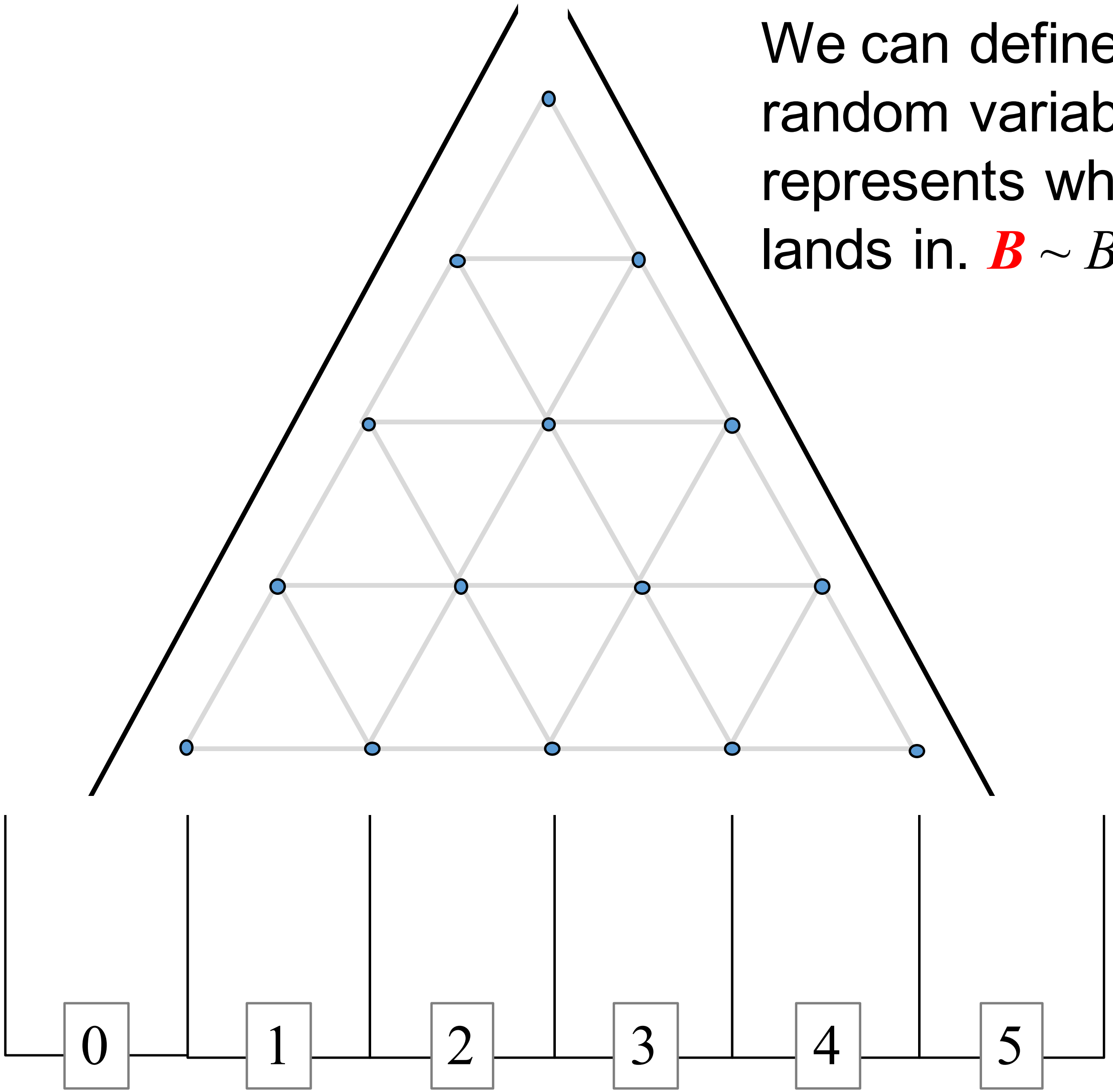


Galton Board



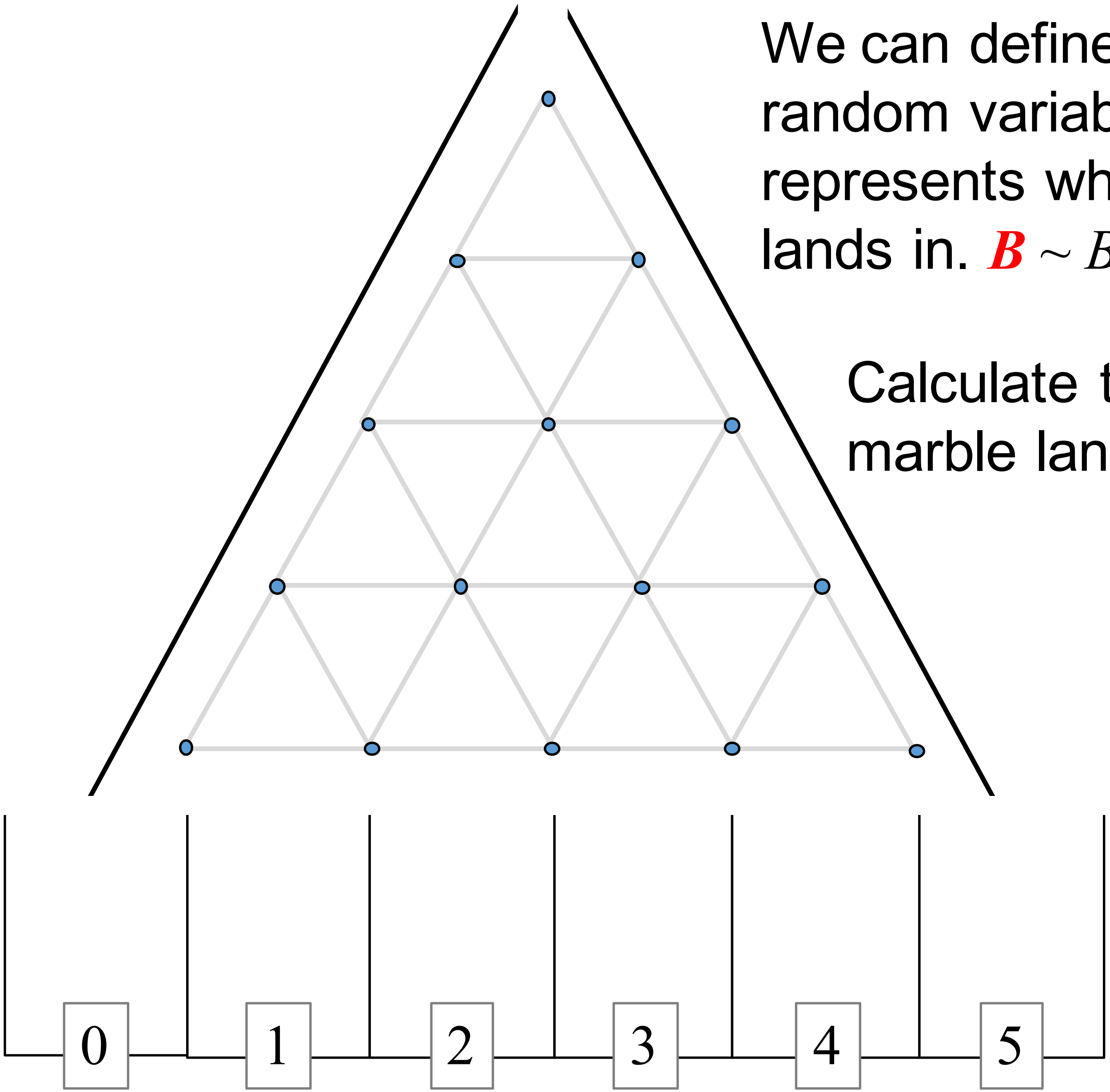
We can define an indicator random variable (B) which represents what bucket a marble lands in. $B \sim Bin(\text{levels}, 0.5)$

