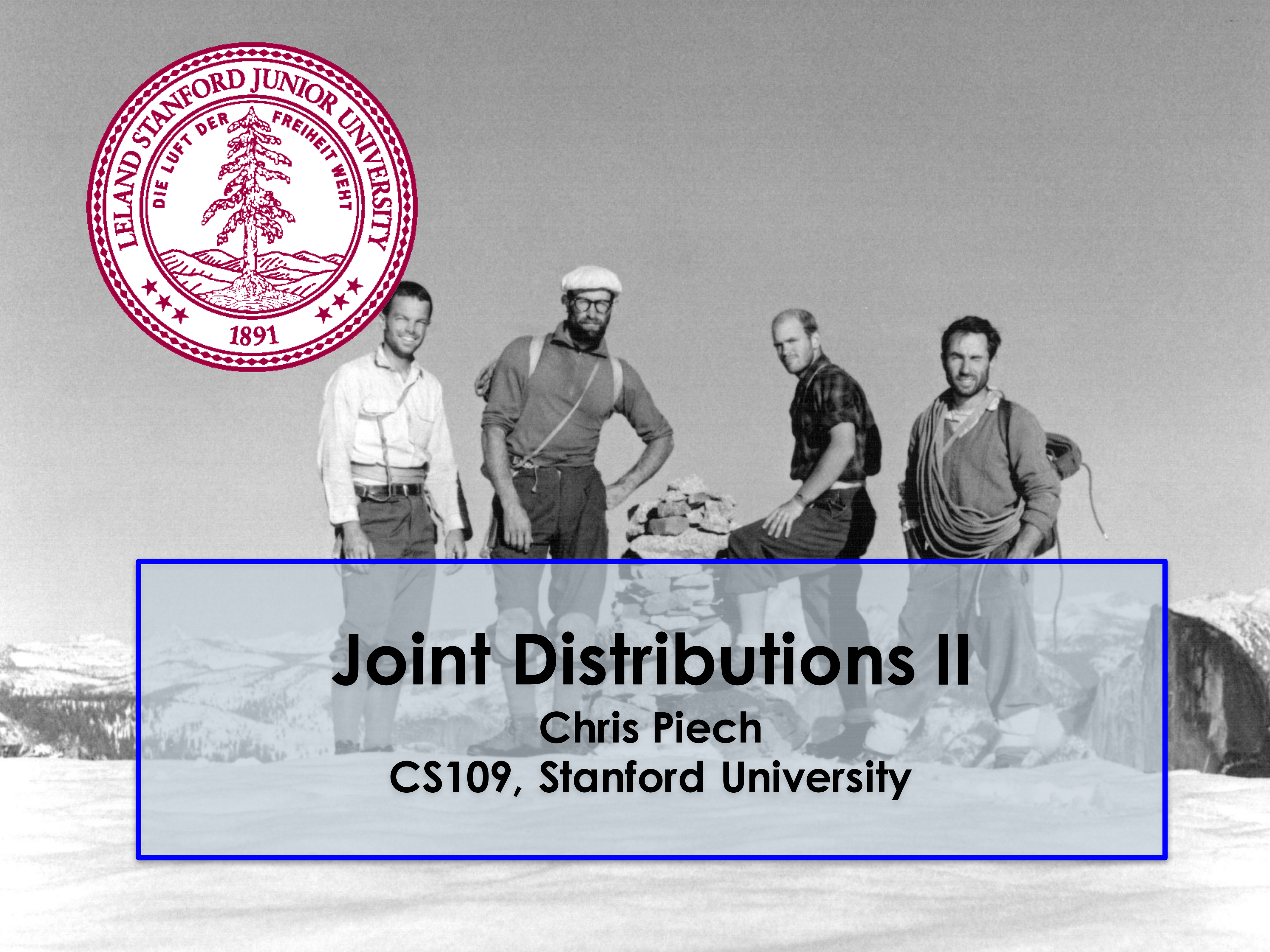


Joint Distributions II

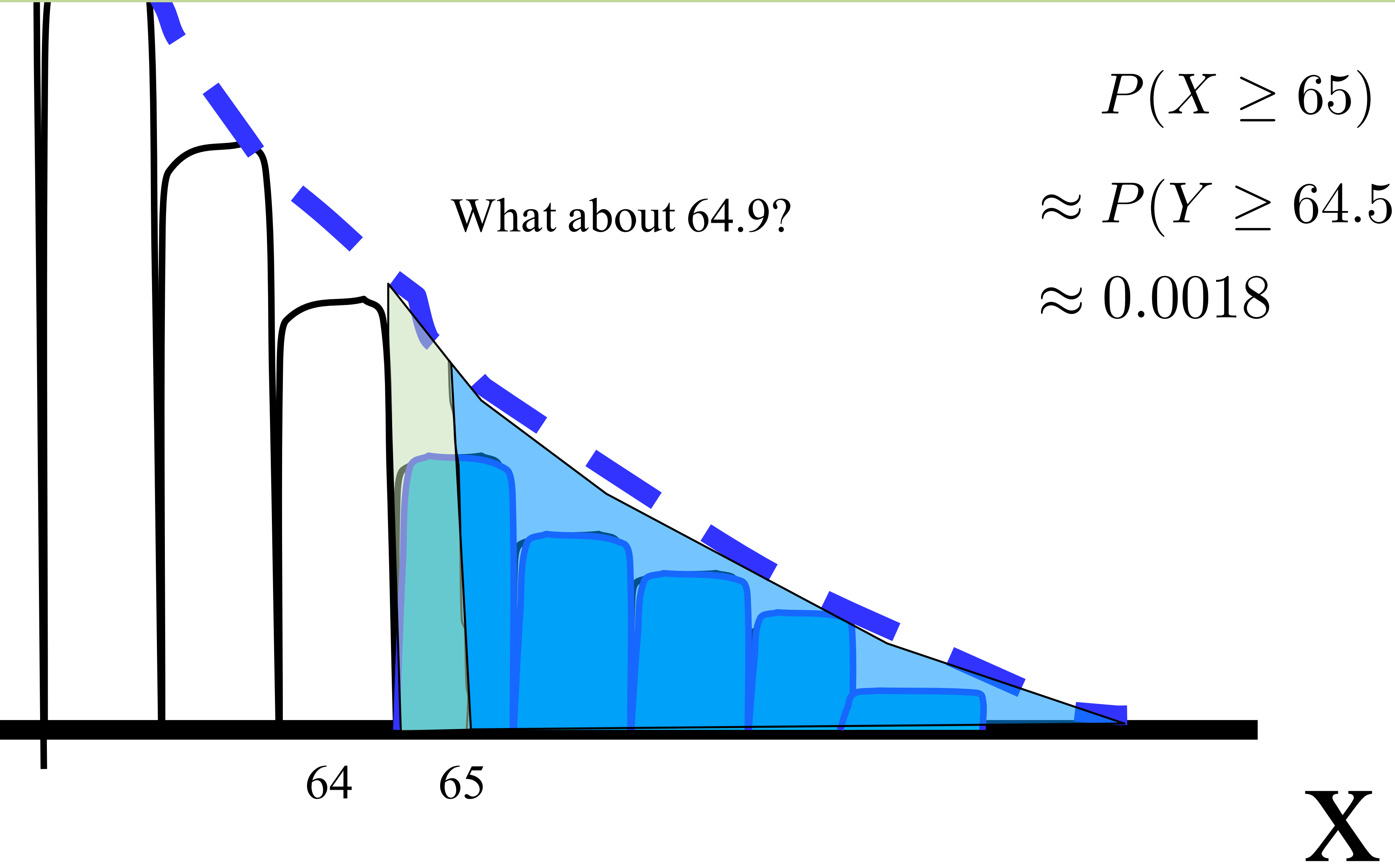
Chris Piech
CS109, Stanford University

Today:

