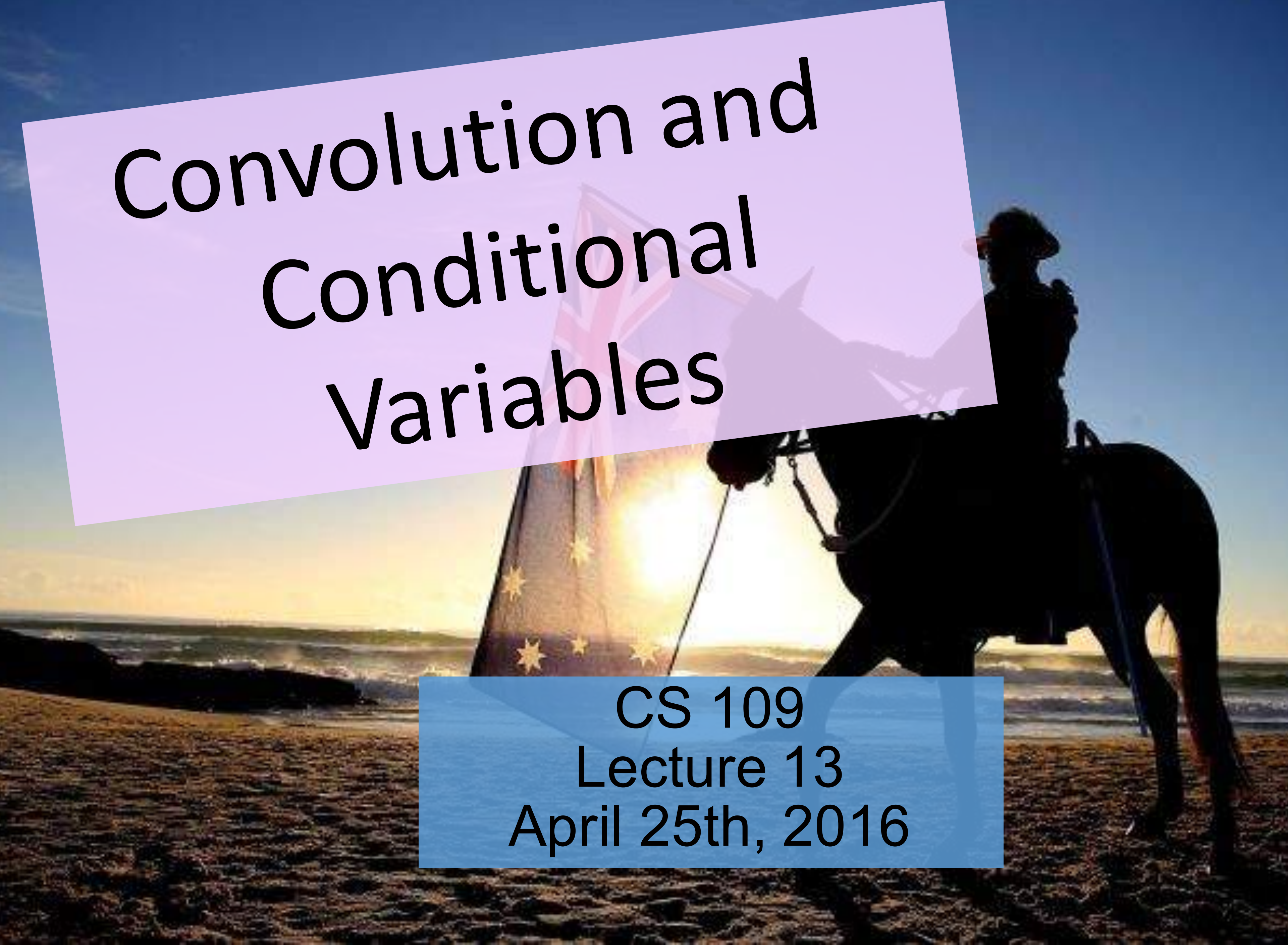


Convolution and Conditional Variables

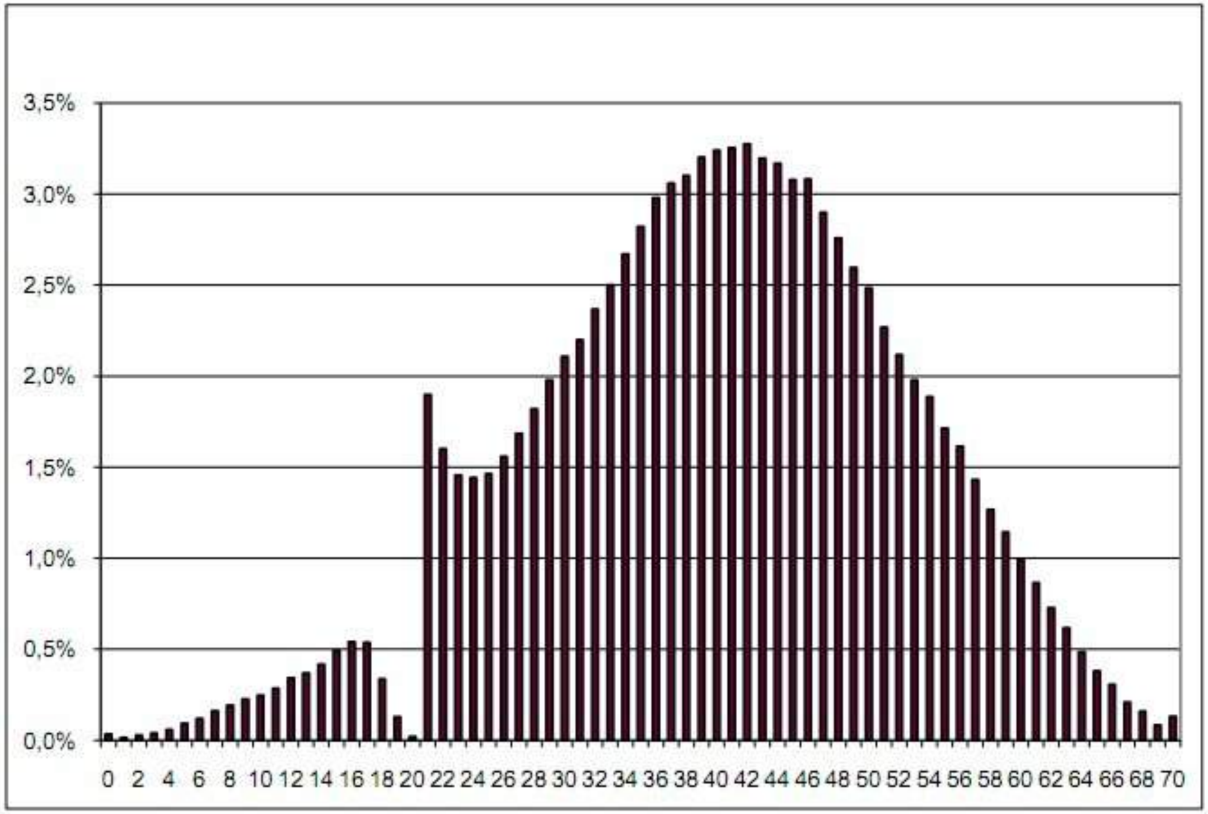
CS 109
Lecture 13
April 25th, 2016

Altruism?

Scores for a standardized test that students in Poland are required to pass before moving on in school

See if you can guess the minimum score to pass the test.

2.1. Poziom podstawowy



Wykres 1. Rozkład wyników na poziomie podstawowym

http://freakonomics.com/2011/07/07/another-case-of-teacher-cheating-or-is-it-just-altruism/comment-page-2/

Independent Discrete Variables

- Two discrete random variables X and Y are called **<u>independent</u>** if:

$$p(x, y) = p_X(x) p_Y(y) \quad \text{for all } x, y$$

- Intuitively: knowing the value of X tells us nothing about the distribution of Y (and vice versa)
  - If two variables are **<u>not</u>** independent, they are called ***<u>dependent</u>***
- Similar conceptually to independent *events*, but we are dealing with multiple ***<u>variables</u>***
  - Keep your events and variables distinct (and clear)!