Galton Board



We can define an indicator random variable (***B***) which represents what bucket a marble lands in. $B \sim Bin(5, 0.5)$

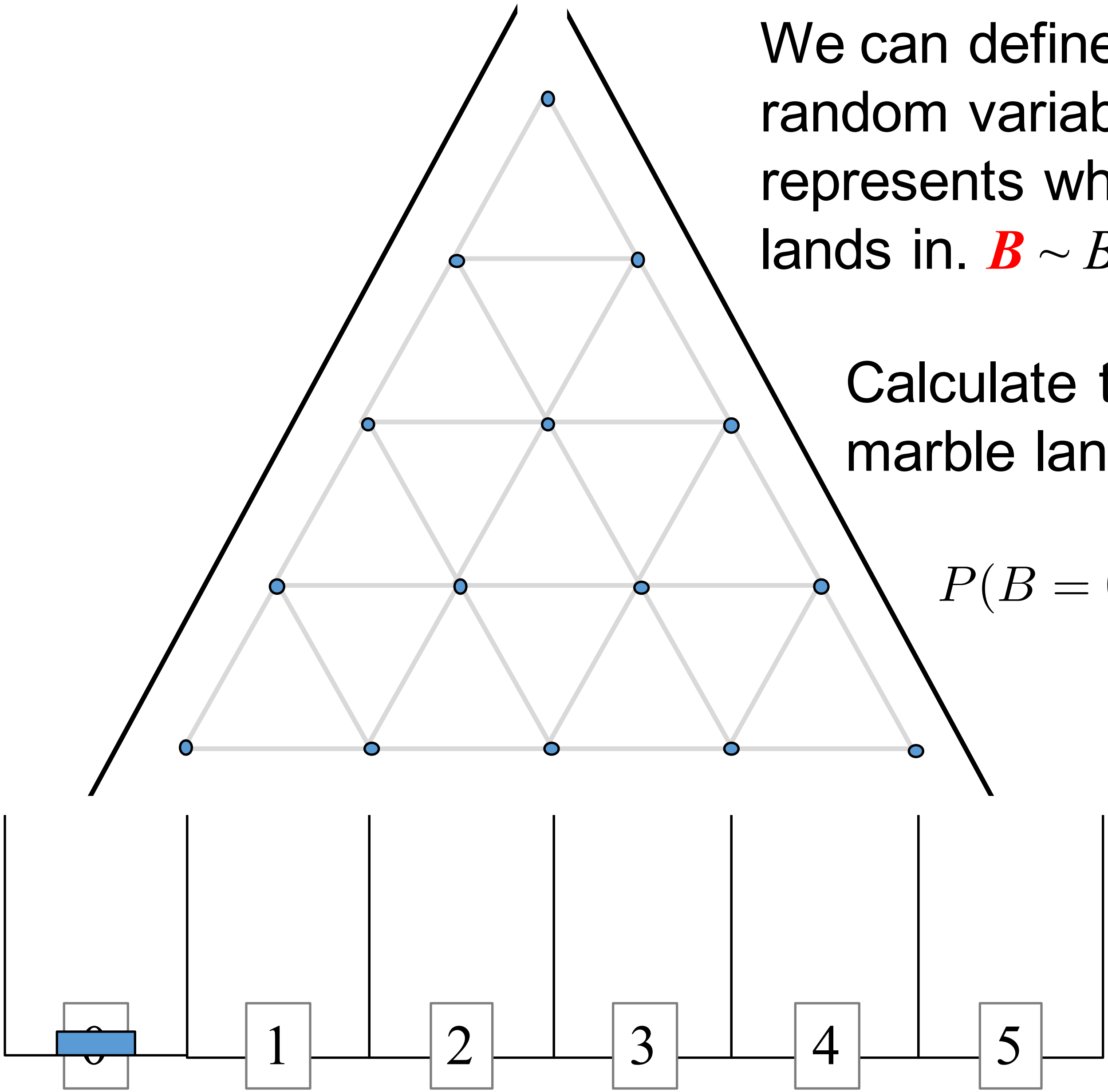
Galton Board



We can define an indicator random variable (***B***) which represents what bucket a marble lands in. ***B*** $\sim Bin(5, 0.5)$

Calculate the probability of a marble landing in a bucket.

