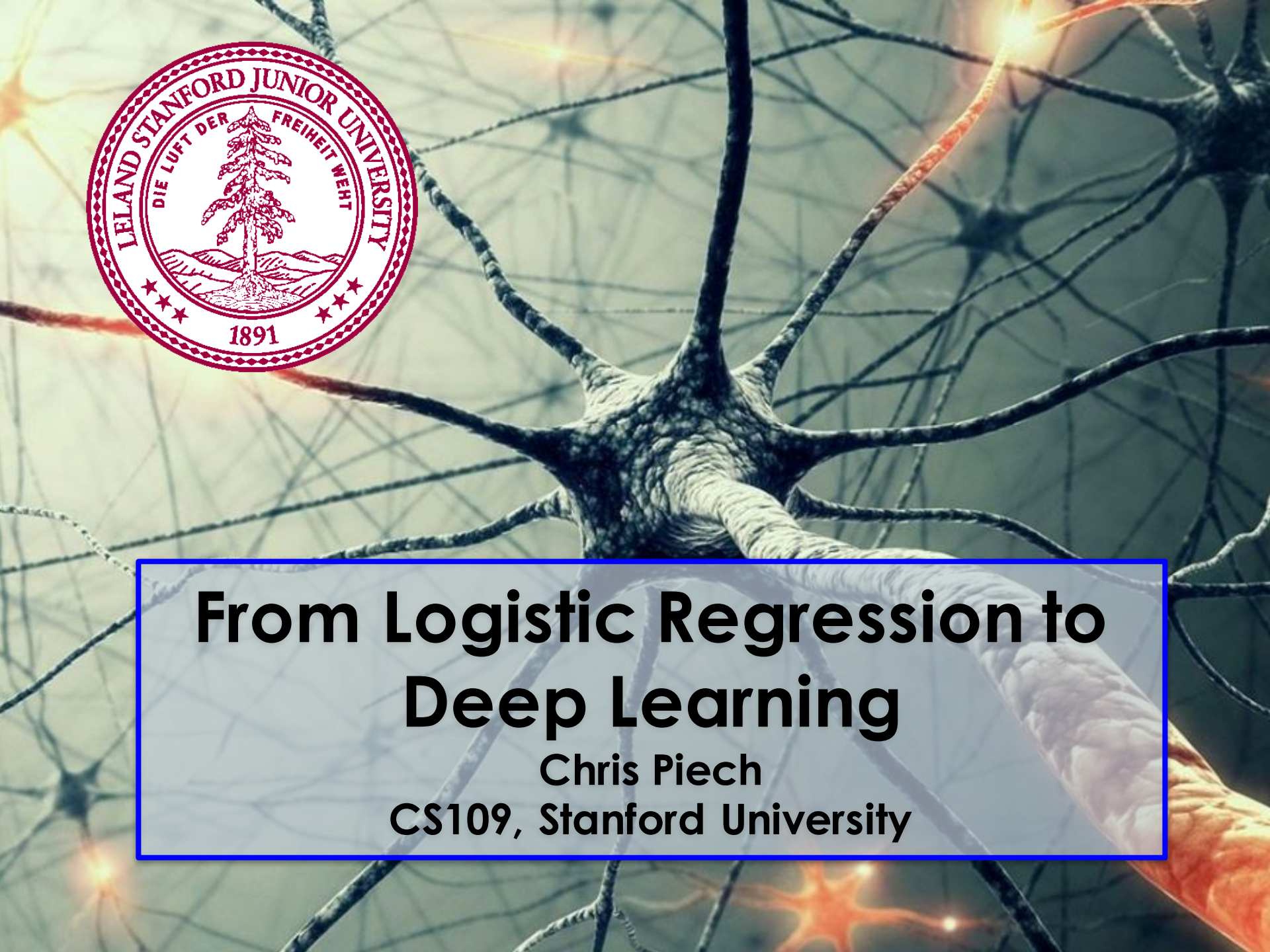


From Logistic Regression to Deep Learning

Chris Piech
CS109, Stanford University

Important Mathematical Journey

Notation

$$\sigma(z) = \frac{1}{1 + e^{-z}}$$

Sigmoid function

$$\theta^T \mathbf{x} = \sum_{i=1}^{n} \theta_i x_i$$

$$= \theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_n x_n$$

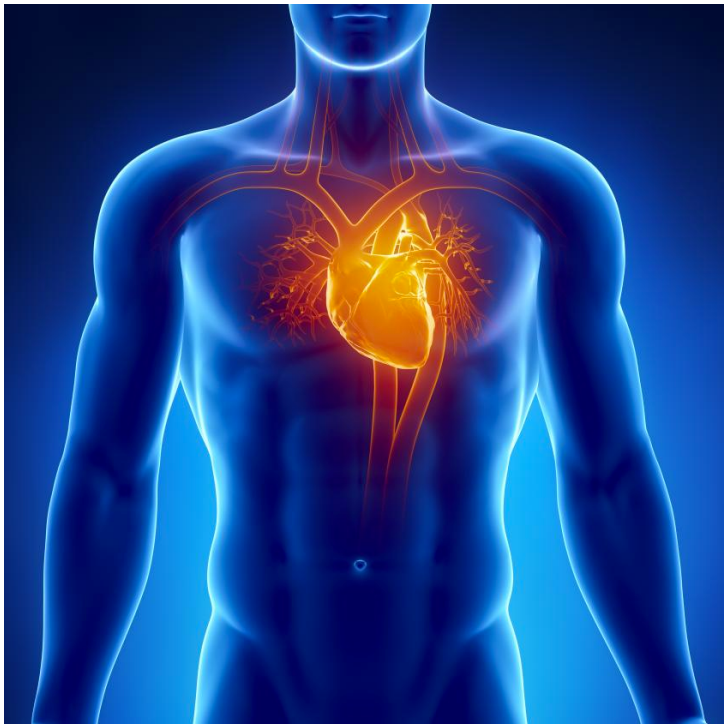
Weighted sum
