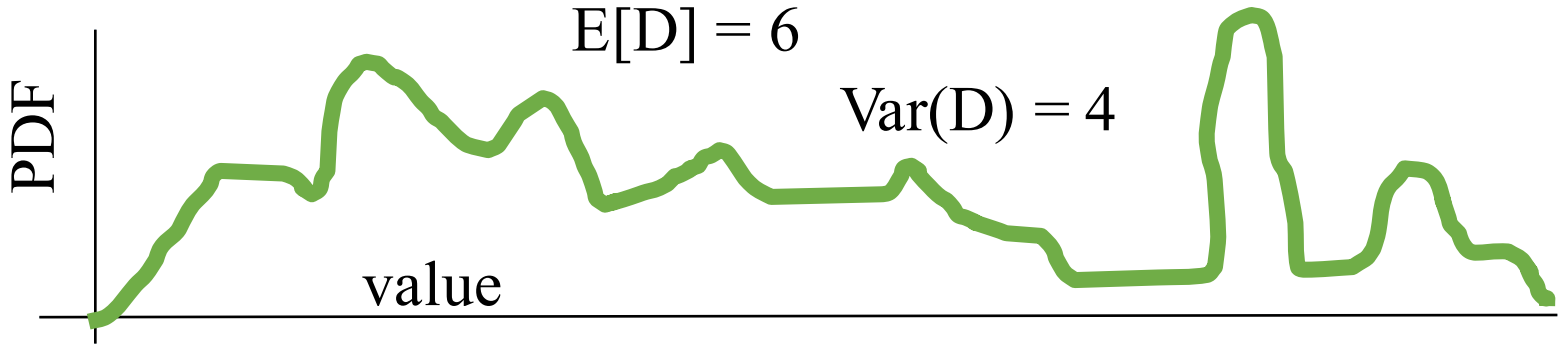
How To Find Central Limit Theorem?

Are you averaging many (>10) I.I.D. random variables?

Are you adding many (>10) I.I.D. random variables?

You are tracking an object on a 1D line and know its location X. Your radar goes down and you don't get to observe it for 20 time steps. Each time step you assume that its change in position is IID with this this pdf:



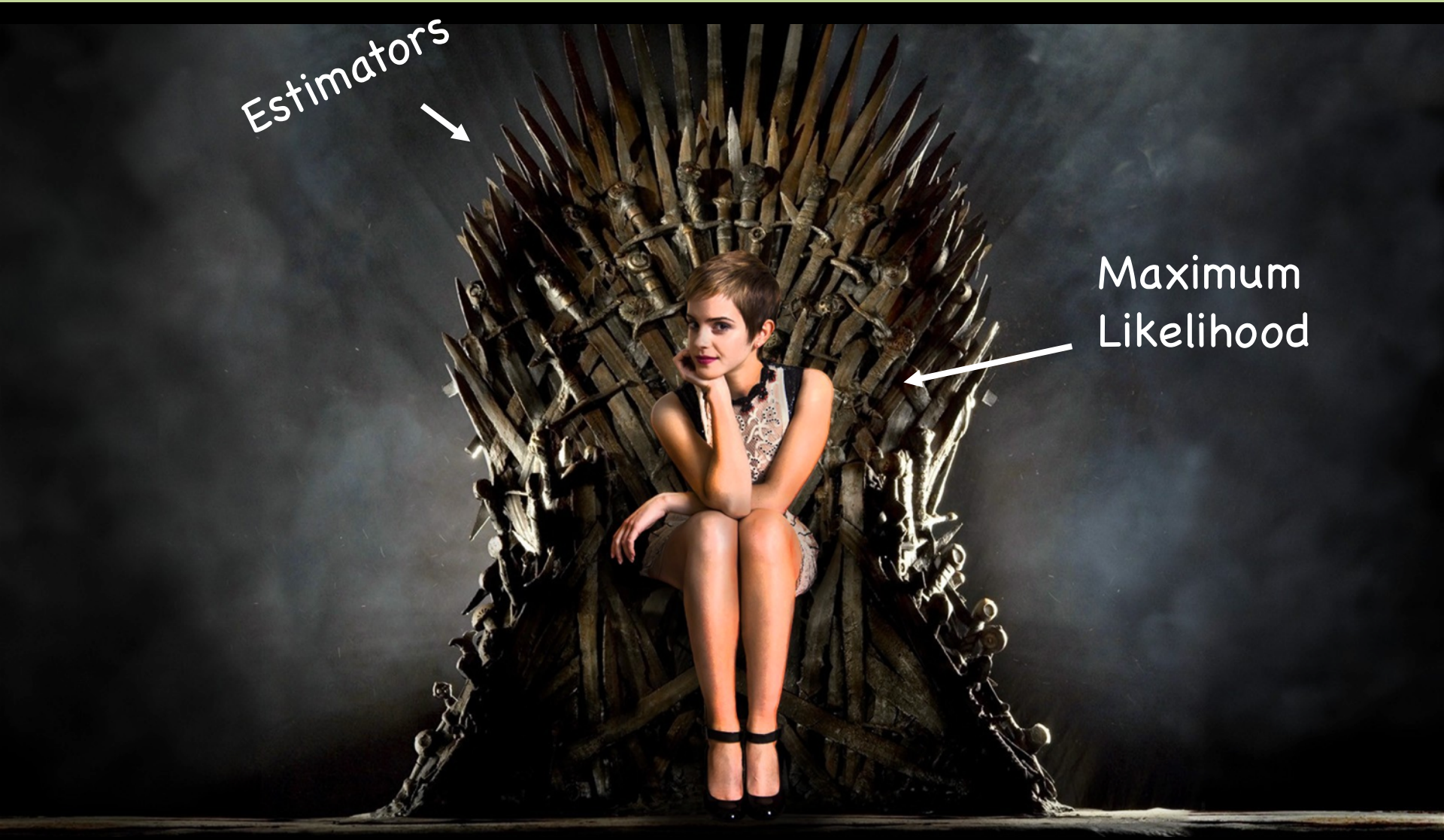
What is the distribution of your belief about the location of the object after 20 time steps?

Maximum A Posteriori

Chris Piech
CS109, Stanford University

Previously in CS109…

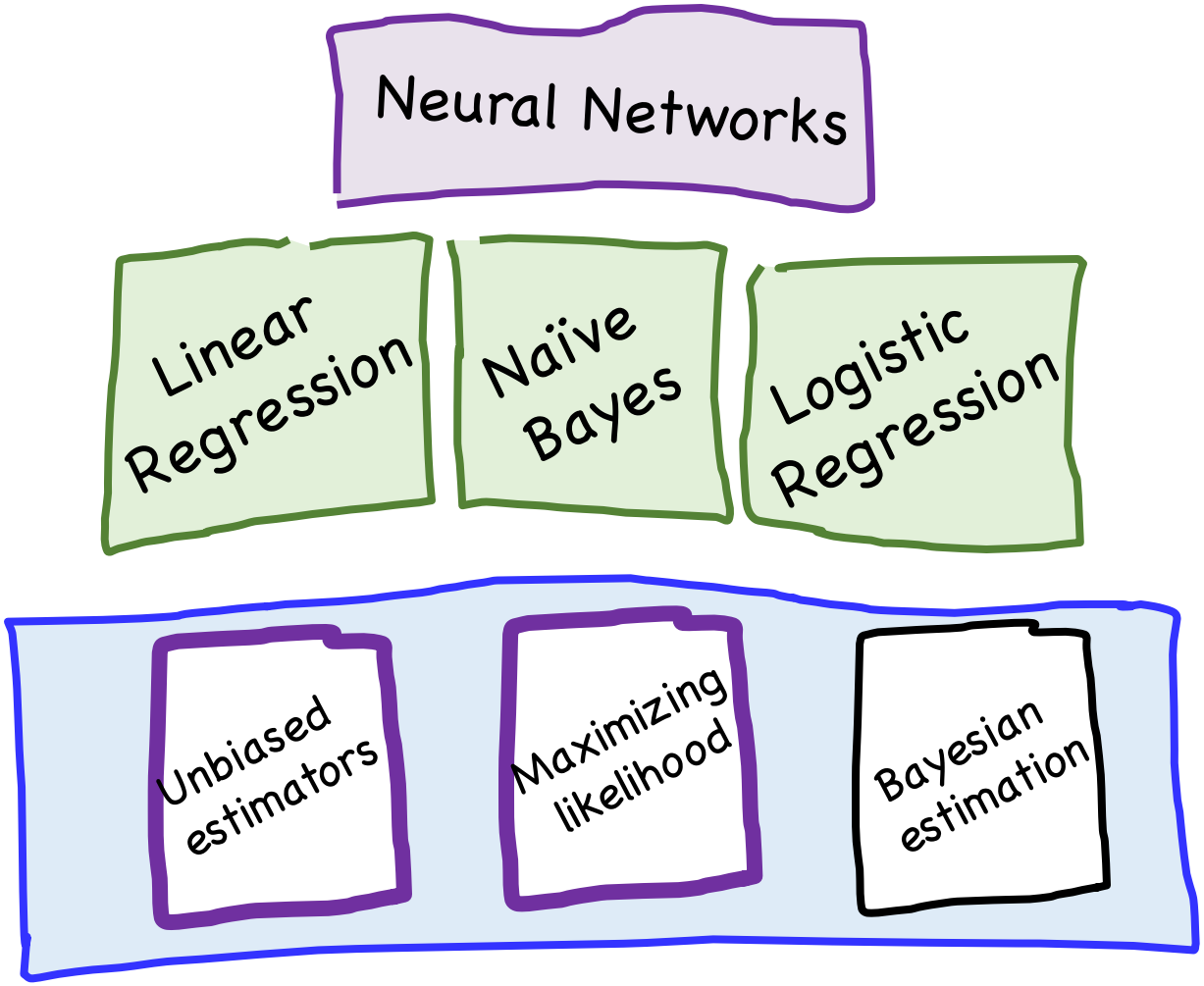
Game of Estimators



Non spoiler: this didn't happen in game of thrones

Side Plot

Our Path


Neural Networks
Linear Regression
Naïve Bayes
Logistic Regression
Unbiased estimators
Maximizing likelihood
Bayesian estimation