1. Multi variable RVs
2. Expectation with multiple RVs

Review

Continuity Correction



$$P(X \geq 65)$$

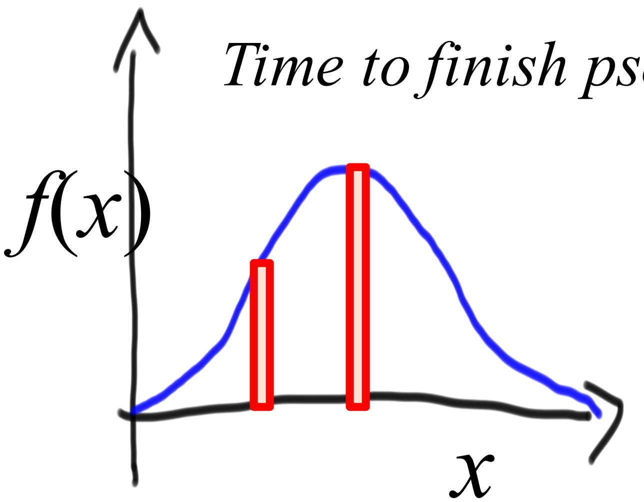
$$\approx P(Y \geq 64.5)$$

$$\approx 0.0018$$

Continuous RV Relative Probability

X = time to finish pset 3

X ~ N(10, 2)



How much more likely are you to complete in 10 hours than in 5?

$$\frac{P(X=10)}{P(X=5)} = \frac{\varepsilon f(X=10)}{\varepsilon f(X=5)}$$

$$= \frac{f(X=10)}{f(X=5)}$$

$$= \frac{\frac{1}{\sqrt{2\sigma^2\pi}} e^{-\frac{(10-\mu)^2}{2\sigma^2}}}{\frac{1}{\sqrt{2\sigma^2\pi}} e^{-\frac{(5-\mu)^2}{2\sigma^2}}}$$

$$= \frac{\frac{1}{\sqrt{4\pi}} e^{-\frac{(10-10)^2}{4}}}{\frac{1}{\sqrt{4\pi}} e^{-\frac{(5-10)^2}{4}}}$$

$$= \frac{e^0}{e^{-\frac{25}{4}}} = 518$$

Discrete Joint Mass Function

- For two discrete random variables *X* and *Y*, the **Joint Probability Mass Function** is:

$$p_{X,Y}(a,b) = P(X = a, Y = b)$$

- Marginal distributions:

$$p_X(a) = P(X = a) = \sum_y p_{X,Y}(a, y)$$

$$p_Y(b) = P(Y = b) = \sum_x p_{X,Y}(x, b)$$

- Example: X = value of die D_1, Y = value of die D_2

$$P(X = 1) = \sum_{y=1}^{6} p_{X,Y}(1, y) = \sum_{y=1}^{6} \frac{1}{36} = \frac{1}{6}$$

Joint Probability Table for Discrete

- States all possible outcomes with several discrete variables
- Often is not "parametric"
- If #variables is > 2, you can have a probability table, but you can't draw it on a slide

All values of A

All values of B

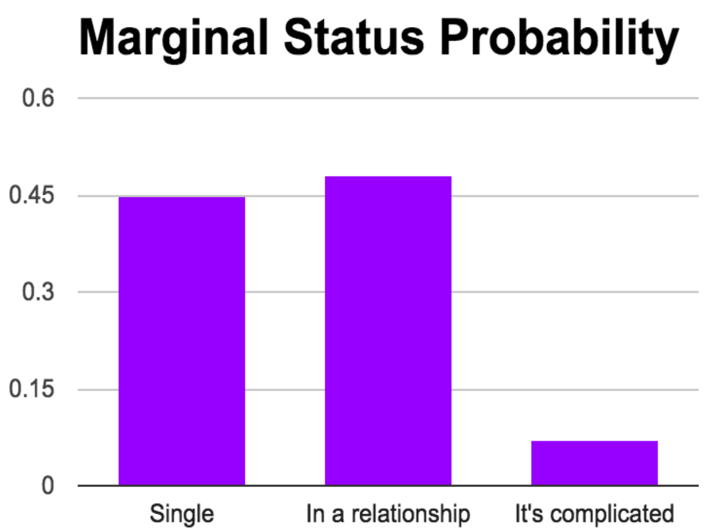
	0	1	2
0			
1		$P(A = 1, B = 1)$	
2			

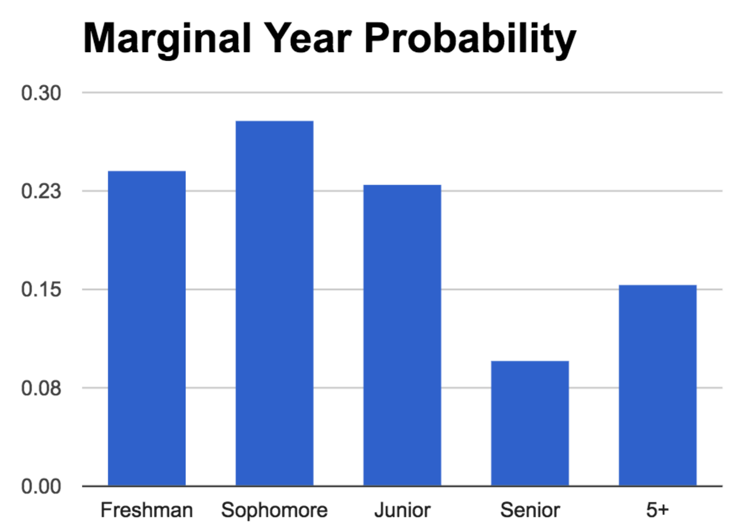
Every outcome falls into a bucket

Remember "," means "and"

Probability Table

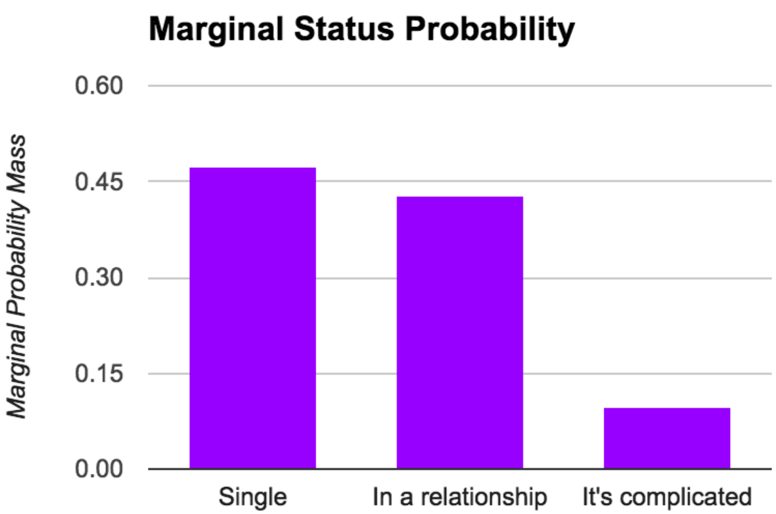
Joint Probability Table				
	Single	In a relationship	It's complicated	**Marginal Year**
Freshman	0.13	0.09	0.02	0.24
Sophomore	0.16	0.10	0.02	0.28
Junior	0.12	0.10	0.02	0.23
Senior	0.01	0.09	0.00	0.10
5+	0.03	0.12	0.01	0.15
Marginal Status	0.45	0.48	0.07	