Independent Continuous Variables

- Two continuous random variables X and Y are called **independent** if:

  $P(X \leq a, Y \leq b) = P(X \leq a)\,P(Y \leq b)$ for any a, b

- Equivalently:

$$F_{X,Y}(a,b) = F_X(a)F_Y(b) \quad \text{for all } a,b$$

$$f_{X,Y}(a,b) = f_X(a)f_Y(b) \quad \text{for all } a,b$$

- More generally, joint density factors separately:

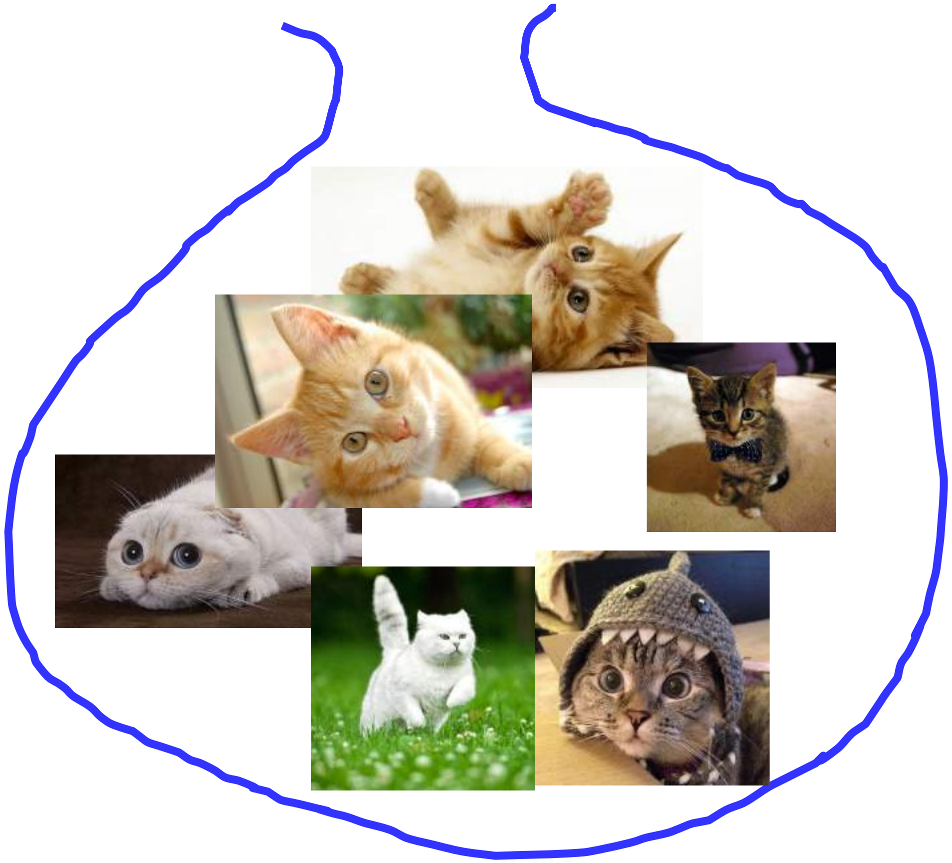
$$f_{X,Y}(x,y) = h(x)g(y) \quad \text{where } -\infty < x, y < \infty$$

Independence is Symmetric

- If random variables X and Y independent, then
  - X independent of Y, and Y independent of X
- Duh!? Duh, indeed...
  - Let X_1, X_2, ... be a sequence of independent and identically distributed (I.I.D.) continuous random vars
  - Say $X_n > X_i$ for all $i = 1,\ldots, n - 1$ (i.e. $X_n = \max(X_1, \ldots, X_n)$)
    - Call X_n a "record value"
  - Let event A_i indicate X_i is "record value"
    - Is A_{n+1} independent of A_n?
    - Is A_n independent of A_{n+1}?
    - Easier to answer: Yes!
    - By symmetry, $P(A_n) = 1/n$ and $P(A_{n+1}) = 1/(n+1)$
    - $P(A_n A_{n+1}) = (1/n)(1/(n+1)) = P(A_n)P(A_{n+1})$

Choosing a Random Subset

Original Set (size *n*)



Subset (size *k*)

Choosing a Random Subset

- From set of *n* elements, choose a subset of size *k* such that all $\binom{n}{k}$ possibilities are equally likely
  - Only have `random()`, which simulates X ~ Uni(0, 1)
- Brute force:
  - Generate (an ordering of) all subsets of size *k*
  - Randomly pick one (divide (0, 1) into $\binom{n}{k}$ intervals)
  - Expensive with regard to time and space
  - Bad times!

(Happily) Choosing a Random Subset

- Good times:

```
int indicator(double p) {
    if (random() < p) return 1; else return 0;
}

subset rSubset(k, set of size n) {
    subset_size = 0;
    I[1] = indicator((double)k/n);
    for(i = 1; i < n; i++) {
        subset_size += I[i];
        I[i+1] = indicator((k – subset_size)/(n – i));
    }
    return (subset containing element[i] iff I[i] == 1);
}
```

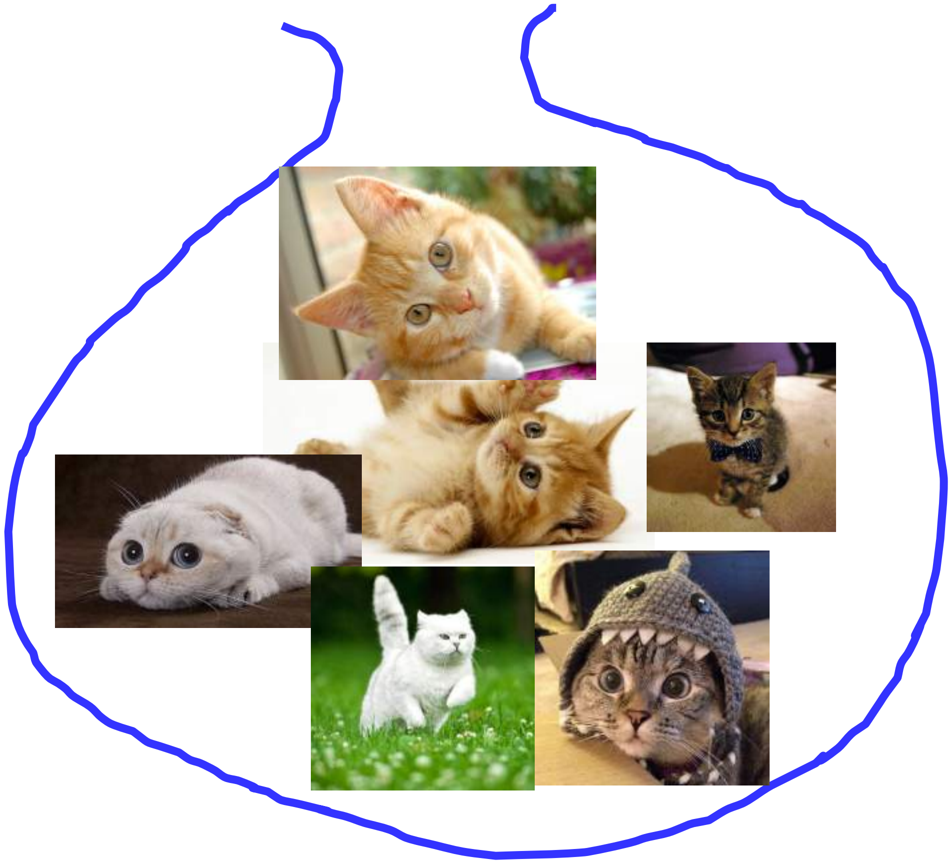
$P(I[1]=1) = \frac{k}{n}$ and $P(I[i+1]=1 \mid I[1],\ldots,I[i]) = \frac{k - \sum_{j=1}^{i} I[j]}{n-i}$ where $1 < i < n$