Maximum Likelihood of Data

- Consider *n* I.I.D. random variables X_1, X_2, ..., X_n
  - X_i is a sample from density function $f(X_i \mid \theta)$

$$L(\theta) = \prod_{i=1}^{n} f(X_i \mid \theta)$$

$$LL(\theta) = \log L(\theta) = \log \prod_{i=1}^{n} f(X_i \mid \theta) = \sum_{i=1}^{n} \log f(X_i \mid \theta)$$

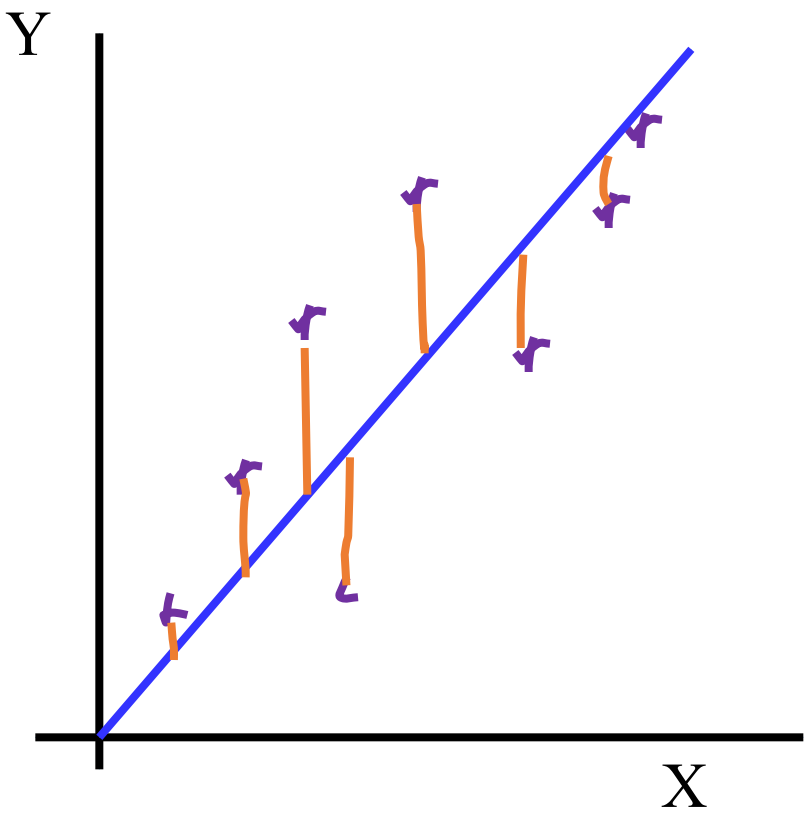
$$\theta_{\mathrm{MLE}} = \underset{\theta}{\operatorname{argmax}} LL(\theta)$$

Ok sure, we can use unbiased estimator for beta ☺

MLE motivation: how could we use an unbiased mean and variance for fitting a beta?

MLE to Linear Regression

How do you fit this line?



Assume:

$$Y = \theta X + Z$$

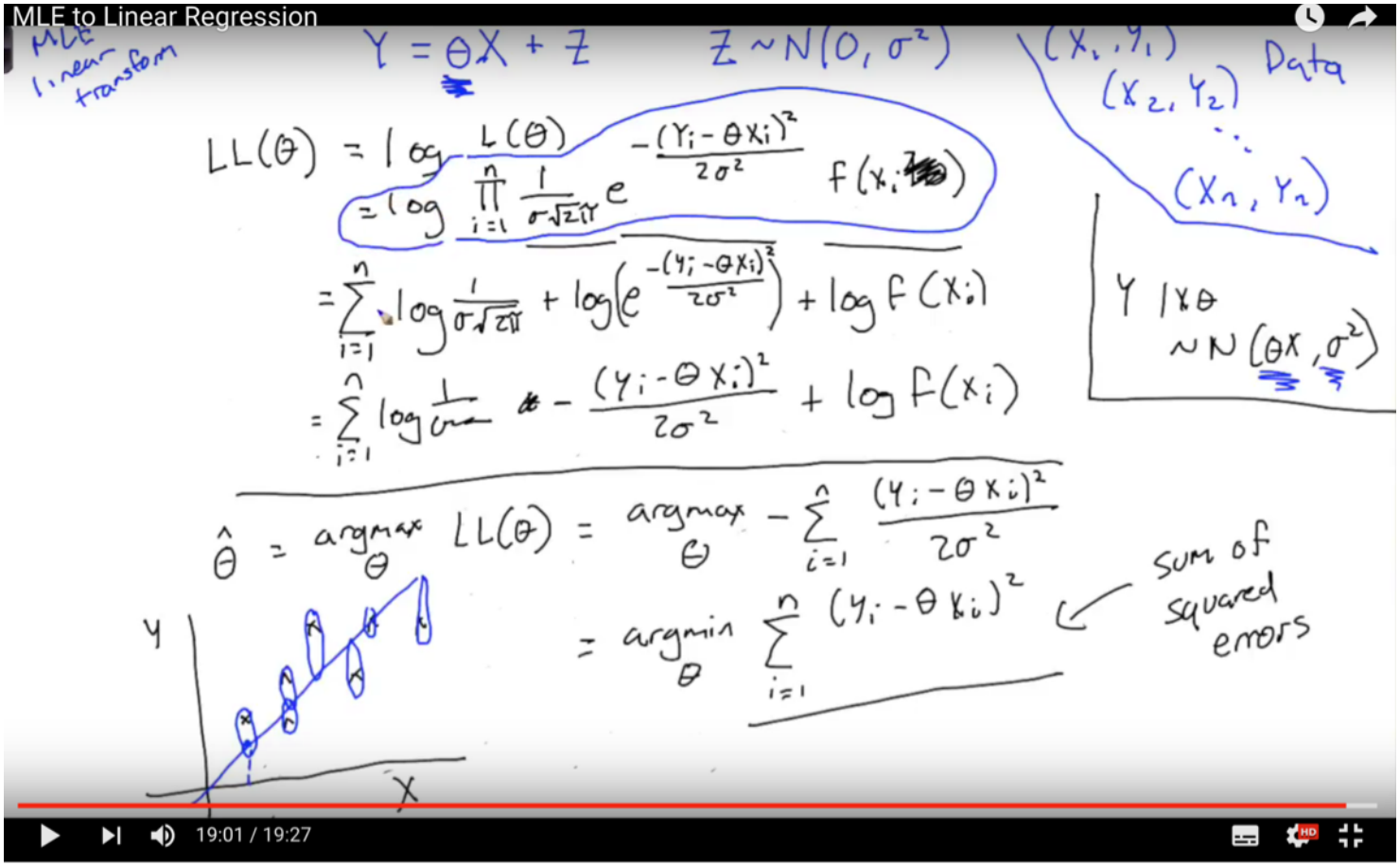
$$Z \sim N(0, \sigma^2)$$

Calculate MLE of θ

$$\hat{\theta} = \underset{\theta}{\operatorname{argmin}} \sum_{i=1}^{m} (Y_i - \theta X_i)^2$$