(aka dot product)

$$\sigma(\theta^T \mathbf{x}) = \frac{1}{1 + e^{-\theta^T \mathbf{x}}}$$

Sigmoid function of
weighted sum

Classification Task

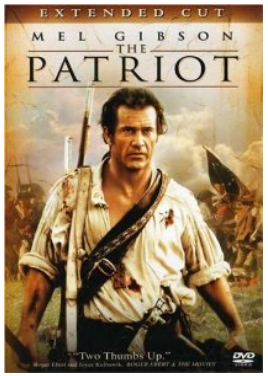
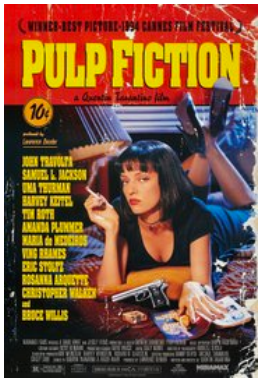
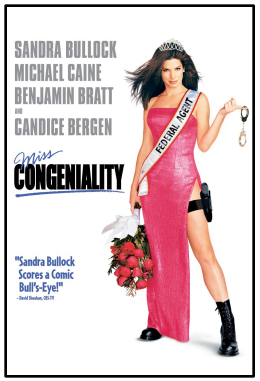
Heart



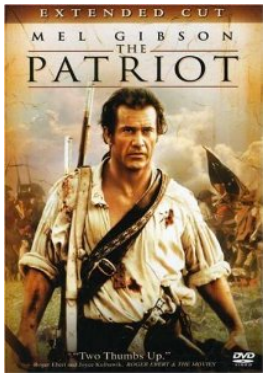
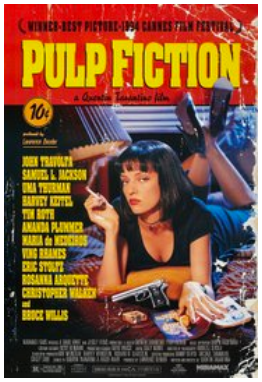
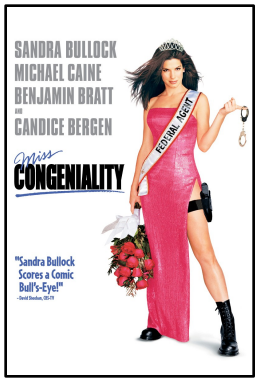
Ancestry

