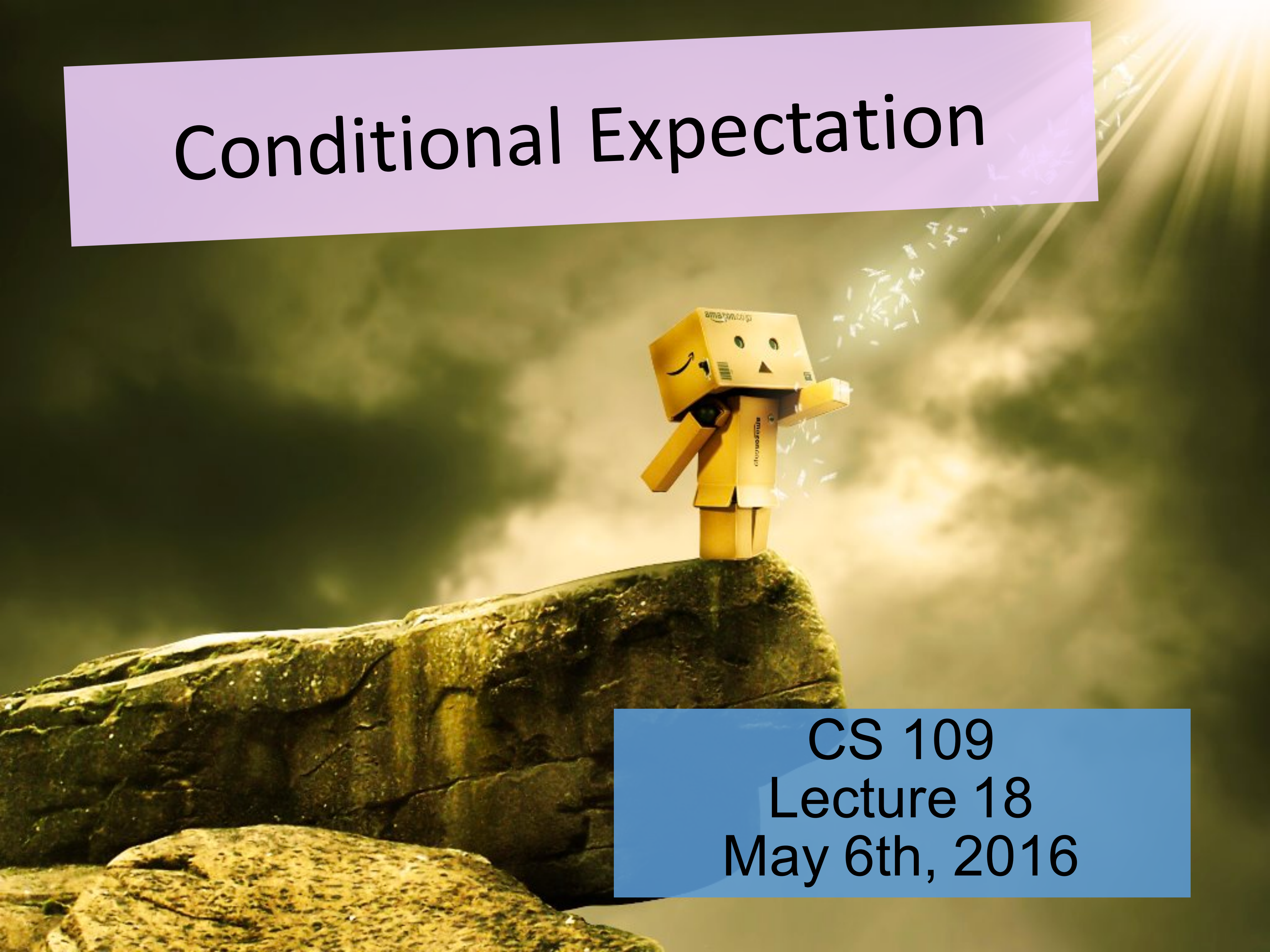
