Joint Distributions
(and Exponential)

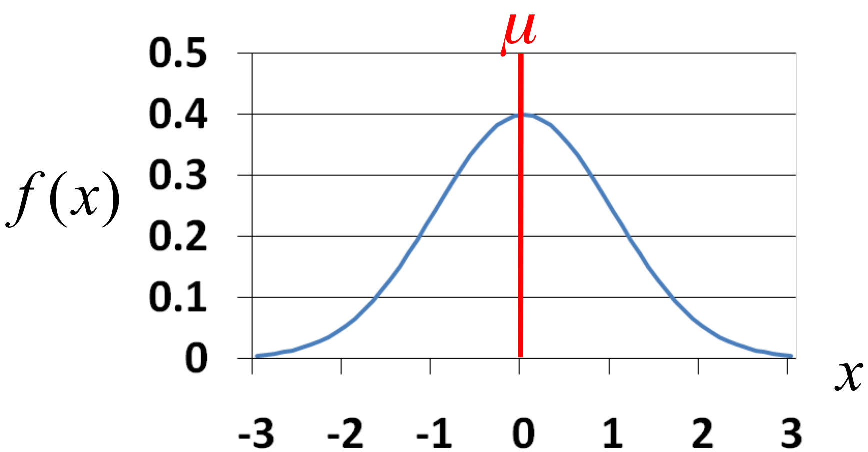
CS 109
Lecture 11
April 20th, 2016

Review

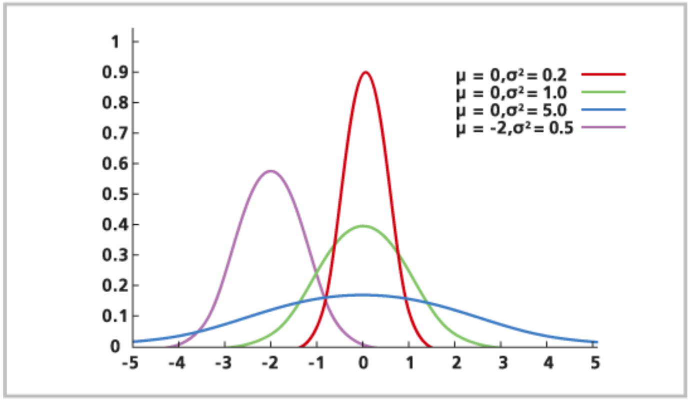
The Normal Distribution

- X is a **Normal Random Variable**: $X \sim N(\mu, \sigma^2)$
  - Probability Density Function (PDF):

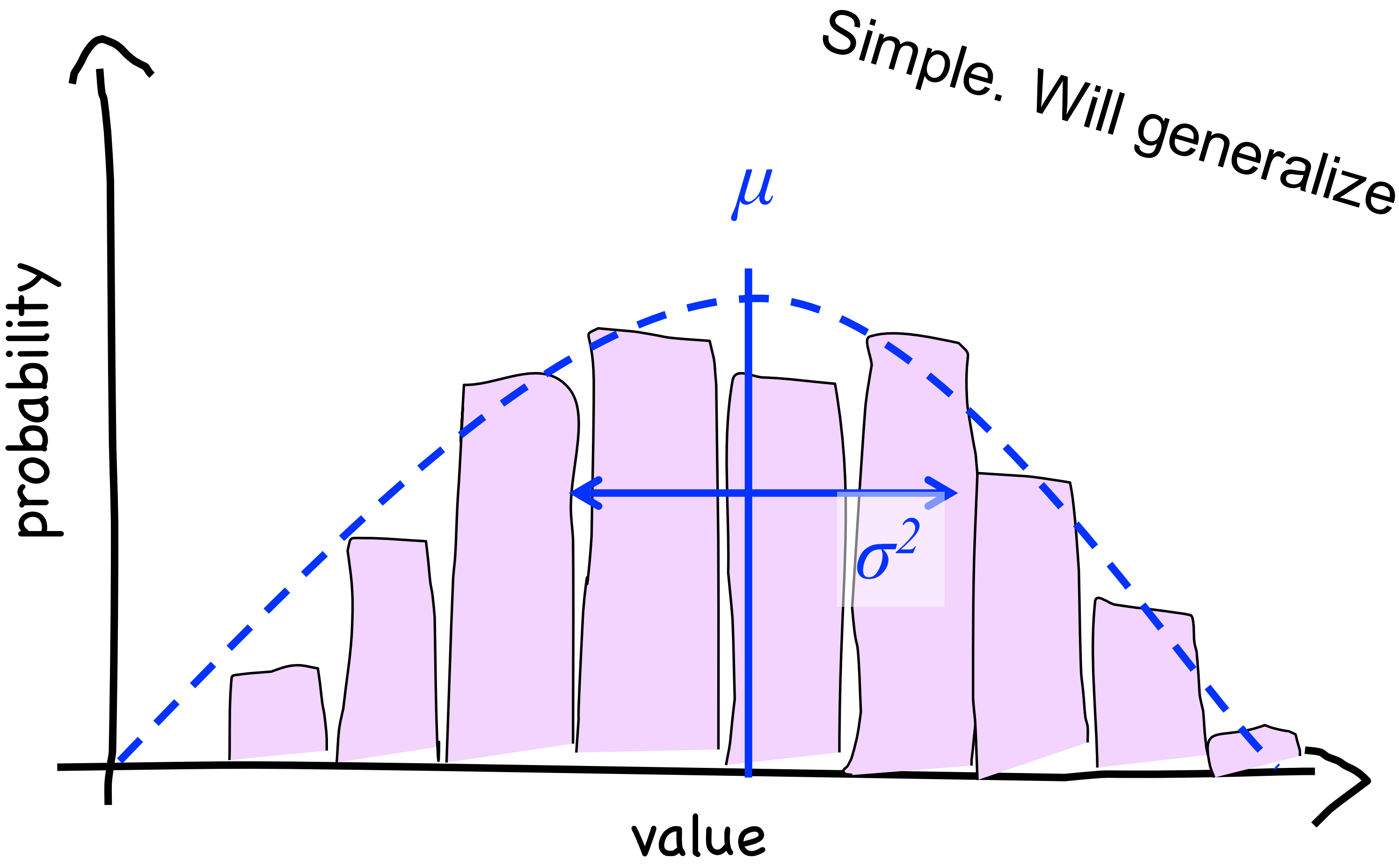
$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} \quad \text{where} \quad -\infty < x < \infty$$



  - $E[X] = \mu$
  - $Var(X) = \sigma^2$
  - Also called “Gaussian”
  - Note: $f(x)$ is symmetric about μ