Netflix

Target Movie "Like" Classification

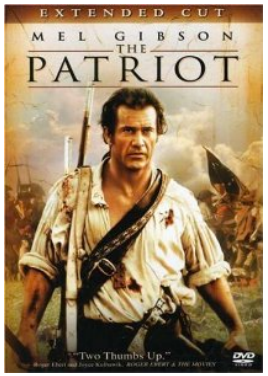
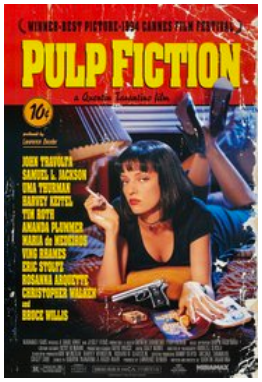
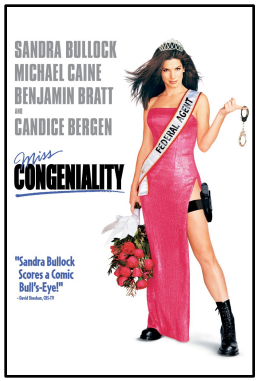
	Movie 1	Movie 2	…	Movie m	Output
					
User 1	1	0		1	1
User 2	1	1		0	0
		⋮			⋮
User n	0	0		1	1

Single Instance

	Movie 1	Movie 2	…	Movie m	Output
					
User 1	1	0		1	1
User 2	1	1		0	0
	⋮	⋮			⋮
User n	0	0		1	1

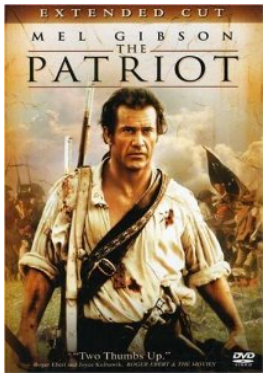
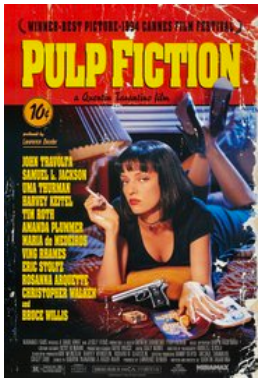
$$(\mathbf{x}^{(i)}, y^{(i)}) \text{ such that } 1 \leq i \leq n$$

Feature Vector

	Movie 1	Movie 2	…	Movie m	Output
					
User 1	1	0		1	1
User 2	1	1		0	0
	⋮	⋮			⋮
User n	0	0		1	1

$(\mathbf{x}^{(i)}, y^{(i)})$ such that $1 \leq i \leq n$

Output Value

	Movie 1	Movie 2	…	Movie m	Output
					
User 1	1	0		1	1
User 2	1	1		0	0
	⋮	⋮			⋮
User n	0	0		1	1

$$(\mathbf{x}^{(i)}, y^{(i)}) \text{ such that } 1 \leq i \leq n$$

Training Data

Assume IID data:

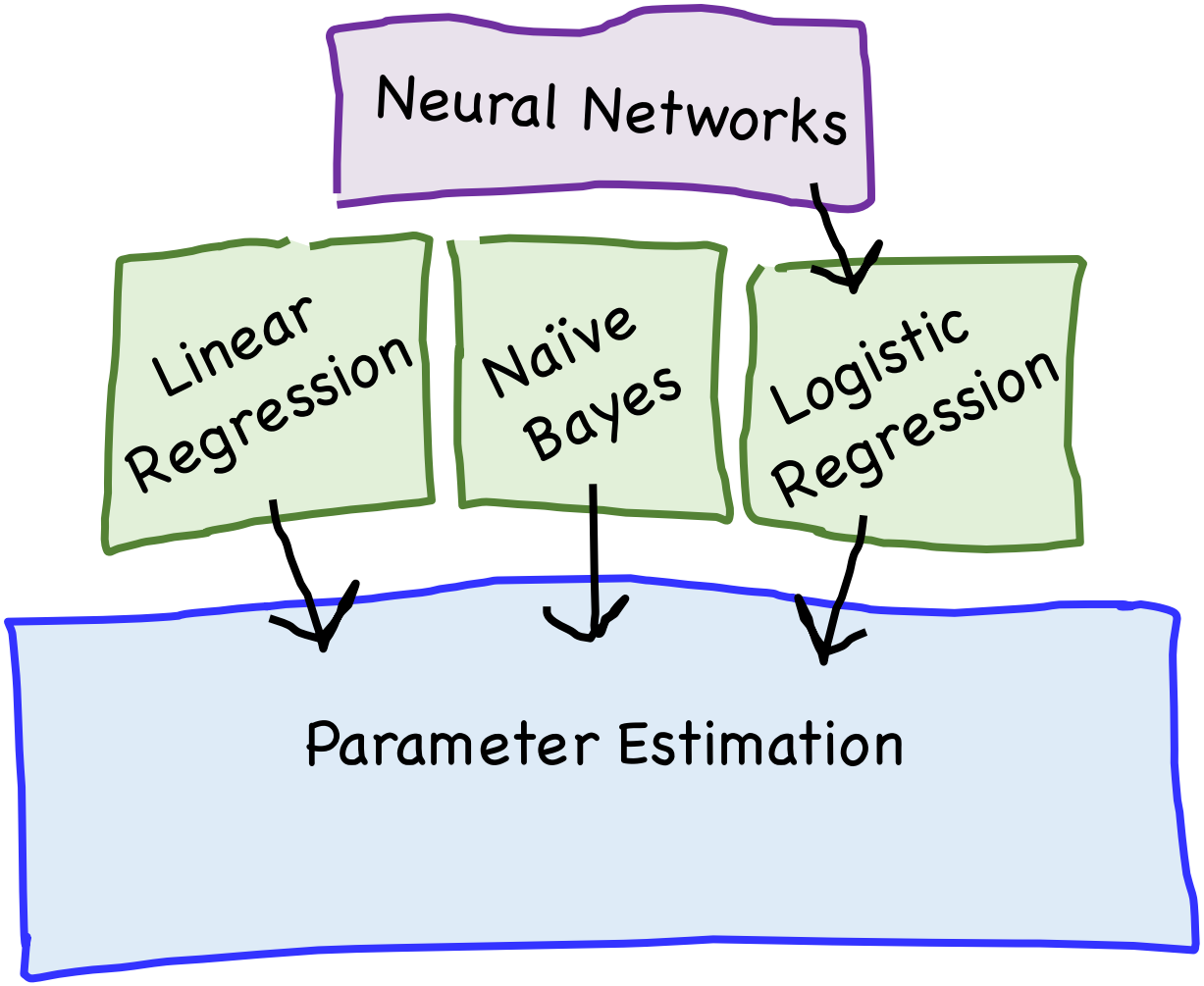
N training datapoints

$$(\mathbf{x}^{(1)}, y^{(1)}), (\mathbf{x}^{(2)}, y^{(2)}), \ldots (\mathbf{x}^{(n)}, y^{(n)})$$

$$m = |\mathbf{x}^{(i)}|$$

Each datapoint has m features and a single output

Knowledge Dependency



Logistic Regression:

Define a function that predicts P(Y = 1) directly

Logistic Regression is like the Harry Pottery Sorting Hat

$$[1, 1, 0, 0]$$

Logistic Regression is like the Harry Pottery Sorting Hat

$$[1, 1, 0, 0]$$

Logistic Regression is like the Harry Pottery Sorting Hat

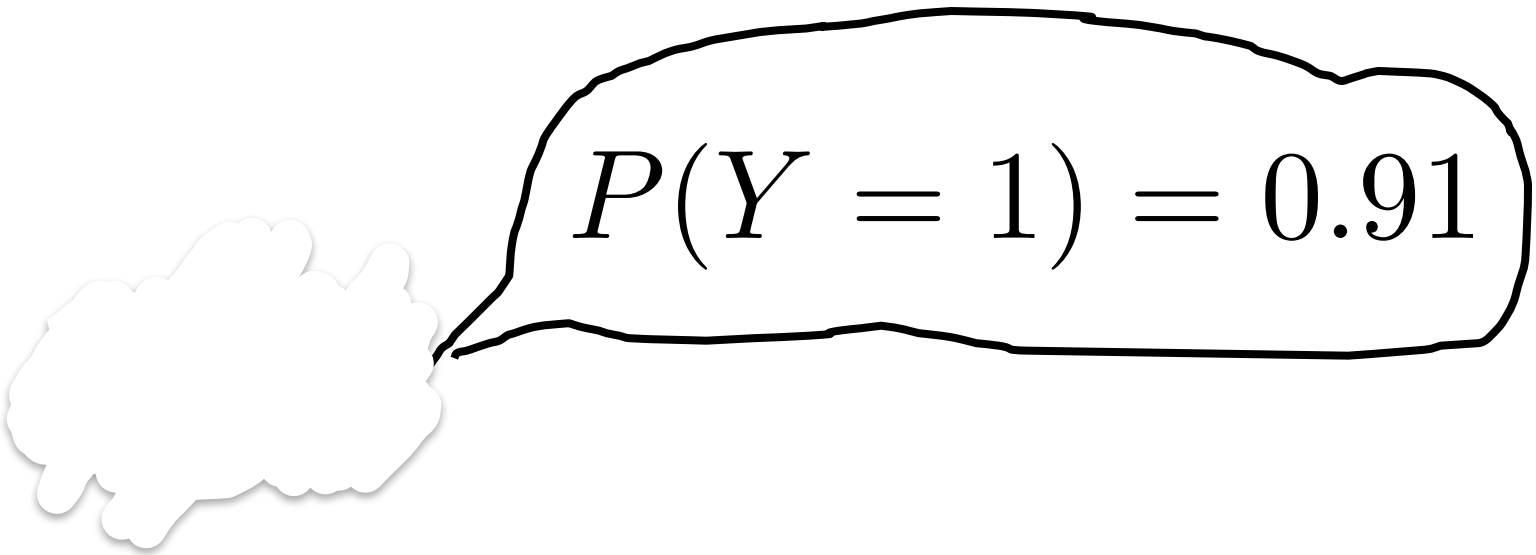
$$X$$

Logistic Regression is like the Harry Pottery Sorting Hat

$$P(Y = 1) = 0.91$$

$$\mathbf{X}$$

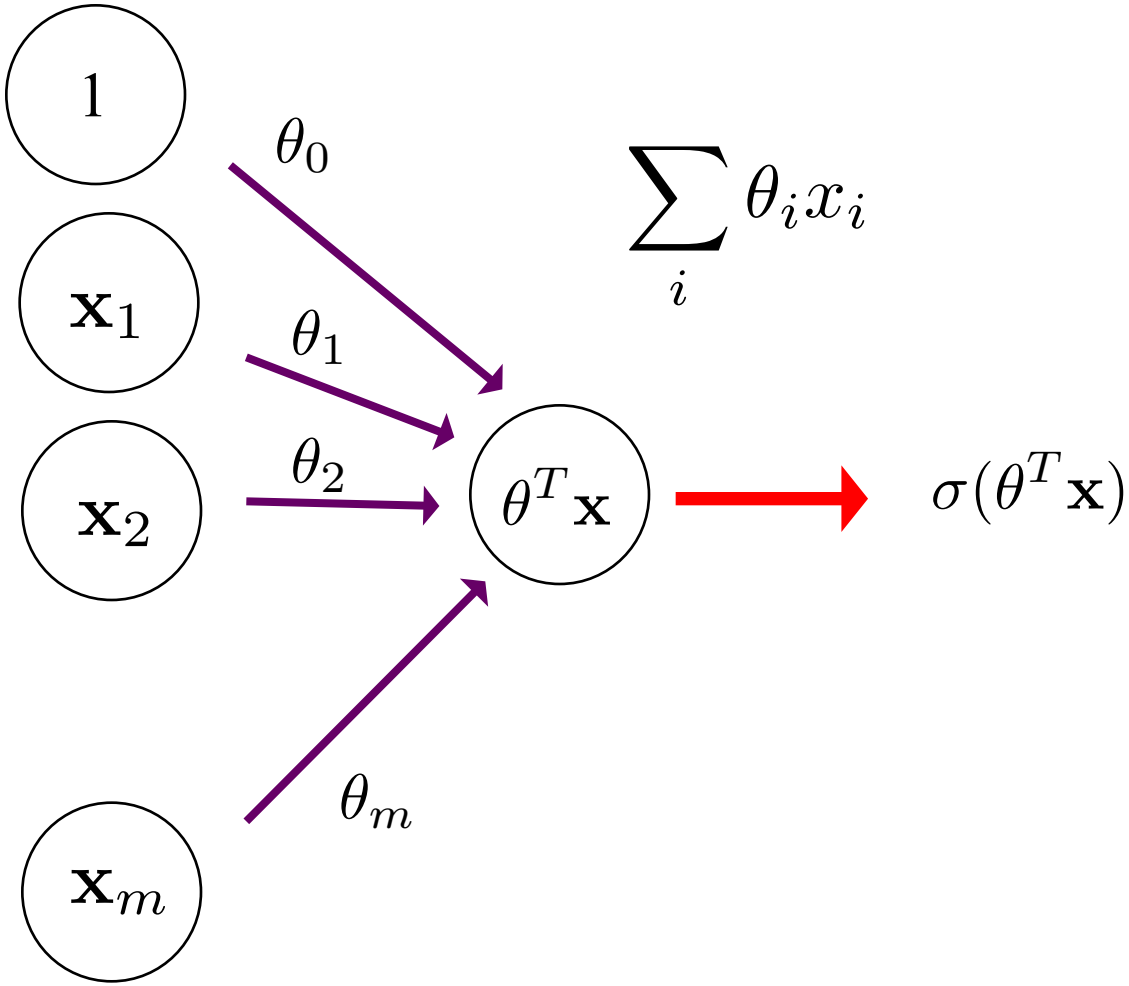
Logistic Regression is like the Harry Pottery Sorting Hat