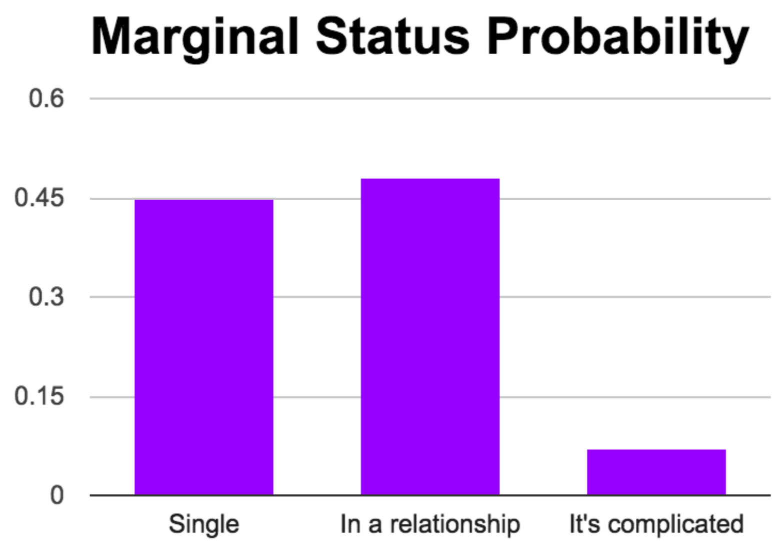


Quite Consistent

2016



2017

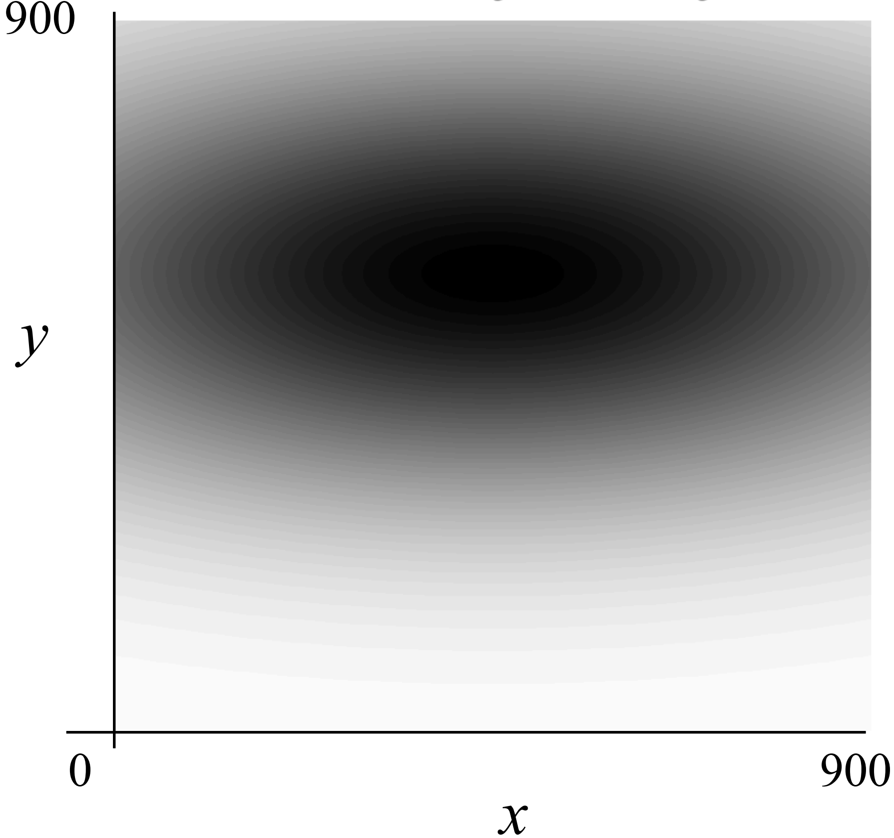


End Review

Our CS109 Logo

Joint Dart Distribution

Dart Probability Density

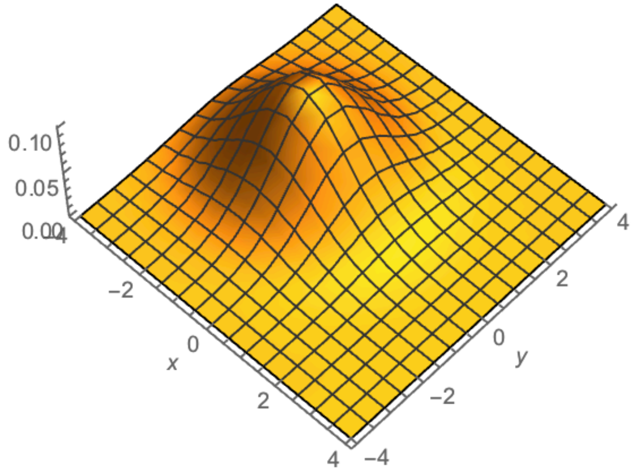


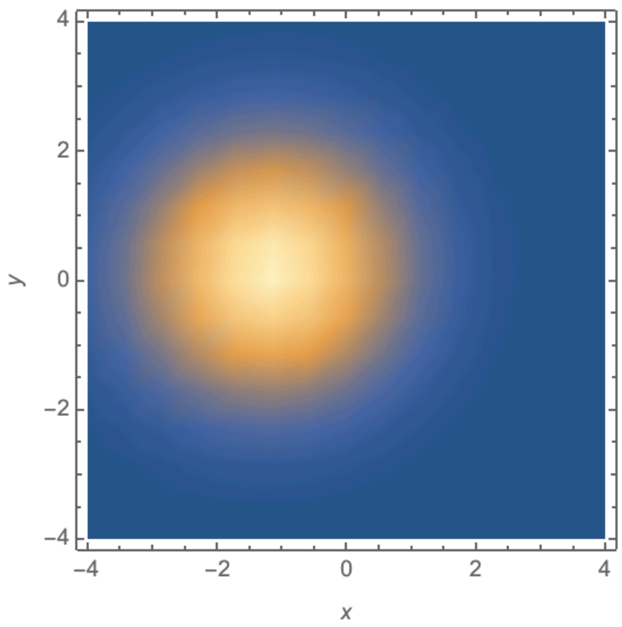
Dart Results

Jointly Continuous

- Random variables *X* and *Y*, are **<u>Jointly Continuous</u>** if there exists PDF $f_{X,Y}(x, y)$ defined over $-\infty < x, y < \infty$ such that:

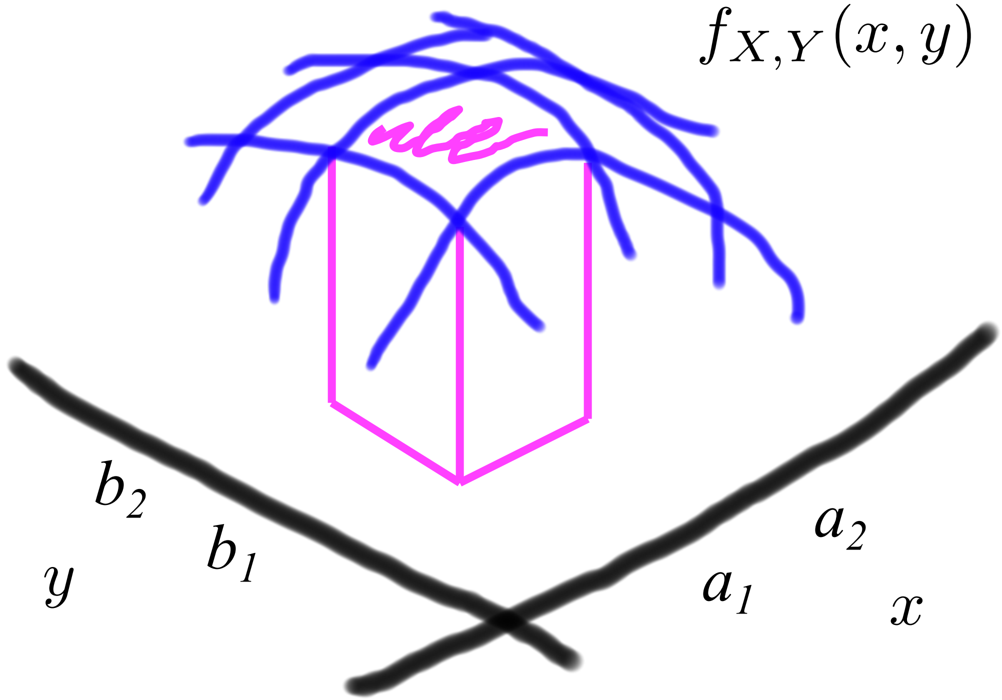
$$P(a_1 < X \leq a_2, b_1 < Y \leq b_2) = \int_{a_1}^{a_2} \int_{b_1}^{b_2} f_{X,Y}(x, y)\, dy\, dx$$





Jointly Continuous

$$P(a_1 < X \leq a_2, b_1 < Y \leq b_2) = \int_{a_1}^{a_2} \int_{b_1}^{b_2} f_{X,Y}(x, y)\, dy\, dx$$



Jointly Continuous

$$\mathrm{P}(a_1 < X \leq a_2, b_1 < Y \leq b_2) = \int_{a_1}^{a_2} \int_{b_1}^{b_2} f_{X,Y}(x, y)\, dy\, dx$$

- Cumulative Density Function (CDF):

$$F_{X,Y}(a,b) = \int_{-\infty}^{a} \int_{-\infty}^{b} f_{X,Y}(x, y)\, dy\, dx \qquad f_{X,Y}(a,b) = \frac{\partial^2}{\partial a\, \partial b} F_{X,Y}(a,b)$$

- Marginal density functions:

$$f_X(a) = \int_{-\infty}^{\infty} f_{X,Y}(a, y)\, dy \qquad f_Y(b) = \int_{-\infty}^{\infty} f_{X,Y}(x,b)\, dx$$

Continuous Joint Distribution Functions

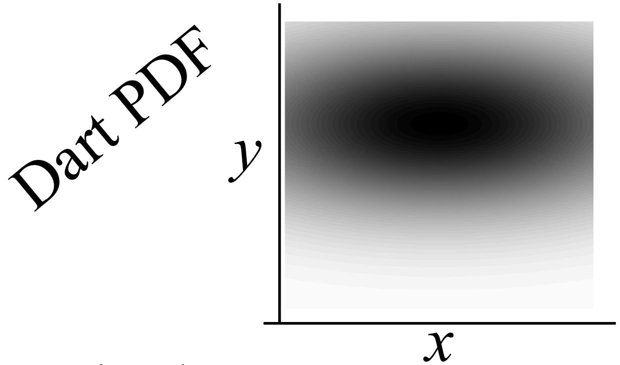
$$\mathrm{P}(a_1 < X \le a_2, b_1 < Y \le b_2) = \int_{a_1}^{a_2}\int_{b_1}^{b_2} f_{X,Y}(x,y)\,dy\,dx$$

- Marginal distributions:

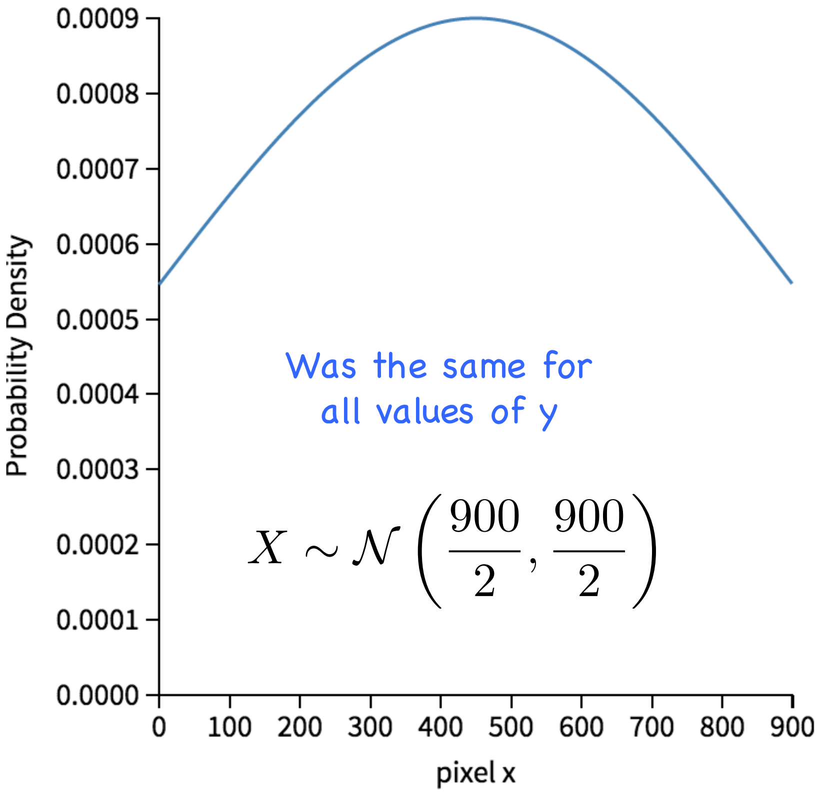
$$F_X(a) = P(X \le a) = \mathrm{P}(X \le a, Y < \infty) = F_{X,Y}(a,\infty)$$

$$F_Y(b) = P(Y \le b) = \mathrm{P}(X < \infty, Y \le b) = F_{X,Y}(\infty,b)$$

Darts!