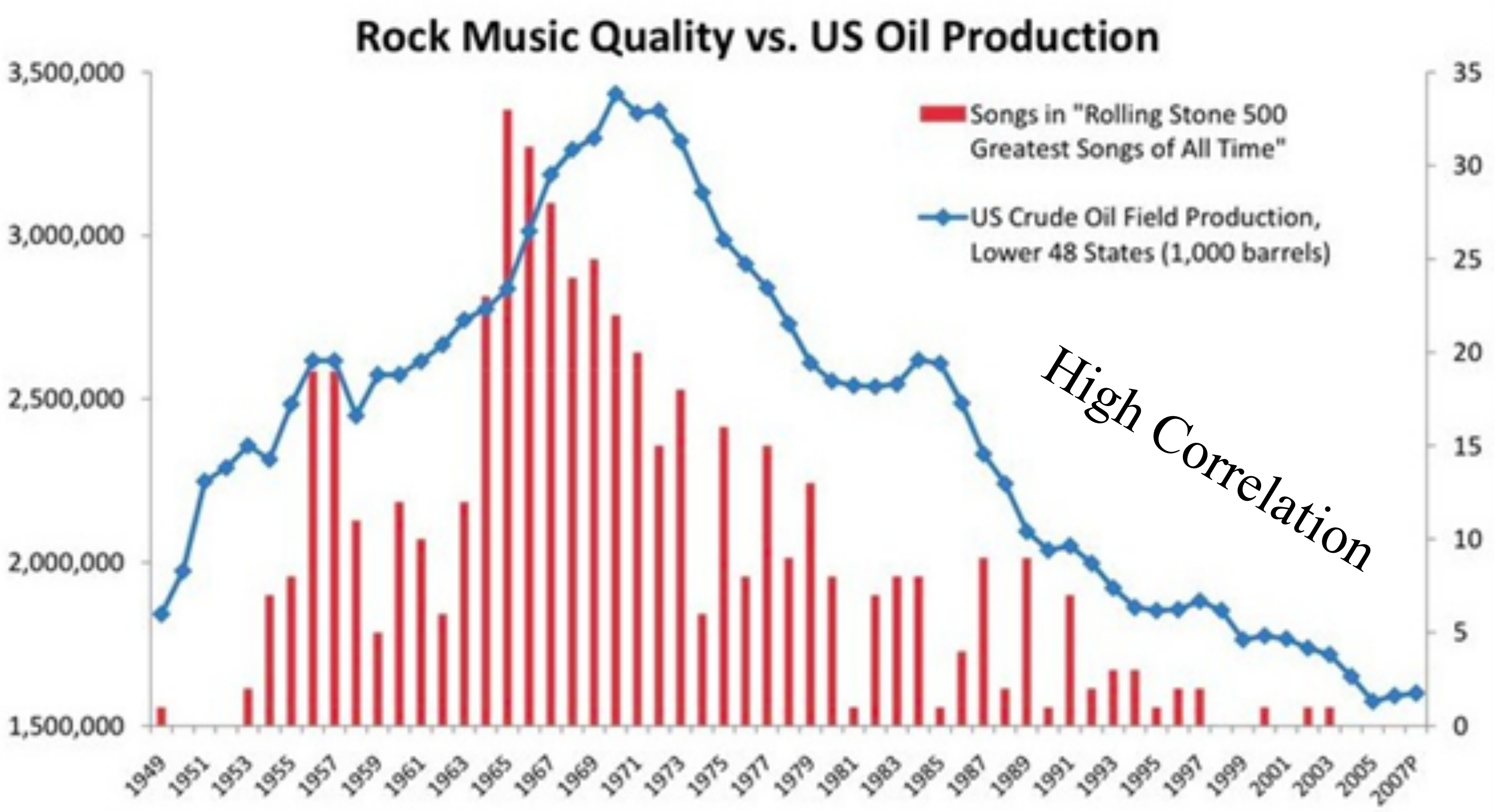
Conditional Expectation