This is an algorithm called linear regression. Learn more about it later…

Watch it Online

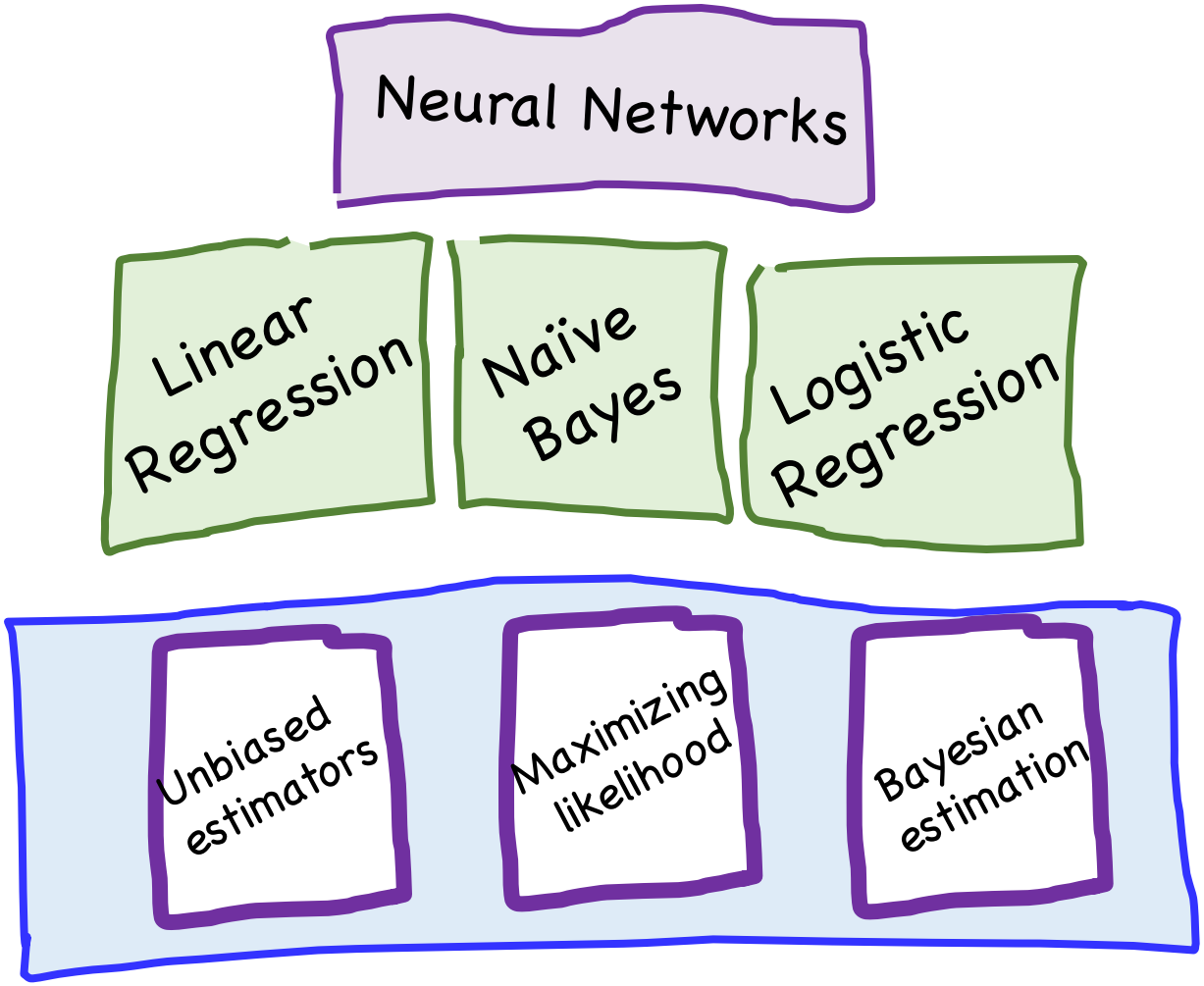


Properties of MLE

- Maximum Likelihood Estimators are generally:
  - Consistent: $\lim_{n \to \infty} P(|\hat{\theta} - \theta| < \varepsilon) = 1$ for $\varepsilon > 0$
  - Potentially biased (though asymptotically less so)
  - Asymptotically optimal
    - Has smallest variance of "good" estimators for large samples
  - Often used in practice where sample size is large relative to parameter space
    - But be careful, there are some very large parameter spaces

Episode 22
The Song of The Last Estimator

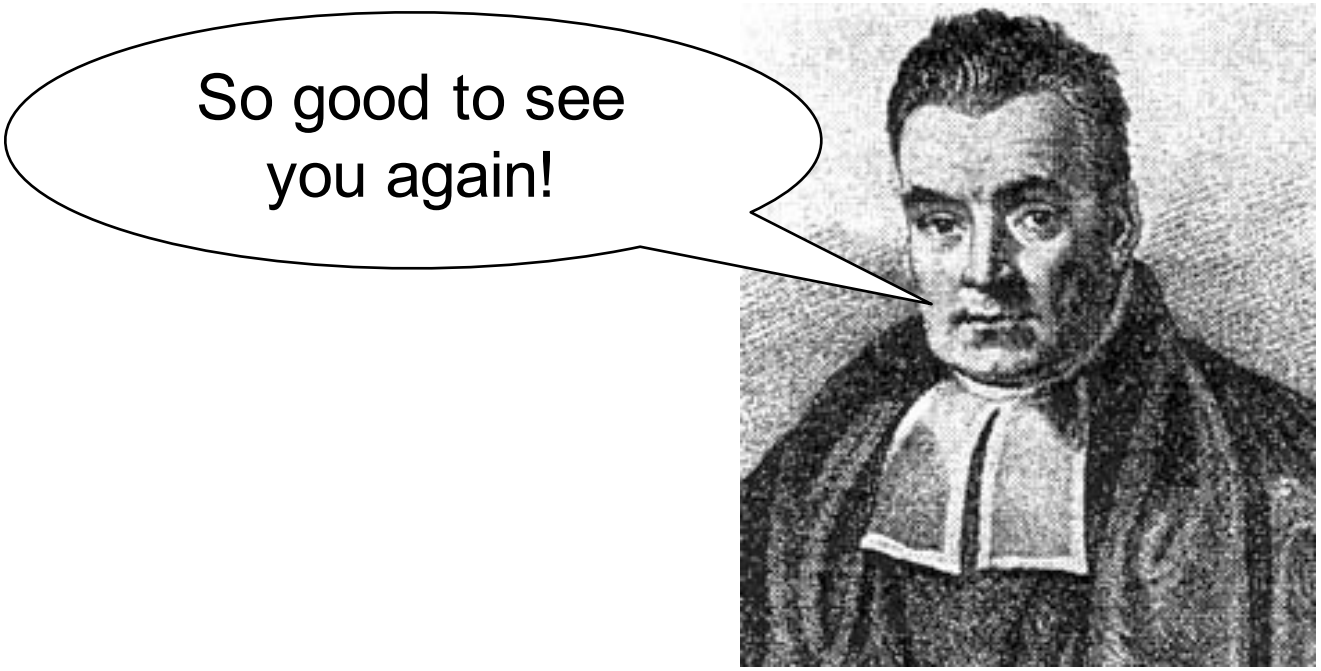
The Song of the Last Estimator



Something rotten
in the world of MLE

Foreshadowing..

Need a Volunteer



Two Envelopes

- I have two envelopes, will allow you to have one
  - One contains $X, the other contains $2X
  - Select an envelope
    - Open it!
  - Now, would you like to switch for other envelope?
  - To help you decide, compute E[$ in other envelope]
    - Let Y = $ in envelope you selected

$$E[\$ \text{ in other envelope}] = \frac{1}{2} \cdot \frac{Y}{2} + \frac{1}{2} \cdot 2Y = \frac{5}{4} Y$$

  - Before opening envelope, think either <u>equally</u> good
  - So, what happened by opening envelope?
    - And does it really make sense to switch?