Galton Board

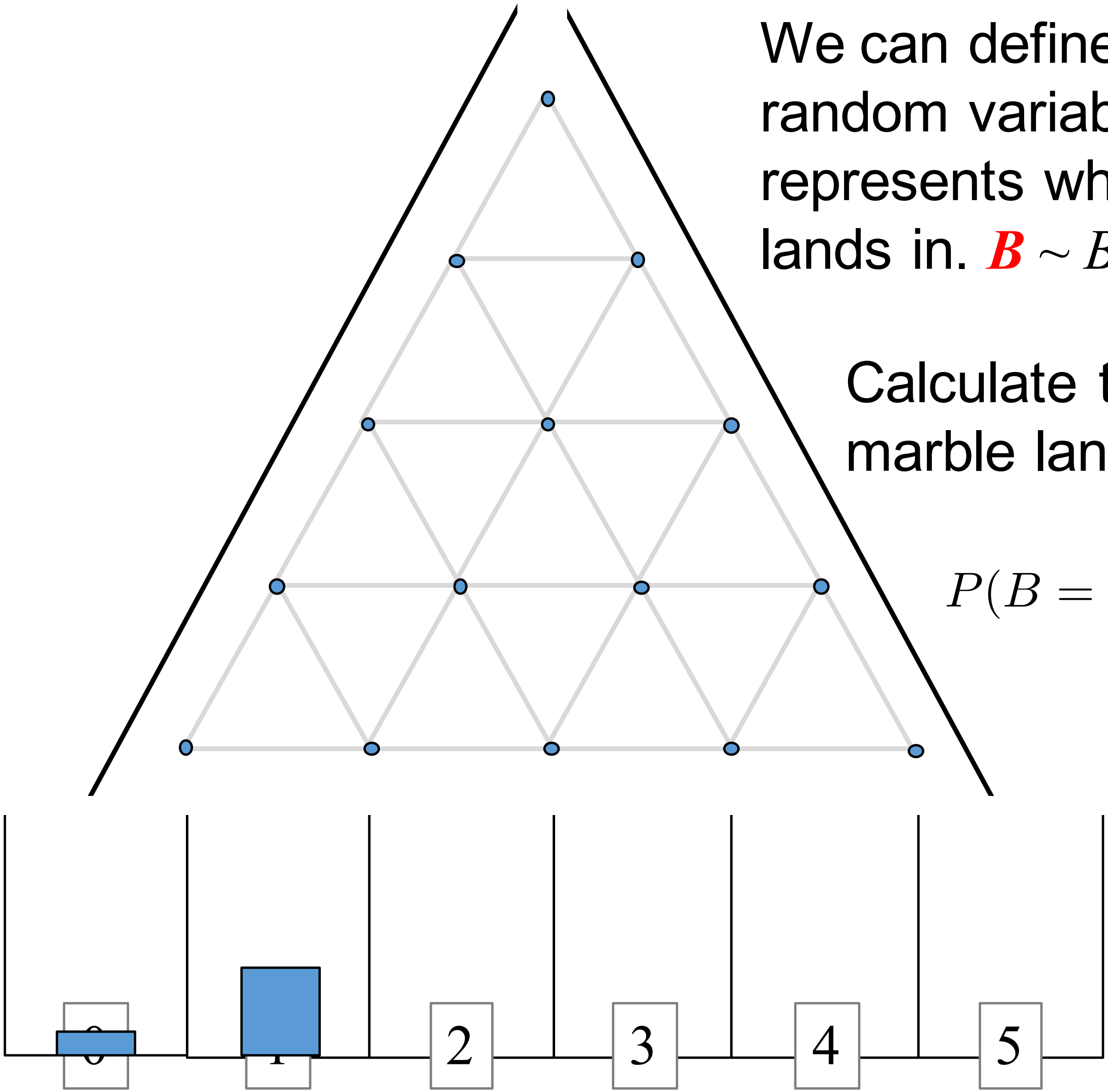


We can define an indicator random variable (***B***) which represents what bucket a marble lands in. ***B*** $\sim Bin(5, 0.5)$

Calculate the probability of a marble landing in a bucket.