Simplicity is Humble



* A Gaussian maximizes entropy for a given mean and variance

Anatomy of a beautiful equation

$\mathcal{N}(\mu, \sigma^2)$

"exponential"

the distance to the mean

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

probability density at x

a constant

sigma shows up twice

And here we are

$\mathcal{N}(\mu, \sigma^2)$

CDF of Standard Normal: A function that has been solved for numerically

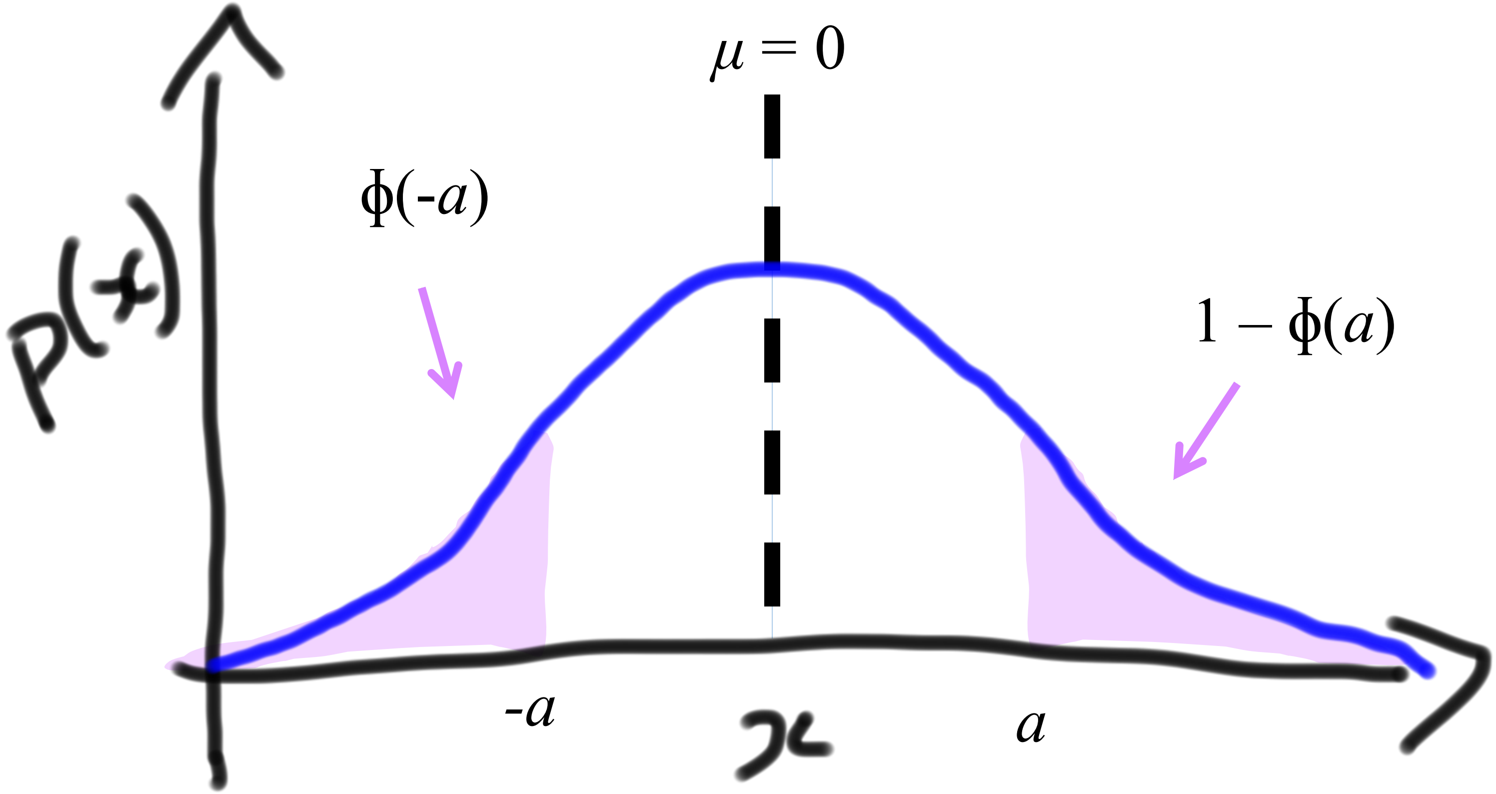
$$F(x) = \Phi\left(\frac{x - \mu}{\sigma}\right)$$

The cumulative density function (CDF) of any normal

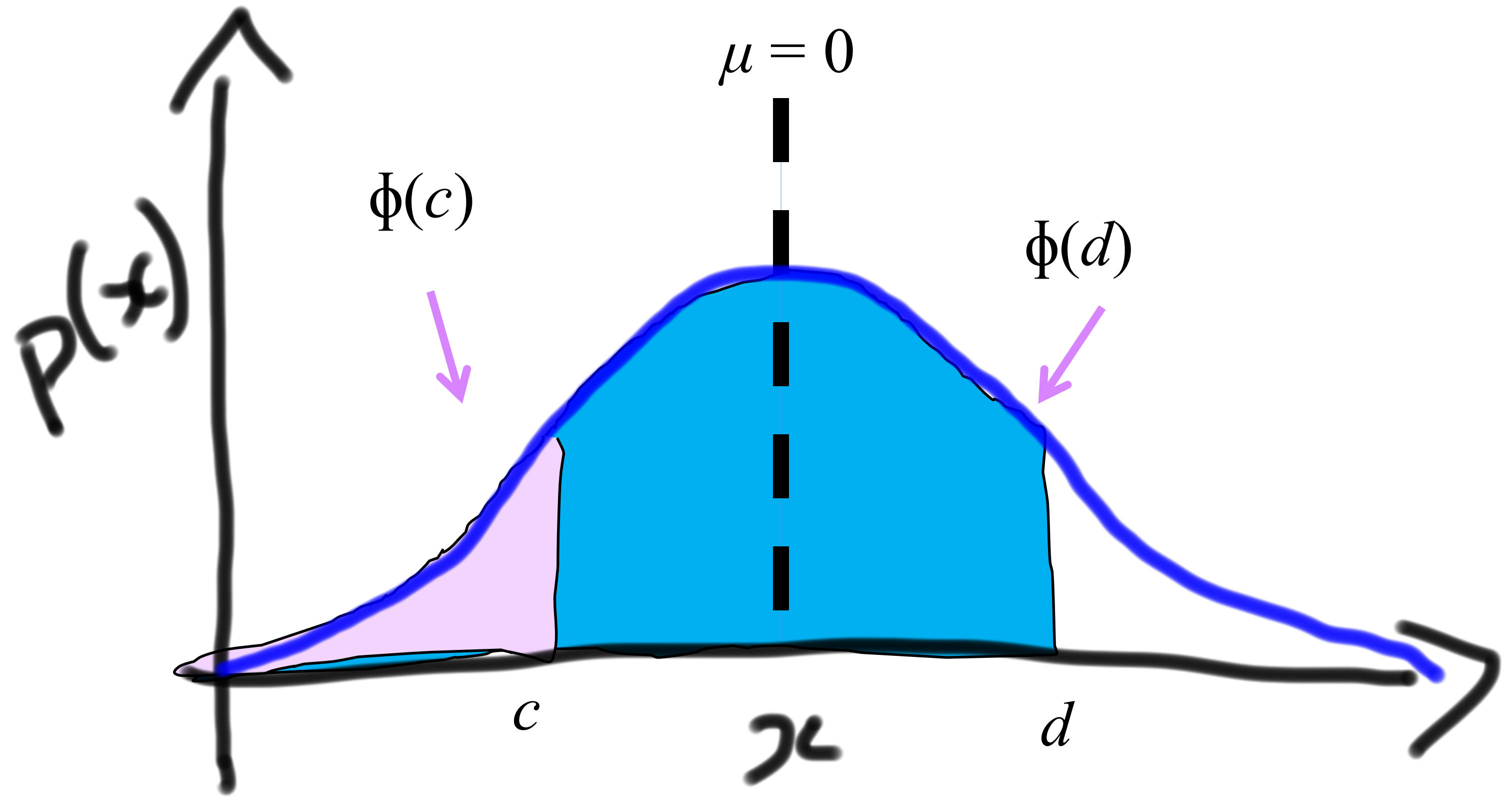
Table of $\Phi(z)$ values in textbook, p. 201 and handout