Random Subsets the Happy Way

- Proof (Induction on (k + n)): (i.e., why this algorithm works)
  - Base Case: k = 1, n = 1, Set S = {*a*}, **`rSubset`** returns {*a*} with p= $1 \Big/ \binom{1}{1}$
  - Inductive Hypoth. (IH): for k + x ≤ c, Given set S, |S| = x and k ≤ x, **`rSubset`** returns any subset S' of S, where |S'| = k, with p = $1 \Big/ \binom{x}{k}$
  - Inductive Case 1: (where k + n ≤ c + 1) |S| = n (= x + 1), **`I[1] = 1`**
    - Elem 1 in subset, choose k – 1 elems from remaining n – 1
    - By IH: **`rSubset`** returns subset S' of size k – 1 with p = $1 \Big/ \binom{n-1}{k-1}$
    - P(I[1] = 1, subset S') = $\frac{k}{n} \cdot 1 \Big/ \binom{n-1}{k-1} = 1 \Big/ \binom{n}{k}$
  - Inductive Case 2: (where k + n ≤ c + 1) |S| = n (= x + 1), **`I[1] = 0`**
    - Elem 1 not in subset, choose k elems from remaining n – 1
    - By IH: **`rSubset`** returns subset S' of size k with p = $1 \Big/ \binom{n-1}{k}$
    - P(I[1] = 0, subset S') = $\left(1 - \frac{k}{n}\right) \cdot 1 \Big/ \binom{n-1}{k} = \left(\frac{n-k}{n}\right) \cdot 1 \Big/ \binom{n-1}{k} = 1 \Big/ \binom{n}{k}$

Choosing a Random Subset

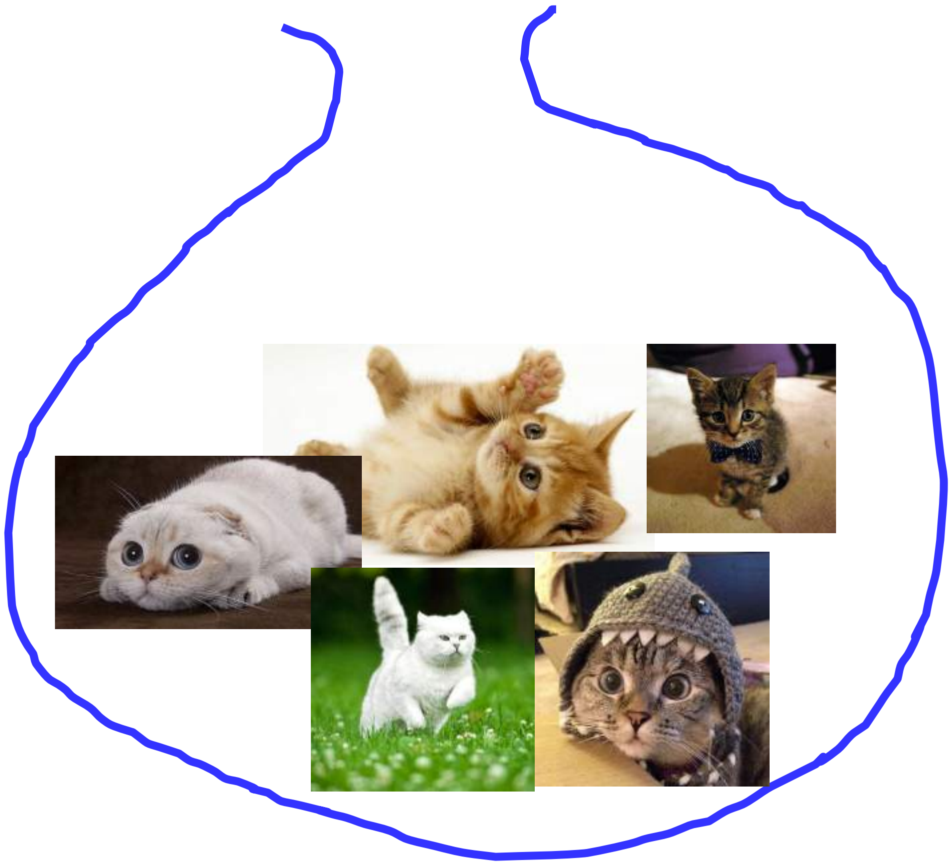
Original Set (size n)



Subset (size k)

Case 1

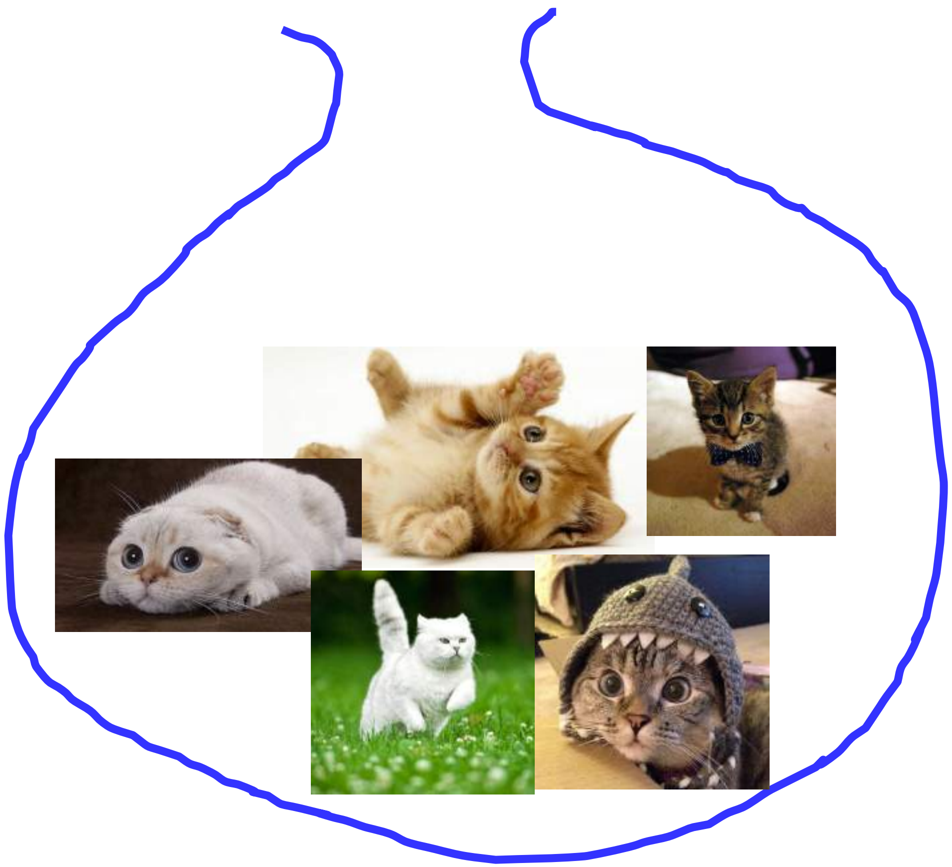
Original Set (size n)



Subset (size k)

Case 1

Original Set (size *n-1*)