CS 109
Lecture 18
May 6th, 2016

Philosophy

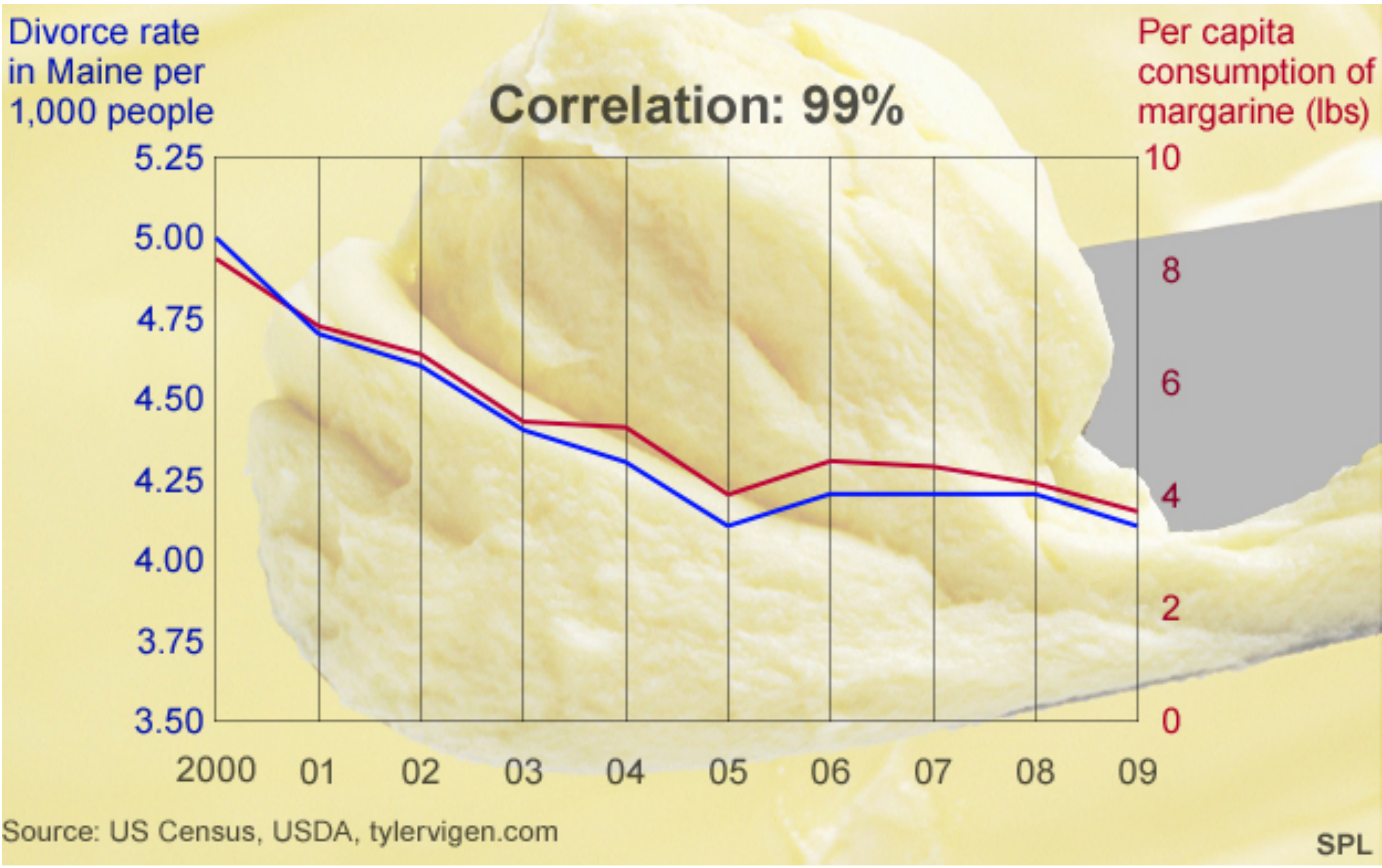
Rock Music Vs Oil?