$$P(B = 0) = \binom{5}{0} \frac{1}{2}^5 \approx 0.03$$

Galton Board

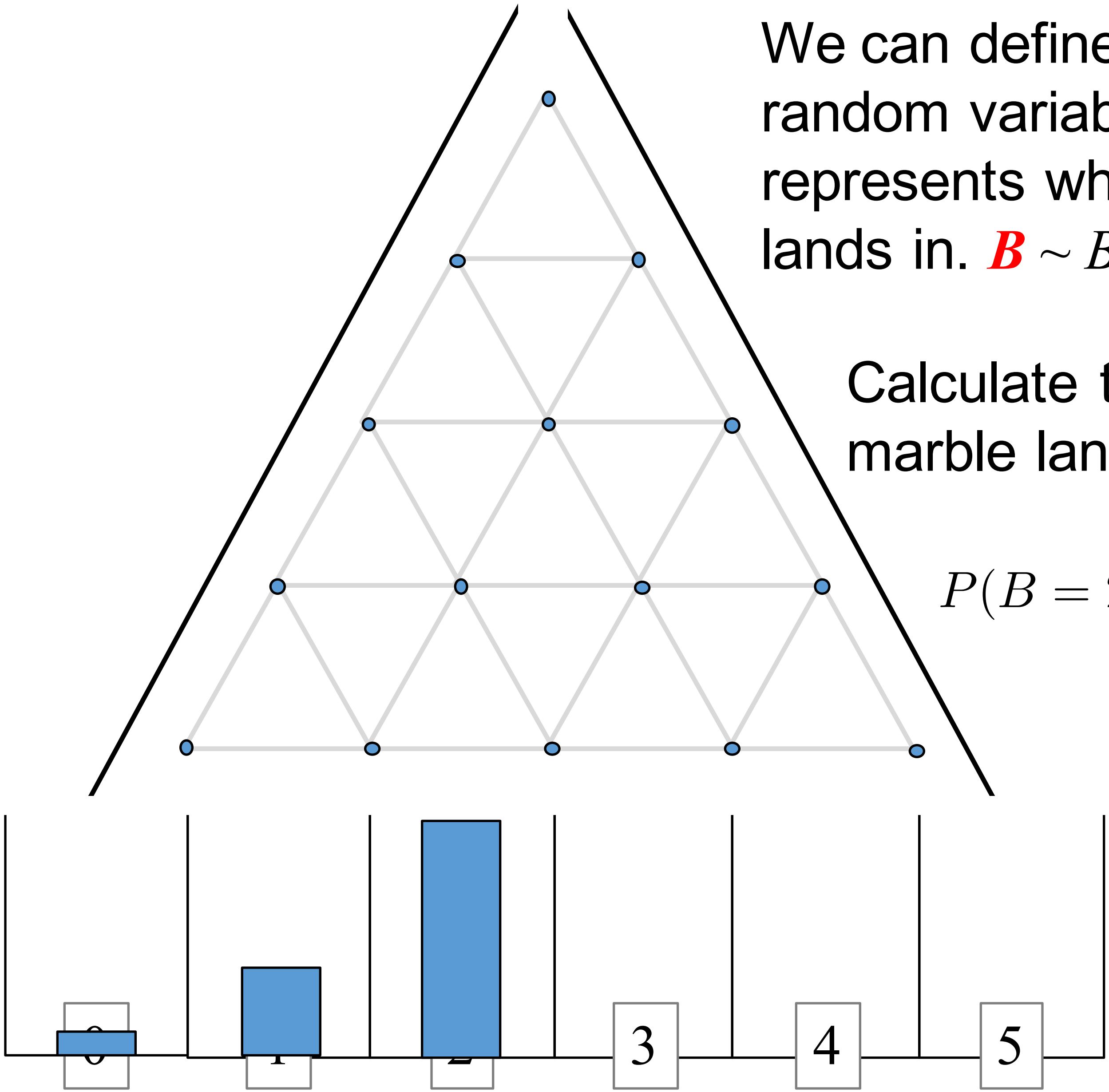


We can define an indicator random variable (***B***) which represents what bucket a marble lands in. ***B*** $\sim Bin(5, 0.5)$

Calculate the probability of a marble landing in a bucket.

$$P(B = 1) = \binom{5}{1} \frac{1}{2}^{5} \approx 0.16$$

Galton Board

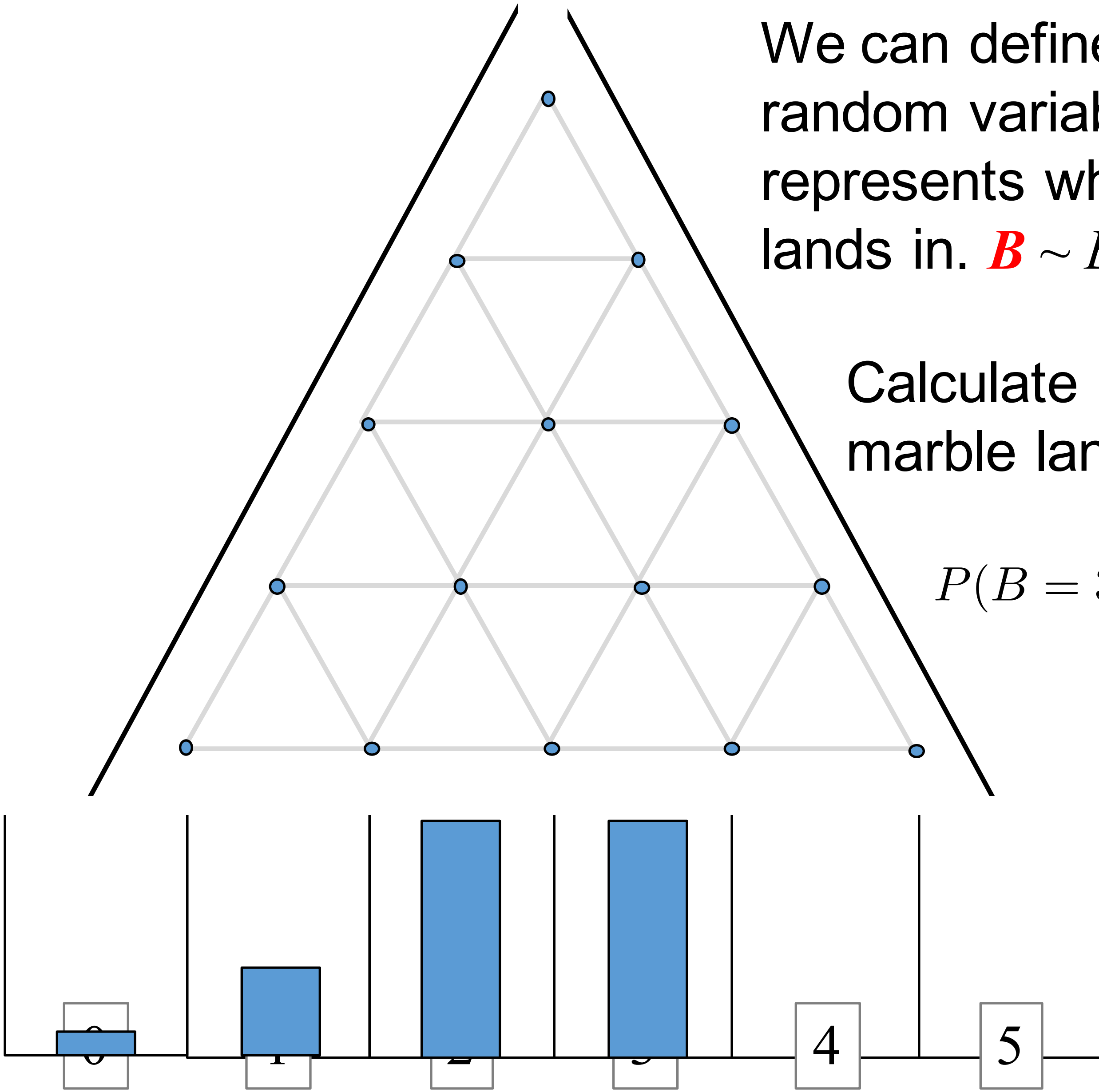


We can define an indicator random variable (***B***) which represents what bucket a marble lands in. ***B*** *~ Bin(5, 0.5)*

Calculate the probability of a marble landing in a bucket.

$$P(B = 2) = \binom{5}{2} \frac{1}{2}^{5} \approx 0.31$$

Galton Board



We can define an indicator random variable (***B***) which represents what bucket a marble lands in. ***B*** ~ *Bin(5, 0.5)*