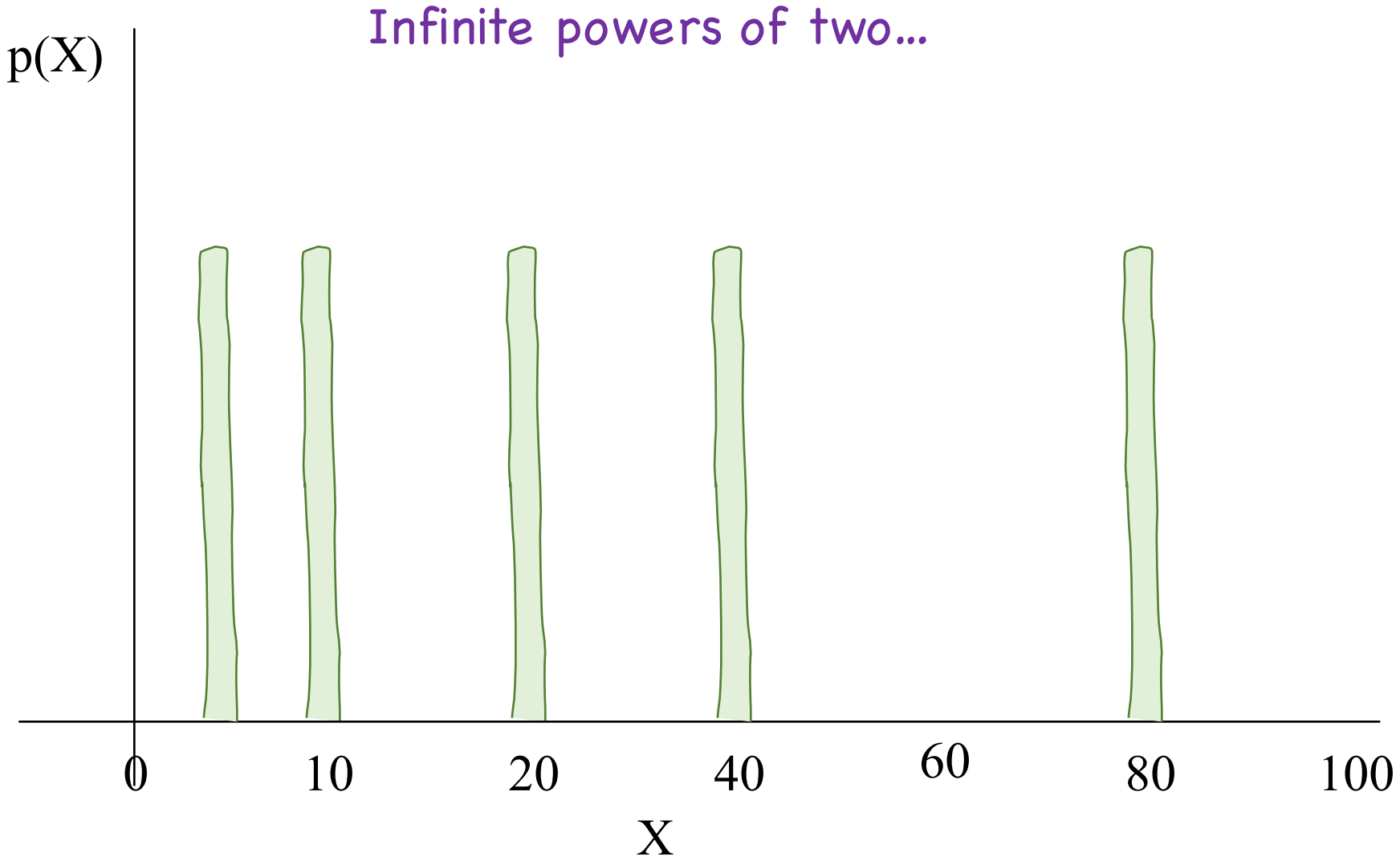
Thinking Deeper About Two Envelopes

- The "two envelopes" problem set-up
  - Two envelopes: one contains $X, other contains $2X
  - You select an envelope and open it
    - Let Y = $ in envelope you selected
    - Let Z = $ in other envelope

$$E[Z \mid Y] = \frac{1}{2} \cdot \frac{Y}{2} + \frac{1}{2} \cdot 2Y = \frac{5}{4} Y$$

  - E[Z | Y] above assumes all values X (where $0 < X < \infty$) are equally likely
    - Note: there are infinitely many values of X
    - So, not true probability distribution over X (doesn't integrate to 1)

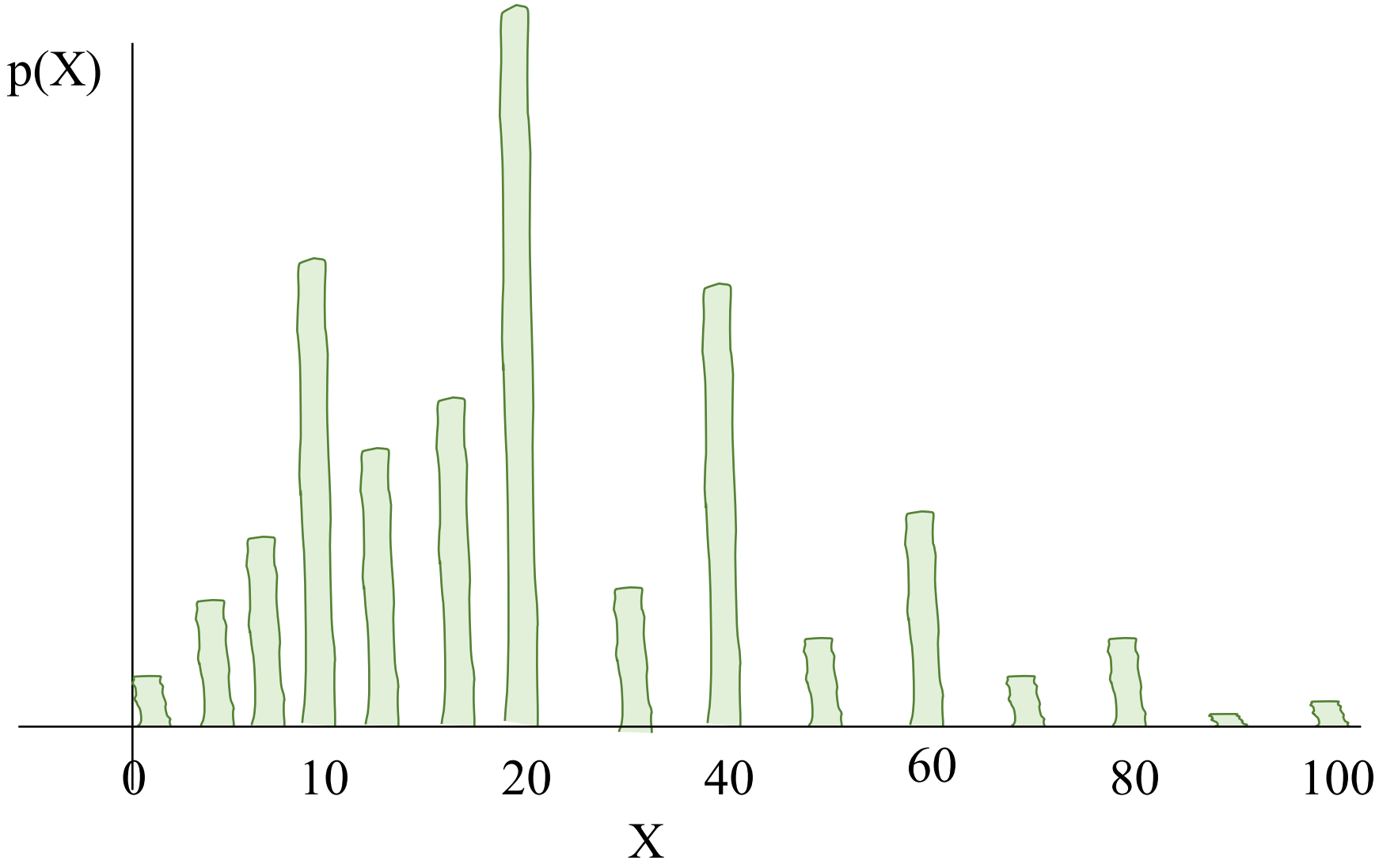
All Values are Equally Likely?



Subjectivity of Probability

- Belief about contents of envelopes
  - Since implied distribution over X is not a true probability distribution, what is our distribution over X?
    - *Frequentist*: play game infinitely many times and see how often different values come up.
    - Problem: I only allow you to play the game *once*
  - Bayesian probability
    - Have prior belief of distribution for X (or anything for that matter)
    - Prior belief is a *subjective* probability
      - By extension, *all* probabilities are subjective
    - Allows us to answer question when we have no/limited data
      - E.g., probability a coin you've never flipped lands on heads
    - As we get more data, prior belief is "swamped" by data

Subjectivity of Probability



The Envelope, Please

- *Bayesian*: have prior distribution over X, P(X)
  - Let Y = $ in envelope you selected
  - Let Z = $ in other envelope
  - Open your envelope to determine Y
  - If Y > E[Z | Y], keep your envelope, otherwise switch
    - No inconsistency!
  - Opening envelope provides data to compute P(X | Y) and thereby compute E[Z | Y]
  - Of course, there's the issue of how you determined your prior distribution over X…
    - Bayesian: Doesn't matter how you determined prior, but you *must* have one (whatever it is)
    - Imagine if envelope you opened contained $20.01

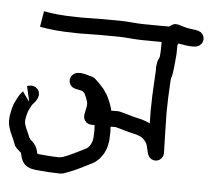
The Dreaded Loaded Die

Envelope Summary:
Probabilities are beliefs
Incorporating prior beliefs is useful

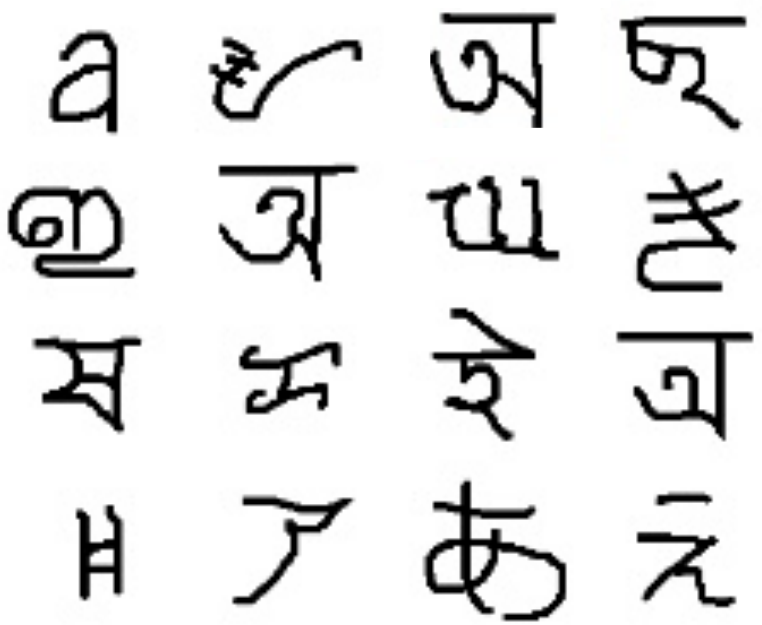
Especially for one shot learning…

One Shot Learning

Single training example:



Test set:



Priors for Parameter Estimation?