Hubbert Peak Theory

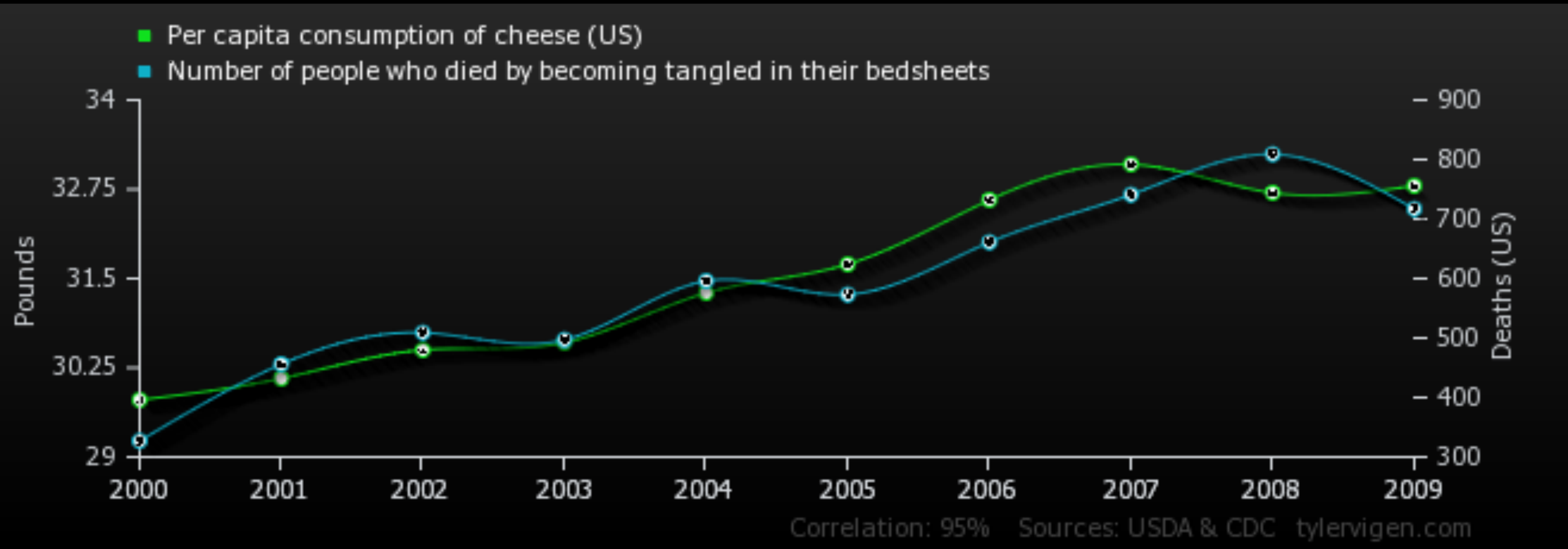
http://www.aei.org/publication/blog/

Divorce Vs Butter?



http://www.bbc.com/news/magazine-27537142

Cheese Vs Bedsheets?



http://www.tylervigen.com/view_correlation?id=7

Hidden Cause?
Correlation != Causation

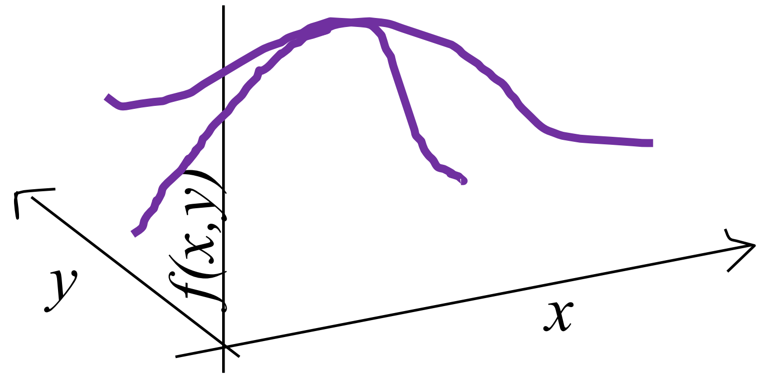
Multiple Hypothesis Testing!