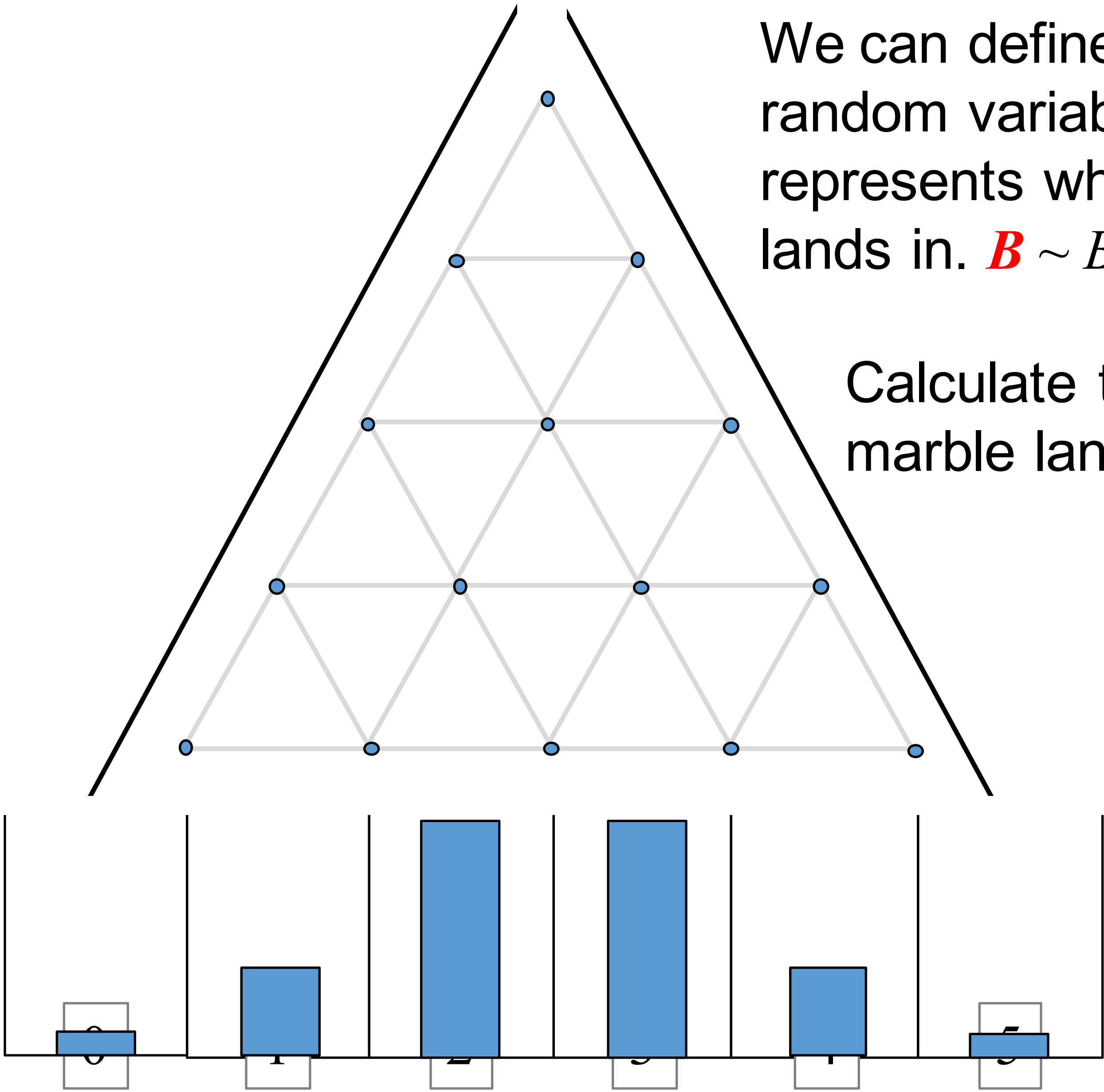
Calculate the probability of a marble landing in a bucket.

$$P(B = 3) = \binom{5}{2} \frac{1}{2}^{5} \approx 0.31$$

Galton Board

We can define an indicator random variable ($\boldsymbol{B}$) which represents what bucket a marble lands in. $\boldsymbol{B} \sim Bin(5, 0.5)$

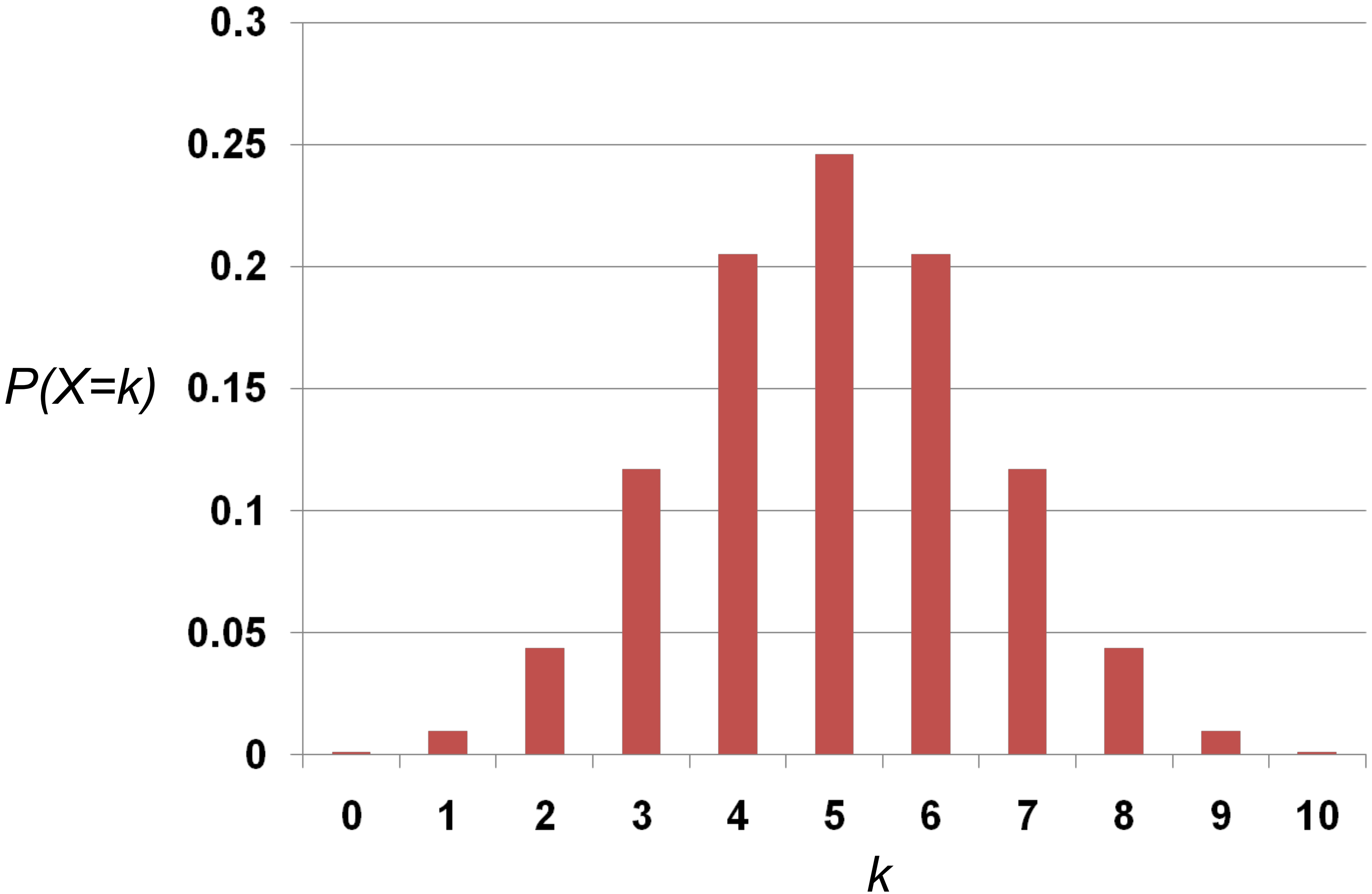
Calculate the probability of a marble landing in a bucket.