Symmetry of Phi

$$\phi(-a) = 1 - \phi(a)$$

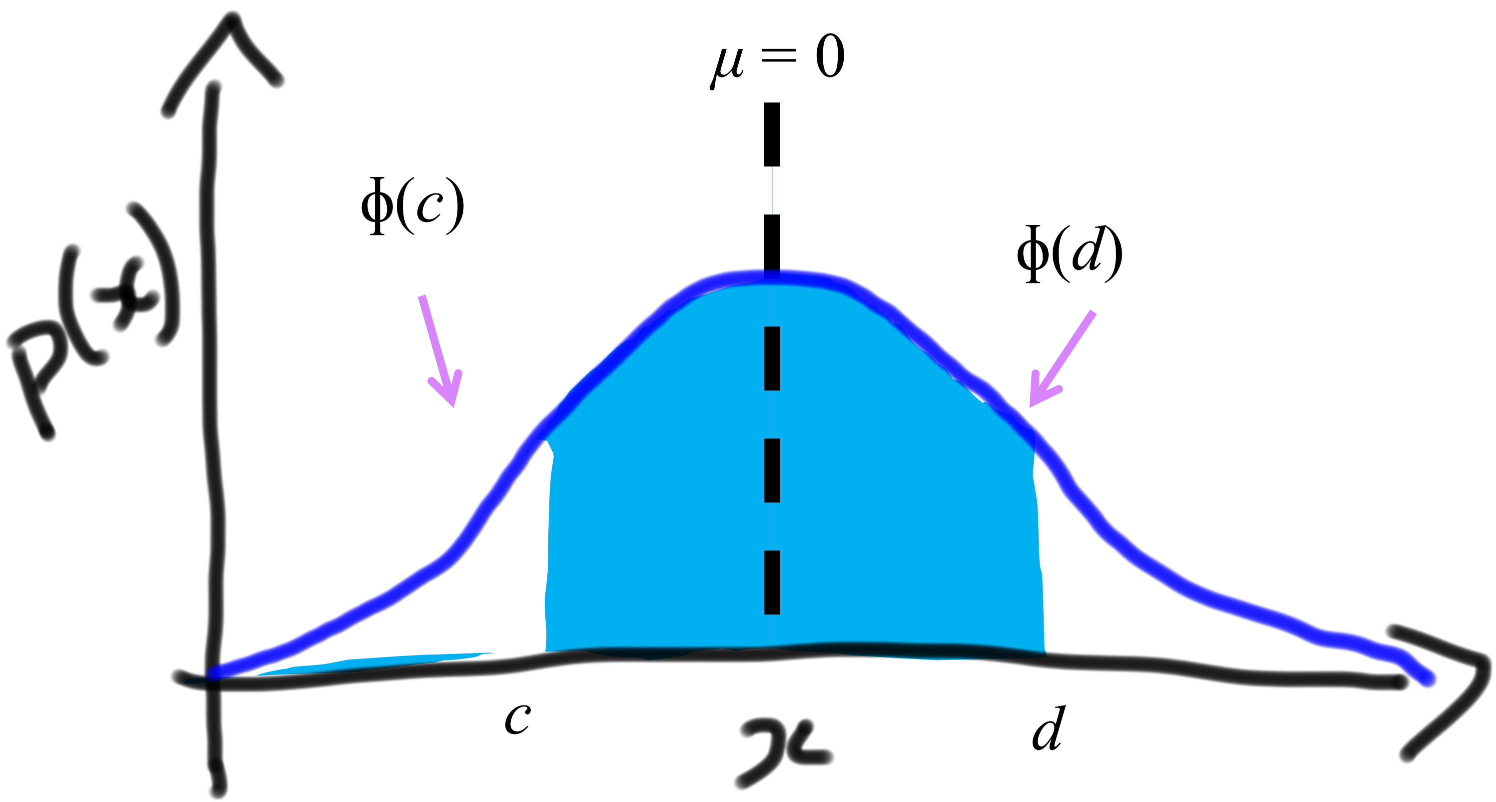


Interval of Phi



Interval of Phi

$$\phi(d) - \phi(c)$$



Great in class questions

68% rule only for Gaussians?

68% Rule?

What is the probability that a normal variable $X \sim N(\mu, \sigma^2)$ has a value within one standard deviation of its mean?

$$P(\mu - \sigma < X < \mu + \sigma) = P\left(\frac{\mu - \sigma - \mu}{\sigma} < \frac{X - \mu}{\sigma} < \frac{\mu + \sigma - \mu}{\sigma}\right)$$

$$= P(-1 < Z < 1)$$

$$= \Phi(1) - \Phi(-1)$$

$$= \Phi(1) - [1 - \Phi(1)]$$

$$= 2\Phi(1) - 1$$

$$= 2[0.8413] - 1 = 0.683$$

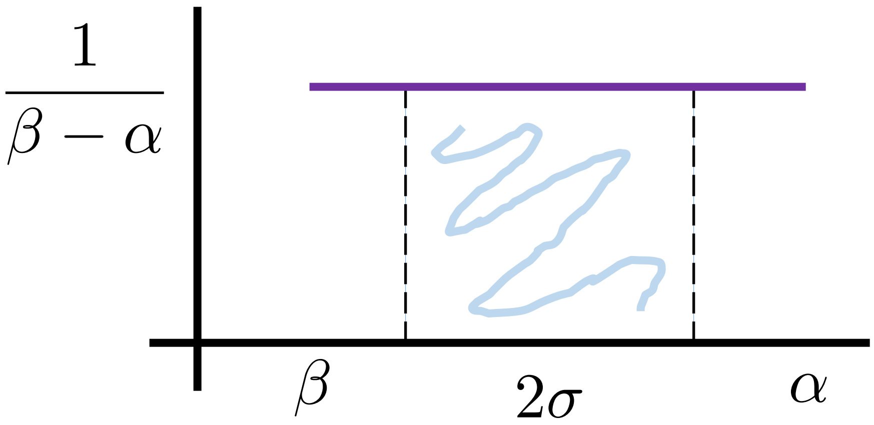
Only applies to normal

68% Rule?