Old School Review

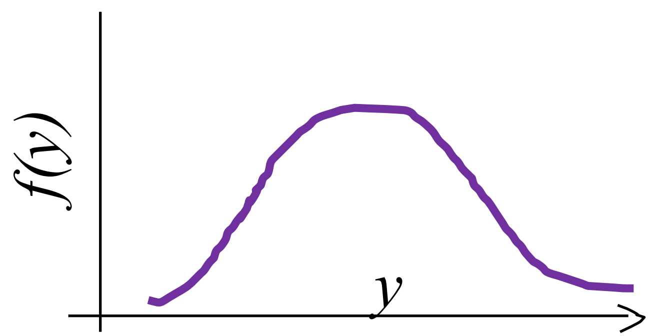
Continuous Conditional Distributions

- Let X and Y be continuous random variables
  - Conditional PDF of X given Y (where $f_Y(y) > 0$):

$$f_{X|Y}(x \mid y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$



$$f_{X|Y}(x \mid y)\, dx = \frac{f_{X,Y}(x,y)\, dx\, dy}{f_Y(y)\, dy}$$



$$P(x \le X \le x+dx \mid y \le Y \le y+dy) = \frac{P(x \le X \le x+dx, y \le Y \le y+dy)}{P(y \le Y \le y+dy)}$$

Bayes Theorem

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

$$f_{X|Y}(x|y) = \frac{f_{Y|X}(y|x) f_X(x)}{f_Y(y)}$$

Conditional Expectation Review

Conditional Expectation

- X and Y are jointly discrete random variables
  - Recall conditional PMF of X given Y = y:

$$p_{X|Y}(x \mid y) = P(X = x \mid Y = y) = \frac{p_{X,Y}(x, y)}{p_Y(y)}$$

- Define conditional expectation of X given Y = y:

$$E[X \mid Y = y] = \sum_x x P(X = x \mid Y = y) = \sum_x x\, p_{X|Y}(x \mid y)$$

- Analogously, jointly continuous random variables:

$$f_{X|Y}(x \mid y) = \frac{f_{X,Y}(x, y)}{f_Y(y)} \qquad E[X \mid Y = y] = \int_{-\infty}^{\infty} x\, f_{X|Y}(x \mid y)\, dx$$

Rolling Dice

- Roll two 6-sided dice D_1 and D_2
  - X = value of $D_1 + D_2$ Y = value of D_2
  - What is E[X | Y = 6]?

$$E[X \mid Y = 6] = \sum_x x P(X = x \mid Y = 6)$$

$$= \left(\frac{1}{6}\right)(7 + 8 + 9 + 10 + 11 + 12) = \frac{57}{6} = 9.5$$

  - Intuitively makes sense: 6 + E[value of D_1] = 6 + 3.5

End Review

Properties of Conditional Expectation

- X and Y are jointly distributed random variables

$$E[g(X) \mid Y = y] = \sum_x g(x) p_{X|Y}(x \mid y) \quad \text{or} \quad \int_{-\infty}^{\infty} g(x) f_{X|Y}(x \mid y)\, dx$$

- Expectation of conditional sum:

$$E\left[\sum_{i=1}^{n} X_i \mid Y = y\right] = \sum_{i=1}^{n} E[X_i \mid Y = y]$$