Flash Back: Bayes Theorem

- Bayes' Theorem (θ = model parameters, D = data):

"Posterior" "Likelihood" "Prior"

$$P(\theta \mid D) = \frac{P(D \mid \theta)\, P(\theta)}{P(D)}$$

  - Likelihood: you've seen this before (in context of MLE)
    - Probability of data given probability model (parameter θ)
  - Prior: before seeing any data, what is belief about model
    - I.e., what is *distribution* over parameters θ
  - Posterior: after seeing data, what is belief about model
    - After data D observed, have posterior distribution $p(\theta \mid D)$ over parameters θ conditioned on data. Use this to predict new data.

Computing $P(\theta \mid D)$

- Bayes' Theorem (θ = model parameters, D = data):

$$P(\theta \mid D) = \frac{P(D \mid \theta)\, P(\theta)}{P(D)}$$

- We have prior $P(\theta)$ and can compute $P(D \mid \theta)$
- But how do we calculate P(D)?
  - Complicated answer: $P(D) = \int P(D \mid \theta) P(\theta)\, d\theta$
  - Easy answer: It does not depend on θ, so ignore it
    - Just a constant that forces $P(\theta \mid D)$ to integrate to 1

Maximum A Posteriori

Maximum Likelihood Estimation

$$\theta_{\text{MLE}} = \underset{\theta}{\text{argmax}}\ f(X_1, X_2, \ldots, X_n|\theta)$$

$$= \underset{\theta}{\text{argmax}} \sum_i \log f(X_i|\theta)$$

Maximum A Posteriori

$$\theta_{\text{MAP}} = \underset{\theta}{\text{argmax}}\ f(\theta|X_1, X_2, \ldots, X_n)$$

Most important slide of today

Maximum A Posteriori

- Recall Maximum Likelihood Estimator (MLE) of θ

$$\theta_{MLE} = \arg\max_{\theta} \prod_{i=1}^{n} f(X_i \mid \theta)$$

- Maximum A Posteriori (MAP) estimator of θ:

$$\theta_{MAP} = \arg\max_{\theta} f(\theta \mid X_1, X_2, \ldots, X_n) = \arg\max_{\theta} \frac{f(X_1, X_2, \ldots, X_n \mid \theta)\, g(\theta)}{h(X_1, X_2, \ldots, X_n)}$$

$$= \arg\max_{\theta} \frac{\left(\prod_{i=1}^{n} f(X_i \mid \theta) \right) g(\theta)}{h(X_1, X_2, \ldots, X_n)} = \arg\max_{\theta} g(\theta) \prod_{i=1}^{n} f(X_i \mid \theta)$$

where g(θ) is prior distribution of θ.

  - As before, can often be more convenient to use log:

$$\theta_{MAP} = \arg\max_{\theta} \left(\log(g(\theta)) + \sum_{i=1}^{n} \log(f(X_i \mid \theta)) \right)$$

  - MAP estimate is the mode of the posterior distribution

Maximum A Posteriori

Estimated parameter

Log prior

$$\theta_{\mathrm{MAP}} = \operatorname*{argmax}_{\theta} \left(\log(g(\theta)) + \sum_{i=1}^{n} \log(f(X_i|\theta)) \right)$$

Chose the value of theta that maximizes:

Sum of log likelihood

MLE vs MAP

Maximum Likelihood Estimation

$$\theta_{\text{MLE}} = \underset{\theta}{\text{argmax}}\ f(X_1, X_2, \ldots, X_n|\theta)$$

$$= \underset{\theta}{\text{argmax}} \sum_i \log f(X_i|\theta)$$

Maximum A Posteriori

$$\theta_{\text{MAP}} = \underset{\theta}{\text{argmax}}\ f(\theta|X_1, X_2, \ldots, X_n)$$

$$= \underset{\theta}{\text{argmax}} \left(\log g(\theta) + \sum_i \log f(X_i|\theta) \right)$$

Gotta get that intuition

$P(\theta \mid D)$ For Bernoulli

- Prior: $\theta \sim \text{Beta}(a, b)$; D = {*n* heads, *m* tails}

$$f_{\theta|D}(\theta = p \mid D) = \frac{f_{D|\theta}(D \mid \theta = p) f_{\theta}(\theta = p)}{f_D(D)}$$

$$= \frac{\binom{n+m}{n} p^n (1-p)^m \cdot \frac{p^{a-1}(1-p)^{b-1}}{C_1}}{C_2} = \frac{\binom{n+m}{n}}{C_1 C_2} p^n (1-p)^m \cdot p^{a-1}(1-p)^{b-1}$$

$$= C_3 p^{n+a-1}(1-p)^{m+b-1}$$

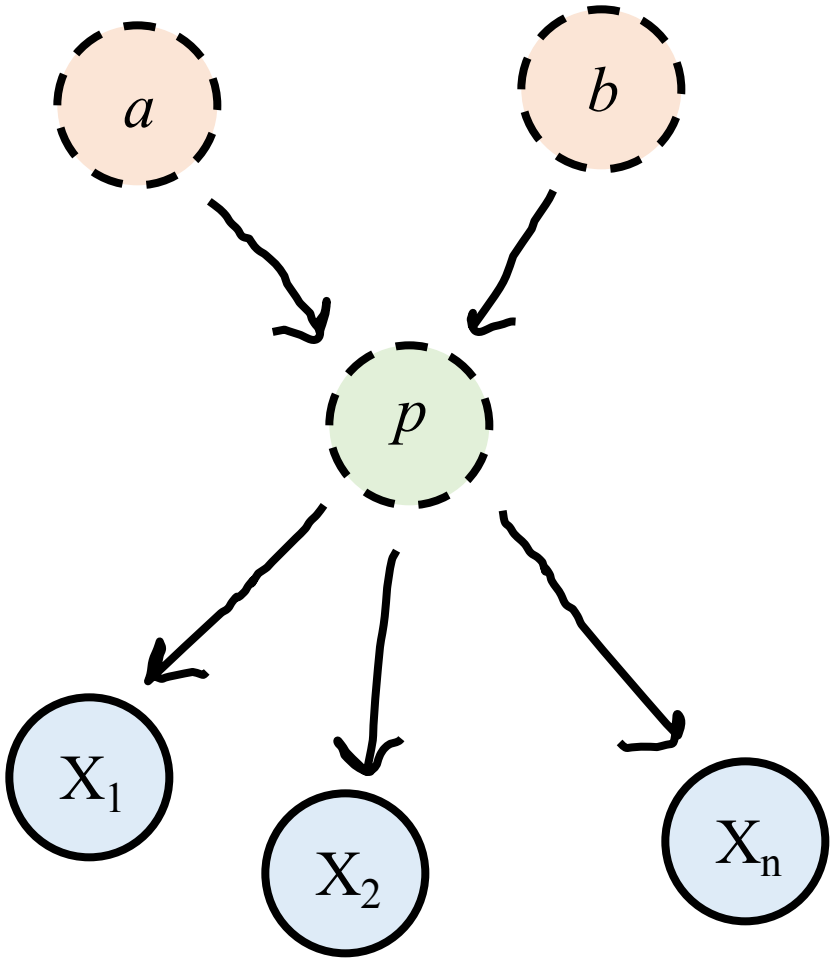
- Estimate *p*, *aka* θ

$$\theta_{\text{MAP}} = \underset{\theta}{\text{argmax}}\ f(\theta|D)$$

$$= \underset{\theta}{\text{argmax}}\ (n+a-1)\log\theta + (m+b-1)\log(1-\theta)$$