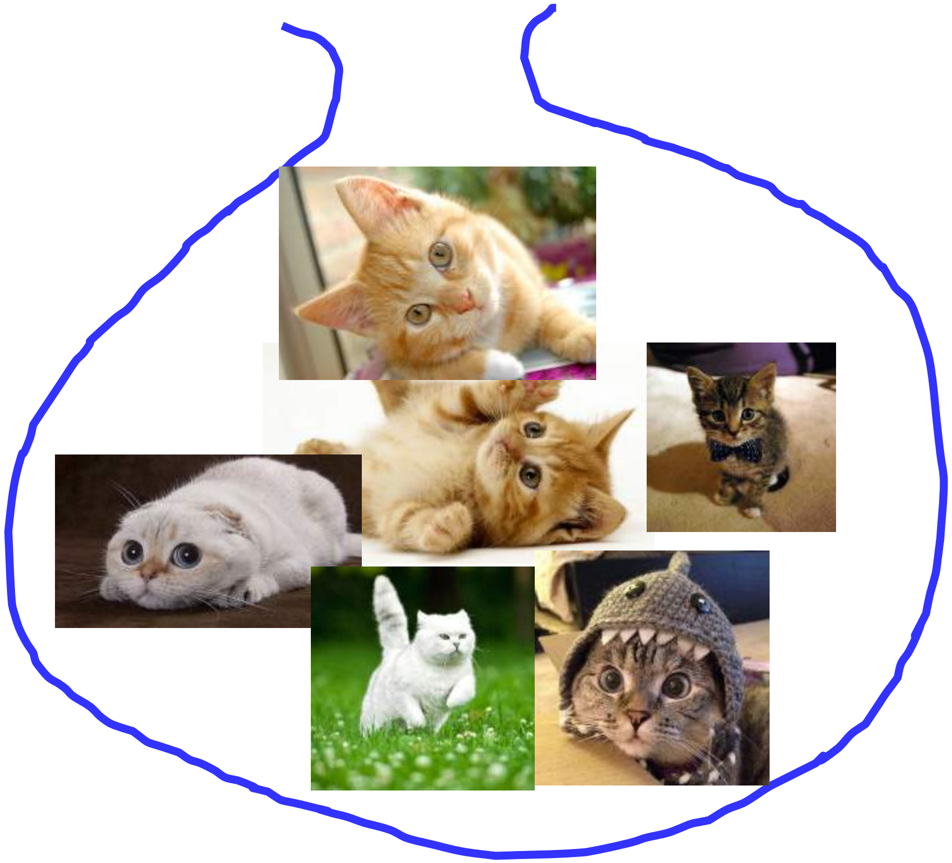
Subset (size *k-1*)

By induction we know that all subsamples of size k-1 from n-1 are equally likely

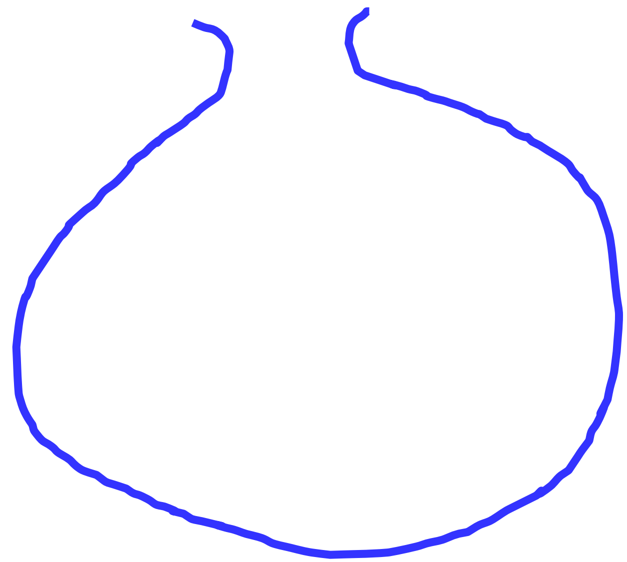
$$\mathrm{P(subset)} = \frac{k}{n} \cdot 1 \Big/ \binom{n-1}{k-1} = 1 \Big/ \binom{n}{k}$$

Choosing a Random Subset

Original Set (size *n*)



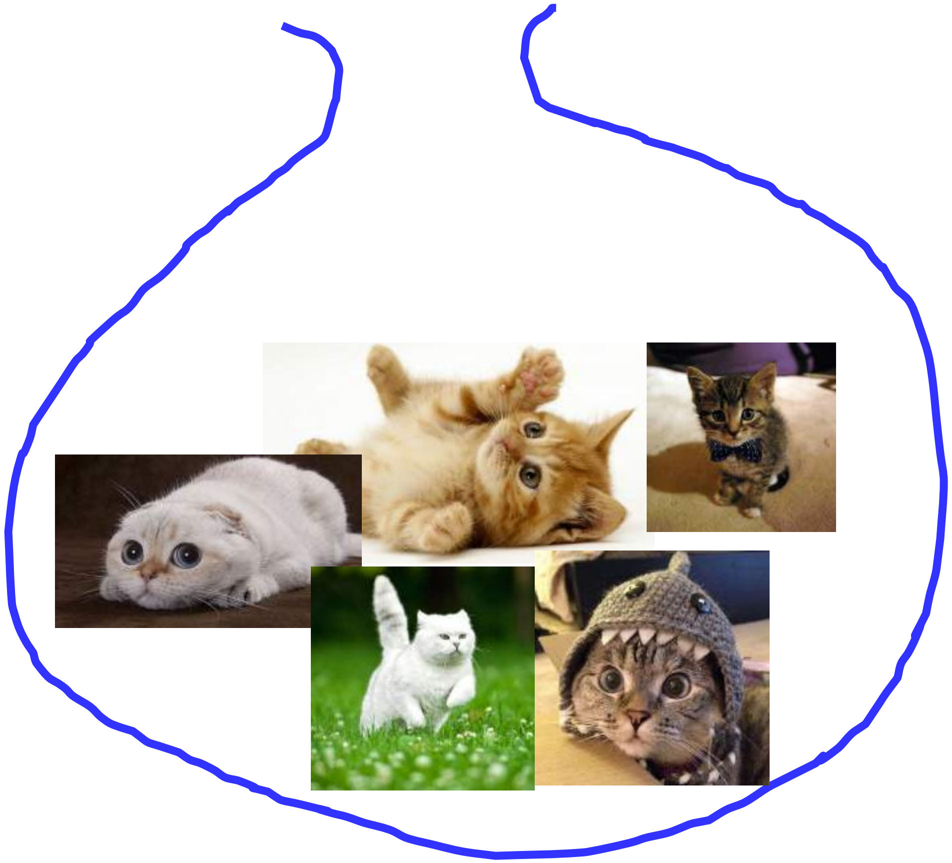
Subset (size *k*)

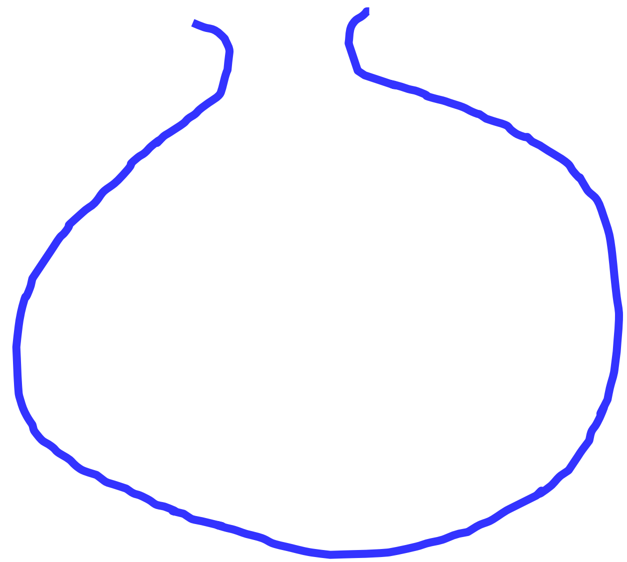


Case 2

Original Set (size *n-1*)

Subset (size *k*)





By induction we know that all subsamples of size k from n-1 are equally likely

$$P(\text{subset}) = \left(1-\frac{k}{n}\right)\cdot 1\Big/\binom{n-1}{k} = \left(\frac{n-k}{n}\right)\cdot 1\Big/\binom{n-1}{k} = 1\Big/\binom{n}{k}$$

All combinations are in either case. Each combination in the cases are equally likely

End Review

What happens when you add random variables?