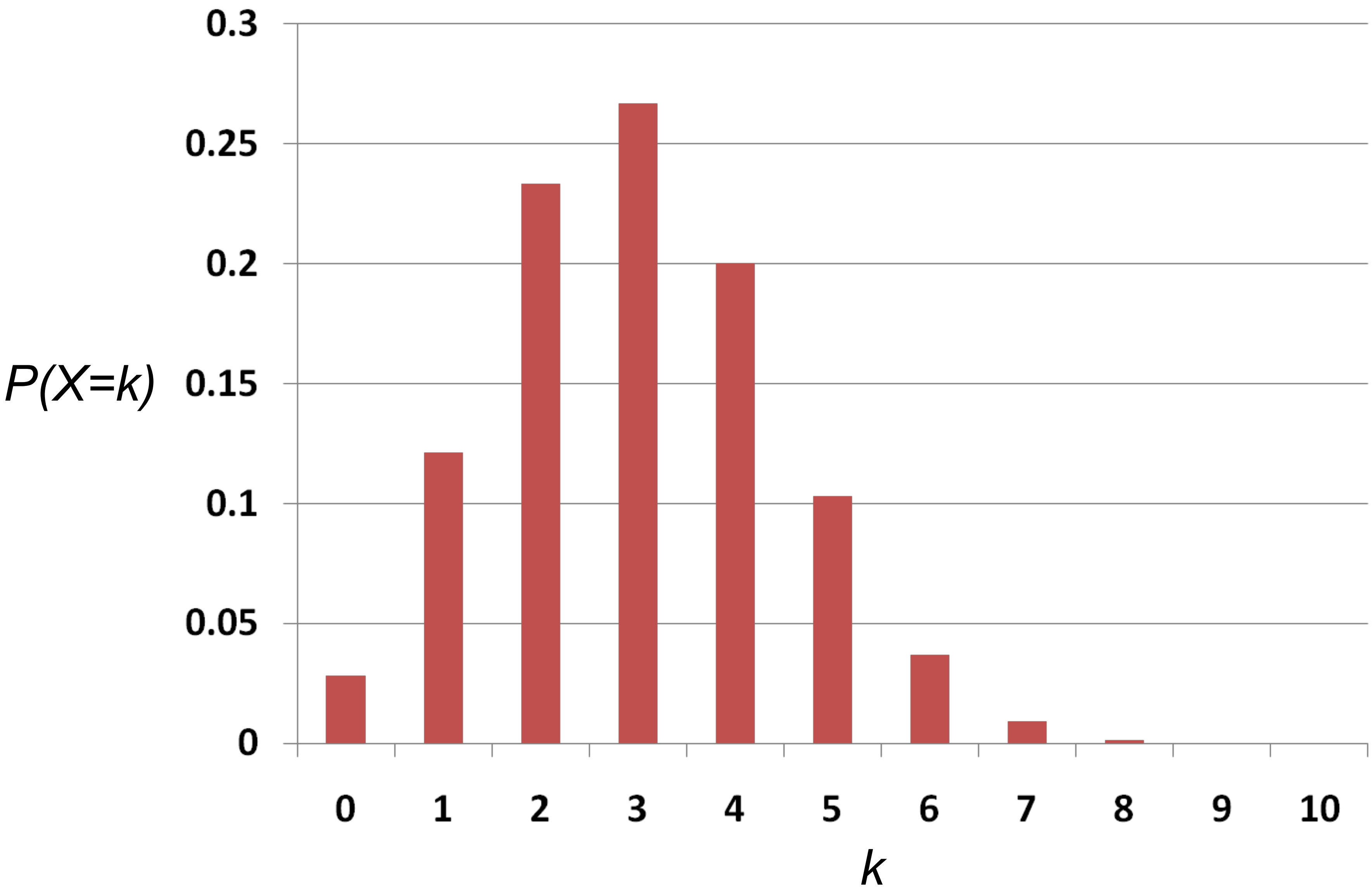
PDF

https://www.youtube.com/watch?v=p65aYYuAz-s

PMF for X ~ Bin(10, 0.5)



PMF for X ~ Bin(10, 0.3)



Power of Your Vote

- Is it better to vote in small or large state?
  - Small: more likely your vote changes outcome
  - Large: larger outcome (electoral votes) if state swings
  - a (= 2n) voters equally likely to vote for either candidate
  - You are deciding (a + 1)st vote

$$P(2n \text{ voters tie}) = \binom{2n}{n}\left(\frac{1}{2}\right)^n\left(\frac{1}{2}\right)^n = \frac{(2n)!}{n!n!2^{2n}}$$