$$\theta_{\text{MAP}} = \frac{n+a-1}{n+m+a+b-2}$$

Hyper Parameters



Hyperparameter

a, b are fixed

Prior

$p \sim \text{Beta}(a, b)$

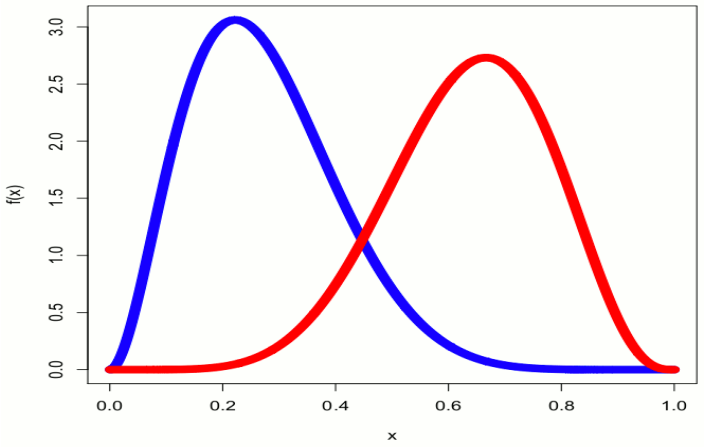
Data distribution

$X_i \sim \text{Bern}(p)$

MAP will estimate the most likely value of p for this model

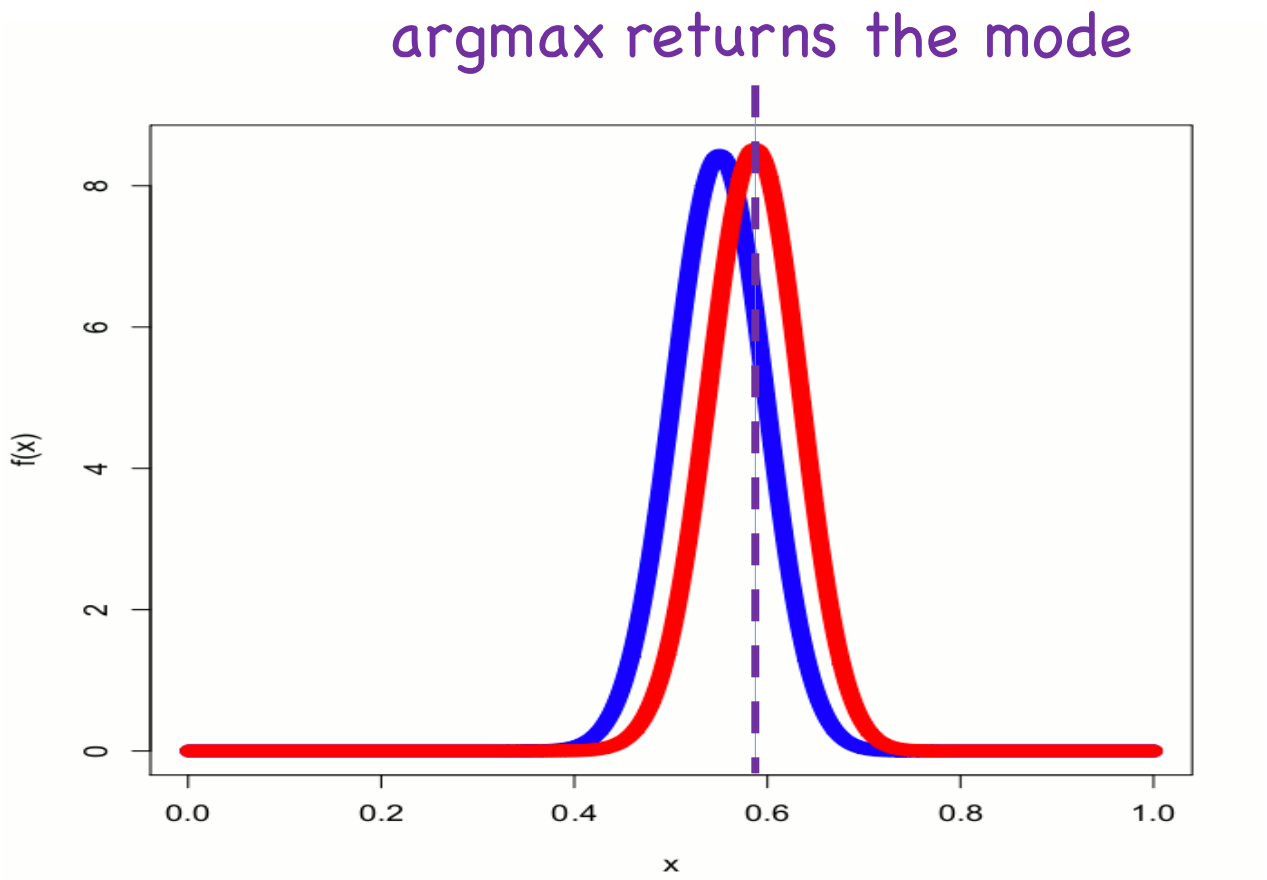
Where'd Ya Get Them P(θ)?

- θ is the probability a coin turns up heads
- Model θ with 2 different priors:
  - $P_1(\theta)$ is Beta(3,8) (blue)
  - $P_2(\theta)$ is Beta(7,4) (red)
- They look pretty different!



- Now flip 100 coins; get 58 heads and 42 tails
  - What do posteriors look like?

It's Like Having Twins

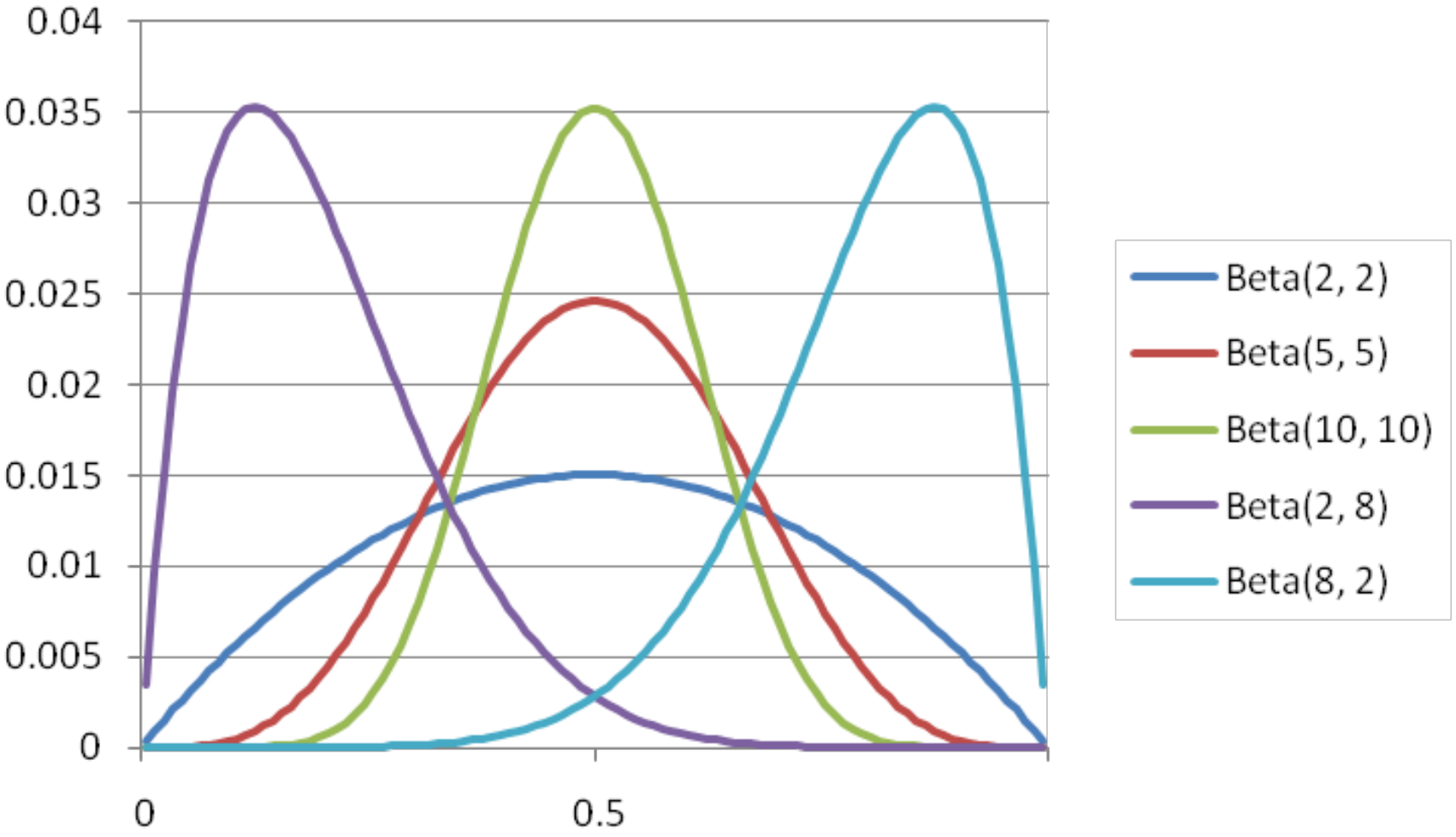


- As long as we collect enough data, posteriors will converge to the true value!