Expectations of Conditional Expectation

- Define $g(Y) = E[X \mid Y]$
  - For any $Y = y$, $g(Y) = E[X \mid Y = y]$
    - This is just function of Y, since we sum over all values of X
  - What is $E[E[X \mid Y]] = E[g(Y)]$? (Consider discrete case)

$$E[E[X \mid Y]] = \sum_y E[X \mid Y = y] P(Y = y)$$

$$= \sum_y \left[\sum_x x P(X = x \mid Y = y)\right] P(Y = y)$$

$$= \sum_y \sum_x x P(X = x, Y = y) = \sum_x x \sum_y P(X = x, Y = y)$$

$$= \sum_x x P(X = x) = E[X]$$

(Same for continuous)

Analyzing Recursive Code

```
int Recurse() {
   int x = randomInt(1, 3);  // Equally likely values
   if (x == 1) return 3;
   else if (x == 2) return (5 + Recurse());
   else return (7 + Recurse());
}
```

- Let Y = value returned by `Recurse()`. What is E[Y]?

$$E[Y] = E[Y \mid X = 1]P(X = 1) + E[Y \mid X = 2]P(X = 2) + E[Y \mid X = 3]P(X = 3)$$

$$E[Y \mid X = 1] = 3$$

$$E[Y \mid X = 2] = E[5 + Y] = 5 + E[Y]$$

$$E[Y \mid X = 3] = E[7 + Y] = 7 + E[Y]$$

$$E[Y] = 3(1/3) + (5 + E[Y])(1/3) + (7 + E[Y])(1/3) = (1/3)(15 + 2E[Y])$$

$$E[Y] = 15$$

Protip: do this in CS161

Funny thought: variance of runtime?

Random Number of Random Variables

- Say you have a web site: `PimentoLoaf.com`
  - X = Number of people/day visit your site. $X \sim N(50, 25)$
  - Y_i = Number of minutes spent by visitor *i*. $Y_i \sim \text{Poi}(8)$
  - X and all Y_i are independent
  - Time spent by all visitors/day: $W = \sum_{i=1}^{X} Y_i$. What is E[W]?

$$E[W] = E\left[\sum_{i=1}^{X} Y_i\right] = E\left[E\left[\sum_{i=1}^{X} Y_i \mid X\right]\right] = E\left[X \cdot E[Y_i]\right] = E[X]E[Y_i] = 50 \cdot 8$$

$$E\left[\sum_{i=1}^{X} Y_i \mid X = n\right] = \sum_{i=1}^{n} E[Y_i \mid X = n] = \sum_{i=1}^{n} E[Y_i] = nE[Y_i]$$

$$E\left[\sum_{i=1}^{X} Y_i \mid X\right] = X \cdot E[Y_i]$$

Making Predictions

- We observe random variable X
  - Want to make prediction about Y
  - E.g., X = stock price at 9am, Y = stock price at 10am
  - Let $g(X)$ be function we use to predict Y, i.e.: $\hat{Y} = g(X)$
  - Choose $g(X)$ to minimize $E[(Y - g(X))^2]$
  - Best predictor: $g(X) = E[Y | X]$
  - Intuitively: $E[(Y - c)^2]$ is minimized when $c = E[Y]$
    - Now, you observe X, and Y depends on X, then use $c = E[Y | X]$
  - You just got your first baby steps into Machine Learning
    - We'll go into this more rigorously in a few weeks

Speaking of Babies…

Baby Height

- My sister's height is X inches ($x = 67$)
  - Alyssa:  Perhaps a bit like: 
  - Say, historically, daughters grow to heights Y where Y ~ N(X + 1, 4), and X is height of mother
    - $Y = (X + 1) + C$ where $C \sim N(0, 4)$
  - What should I predict for the eventual height of Alyssa (my niece)?
  - $E[Y \mid X = 71] = E[X + 1 + C \mid X = 67]$