Sum of Independent Binomials

- Let X and Y be independent random variables
  - $X \sim \text{Bin}(n_1, p)$ and $Y \sim \text{Bin}(n_2, p)$
  - $X + Y \sim \text{Bin}(n_1 + n_2, p)$
- Intuition:
  - X has n_1 trials and Y has n_2 trials
    - Each trial has same "success" probability p
  - Define Z to be $n_1 + n_2$ trials, each with success prob. p
  - $Z \sim \text{Bin}(n_1 + n_2, p)$, and also $Z = X + Y$
- More generally: $X_i \sim \text{Bin}(n_i, p)$ for $1 \leq i \leq N$

$$\left(\sum_{i=1}^{N} X_i\right) \sim \text{Bin}\left(\sum_{i=1}^{N} n_i,\ p\right)$$

Sum of Independent Poissons

- Let X and Y be independent random variables
  - X ~ Poi(λ_1) and Y ~ Poi(λ_2)
  - X + Y ~ Poi(λ_1 + λ_2)
- Proof: (just for reference)
  - Rewrite (X + Y = *n*) as (X = *k*, Y = *n* – *k*) where 0 ≤ *k* ≤ *n*

$$P(X+Y=n)=\sum_{k=0}^{n} P(X=k, Y=n-k)=\sum_{k=0}^{n} P(X=k)P(Y=n-k)$$

$$=\sum_{k=0}^{n} e^{-\lambda_1}\frac{\lambda_1^k}{k!}e^{-\lambda_2}\frac{\lambda_2^{n-k}}{(n-k)!}=e^{-(\lambda_1+\lambda_2)}\sum_{k=0}^{n}\frac{\lambda_1^k\lambda_2^{n-k}}{k!(n-k)!}=\frac{e^{-(\lambda_1+\lambda_2)}}{n!}\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\lambda_1^k\lambda_2^{n-k}$$

  - Noting Binomial theorem: $(\lambda_1+\lambda_2)^n=\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\lambda_1^k\lambda_2^{n-k}$
  - $P(X+Y=n)=\frac{e^{-(\lambda_1+\lambda_2)}}{n!}(\lambda_1+\lambda_2)^n$ so, X + Y = n ~ Poi(λ_1 + λ_2)

Reference: Sum of Independent RVs

- Let X and Y be independent Binomial RVs
  - $X \sim \text{Bin}(n_1, p)$ and $Y \sim \text{Bin}(n_2, p)$
  - $X + Y \sim \text{Bin}(n_1 + n_2, p)$
  - More generally, let $X_i \sim \text{Bin}(n_i, p)$ for $1 \leq i \leq N$, then

$$\left(\sum_{i=1}^{N} X_i \right) \sim \text{Bin}\left(\sum_{i=1}^{N} n_i,\ p \right)$$

- Let X and Y be independent Poisson RVs
  - $X \sim \text{Poi}(\lambda_1)$ and $Y \sim \text{Poi}(\lambda_2)$
  - $X + Y \sim \text{Poi}(\lambda_1 + \lambda_2)$
  - More generally, let $X_i \sim \text{Poi}(\lambda_i)$ for $1 \leq i \leq N$, then

$$\left(\sum_{i=1}^{N} X_i \right) \sim \text{Poi}\left(\sum_{i=1}^{N} \lambda_i \right)$$

If only it were always that simple

Convolution of Probability Distributions

We talked about sum of Binomial and Poisson…who's missing from this party?

Uniform.

Summation: not just for the 1%

Dance, Dance Convolution

- Let X and Y be independent random variables
  - Cumulative Distribution Function (CDF) of X + Y:

$$F_{X+Y}(a) = P(X+Y \leq a)$$

$$= \iint_{x+y \leq a} f_X(x) f_Y(y)\, dx\, dy = \int_{y=-\infty}^{\infty} \int_{x=-\infty}^{a-y} f_X(x)\, dx\, f_Y(y)\, dy$$

$$= \int_{y=-\infty}^{\infty} F_X(a-y)\, f_Y(y)\, dy$$

CDF of X

PDF of Y

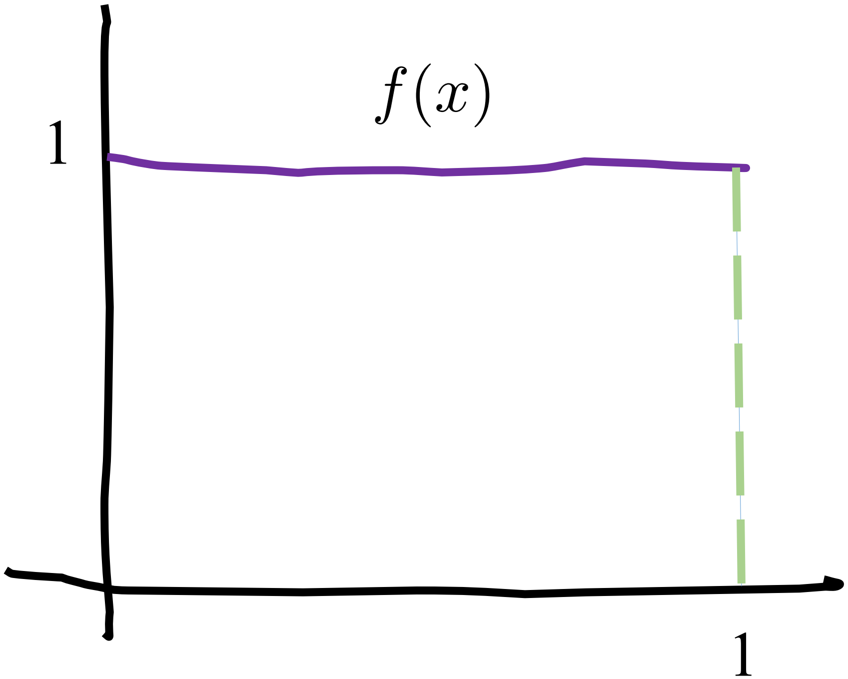
  - F_{X+Y} is called **convolution** of F_X and F_Y
  - Probability Density Function (PDF) of X + Y, analogous:

$$f_{X+Y}(a) = \int_{y=-\infty}^{\infty} f_X(a-y)\, f_Y(y)\, dy$$

  - In discrete case, replace $\int_{y=-\infty}^{\infty}$ with $\sum_y$, and $f(y)$ with $p(y)$

Sum of Independent Uniforms

- Let X and Y be independent random variables
  - X ~ Uni(0, 1) and Y ~ Uni(0, 1) → $f(x) = 1$ for $0 \leq x \leq 1$



For both X and Y

Sum of Independent Uniforms

- Let X and Y be independent random variables
  - X ~ Uni(0, 1) and Y ~ Uni(0, 1) → $f(x) = 1$ for $0 \le x \le 1$
  - What is PDF of X + Y?

$$f_{X+Y}(a) = \int_{y=0}^{1} f_X(a-y)\, f_Y(y)\, dy = \int_{y=0}^{1} f_X(a-y)\, dy$$

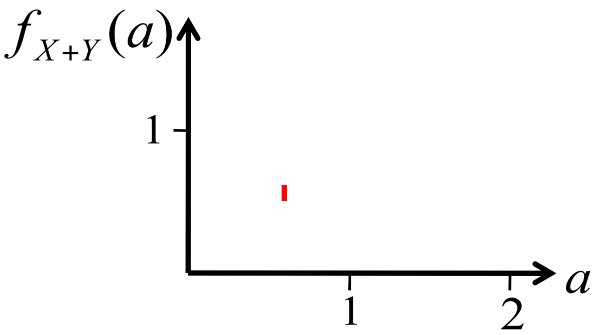
When $a = 0.5$:

$$f_{X+Y}(0.5) = \int_{y=?}^{y=?} f_X(0.5-y)dy$$

$$= \int_0^{0.5} f_X(0.5-y)dy$$

$$= \int_0^{0.5} 1dy$$

$$= 0.5$$



Sum of Independent Uniforms

- Let X and Y be independent random variables
  - X ~ Uni(0, 1) and Y ~ Uni(0, 1) → $f(x) = 1$ for $0 \le x \le 1$
  - What is PDF of X + Y?