Conjugate Distributions Without Tears

- Just for review…
- Have coin with unknown probability θ of heads
  - Our prior (subjective) belief is that $\theta \sim \text{Beta}(a, b)$
  - Now flip coin $k = n + m$ times, getting n heads, m tails
  - Posterior density: $(\theta \mid n \text{ heads}, m \text{ tails}) \sim \text{Beta}(a+n, b+m)$
    - Beta is conjugate for Bernoulli, Binomial, Geometric, and Negative Binomial
  - a and b are called "hyperparameters"
    - Saw $(a + b - 2)$ imaginary trials, of those $(a - 1)$ are "successes"
  - For a coin you never flipped before, use Beta(x, x) to denote you think coin likely to be fair
    - How strongly you feel coin is fair is a function of x

Mo' Beta

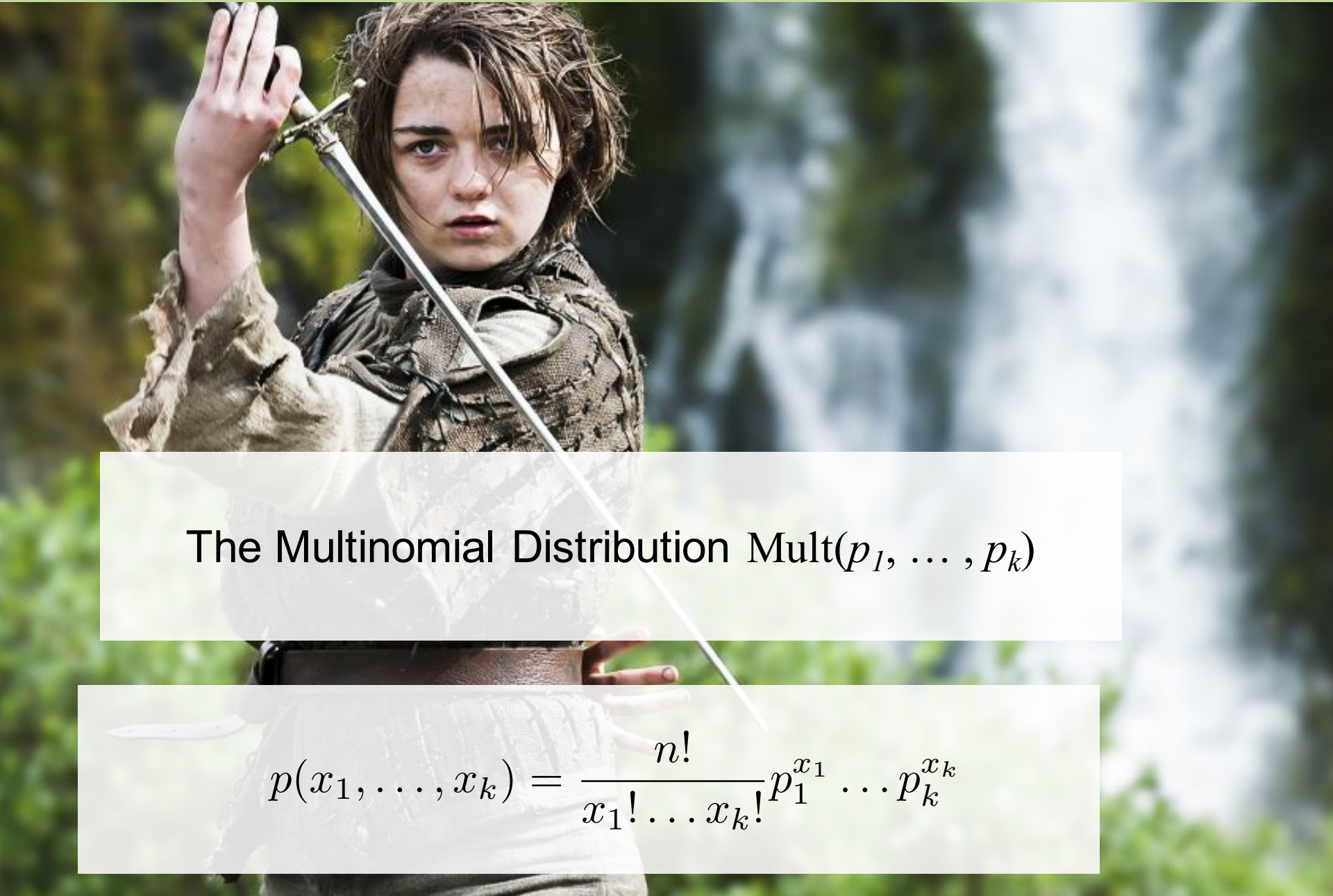

0.04
0.035
0.03
0.025
0.02
0.015
0.01
0.005
0
0
0.5
Beta(2, 2)
Beta(5, 5)
Beta(10, 10)
Beta(2, 8)
Beta(8, 2)

Gonna Need Priors

Parameter	Distribution for Parameter
Bernoulli p	Beta
Binomial p	Beta
Poisson λ	Gamma
Exponential λ	Gamma
Multinomial p_i	Dirichlet
Normal μ	Normal
Normal σ^2	Inverse Gamma

Don't need to know Inverse Gamma. But it will know you...

Reviving an Old Story Line



The Multinomial Distribution $\text{Mult}(p_1, \ldots, p_k)$

$$p(x_1, \ldots, x_k) = \frac{n!}{x_1! \ldots x_k!} p_1^{x_1} \ldots p_k^{x_k}$$

Multinomial is Multiple Times the Fun

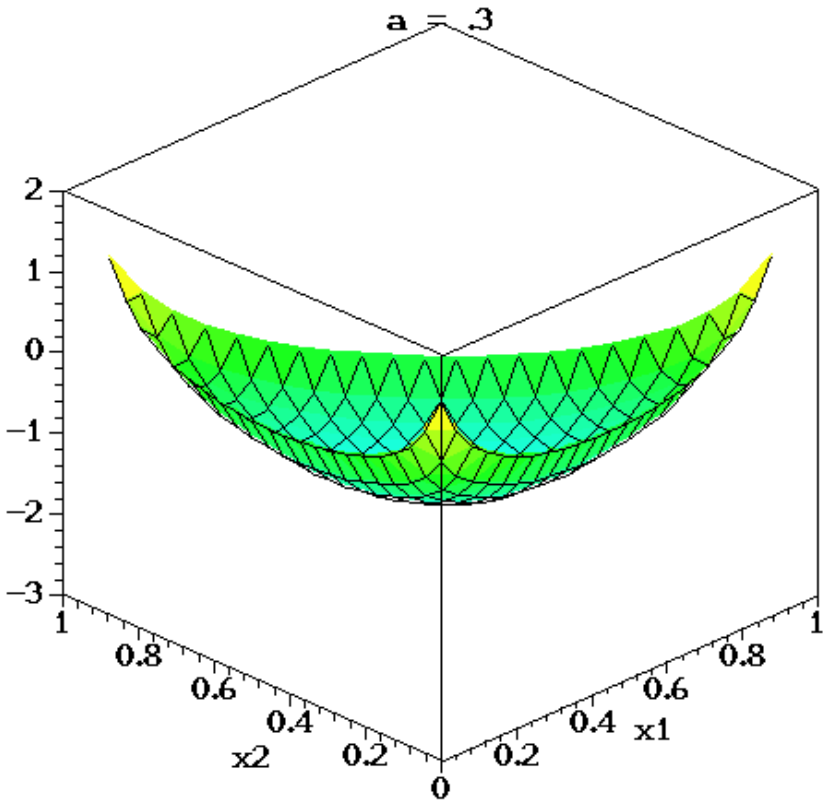
- Dirichlet(a_1, a_2, …, a_m) distribution
  - Conjugate for Multinomial
    - Dirichlet generalizes Beta in same way Multinomial generalizes Bernoulli

$$f(X_1 = x_1, X_2 = x_2, \ldots, X_m = x_m) = K \prod_{i=1}^{m} x_i^{a_i - 1}$$

  - Intuitive understanding of hyperparameters:
    - Saw $\sum_{i=1}^{m} a_i - m$ imaginary trials, with (a_i – 1) of outcome i
  - Updating to get the posterior distribution
    - After observing $n_1 + n_2 + \ldots + n_m$, new trials with n_i of outcome i…
    - … posterior distribution is Dirichlet($a_1 + n_1$, $a_2 + n_2$, …, $a_m + n_m$)

Best Short Film in the Dirichlet Category

- And now a cool animation of Dirichlet(*a*, *a*, *a*)
  - This is actually *log* density (but you get the idea…)



Thanks Wikipedia!