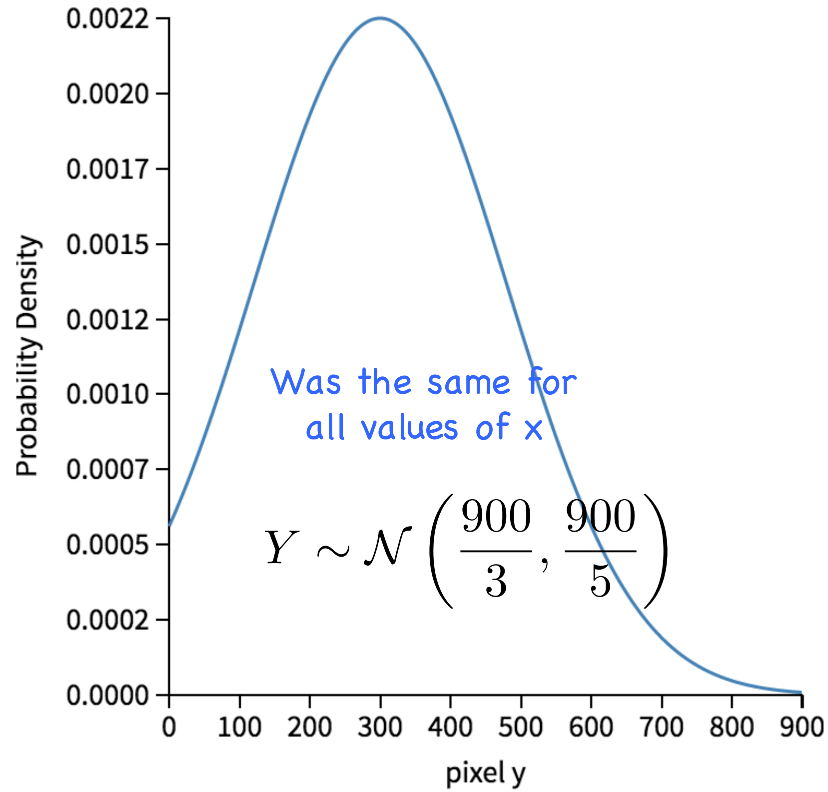


X-Pixel Marginal



$$X \sim \mathcal{N}\left(\frac{900}{2}, \frac{900}{2}\right)$$

Y-Pixel Marginal



$$Y \sim \mathcal{N}\left(\frac{900}{3}, \frac{900}{5}\right)$$

Multiple Integrals Without Tears

- Let X and Y be two continuous random variables
  - where $0 \le X \le 1$ and $0 \le Y \le 2$
- We want to integrate $g(x,y) = xy$ w.r.t. X and Y:
  - First, do "innermost" integral (treat y as a constant):

$$\int_{y=0}^{2}\int_{x=0}^{1} xy\,dx\,dy = \int_{y=0}^{2}\left(\int_{x=0}^{1} xy\,dx\right)dy = \int_{y=0}^{2} y\left[\frac{x^2}{2}\right]_0^1 dy = \int_{y=0}^{2} y\frac{1}{2}\,dy$$

  - Then, evaluate remaining (single) integral:

$$\int_{y=0}^{2} y\frac{1}{2}\,dy = \left[\frac{y^2}{4}\right]_0^2 = 1 - 0 = 1$$

Computing Joint Probabilities

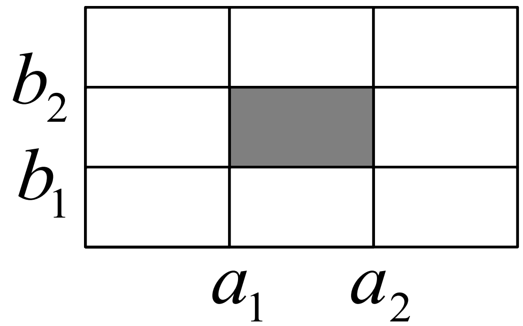
Let $F_{X,Y}(x, y)$ be joint CDF for X and Y

$$
\begin{aligned}
\mathrm{P}(X > a, Y > b) &= 1 - P((X > a, Y > b)^c) \\
&= 1 - P((X > a)^c \cup (Y > b)^c) \\
&= 1 - P((X \le a) \cup (Y \le b)) \\
&= 1 - (P(X \le a) + P(Y \le b) - P(X \le a, Y \le b)) \\
&= 1 - F_X(a) - F_Y(b) + F_{X,Y}(a, b)
\end{aligned}
$$

The General Rule Given Joint CDF

Let $F_{X,Y}(x,y)$ be joint CDF for X and Y

$$\mathrm{P}(a_1 < X \le a_2, b_1 < Y \le b_2)$$
$$= F(a_2, b_2) - F(a_1, b_2) + F(a_1, b_1) - F(a_2, b_1)$$



For joint, continuous random variables:

We still have a PDF and a CDF. They are still key.

But they are now multi-dimensional

Lovely Lemma

- Y is a non-negative continuous random variable
  - Probability Density Function: $f_Y(y)$
  - Already knew that:

$$E[Y] = \int_{-\infty}^{\infty} y \, f_Y(y) \, dy$$

  - But, did you know that:

$$E[Y] = \int_{0}^{\infty} P(Y > y) \, dy \text{ ?!?}$$

  - Analogously, in the discrete case, where $X = 1, 2, \ldots, n$

$$E[X] = \sum_{i=1}^{n} P(X \geq i)$$

How this lemma was made

In the discrete case, where $X = 1, 2, \ldots, n$

$$E[X] = \sum_{i=1}^{n} P(X \geq i)$$

Each row is an expansion for one value of i

$$\begin{aligned}
\sum_{i=1}^{n} P(X \geq i) = \\
P(X = 1) + P(X = 2) + P(X = 3) + \cdots + P(X = n) \\
+ P(X = 2) + P(X = 3) + \cdots + P(X = n) \\
+ P(X = 3) + \cdots + P(X = n) \\
\vdots \\
+ P(X = n)
\end{aligned}$$

$$\begin{aligned}
&= 1P(X = 1) + 2P(X = 2) + \cdots + n(PX = n) \\
&= E[X]
\end{aligned}$$

Life gives you lemmas,
make lemmanade!

Imperfections on a Disk

- Disk surface is a circle of radius R
  - A single point imperfection uniformly distributed on disk

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{\pi R^2} & \text{if } x^2 + y^2 \le R^2 \\ 0 & \text{if } x^2 + y^2 > R^2 \end{cases} \quad \text{where} \quad -\infty < x, y < \infty$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy = \frac{1}{\pi R^2} \int_{x^2+y^2 \le R^2} dy$$

$$= \frac{1}{\pi R^2} \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} dy$$

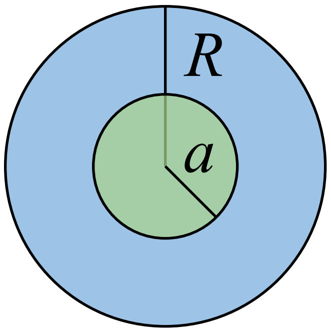
$$= \frac{2\sqrt{R^2 - x^2}}{\pi R^2}$$

Only integrate over the support range

Marginal of Y is the same by symmetry

Imperfections on a Disk

- Disk surface is a circle of radius R
  - A single point imperfection uniformly distributed on disk
  - Distance to origin: $D = \sqrt{X^2 + Y^2}$
  - What is *E[D]*?