Counter example: Uniform $X \sim Uni(\alpha, \beta)$

$$Var(X) = \frac{(\beta - \alpha)^2}{12}$$

$$\sigma = \sqrt{Var(X)} = \frac{\beta - \alpha}{\sqrt{12}}$$

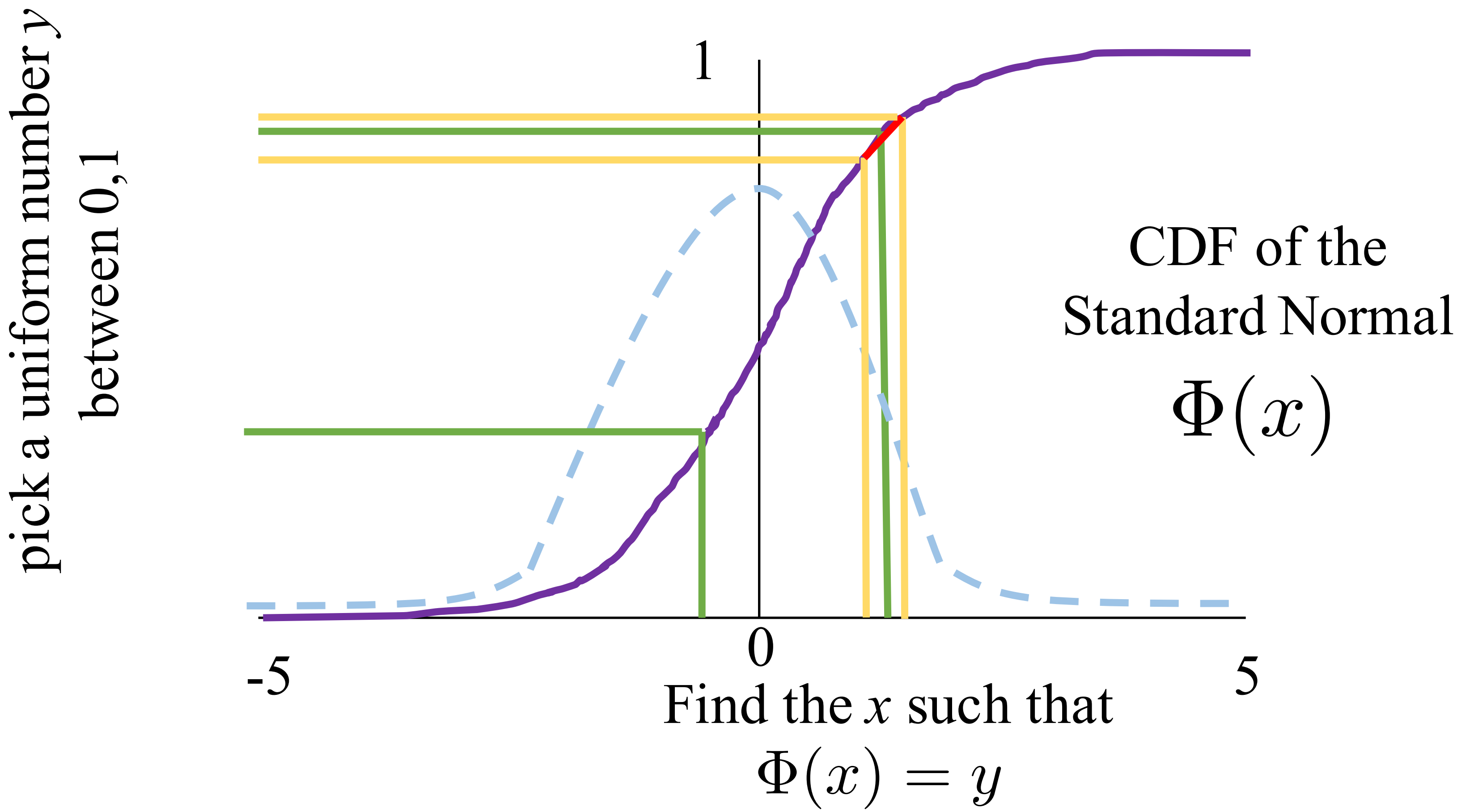


$$P(\mu - \sigma < X < \mu + \sigma) = \frac{1}{\beta - \alpha}\left[\frac{2(\beta - \alpha)}{\sqrt{12}}\right] = \frac{2}{\sqrt{12}} = 0.58$$

How do you sample from a Gaussian?

How Does a Computer Sample Normal?

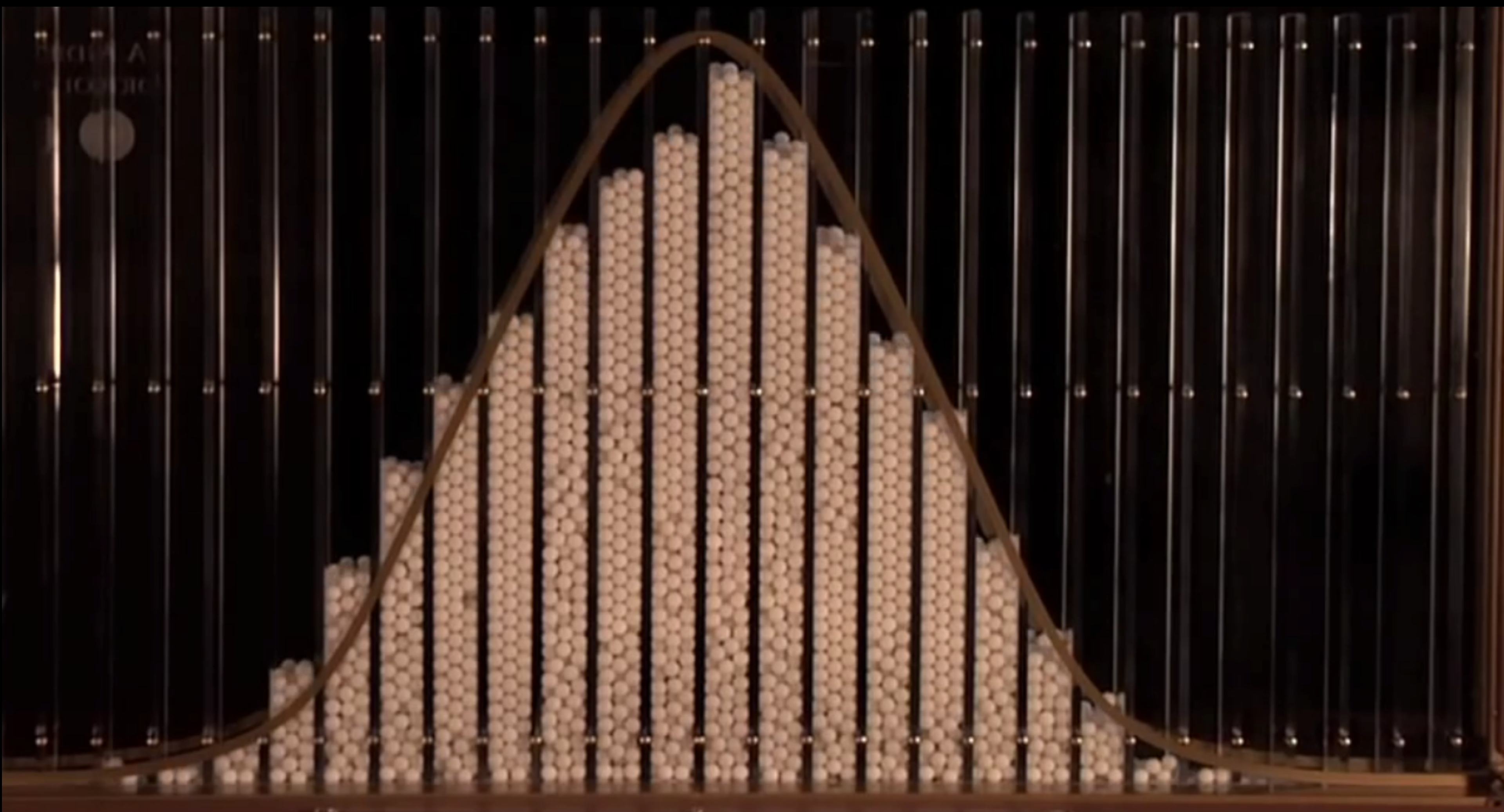
Inverse Transform Sampling



Further reading: Box–Muller transform

Where we left off…

Normal Approximates Binomial



There is a deep reason for the Binomial/Normal approximation…

Stanford Admissions

- Stanford accepts 2480 students
  - Each accepted student has 68% chance of attending
  - X = # students who will attend. X ~ Bin(2480, 0.68)
  - What is P(X > 1745)?

$$np = 1686.4 \quad np(1-p) \approx 539.65 \quad \sqrt{np(1-p)} \approx 23.23$$

  - Use Normal approximation: Y ~ N(1686.4, 539.65)

$$P(X > 1745) \approx P(Y > 1745.5)$$

$$P(Y > 1745.5) = P\left(\frac{Y - 1686.4}{23.23} > \frac{1745.5 - 1686.4}{23.23}\right) = 1 - \Phi(2.54) \approx 0.0055$$

  - Using Binomial:

$$P(X > 1745) \approx 0.0053$$

Changes in Stanford Admissions

- Stanford Daily, March 28, 2014

  "Class of 2018 Admit Rates Lowest in University History" by Alex Zivkovic

  "Fewer students were admitted to the Class of 2018 than the Class of 2017, due to the increase in Stanford's yield rate which has increased over 5 percent in the past four years, according to Colleen Lim M.A. '80, Director of Undergraduate Admission."