Example: Estimating Die Parameters

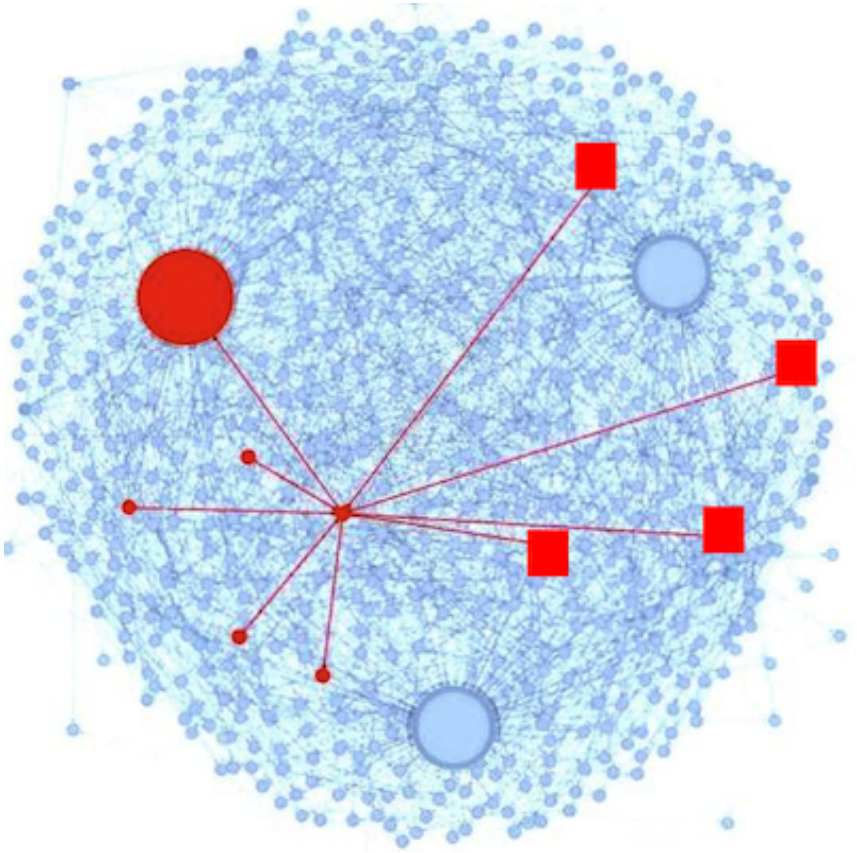
Your Happy Laplace

- Recall example of 6-sides die rolls:
  - X ~ Multinomial(p_1, p_2, p_3, p_4, p_5, p_6)
  - Roll n = 12 times
  - Result: 3 ones, 2 twos, 0 threes, 3 fours, 1 fives, 3 sixes
    - MLE: p_1=3/12, p_2=2/12, p_3=0/12, p_4=3/12, p_5=1/12, p_6=3/12
  - Dirichlet prior allows us to pretend we saw each outcome k times before. MAP estimate: $p_i = \frac{X_i + k}{n + mk}$
    - Laplace's "law of succession": idea above with k = 1
    - Laplace estimate: $p_i = \frac{X_i + 1}{n + m}$
    - Laplace: p_1=4/18, p_2=3/18, p_3=1/18, p_4=4/18, p_5=2/18, p_6=4/18
    - No longer have 0 probability of rolling a three!

Good Times with Gamma

- Gamma(k, θ) distribution
  - Conjugate for Poisson
    - Also conjugate for Exponential, but we won't delve into that
  - Intuitive understanding of hyperparameters:
    - Saw k total imaginary events during θ prior time periods
  - Updating to get the posterior distribution
    - After observing *n* events during next *t* time periods…
    - … posterior distribution is Gamma(k + *n*, θ + *t*)
    - Example: Gamma(10, 5)
    - Saw 10 events in 5 time periods. Like observing at rate = 2
    - Now see 11 events in next 2 time periods → Gamma(21, 7)
    - Equivalent to updated rate = 3

Is Peer Grading Accurate Enough?

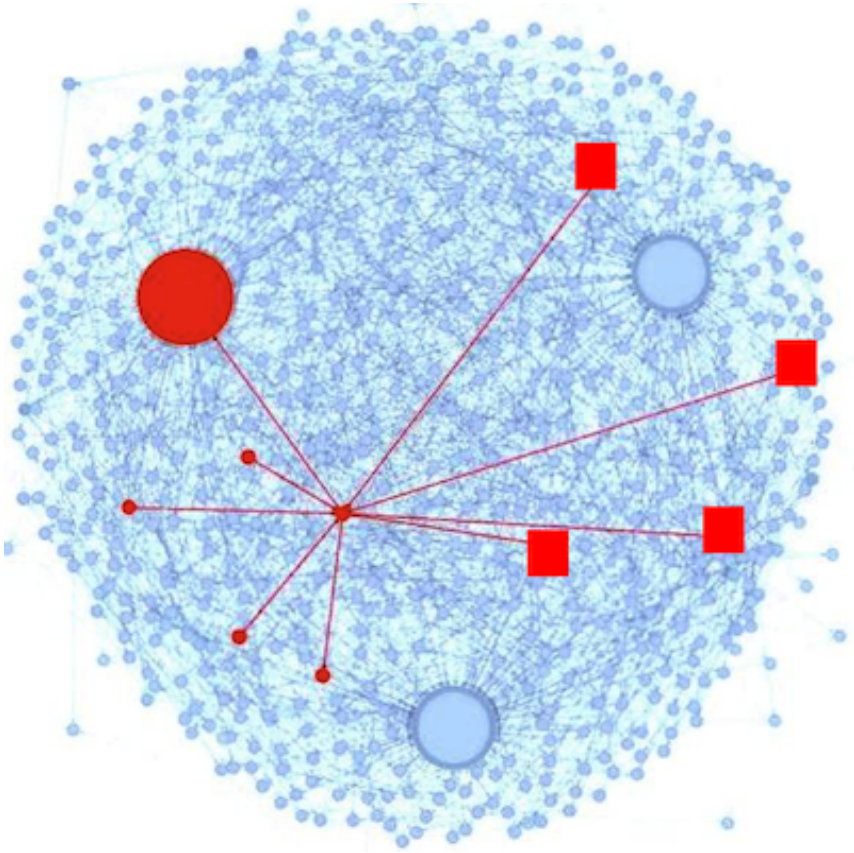


Peer Grading on Coursera HCI.

31,067 peer grades for 3,607 students.

Tuned Models of Peer Assessment. C Piech, J Huang, A Ng, D Koller

Is Peer Grading Accurate Enough?



= hyperparameter

1. Defined random variables for:

- True grade (s_i) for assignment i
- Observed (z_i^j) score for assign i
- Bias (b_j) for each grader j
- Variance (r_j) for each grader j

2. Designed a probabilistic model that defined the distributions for all random variables

$$z_i^j \sim \mathcal{N}(\mu = s_i + b_j, \sigma = \sqrt{r_j})$$

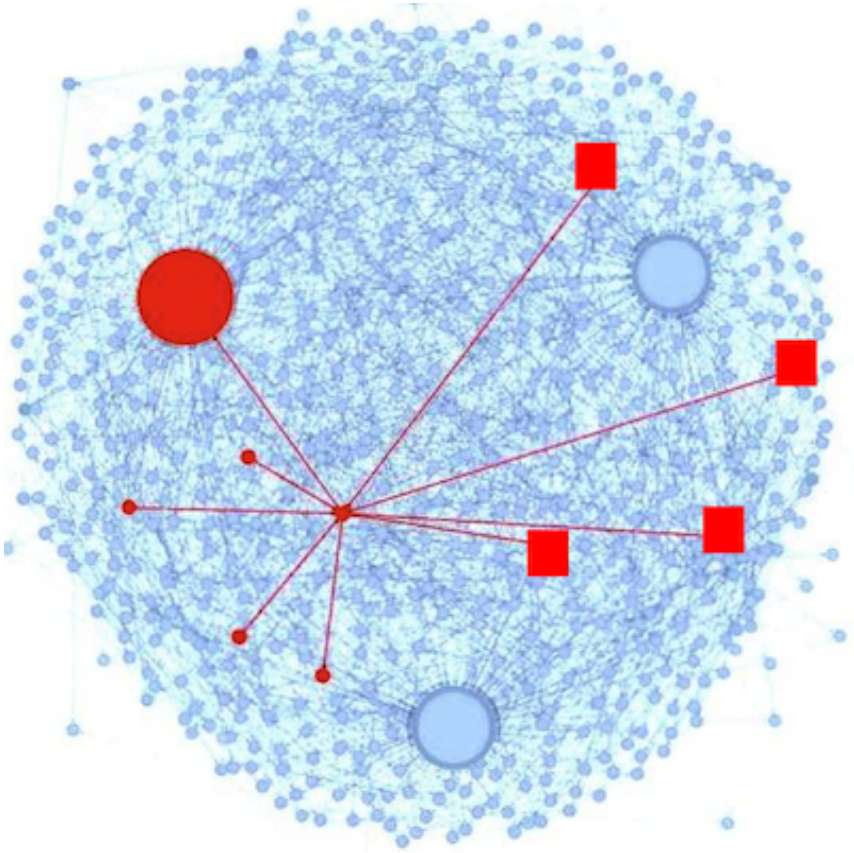
$$s_i \sim N(\mu_0, \sigma_0)$$

$$b_i \sim N(0, \eta_0)$$

$$r_i \sim \text{InvGamma}(\alpha_0, \theta_0)$$

Tuned Models of Peer Assessment. C Piech, J Huang, A Ng, D Koller

Is Peer Grading Accurate Enough?



1. Defined random variables for:
- True grade (s_i) for assignment i
- Observed (z_i^j) score for assign i
- Bias (b_j) for each grader j
- Variance (r_j) for each grader j

2. Designed a probabilistic model that defined the distributions for all random variables

3. Found variable assignments using MAP estimation given the observed data

Inference or Machine Learning

Tuned Models of Peer Assessment. C Piech, J Huang, A Ng, D Koller

The last estimator has risen…

Next time: Machine Learning algorithms

It's Normal to Be Normal

- Normal(μ_0, σ_0^2) distribution
  - Conjugate for Normal (with unknown μ, known σ^2)
  - Intuitive understanding of hyperparameters:
    - A priori, believe true μ distributed ~ N(μ_0, σ_0^2)
  - Updating to get the posterior distribution
    - After observing *n* data points...
    - ... posterior distribution for μ is:

$$N\left(\left(\frac{\mu_0}{\sigma_0^2} + \frac{\sum_{i=1}^{n} x_i}{\sigma^2}\right) \Big/ \left(\frac{1}{\sigma_0^2} + \frac{n}{\sigma^2}\right), \quad \left(\frac{1}{\sigma_0^2} + \frac{n}{\sigma^2}\right)^{-1}\right)$$