$$f_{X+Y}(a) = \int_{y=0}^{1} f_X(a-y)\, f_Y(y)\, dy = \int_{y=0}^{1} f_X(a-y)\, dy$$

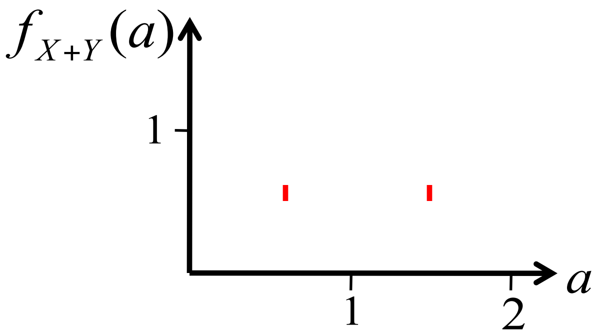
When $a = 1.5$:

$$f_{X+Y}(1.5) = \int_{y=?}^{y=?} f_X(1.5-y)dy$$

$$= \int_{0.5}^{1} f_X(1.5-y)dy$$

$$= \int_{0.5}^{1} 1dy$$

$$= 0.5$$



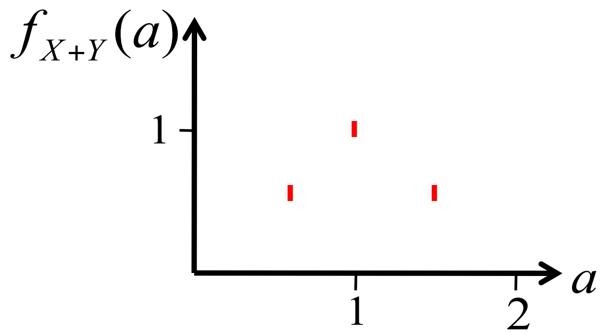
Sum of Independent Uniforms

- Let X and Y be independent random variables
  - X ~ Uni(0, 1) and Y ~ Uni(0, 1) → $f(x) = 1$ for $0 \le x \le 1$
  - What is PDF of X + Y?

$$f_{X+Y}(a) = \int_{y=0}^{1} f_X(a-y)\, f_Y(y)\, dy = \int_{y=0}^{1} f_X(a-y)\, dy$$

When $a = 1$:

$$\begin{aligned} f_{X+Y}(1) &= \int_{y=?}^{y=?} f_X(1-y)dy \\ &= \int_0^1 f_X(1-y)dy \\ &= \int_0^1 1dy \\ &= 1 \end{aligned}$$



Sum of Independent Uniforms

- Let X and Y be independent random variables
  - X ~ Uni(0, 1) and Y ~ Uni(0, 1) → $f(x) = 1$ for $0 \le x \le 1$
  - What is PDF of X + Y?

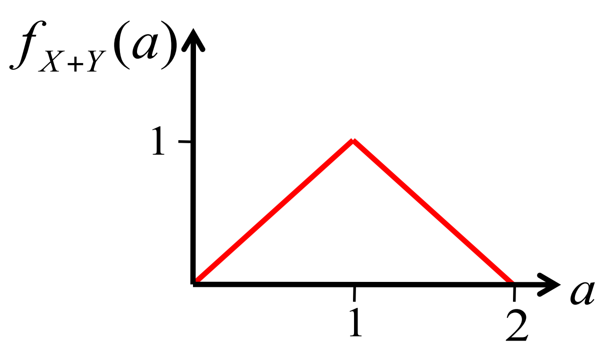
$$f_{X+Y}(a) = \int_{y=0}^{1} f_X(a-y)\, f_Y(y)\, dy = \int_{y=0}^{1} f_X(a-y)\, dy$$

  - When $0 \le a \le 1$ and $0 \le y \le a$, $0 \le a - y \le 1$ → $f_X(a-y) = 1$

$$f_{X+Y}(a) = \int_{y=0}^{a} dy = a$$

  - When $1 \le a \le 2$ and $a-1 \le y \le 1$, $0 \le a - y \le 1$ → $f_X(a-y) = 1$

$$f_{X+Y}(a) = \int_{y=a-1}^{1} dy = 2 - a$$



  - Combining: $f_{X+Y}(a) = \begin{cases} a & 0 \le a \le 1 \\ 2 - a & 1 < a \le 2 \\ 0 & \text{otherwise} \end{cases}$

Sum of Independent Normals

- Let X and Y be independent random variables
  - $X \sim N(\mu_1, \sigma_1^2)$ and $Y \sim N(\mu_2, \sigma_2^2)$
  - $X + Y \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$

- Generally, have *n* independent random variables $X_i \sim N(\mu_i, \sigma_i^2)$ for $i = 1, 2, ..., n$:

$$\left(\sum_{i=1}^{n} X_i \right) \sim N\left(\sum_{i=1}^{n} \mu_i , \sum_{i=1}^{n} \sigma_i^2 \right)$$

Virus Infections

- Say you are working with the WHO to plan a response to a the initial conditions of a virus:
  - Two exposed groups
  - P1: 50 people, each independently infected with $p = 0.1$
  - P2: 100 people, each independently infected with $p = 0.4$
  - Question: Probability of more than 40 infections?

Sanity check: Should we use the Binomial Sum-of-RVs shortcut?

A. YES!
B. NO!
C. Other/none/more

Virus Infections

- Say you are working with the WHO to plan a response to a the initial conditions of a virus:
  - Two exposed groups
  - P1: 50 people, each independently infected with $p = 0.1$
  - P2: 100 people, each independently infected with $p = 0.4$
  - A = # infected in P1 A ~ Bin(50, 0.1) ≈ X ~ N(5, 4.5)
  - B = # infected in P2 B ~ Bin(100, 0.4) ≈ Y ~ N(40, 24)
  - What is P(≥ 40 people infected)?
  - P(A + B ≥ 40) ≈ P(X + Y ≥ 39.5)
  - X + Y = W ~ N(5 + 40 = 45, 4.5 + 24 = 28.5)

$$P(W \geq 39.5) = P\left(\frac{W - 45}{\sqrt{28.5}} > \frac{39.5 - 45}{\sqrt{28.5}}\right) = 1 - \Phi(-1.03) \approx 0.8485$$