68% 10 years ago 80% last year

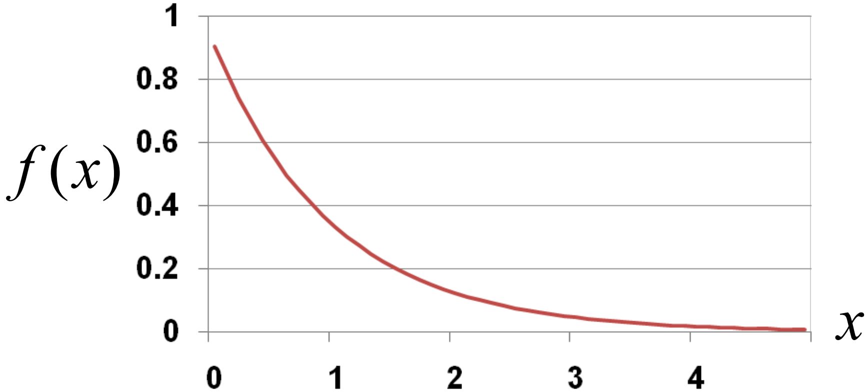
Next distribution: Exponential

Exponential Random Variable

- *X* is an **<u>Exponential RV</u>**: $X \sim \text{Exp}(\lambda)$ Rate: $\lambda > 0$
  - Probability Density Function (PDF):

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} \quad \text{where} \quad -\infty < x < \infty$$

  - $E[X] = \dfrac{1}{\lambda}$
  - $Var(X) = \dfrac{1}{\lambda^2}$



  - Cumulative distribution function (CDF), *F*(X) = P(*X* ≤ *x*):

$$F(x) = 1 - e^{-\lambda x} \quad \text{where } x \geq 0$$

  - Represents time until some event
    - Earthquake, request to web server, end cell phone contract, etc.

Exponential is Memoryless

- X = time until some event occurs
  - X ~ Exp(λ)
  - What is P(X > s + t | X > s)?

$$P(X > s + t \mid X > s) = \frac{P(X > s + t \text{ and } X > s)}{P(X > s)} = \frac{P(X > s + t)}{P(X > s)}$$

$$\frac{P(X > s + t)}{P(X > s)} = \frac{1 - F(s + t)}{1 - F(s)} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{-\lambda t} = 1 - F(t) = P(X > t)$$

$$\text{So, } P(X > s + t \mid X > s) = P(X > t)$$

  - After initial period of time *s*, P(X > t | •) for waiting another *t* units of time until event is same as at start
  - "Memoryless" = no impact from preceding period *s*

E[X] and Var(X) for exponential

A Little Calculus Review

- Product rule for derivatives:

$$d(u \cdot v) = du \cdot v + u \cdot dv$$

- Derivative and integral of exponential:

$$\frac{d(e^u)}{dx} = e^u \frac{du}{dx} \qquad \int e^u du = e^u$$

- Integration by parts:

$$\int d(u \cdot v) = u \cdot v = \int v \cdot du + \int u \cdot dv$$

$$\int u \cdot dv = u \cdot v - \int v \cdot du$$

And Now, Calculus Practice

- Compute *n*-th moment of Exponential distribution

$$E[X^n] = \int_0^\infty x^n \lambda e^{-\lambda x} dx$$

  - Step 1: don't panic, think happy thoughts, recall...
  - Step 2: find *u* and *v* (and *du* and *dv*):

$$u = x^n \qquad v = -e^{-\lambda x}$$

$$du = nx^{n-1} dx \qquad dv = \lambda e^{-\lambda x} dx$$

  - Step 3: substitute (a.k.a. "plug and chug")

$$\int u \cdot dv = \int x^n \cdot \lambda e^{-\lambda x} dx = u \cdot v - \int v \cdot du = -x^n e^{-\lambda x} + \int nx^{n-1} e^{-\lambda x} dx$$

$$E[X^n] = -x^n e^{-\lambda x}\Big|_0^\infty + \int nx^{n-1} e^{-\lambda x} dx = 0 + \frac{n}{\lambda} \int x^{n-1} \lambda e^{-\lambda x} dx = \frac{n}{\lambda} E[X^{n-1}]$$

$$\text{Base case}: E[X^0] = E[1] = 1, \quad \text{so } E[X] = \frac{1}{\lambda}, E[X^2] = \frac{2}{\lambda}\frac{1}{\lambda} = \frac{2}{\lambda^2}, \ldots$$

Visits to a Website

- Say visitor to your web site leaves after X minutes
  - On average, visitors leave site after 5 minutes
  - Assume length of stay is Exponentially distributed
  - X ~ Exp(λ = 1/5), since E[X] = 1/λ = 5
  - What is P(X > 10)?

$$P(X > 10) = 1 - F(10) = 1 - (1 - e^{-\lambda 10}) = e^{-2} \approx 0.1353$$

  - What is P(10 < X < 20)?

$$P(10 < X < 20) = F(20) - F(10) = (1 - e^{-4}) - (1 - e^{-2}) \approx 0.1170$$

Replacing Your Laptop

- X = # hours of use until your laptop dies
  - On average, laptops die after 5000 hours of use
  - X ~ Exp(λ = 1/5000), since E[X] = 1/λ = 5000
  - You use your laptop 5 hours/day.
  - What is P(your laptop lasts 4 years)?
  - That is: P(X > (5)(365)(4) = 7300)

$$P(X > 7300) = 1 - F(7300) = 1 - (1 - e^{-7300/5000}) = e^{-1.46} \approx 0.2322$$

  - Better plan ahead... especially if you are coterming:

$$P(X > 9125) = 1 - F(9125) = e^{-1.825} \approx 0.1612 \quad \text{(5 year plan)}$$

$$P(X > 10950) = 1 - F(10950) = e^{-2.19} \approx 0.1119 \quad \text{(6 year plan)}$$

Continuous Random Variables

Uniform Random Variable $X \sim Uni(\alpha, \beta)$

All values of *x* between alpha and beta are equally likely.

Normal Random Variable $X \sim \mathcal{N}(\mu, \sigma^2)$

Aka Gaussian. Defined by mean and variance. Goldilocks distribution.

Exponential Random Variable $X \sim Exp(\lambda)$

Time until an event happens. Parameterized by lambda (same as Poisson).

Alpha Beta Random Variable

How mysterious and curious. You must wait a few classes ☺.

Joint Distributions

Events occur with other events

Discrete Joint Mass Function

- For two discrete random variables *X* and *Y*, the **Joint Probability Mass Function** is:

$$p_{X,Y}(a,b) = P(X = a, Y = b)$$

- Marginal distributions:

$$p_X(a) = P(X = a) = \sum_y p_{X,Y}(a, y)$$

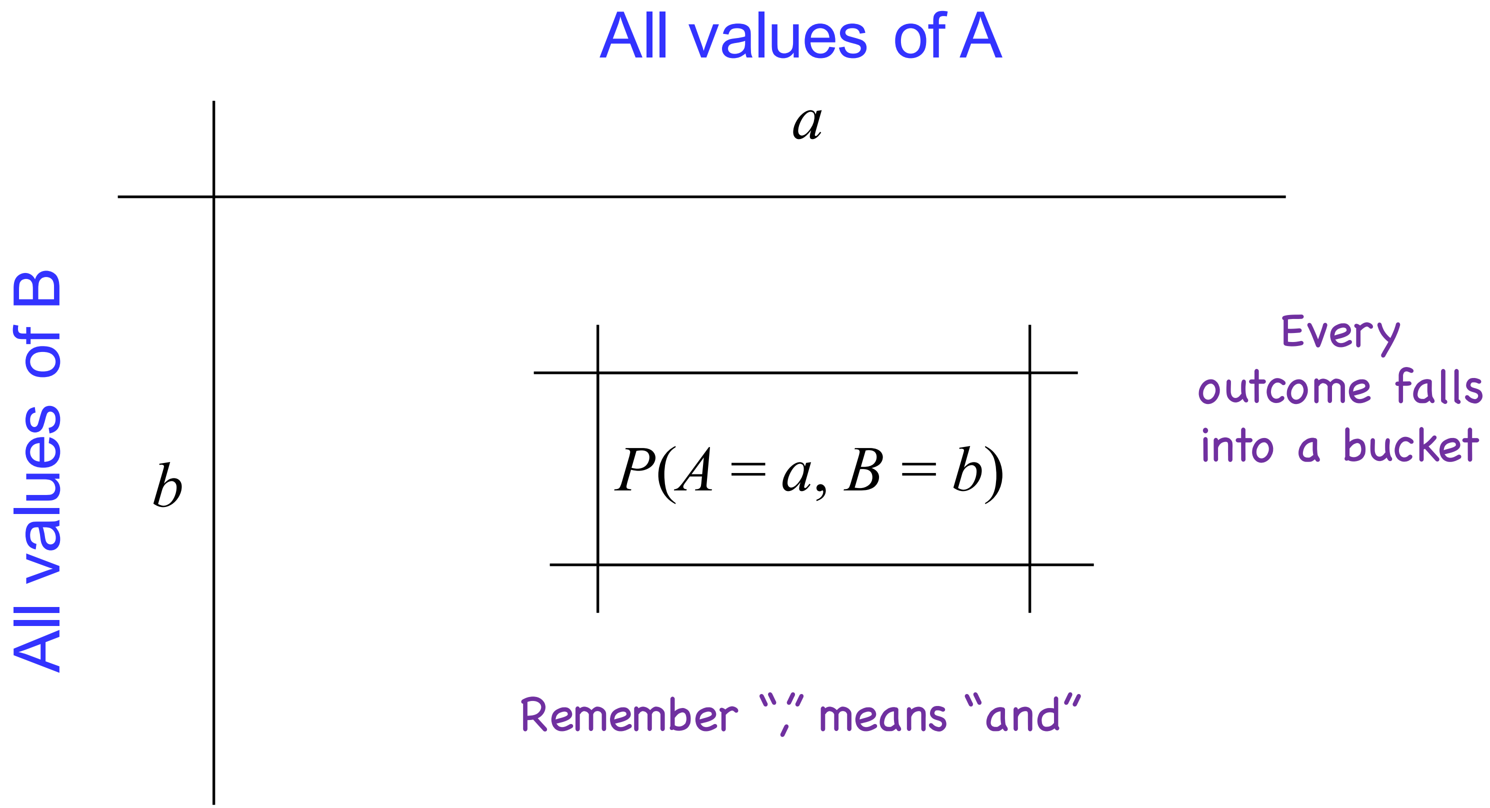
$$p_Y(b) = P(Y = b) = \sum_x p_{X,Y}(x, b)$$

- Example: X = value of die D_1, Y = value of die D_2

$$P(X = 1) = \sum_{y=1}^{6} p_{X,Y}(1, y) = \sum_{y=1}^{6} \frac{1}{36} = \frac{1}{6}$$

Probability Table

- States all possible outcomes with several discrete variables
- Often is not "parametric"
- If #variables is > 2, you can have a probability table, but you can't draw it on a slide



It’s Complicated Demo

Go to this URL: **https://goo.gl/ZNRsqD**

A Computer (or Three) In Every House

- Consider households in Silicon Valley
  - A household has C computers: C = X Macs + Y PCs
  - Assume each computer equally likely to be Mac or PC

$$P(C=c)=\begin{cases}0.16 & c=0\\ 0.24 & c=1\\ 0.28 & c=2\\ 0.32 & c=3\end{cases}$$

Y \ X	0	1	2	3	$p_Y(y)$
0	0.16	0.12	?	0.04	
1	0.12	0.14	0.12	0	
2	0.07	0.12	0	0	
3	0.04	0	0	0	
$p_X(x)$					

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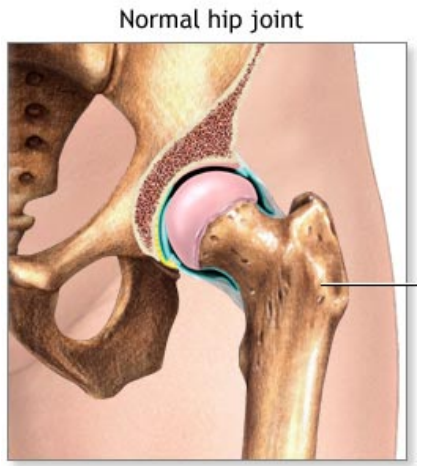
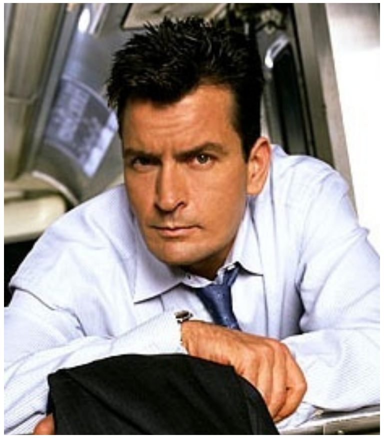
Y \ X	0	1	2	3	$p_Y(y)$
0	0.16	0.12	0.07	0.04	0.39
1	0.12	0.14	0.12	0	0.38
2	0.07	0.12	0	0	0.19
3	0.04	0	0	0	0.04
$p_X(x)$	0.39	0.38	0.19	0.04	1.00

Marginal distributions

Joint

- This is a joint





- A joint is not a mathematician
  - It did not start doing mathematics at an early age
  - It is not the reason we have “joint distributions”
  - And, no, Charlie Sheen does not look like a joint
    - But he does have them…
    - He also has **joint** custody of his children with Denise Richards

What about the continuous world?

Jointly Continuous

- Random variables X and Y, are **Jointly Continuous** if there exists PDF $f_{X,Y}(x, y)$ defined over $-\infty < x, y < \infty$ such that:

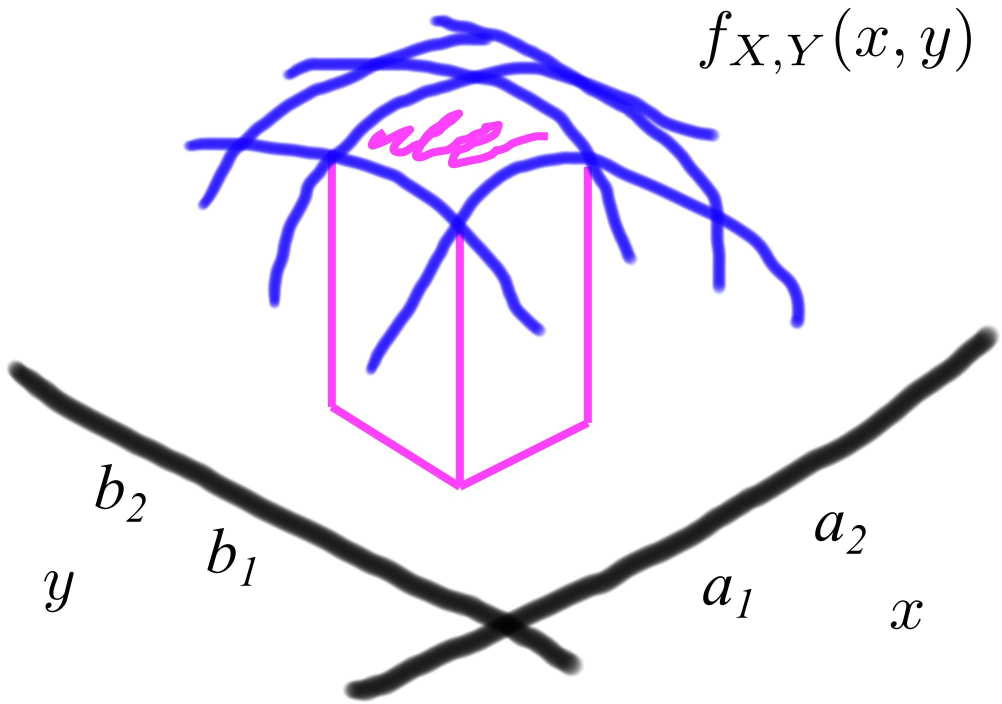
$$P(a_1 < X \leq a_2, b_1 < Y \leq b_2) = \int_{a_1}^{a_2} \int_{b_1}^{b_2} f_{X,Y}(x, y)\, dy\, dx$$