$$P(D \le a) = \frac{\pi a^2}{\pi R^2} = \frac{a^2}{R^2}$$

Because of equally likely outcomes

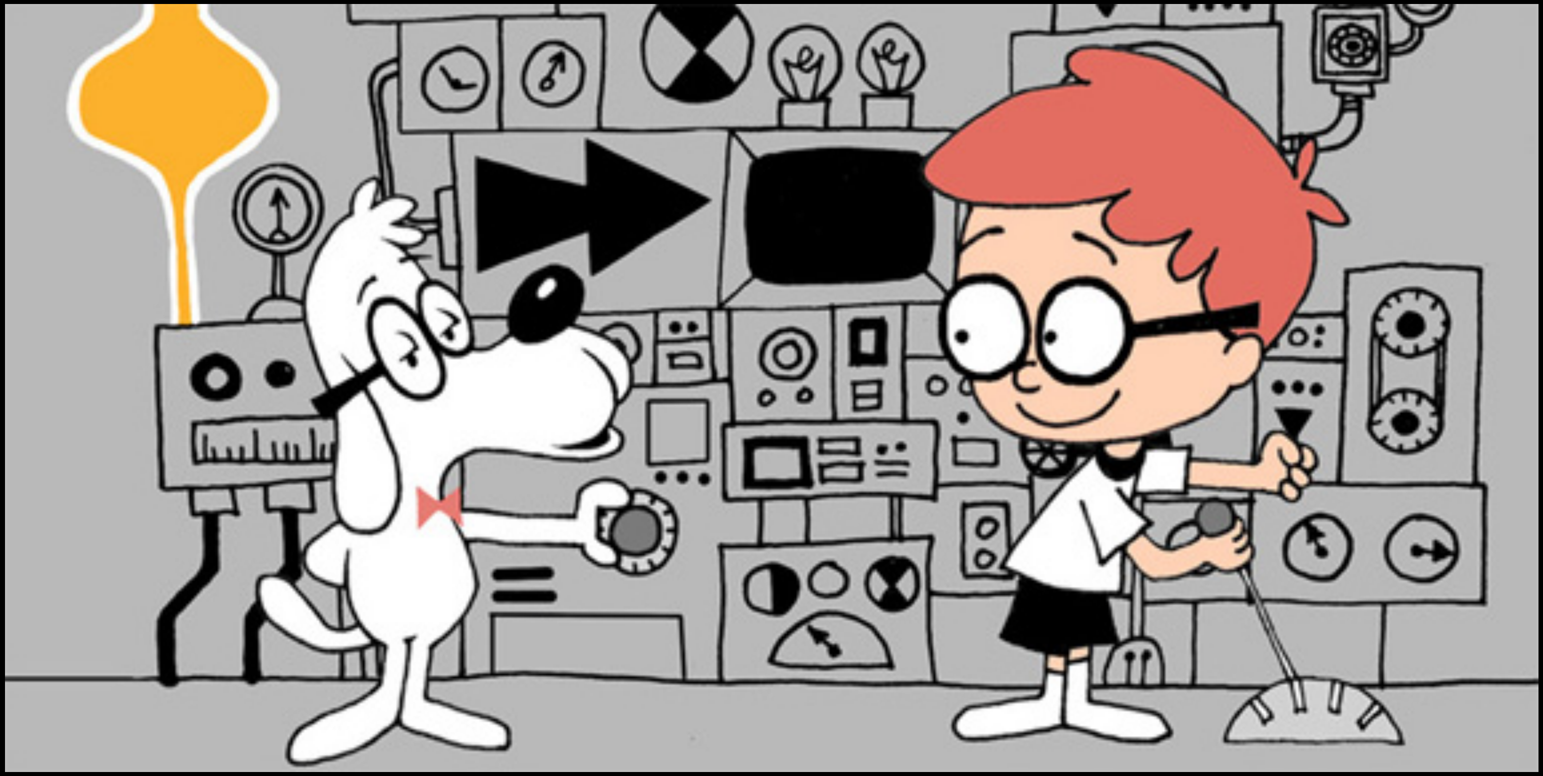
$$E[D] = \int_0^R P(D > a)da = \int_0^R 1 - P(D \le a)da$$

$$= \int_0^R 1 - \frac{a^2}{R^2} da$$

$$= \left[a - \frac{a^3}{3R^2}\right]_0^R = \frac{2R}{3}$$

Transfer Learning

Way Back



Permutations

How many ways are there to order n distinct objects?

$$n!$$

Multinomial

How many ways are there to order n objects such that:

n_1 are the same (indistinguishable)

n_2 are the same (indistinguishable)

$\ldots$

n_r are the same (indistinguishable)?

$$\frac{n!}{n_1!n_2!\ldots n_r!} = \binom{n}{n_1, n_2, \ldots, n_r}$$

Called the "multinomial" because of something from Algebra

Binomial

How many ways are there to make an unordered selection of r objects from n objects?

How many ways are there to order n objects such that:
r are the same (indistinguishable)
$(n-r)$ are the same (indistinguishable)?

$$\frac{n!}{r!(n-r)!} = \binom{n}{r}$$

Called the Binomial (Multi > Bi)

Binomial Distribution

- Consider *n* independent trials of Ber(p) rand. var.
  - X is number of successes in *n* trials
  - X is a **Binomial** Random Variable: X ~ Bin(n, p)

Binomial # ways of ordering the successes

$$P(X = i) = p(i) = \binom{n}{i} p^i (1-p)^{n-i} \quad i = 0,1,\ldots,n$$

Probability of exactly *i* successes

Probability of each ordering of *i* successes is equal + mutually exclusive

End Way Back

Welcome Back the Multinomial

- Multinomial distribution
  - n independent trials of experiment performed
  - Each trial results in one of m outcomes, with respective probabilities: $p_1, p_2, \ldots, p_m$ where $\sum_{i=1}^{m} p_i = 1$
  - X_i = number of trials with outcome i

$$P(X_1 = c_1, X_2 = c_2, \ldots, X_m = c_m) = \binom{n}{c_1, c_2, \ldots, c_m} p_1^{c_1} p_2^{c_2} \ldots p_m^{c_m}$$

Joint distribution

Multinomial # ways of ordering the successes

Probabilities of each ordering are equal and mutually exclusive

where $\sum_{i=1}^{m} c_i = n$ and $\binom{n}{c_1, c_2, \ldots, c_m} = \dfrac{n!}{c_1! c_2! \cdots c_m!}$

Hello Die Rolls, My Old Friends

- 6-sided die is rolled 7 times
  - Roll results: 1 one, 1 two, 0 three, 2 four, 0 five, 3 six

$$P(X_1 = 1, X_2 = 1, X_3 = 0, X_4 = 2, X_5 = 0, X_6 = 3)$$

$$= \frac{7!}{1!1!0!2!0!3!}\left(\frac{1}{6}\right)^1\left(\frac{1}{6}\right)^1\left(\frac{1}{6}\right)^0\left(\frac{1}{6}\right)^2\left(\frac{1}{6}\right)^0\left(\frac{1}{6}\right)^3 = 420\left(\frac{1}{6}\right)^7$$

- This is generalization of Binomial distribution
  - Binomial: each trial had 2 possible outcomes
  - Multinomial: each trial has *m* possible outcomes

Probabilistic Text Analysis

- Ignoring order of words, what is probability of any given word you write in English?
  - P(word = “the”) > P(word = “transatlantic”)
  - P(word = “Stanford”) > P(word = “Cal”)
  - Probability of each word is just multinomial distribution
- What about probability of those same words in someone else’s writing?
  - P(word = “probability” | writer = you) >

    P(word = “probability” | writer = non-CS109 student)
  - After estimating P(word | writer) from known writings, use Bayes’ Theorem to determine P(writer | word) for new writings!