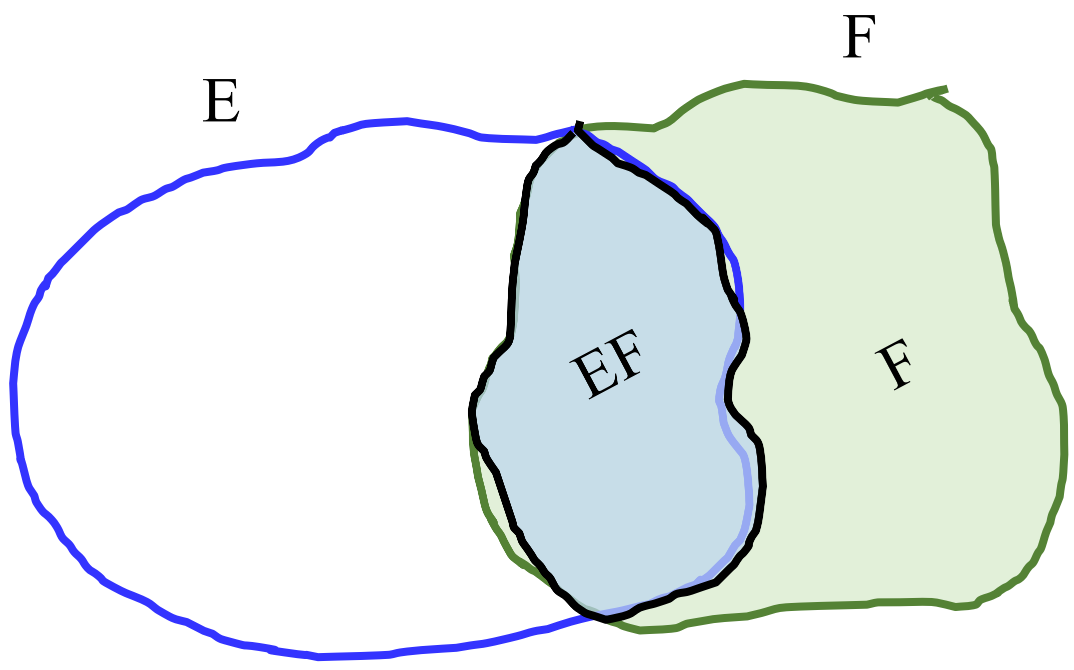
End sum of independent vars

Conditionals with multiple variables

Discrete Conditional Distribution

- Recall that for *events* E and F:

$$P(E \mid F) = \frac{P(EF)}{P(F)} \quad \text{where } P(F) > 0$$



Discrete Conditional Distributions

- Recall that for *events* E and F:

$$P(E \mid F) = \frac{P(EF)}{P(F)} \quad \text{where } P(F) > 0$$

- Now, have X and Y as discrete random variables
  - Conditional PMF of X given Y (where $p_Y(y) > 0$):

$$P_{X|Y}(x \mid y) = P(X = x \mid Y = y) = \frac{P(X = x, Y = y)}{P(Y = y)} = \frac{p_{X,Y}(x, y)}{p_Y(y)}$$

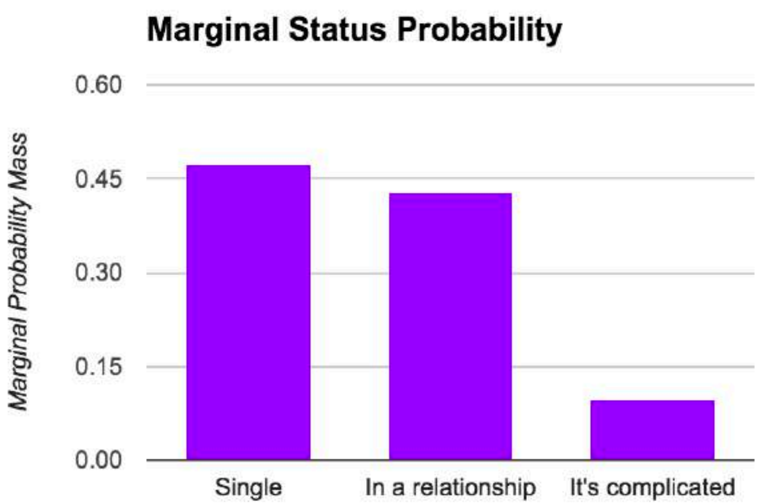
  - Conditional CDF of X given Y (where $p_Y(y) > 0$):

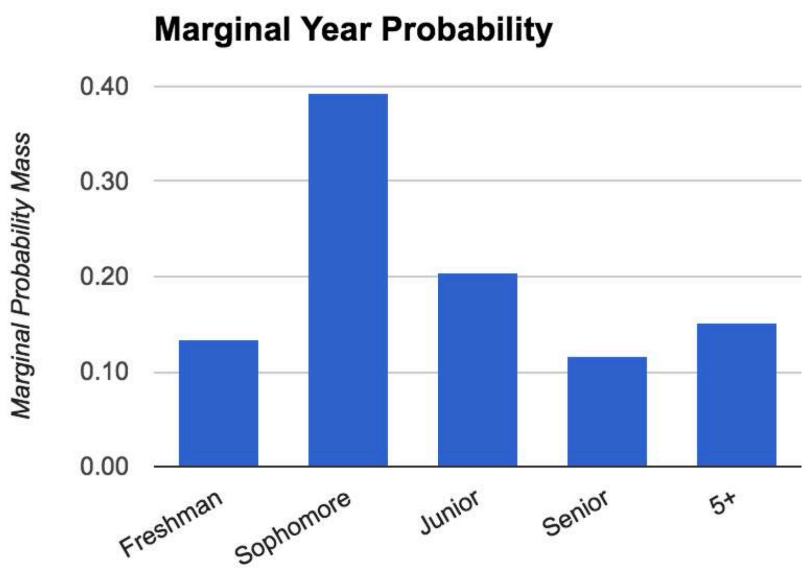
$$F_{X|Y}(a \mid y) = P(X \le a \mid Y = y) = \frac{P(X \le a, Y = y)}{P(Y = y)}$$

$$= \frac{\sum_{x \le a} p_{X,Y}(x, y)}{p_Y(y)} = \sum_{x \le a} p_{X|Y}(x \mid y)$$