Let's look at one:

Demo

Jointly Continuous

$$P(a_1 < X \le a_2, b_1 < Y \le b_2) = \int_{a_1}^{a_2} \int_{b_1}^{b_2} f_{X,Y}(x, y)\, dy\, dx$$



Jointly Continuous

- Cumulative Density Function (CDF):

$$F_{X,Y}(a,b) = \int_{-\infty}^{a} \int_{-\infty}^{b} f_{X,Y}(x, y)\, dy\, dx \qquad f_{X,Y}(a,b) = \frac{\partial^2}{\partial a\, \partial b} F_{X,Y}(a,b)$$

- Marginal density functions:

$$f_X(a) = \int_{-\infty}^{\infty} f_{X,Y}(a, y)\, dy \qquad f_Y(b) = \int_{-\infty}^{\infty} f_{X,Y}(x,b)\, dx$$

Continuous Joint Distribution Functions

- For two continuous random variables *X* and *Y*, the **Joint Cumulative Probability Distribution** is:

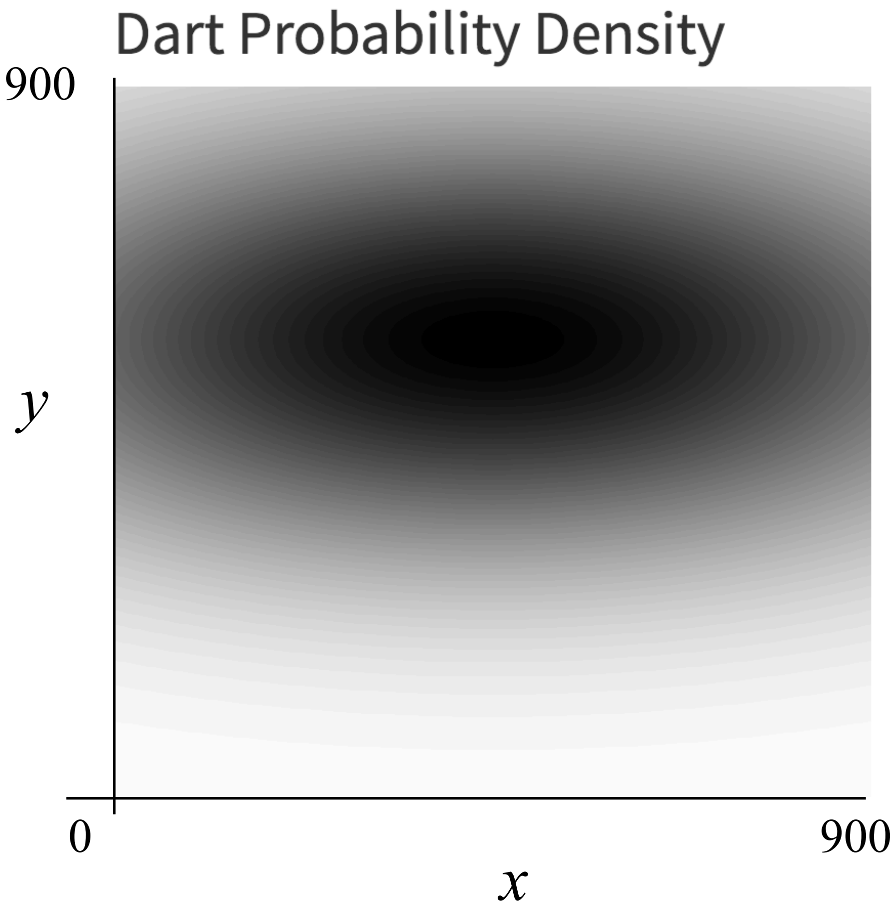
$$F_{X,Y}(a,b) = F(a,b) = P(X \le a, Y \le b) \quad \text{where } -\infty < a,b < \infty$$

- Marginal distributions:

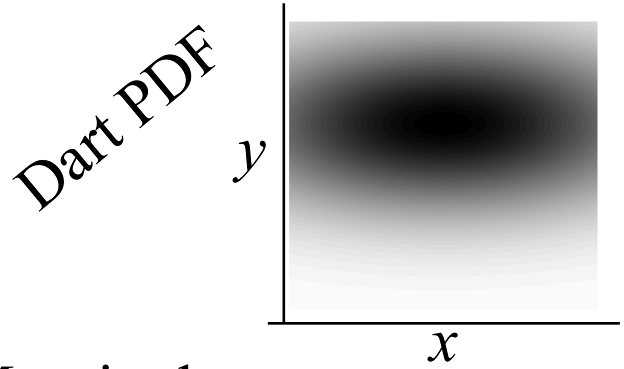
$$F_X(a) = P(X \le a) = \mathrm{P}(X \le a, Y < \infty) = F_{X,Y}(a,\infty)$$