    $= E[68 + C] = E[68] + E[C] = 68 + 0$

    $= 68$ inches

Computing Probabilities by Conditioning

- X = indicator variable for event A: $X = \begin{cases} 1 & \text{if } A \text{ occurs} \\ 0 & \text{otherwise} \end{cases}$
  - E[X] = P(A)
  - Similarly, E[X | Y = y] = P(A | Y = y) for any Y
  - So: $E[X] = E_Y[\, E_X[X \mid Y]\,] = E[\, E[X \mid Y]\,] = E[\, P(A \mid Y)\,]$
  - In discrete case:

$$E[X] = \sum_y P(A \mid Y = y) P(Y = y) = P(A)$$

    - Also holds analogously in continuous case
  - "Law of total probability"

$$P(A) = \sum_y P(A|Y = y) P(Y = y)$$

Hiring Software Engineers

- Interviewing *n* software engineer candidates
  - All *n*! orderings equally likely, but only hiring 1 candidate
    - Claim: There is α-to-1 factor difference in productivity between the "best" and "average" software engineer
    - Steve Jobs set $\alpha = 25$, Mark Zuckerberg claimed $\alpha = 100$
  - Right after each interview must decide hire/no hire
  - Feedback from interview of candidate *i* is just relative ranking with respect to previous *i* – 1 candidates
    - Strategy: first interview *k* (of *n*) candidates, then hire next candidate better than all of first *k* candidates
    - P_k(best) = probability that best of all *n* candidates is hired
    - X = position of best candidate (1, 2, …, *n*)

$$P_k(\text{Best}) = \sum_{i=1}^{n} P_k(\text{Best} \mid X = i) P(X = i) = \frac{1}{n} \sum_{i=1}^{n} P_k(\text{Best} \mid X = i)$$

Hiring Software Engineers (cont.)

- Note: $P_k(\text{Best} \mid X = i) = 0 \quad \text{if} \quad i \le k$
- We will select best candidate (in position *i*) if best of first *i* – 1 candidates is among the first *k* interviewed

$P_k(\text{Best} \mid X = i)$

Hiring Software Engineers (cont.)

- Note: $P_k(\text{Best} \mid X = i) = 0 \quad \text{if} \quad i \le k$
- We will select best candidate (in position *i*) if best of first *i* – 1 candidates is among the first *k* interviewed

$$P_k(\text{Best} \mid X = i) = P_k(\text{best of first } i-1 \text{ in first } k \mid X = i) = \frac{k}{i-1} \quad \text{if} \quad i > k$$

$$P_k(\text{Best}) = \frac{1}{n}\sum_{i=1}^{n} P_k(\text{Best} \mid X = i) = \frac{1}{n}\sum_{i=k+1}^{n} \frac{k}{i-1}$$

$$\approx \frac{k}{n}\int_{i=k+1}^{n} \frac{1}{i-1}\,di = \frac{k}{n}\ln(i-1)\Big|_{k+1}^{n} = \frac{k}{n}\ln\frac{n-1}{k} \approx \frac{k}{n}\ln\frac{n}{k}$$

- To maximize, differentiate $P_k(\text{Best})$ with respect to *k*:

$$g(k) = \frac{k}{n}\ln\frac{n}{k} \qquad g'(k) = \frac{1}{n}\ln\frac{n}{k} + \frac{k}{n}\left(\frac{-1}{k}\right) = \frac{1}{n}\ln\frac{n}{k} - \frac{1}{n}$$

- Set $g'(k) = 0$ and solve for *k*:

$$\frac{1}{n}\ln\frac{n}{k} - \frac{1}{n} = 0 \;\Rightarrow\; \ln\frac{n}{k} = 1 \;\Rightarrow\; \frac{n}{k} = e \;\Rightarrow\; k = \frac{n}{e}$$

- Interview n/e candidates, then pick best: $P_k(\text{Best}) \approx 1/e \approx 0.368$

Also called the Marriage Problem

But people are
not single
statistics…

One more song?