Text is a Multinomial

Example document:
"Pay for Viagra with a credit-card. Viagra is great. So are credit-cards. Risk free Viagra. Click for free."
n = 18

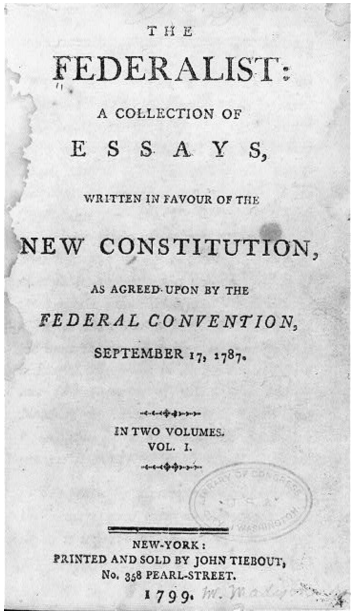
$$\underbrace{P\left(\begin{array}{l}\text{Viagra} = 2 \\ \text{Free} = 2 \\ \text{Risk} = 1 \\ \text{Credit-card: } 2 \\ \ldots \\ \text{For} = 2\end{array} \middle| \text{spam}\right)}_{\text{Probability of seeing this document | spam}} = \overbrace{\frac{n!}{2!2!\ldots2!} p_{\text{viagra}}^2 p_{\text{free}}^2 \ldots p_{\text{for}}^2}^{\text{It's a Multinomial!}}$$

The probability of a word in spam email being viagra

Who wrote the federalist papers?

Old and New Analysis

- Authorship of "Federalist Papers"
  - 85 essays advocating ratification of US constitution
  - Written under pseudonym "Publius"
    - Really, Alexander Hamilton, James Madison and John Jay
  - Who wrote which essays?
    - Analyzed probability of words in each essay versus word distributions from known writings of three authors



THE
FEDERALIST:
A COLLECTION OF
ESSAYS,
WRITTEN IN FAVOUR OF THE
NEW CONSTITUTION,
AS AGREED UPON BY THE
FEDERAL CONVENTION,
SEPTEMBER 17, 1787.

IN TWO VOLUMES.
VOL. I.

NEW-YORK:
PRINTED AND SOLD BY JOHN TIEBOUT,
No. 358 PEARL-STREET.
1799.

- Filtering Spam
  - P(word = "Viagra" | writer = you)

  << P(word = "Viagra" | writer = spammer)

Expectation with Multiple Variables?

Joint Expectation

$$E[X] = \sum_x x p(x)$$

- Expectation over a joint isn't nicely defined because it is not clear how to compose the multiple variables:
  - Add them? Multiply them?

- Lemma: For a function g(X,Y) we can calculate the expectation of that function:

$$E[g(X,Y)] = \sum_{x,y} g(x,y)p(x,y)$$

- By the way, this also holds for single random variables:

$$E[g(X)] = \sum_x g(x)p(x)$$

Expected Values of Sums

Big deal lemma: first stated without proof

$$E[X + Y] = E[X] + E[Y]$$

Generalized: $$E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i]$$

Holds regardless of dependency between X_i's

Skeptical Chris Wants a Proof!

Let g(X,Y) = [X + Y]

$$E[X+Y] = E[g(X,Y)] = \sum_{x,y} g(x,y)p(x,y)$$

What a useful lemma

$$= \sum_{x,y} [x+y]p(x,y)$$

By the definition of g(x,y)

Break that sum into parts!

$$= \sum_{x,y} xp(x,y) + \sum_{x,y} yp(x,y)$$

Change the sum of (x,y) into separate sums

$$= \sum_{x} x \sum_{y} p(x,y) + \sum_{y} y \sum_{x} p(x,y)$$

That is the definition of marginal probability

$$= \sum_{x} xp(x) + \sum_{y} yp(y)$$

That is the definition of expectation

$$= E[X] + E[Y]$$

Independence and Random Variables

Independent Discrete Variables

- Two discrete random variables X and Y are called <u>**independent**</u> if:

$$p(x, y) = p_X(x)p_Y(y) \quad \text{for all } x, y$$

- Intuitively: knowing the value of X tells us nothing about the distribution of Y (and vice versa)
  - If two variables are <u>**not**</u> independent, they are called <u>***dependent***</u>
- Similar conceptually to independent *events*, but we are dealing with multiple <u>***variables***</u>
  - Keep your events and variables distinct (and clear)!

Coin Flips

- Flip coin with probability *p* of “heads”
  - Flip coin a total of *n* + *m* times
  - Let X = number of heads in first *n* flips
  - Let Y = number of heads in next *m* flips

$$P(X = x, Y = y) = \binom{n}{x} p^x (1-p)^{n-x} \binom{m}{y} p^y (1-p)^{m-y}$$

$$= P(X = x) P(Y = y)$$

  - X and Y are independent
  - Let Z = number of total heads in *n* + *m* flips
  - Are X and Z independent?
    - What if you are told Z = 0?

Is Year Independent of Status?

Joint Probability Table				
	Single	In a relationship	It's complicated	**Marginal Year**
Freshman	$\boxed{0.06}$	0.04	0.03	$\boxed{0.13}$
Sophomore	0.21	0.16	0.02	0.39
Junior	0.13	0.06	0.02	0.21
Senior	0.04	0.07	0.01	0.12
5+	0.04	0.09	0.03	0.15
Marginal Status	$\boxed{0.47}$	0.43	0.10	1.00

For all values of Year, Status:

$$\underbrace{P(\text{Year} = y, \text{Status} = s)}_{0.06} = \underbrace{P(\text{Year} = y)}_{0.13}\underbrace{P(\text{Status} = s)}_{0.47}$$

Yes!

Is Year Independent of Status?

Joint Probability Table				
	Single	In a relationship	It's complicated	**Marginal Year**
Freshman	0.06	0.04	0.03	0.13
Sophomore	0.21	0.16	0.02	0.39
Junior	0.13	0.06	0.02	0.21
Senior	0.04	0.07	0.01	0.12
5+	0.04	0.09	0.03	0.15
Marginal Status	0.47	0.43	0.10	1.00

For all values of Year, Status:

$$\underbrace{P(\text{Year}=y, \text{Status}=s)}_{0.21} = \underbrace{P(\text{Year}=y)}_{0.39}\underbrace{P(\text{Status}=s)}_{0.47}$$

No ☹

Aside: Butterfly Effect

Web Server Requests

- Let N = # of requests to web server/day
  - Suppose N ~ Poi(λ)
  - Each request comes from a human (probability = *p*) or from a "bot" (probability = (1 – *p*)), independently
  - X = # requests from humans/day   (X | N) ~ Bin(N, *p*)
  - Y = # requests from bots/day   (Y | N) ~ Bin(N, 1 - *p*)

$$P(X=i,Y=j) = P(X=i,Y=j \mid X+Y=i+j)P(X+Y=i+j) + P(X=i,Y=j \mid X+Y \neq i+j)P(X+Y \neq i+j)$$