$$F_Y(b) = P(Y \le b) = \mathrm{P}(X < \infty, Y \le b) = F_{X,Y}(\infty,b)$$

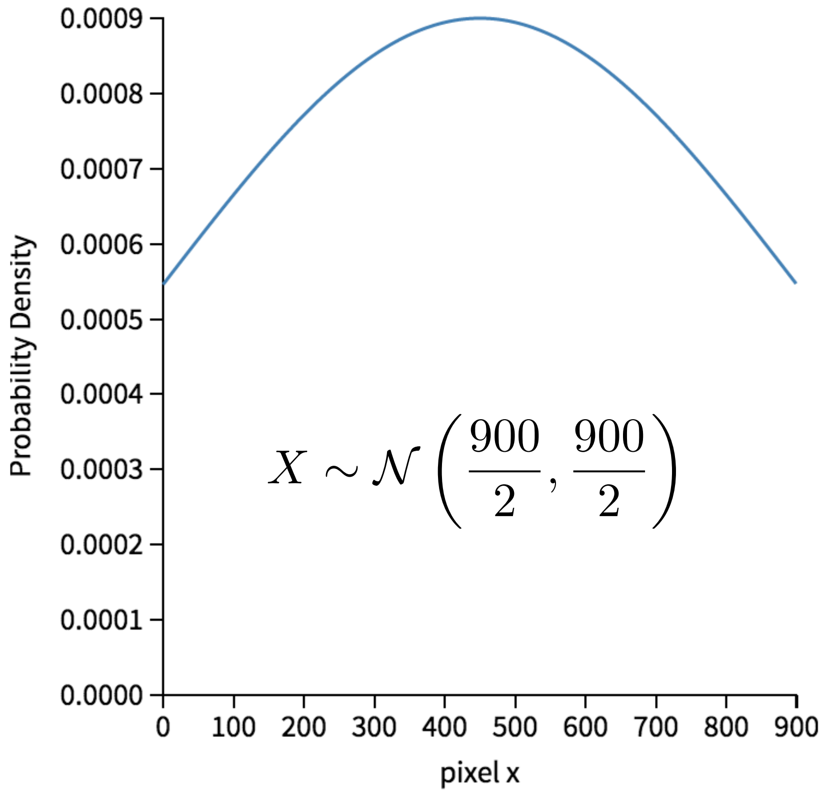
Joint Dart Distribution



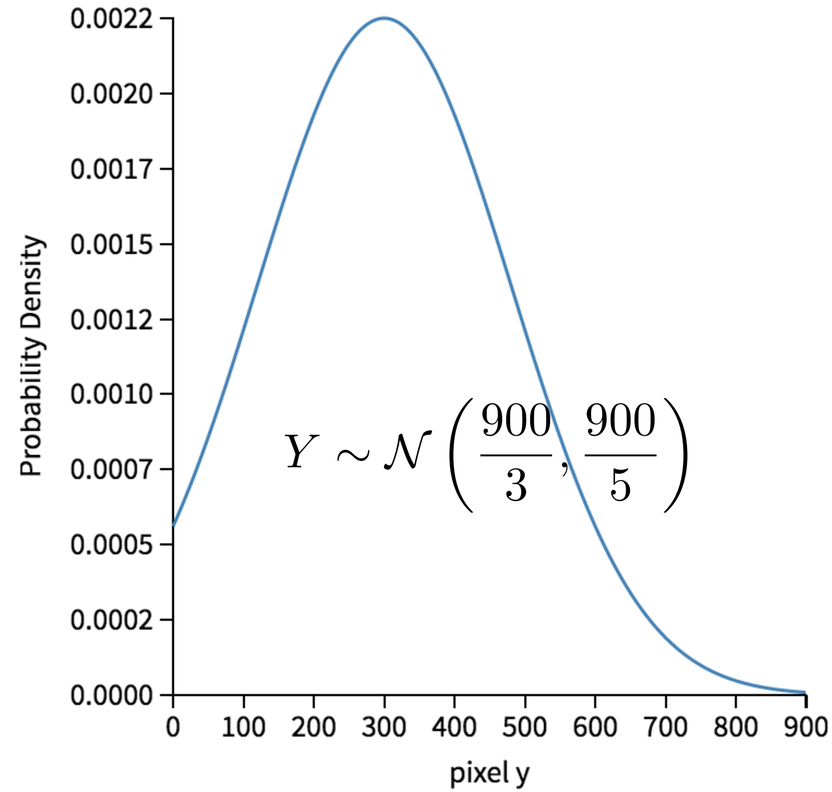
Darts!



X-Pixel Marginal



Y-Pixel Marginal



Multiple Integrals Without Tears

- Let X and Y be two continuous random variables
  - where $0 \le X \le 1$ and $0 \le Y \le 2$
- We want to integrate $g(x,y) = xy$ w.r.t. X and Y:
  - First, do “innermost” integral (treat y as a constant):

$$\int_{y=0}^{2}\int_{x=0}^{1} xy\,dx\,dy = \int_{y=0}^{2}\left(\int_{x=0}^{1} xy\,dx\right)dy = \int_{y=0}^{2} y\left[\frac{x^2}{2}\right]_0^1 dy = \int_{y=0}^{2} y\frac{1}{2}\,dy$$

  - Then, evaluate remaining (single) integral:

$$\int_{y=0}^{2} y\frac{1}{2}\,dy = \left[\frac{y^2}{4}\right]_0^2 = 1 - 0 = 1$$

Computing Joint Probabilities

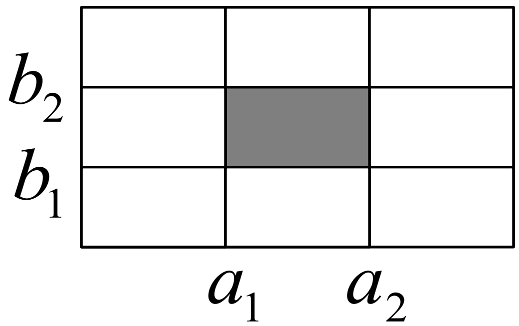
Let $F_{X,Y}(x, y)$ be joint CDF for X and Y

$$
\begin{aligned}
\mathrm{P}(X > a, Y > b) &= 1 - P((X > a, Y > b)^c) \\
&= 1 - P((X > a)^c \cup (Y > b)^c) \\
&= 1 - P((X \le a) \cup (Y \le b)) \\
&= 1 - (P(X \le a) + P(Y \le b) - P(X \le a, Y \le b)) \\
&= 1 - F_X(a) - F_Y(b) + F_{X,Y}(a, b)
\end{aligned}
$$

The General Rule Given Joint CDF

Let $F_{X,Y}(x, y)$ be joint CDF for X and Y

$$
\begin{aligned}
&\mathrm{P}(a_1 < X \le a_2, b_1 < Y \le b_2) \\
&= F(a_2, b_2) - F(a_1, b_2) + F(a_1, b_1) - F(a_2, b_1)
\end{aligned}
$$



Lovely Lemma

- *Y* is a <u>non-negative</u> continuous random variable
  - Probability Density Function: $f_Y(y)$
  - Already knew that:

$$E[Y] = \int_{-\infty}^{\infty} y \, f_Y(y) \, dy$$

  - But, did you know that:

$$E[Y] = \int_{0}^{\infty} P(Y > y) \, dy \text{ ?!?}$$

  - Analogously, in the discrete case, where $X = 1, 2, \ldots, n$

$$E[X] = \sum_{i=1}^{n} P(X \geq i)$$

How this lemma was made

In the discrete case, where $X = 1, 2, \ldots, n$

$$E[X] = \sum_{i=1}^{n} P(X \geq i)$$

Each row is an expansion of

$$\sum_{i=1}^{n} P(X \geq i) =$$

$$P(X = 1) + P(X = 2) + P(X = 3) + \cdots + P(X = n)$$
$$+ P(X = 2) + P(X = 3) + \cdots + P(X = n)$$
$$+ P(X = 3) + \cdots + P(X = n)$$
$$\vdots$$
$$+ P(X = n)$$

$$= 1P(X = 1) + 2P(X = 2) + \cdots + n(PX = n)$$
$$= E[X]$$

Life gives you lemmas,
make lemmanade!

Imperfections on a Disk

- Disk surface is a circle of radius R
  - A single point imperfection uniformly distributed on disk

$$f_{X,Y}(x,y) = \begin{cases} \dfrac{1}{\pi R^2} & \text{if } x^2 + y^2 \le R^2 \\ 0 & \text{if } x^2 + y^2 > R^2 \end{cases} \quad \text{where} \quad -\infty < x, y < \infty$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy = \frac{1}{\pi}R^2 \int_{x^2+y^2 \le R^2} dy$$

$$= \frac{1}{\pi}R^2 \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} dy$$

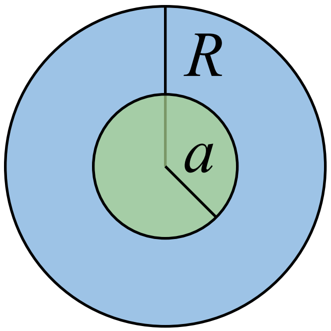
$$= \frac{2\sqrt{R^2 - x^2}}{\pi R^2}$$

Only integrate over the support range

Marginal of Y is the same by symmetry

Imperfections on a Disk

- Disk surface is a circle of radius R
  - A single point imperfection uniformly distributed on disk
  - Distance to origin: $D = \sqrt{X^2 + Y^2}$
  - What is *E[D]*?



$$P(D \le a) = \frac{\pi a^2}{\pi R^2} = \frac{a^2}{R^2}$$

Because of equally likely outcomes

$$E[D] = \int_0^R P(D > a)da = \int_0^R 1 - P(D \le a)da$$

$$= \int_0^R 1 - \frac{a^2}{R^2}da$$

$$= \left[a - \frac{a^3}{3R^2}\right]_0^R = \frac{2R}{3}$$