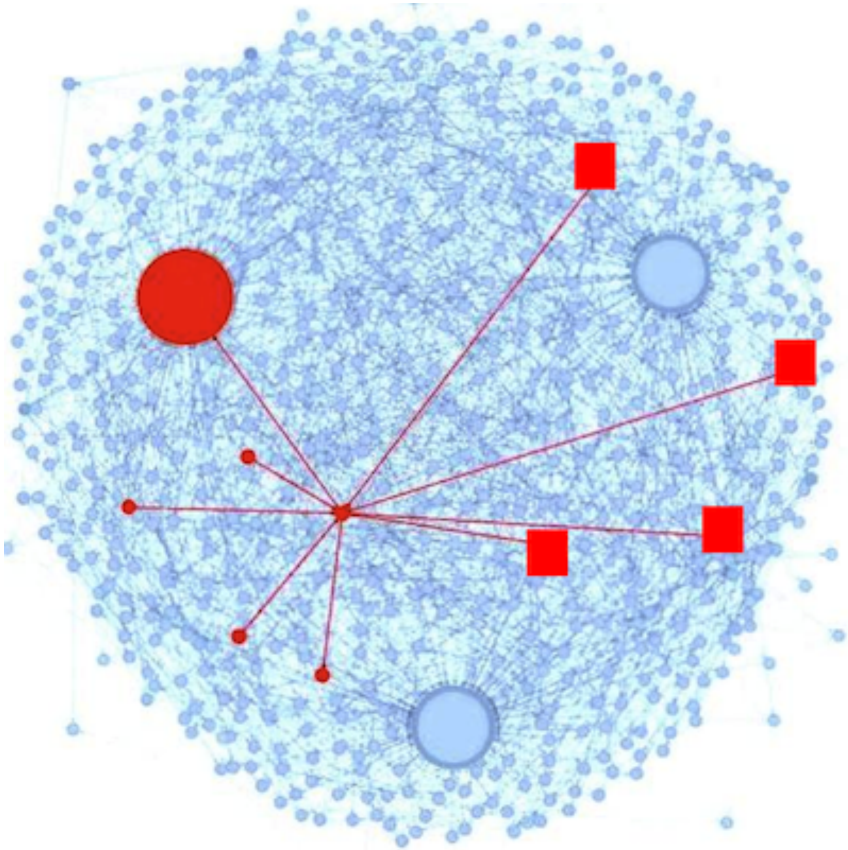
  - Use Stirling's Approximation: $n! \approx n^{n+1/2}e^{-n}\sqrt{2\pi}$

$$P(2n \text{ voters tie}) \approx \frac{(2n)^{2n+1/2}e^{-2n}\sqrt{2\pi}}{n^{2n+1}e^{-2n}2\pi 2^{2n}} = \frac{1}{\sqrt{n\pi}}$$

  - Power = P(tie) * Elec. Votes = $\frac{1}{\sqrt{(a/2)\pi}}(ac) = \frac{c\sqrt{2a}}{\sqrt{\pi}}$
  - Larger state = more power

Is Peer Grading Accurate Enough?

Looking ahead



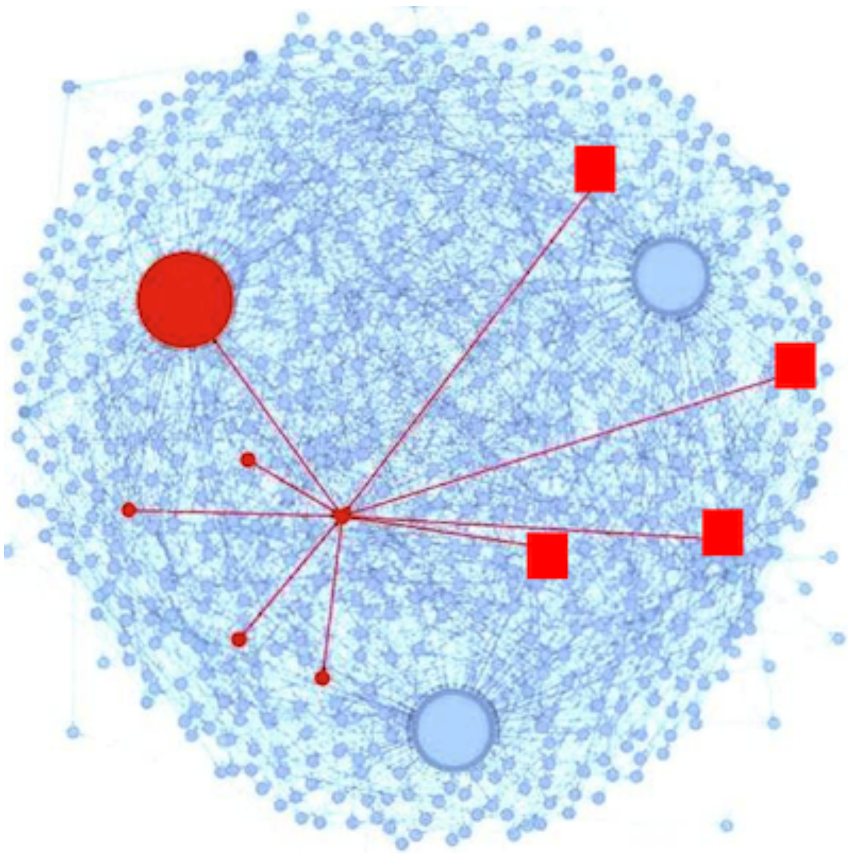
Peer Grading on Coursera HCI.

31,067 peer grades for 3,607 students.

Tuned Models of Peer Assessment. C Piech, J Huang, A Ng, D Koller

Is Peer Grading Accurate Enough?