Probability of *i* human requests and *j* bot requests

Probability of *i* human requests and *j* bot requests | we got *i + j requests*

Probability of number of requests in a day was i + j

Web Server Requests

- Let N = # of requests to web server/day
  - Suppose N ~ Poi(λ)
  - Each request comes from a human (probability = p) or from a "bot" (probability = $(1 – p)$), independently
  - X = # requests from humans/day   (X | N) ~ Bin(N, p)
  - Y = # requests from bots/day   (Y | N) ~ Bin(N, 1 - p)

$$P(X=i,Y=j)=P(X=i,Y=j\mid X+Y=i+j)P(X+Y=i+j)$$
$$+P(X=i,Y=j\mid X+Y\neq i+j)P(X+Y\neq i+j)$$

  - Note: $P(X=i,Y=j\mid X+Y\neq i+j)=0$

You got i human requests and j bot requests

You did not get i + j requests

Web Server Requests

- Let N = # of requests to web server/day
  - Suppose N ~ Poi(λ)
  - Each request comes from a human (probability = *p*) or from a "bot" (probability = (1 – *p*)), independently
  - X = # requests from humans/day (X | N) ~ Bin(N, *p*)
  - Y = # requests from bots/day (Y | N) ~ Bin(N, 1 - *p*)

$$P(X = i, Y = j) = P(X = i, Y = j \mid X + Y = i + j)P(X + Y = i + j)$$

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$$P(X=i, Y=j) = P(X=i, Y=j \mid X+Y=i+j)P(X+Y=i+j)$$

$$P(X=i, Y=j \mid X+Y=i+j) = \binom{i+j}{i} p^i (1-p)^j$$

Binomial

$$P(X+Y=i+j) = e^{-\lambda}\frac{\lambda^{i+j}}{(i+j)!}$$

Poisson

$$P(X=i, Y=j) = \binom{i+j}{i} p^i (1-p)^j e^{-\lambda}\frac{\lambda^{i+j}}{(i+j)!}$$

Joint

Web Server Requests

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  - X = # requests from humans/day (X | N) ~ Bin(N, *p*)
  - Y = # requests from bots/day (Y | N) ~ Bin(N, 1 - *p*)

$$P(X=i,Y=j)=\frac{(i+j)!}{i!\,j!}p^i(1-p)^j e^{-\lambda}\frac{\lambda^{i+j}}{(i+j)!}=e^{-\lambda}\frac{(\lambda p)^i}{i!}\cdot\frac{(\lambda(1-p))^j}{j!}$$

Reorder terms

$$=e^{-\lambda p}\frac{(\lambda p)^i}{i!}\cdot e^{-\lambda(1-p)}\frac{(\lambda(1-p))^j}{j!}=P(X=i)P(Y=j)$$

  - Where X ~ Poi(λp) and Y ~ Poi(λ(1 – *p*))
  - X and Y are independent!

Independent Continuous Variables

- Two continuous random variables X and Y are called **independent** if:

  P(X ≤ *a*, Y ≤ *b*) = P(X ≤ *a*) P(Y ≤ *b*) for any *a*, *b*

- Equivalently:

$$F_{X,Y}(a,b) = F_X(a)F_Y(b) \quad \text{for all } a,b$$
$$f_{X,Y}(a,b) = f_X(a)f_Y(b) \quad \text{for all } a,b$$

- More generally, joint density factors separately:

$$f_{X,Y}(x,y) = h(x)g(y) \quad \text{where } -\infty < x, y < \infty$$

Pop Quiz (just kidding)

- Consider joint density function of X and Y:

$$f_{X,Y}(x,y) = 6e^{-3x}e^{-2y} \quad \text{for } \ 0 < x, y < \infty$$

  - Are X and Y independent? Yes!

Let $h(x) = 3e^{-3x}$ and $g(y) = 2e^{-2y}$, so $f_{X,Y}(x,y) = h(x)g(y)$

- Consider joint density function of X and Y:

$$f_{X,Y}(x,y) = 4xy \quad \text{for } \ 0 < x, y < 1$$

  - Are X and Y independent? Yes!

Let $h(x) = 2x$ and $g(y) = 2y$, so $f_{X,Y}(x,y) = h(x)g(y)$

  - Now add constraint that: $0 < (x + y) < 1$
  - Are X and Y independent? No!
    - Cannot capture constraint on x + y in factorization!

Dating at Stanford

- Two people set up a meeting for 12pm
  - Each arrives independently at time uniformly distributed between 12pm and 12:30pm
  - X = # min. past 12pm person 1 arrives   X ~ Uni(0, 30)
  - Y = # min. past 12pm person 2 arrives   Y ~ Uni(0, 30)
  - What is P(first to arrive waits > 10 min. for other)?

$P(X+10<Y)+P(Y+10<X)=2P(X+10<Y)$ by symmetry

$$2P(X+10<Y)=2\iint_{x+10<y} f(x,y)\,dx\,dy = 2\iint_{x+10<y} f_X(x)f_Y(y)\,dx\,dy$$

$$=2\int_{y=10}^{30}\int_{x=0}^{y-10}\left(\frac{1}{30}\right)^2 dx\,dy = \frac{2}{30^2}\int_{y=10}^{30}\left(\int_{x=0}^{y-10} dx\right)dy = \frac{2}{30^2}\int_{y=10}^{30}\left(x\Big|_0^{y-10}\right)dy = \frac{2}{30^2}\int_{y=10}^{30}(y-10)\,dy$$

$$=\frac{2}{30^2}\left(\frac{y^2}{2}-10y\right)\Bigg|_{10}^{30} = \frac{2}{30^2}\left[\left(\frac{30^2}{2}-300\right)-\left(\frac{10^2}{2}-100\right)\right]=\frac{4}{9}$$

Independence of Multiple Variables

- *n* random variables X_1, X_2, …, X_n are called **<u>independent</u>** if:

$$P(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n) = \prod_{i=1}^{n} P(X_i = x_i) \quad \text{for all subsets of } x_1, x_2, \ldots, x_n$$

- Analogously, for continuous random variables:

$$P(X_1 \le a_1, X_2 \le a_2, \ldots, X_n \le a_n) = \prod_{i=1}^{n} P(X_i \le a_i) \quad \text{for all subsets of } a_1, a_2, \ldots, a_n$$

Independence is Symmetric

- If random variables X and Y independent, then
  - X independent of Y, and Y independent of X
- Duh!? Duh, indeed...
  - Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed (I.I.D.) continuous random vars
  - Say $X_n > X_i$ for all $i = 1, \ldots, n - 1$ (i.e. $X_n = \max(X_1, \ldots, X_n)$)
    - Call X_n a "record value"
  - Let event A_i indicate X_i is "record value"
    - Is A_{n+1} independent of A_n?
    - Is A_n independent of A_{n+1}?
    - Easier to answer: Yes!
    - By symmetry, $P(A_n) = 1/n$ and $P(A_{n+1}) = 1/(n+1)$
    - $P(A_n A_{n+1}) = (1/n)(1/(n+1)) = P(A_n)P(A_{n+1})$