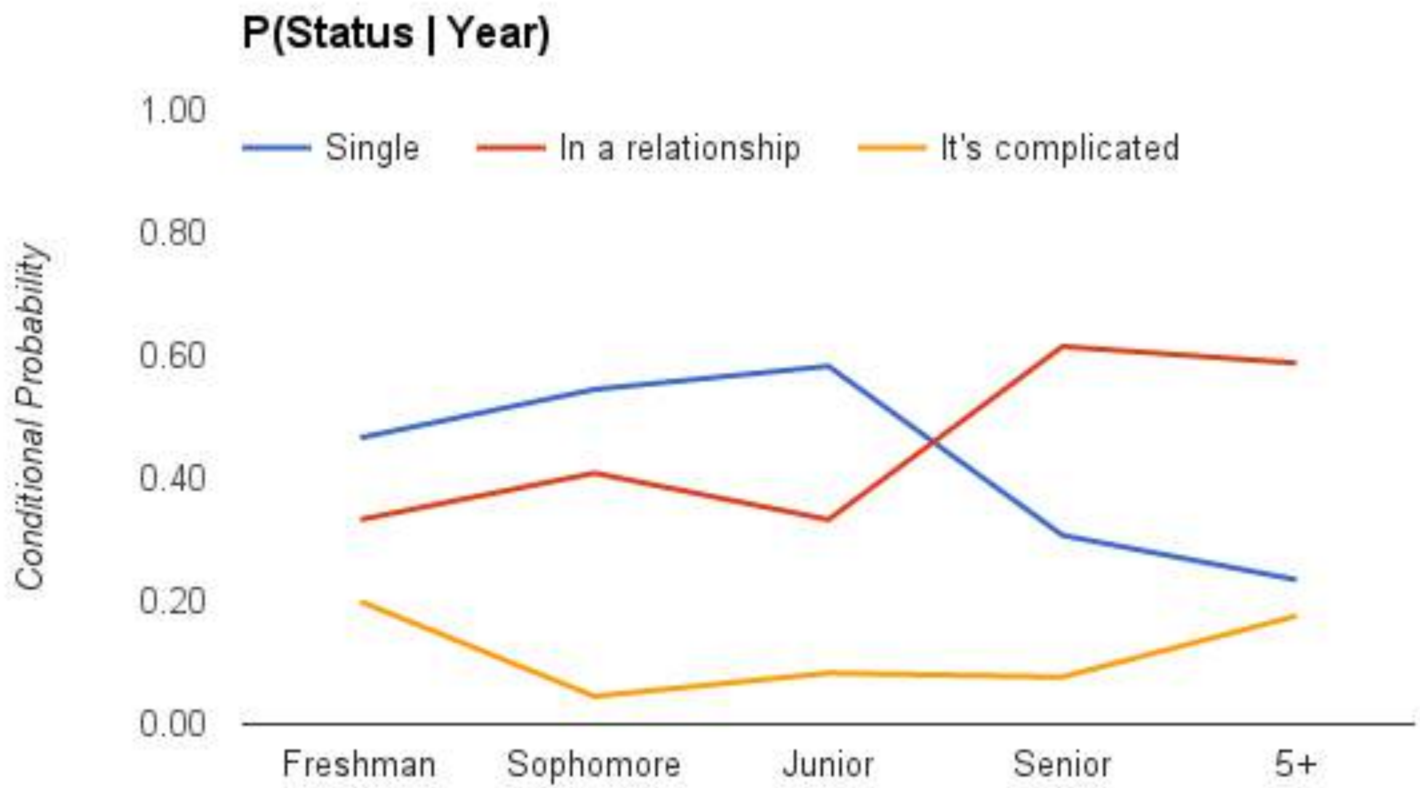
Probability Table

Joint Probability Table				
	Single	In a relationship	It's complicated	**Marginal Year**
Freshman	0.06	0.04	0.03	0.13
Sophomore	0.21	0.16	0.02	0.39
Junior	0.13	0.06	0.02	0.21
Senior	0.04	0.07	0.01	0.12
5+	0.04	0.09	0.03	0.15
Marginal Status	0.47	0.43	0.10	1.00





Relationship Status



Operating System Loyalty

- Consider person buying 2 computers (over time)
  - X = 1st computer bought is a PC (1 if it is, 0 if it is not)
  - Y = 2nd computer bought is a PC (1 if it is, 0 if it is not)
  - Joint probability mass function (PMF):

Y \ X	0	1	$p_Y(y)$
0	0.2	0.3	0.5
1	0.1	0.4	0.5
$p_X(x)$	0.3	0.7	1.0

  - What is P(Y = 0 | X = 0)?

$$P(Y=0 \mid X=0) = \frac{p_{X,Y}(0,0)}{p_X(0)} = \frac{0.2}{0.3} = \frac{2}{3}$$

  - What is P(Y = 1 | X = 0)?

$$P(Y=1 \mid X=0) = \frac{p_{X,Y}(0,1)}{p_X(0)} = \frac{0.1}{0.3} = \frac{1}{3}$$

  - What is P(X = 0 | Y = 1)?

$$P(X=0 \mid Y=1) = \frac{p_{X,Y}(0,1)}{p_Y(1)} = \frac{0.1}{0.5} = \frac{1}{5}$$

And It Applies to Books Too

P(Buy Book Y | Bought Book X)

Web Server Requests Redux

- Requests received at web server in a day
  - X = # requests from humans/day $X \sim Poi(\lambda_1)$
  - Y = # requests from bots/day $Y \sim Poi(\lambda_2)$
  - X and Y are independent → $X + Y \sim Poi(\lambda_1 + \lambda_2)$
  - What is P(X = *k* | X + Y = *n*)?

$$P(X = k \mid X + Y = n) = \frac{P(X = k, Y = n - k)}{P(X + Y = n)} = \frac{P(X = k)P(Y = n - k)}{P(X + Y = n)}$$

$$= \frac{e^{-\lambda_1}\lambda_1^k}{k!} \cdot \frac{e^{-\lambda_2}\lambda_2^{n-k}}{(n-k)!} \cdot \frac{n!}{e^{-(\lambda_1+\lambda_2)}(\lambda_1 + \lambda_2)^n} = \frac{n!}{k!(n-k)!} \cdot \frac{\lambda_1^k \lambda_2^{n-k}}{(\lambda_1 + \lambda_2)^n}$$

$$= \binom{n}{k}\left(\frac{\lambda_1}{\lambda_1 + \lambda_2}\right)^k \left(\frac{\lambda_2}{\lambda_1 + \lambda_2}\right)^{n-k}$$

$$(X|X + Y = n) \sim Bin\left(n, \frac{\lambda_1}{\lambda_1 + \lambda_2}\right)$$

Continuous Conditional Distributions

- Let X and Y be continuous random variables
  - Conditional PDF of X given Y (where $f_Y(y) > 0$):

$$f_{X|Y}(x \mid y) = \frac{f_{X,Y}(x, y)}{f_Y(y)}$$

$$f_{X|Y}(x \mid y)\, dx = \frac{f_{X,Y}(x, y)\, dx\, dy}{f_Y(y)\, dy}$$

$$\approx \frac{P(x \le X \le x + dx, y \le Y \le y + dy)}{P(y \le Y \le y + dy)} = P(x \le X \le x + dx \mid y \le Y \le y + dy)$$

  - Conditional CDF of X given Y (where $f_Y(y) > 0$):

$$F_{X|Y}(a \mid y) = P(X \le a \mid Y = y) = \int_{-\infty}^{a} f_{X|Y}(x \mid y)\, dx$$

  - Note: Even though P(Y = a) = 0, can condition on Y = a
    - Really considering: $P(a - \frac{\varepsilon}{2} \le Y \le a + \frac{\varepsilon}{2}) = \int_{a-\varepsilon/2}^{a+\varepsilon/2} f_Y(y) dy \approx \varepsilon f(a)$

Let's Do an Example

- X and Y are continuous RVs with PDF:

$$f(x,y) = \begin{cases} \frac{12}{5} x(2-x-y) & \text{where } 0 < x,y < 1 \\ 0 & \text{otherwise} \end{cases}$$

  - Compute conditional density: $f_{X|Y}(x \mid y)$

$$f_{X|Y}(x \mid y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{f_{X,Y}(x,y)}{\int_0^1 f_{X,Y}(x,y)\,dx}$$

$$= \frac{\frac{12}{5} x(2-x-y)}{\int_0^1 \frac{12}{5} x(2-x-y)\,dx} = \frac{x(2-x-y)}{\int_0^1 x(2-x-y)\,dx} = \frac{x(2-x-y)}{\left[x^2 - \frac{x^3}{3} - \frac{x^2 y}{2}\right]_0^1}$$

$$= \frac{x(2-x-y)}{\frac{2}{3} - \frac{y}{2}} = \frac{6x(2-x-y)}{4-3y}$$

Independence and Conditioning

- If X and Y are independent discrete RVs:

$$P(X = x \mid Y = y) = \frac{P(X = x, Y = y)}{P(Y = y)} = \frac{P(X = x)P(Y = y)}{P(Y = y)} = P(X = x)$$

$$p_{X|Y}(x \mid y) = \frac{p_{X,Y}(x, y)}{p_Y(y)} = \frac{p_X(x)p_Y(y)}{p_Y(y)} = p_X(x)$$

- Analogously, for independent continuous RVs:

$$f_{X|Y}(x \mid y) = \frac{f_{X,Y}(x, y)}{f_Y(y)} = \frac{f_X(x)f_Y(y)}{f_Y(y)} = f_X(x)$$

Conditional Independence Revisited

- n discrete random variables $X_1, X_2, \ldots, X_n$ are called **conditionally independent** given Y if:

$$P(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n \mid Y = y) = \prod_{i=1}^{n} P(X_i = x_i \mid Y = y) \quad \text{for all } x_1, x_2, \ldots, x_n, y$$

- Analogously, for continuous random variables:

$$P(X_1 \le a_1, X_2 \le a_2, \ldots, X_n \le a_n \mid Y = y) = \prod_{i=1}^{n} P(X_i \le a_i \mid Y = y) \quad \text{for all } a_1, a_2, \ldots, a_n, y$$

- Note: can turn products into sums using logs:

$$\ln \prod_{i=1}^{n} P(X_i = x_i \mid Y = y) = \sum_{\mathrm{i=1}}^{\mathrm{n}} \ln P(X_i = x_i \mid Y = y) = K$$

$$\prod_{i=1}^{n} P(X_i = x_i \mid Y = y) = e^K$$