Looking ahead



1. Defined random variables for:
- True grade (s_i) for assignment i
- Observed (z_i^j) score for assign i
- Bias (b_j) for each grader j
- Variance (r_j) for each grader j

2. Designed a probabilistic model that defined the distributions for all random variables

Problem param

$$s_i \sim \text{Bin}(\text{points}, \theta)$$

$$z_i^j \sim \mathcal{N}(\mu = s_i + b_j, \sigma = \sqrt{r_j})$$

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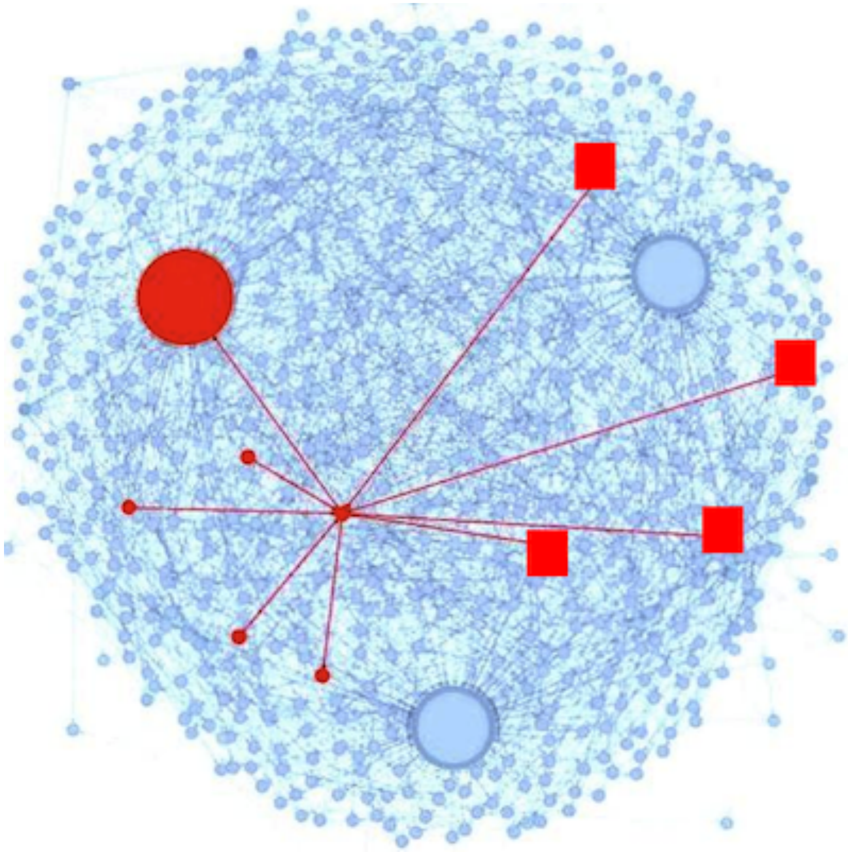
Is Peer Grading Accurate Enough?

Looking ahead



1. Defined random variables for:
- True grade (s_i) for assignment i
- Observed (z_i^j) score for assign i
- Bias (b_j) for each grader j
- Variance (r_j) for each grader j

2. Designed a probabilistic model that defined the distributions for all random variables

3. Found the variable assignments that maximized the probability of our observed data

Inference or Machine Learning

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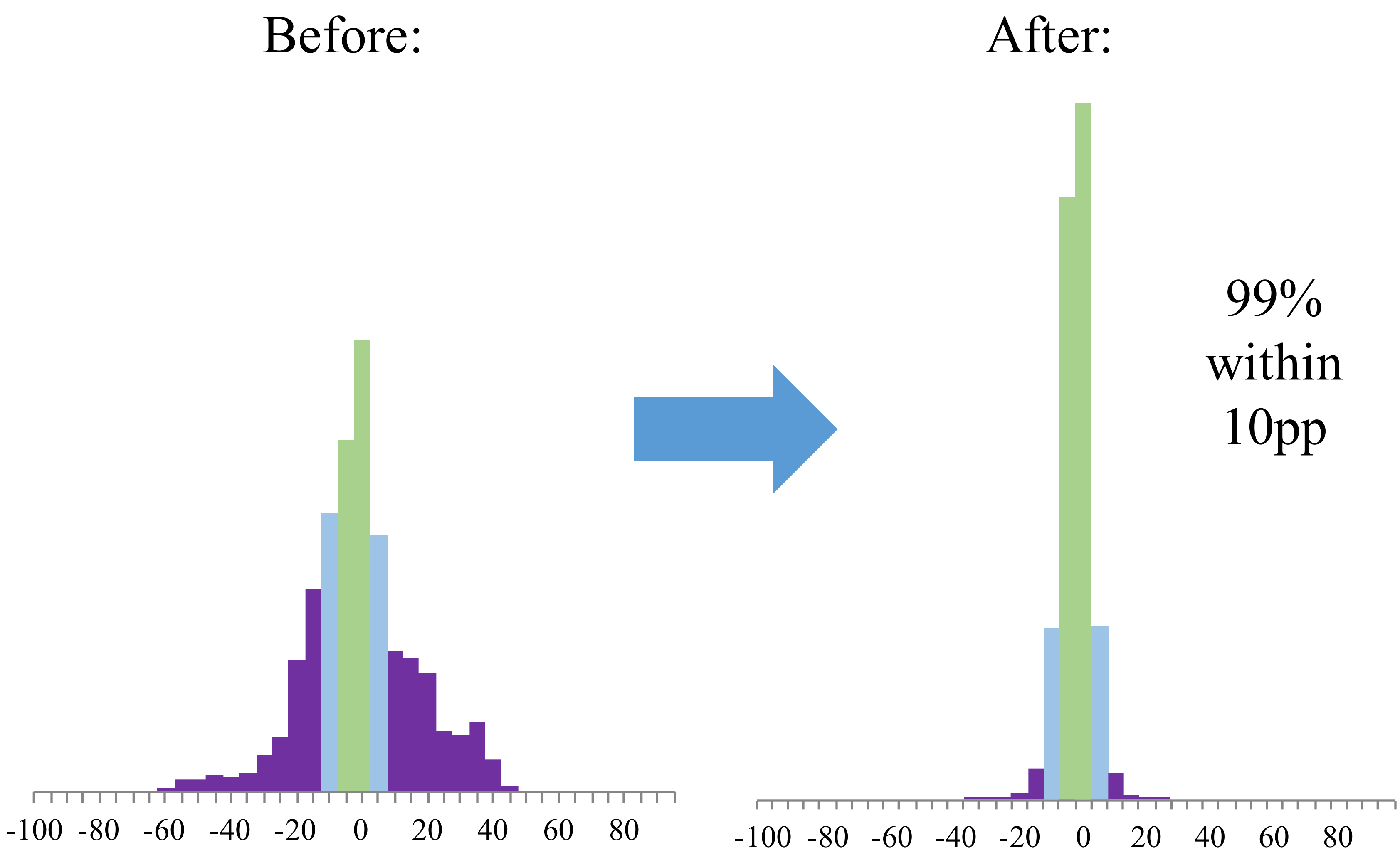
Yes, With Probabilistic Modelling



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Voilà, c'est tout