$\mathbf{X}$

Logistic Regression is like the Harry Pottery Sorting Hat

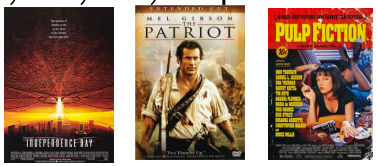
Logistic Regression

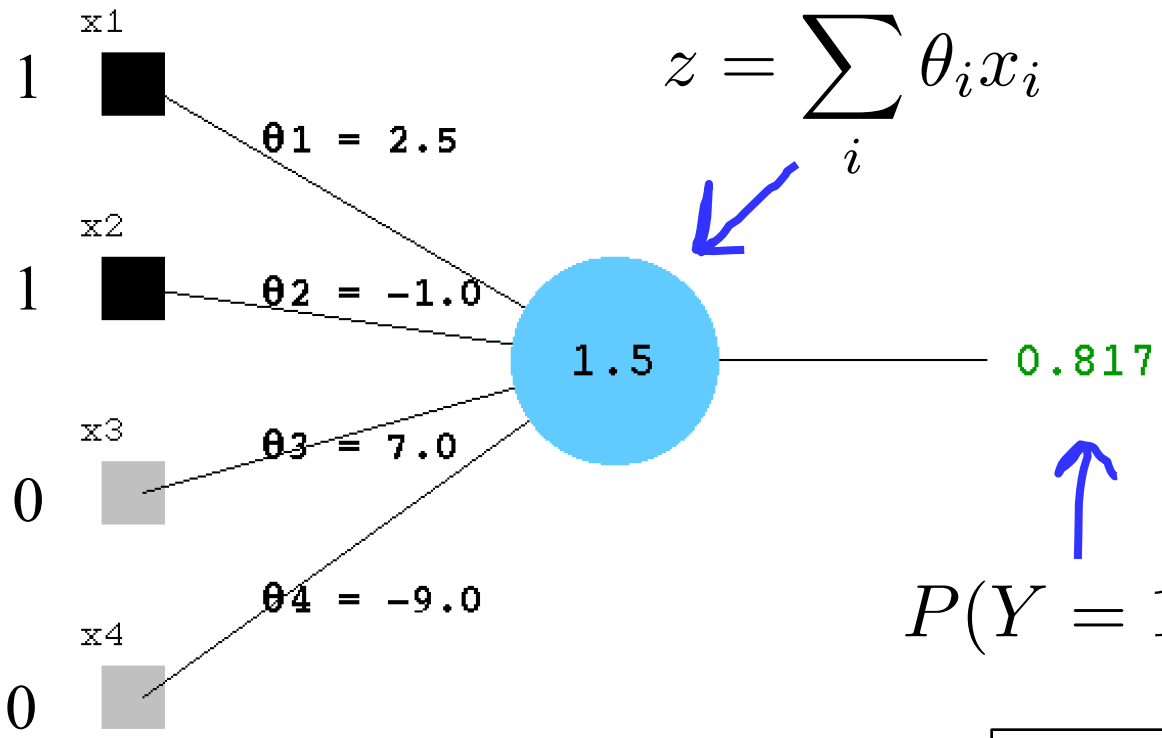


$$P(Y = 1 | X = \mathbf{x}) = \sigma(\theta^T \mathbf{x})$$

Logistic Regression

$$\mathbf{x} = [1, 1, 0, 0]$$



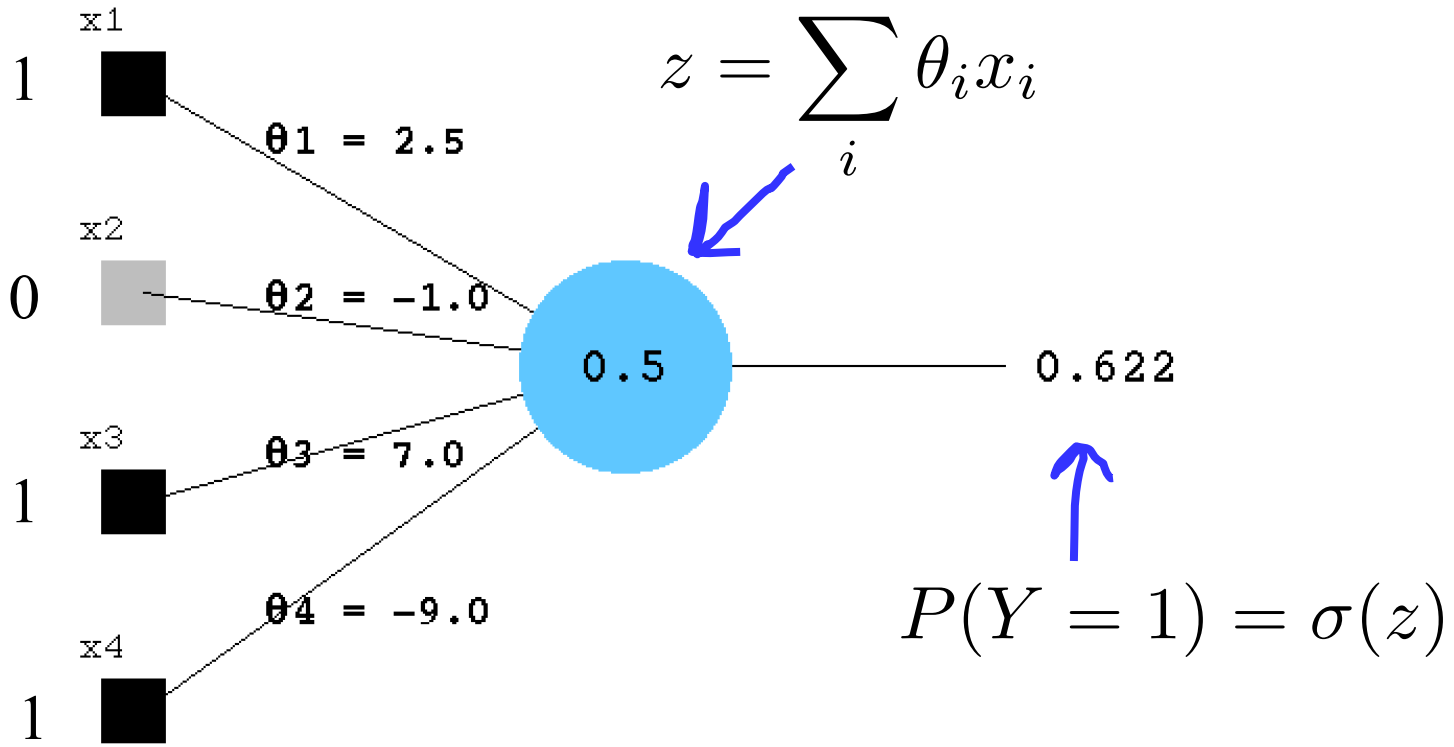


$$z = \sum_i \theta_i x_i$$

$$P(Y = 1) = \sigma(z)$$

Logistic Regression

$$\mathbf{x} = [1, 0, 1, 1]$$



Logistic Regression Assumption

- Model *conditional* likelihood P(Y | **X**) directly
  - Model this probability with *logistic* function:

$$P(Y = 1|\mathbf{X}) = \sigma(z) \text{ where } z = \theta_0 + \sum_{i=1}^{m} \theta_i x_i$$

  - For simplicity define $x_0 = 1$ so $z = \theta^T \mathbf{x}$
  - Since $\mathrm{P}(Y = 0 \mid \mathbf{X}) + \mathrm{P}(Y = 1 \mid \mathbf{X}) = 1$:

$$P(Y = 1|X = \mathbf{x}) = \sigma(\theta^T \mathbf{x})$$

$$P(Y = 0|X = \mathbf{x}) = 1 - \sigma(\theta^T \mathbf{x})$$

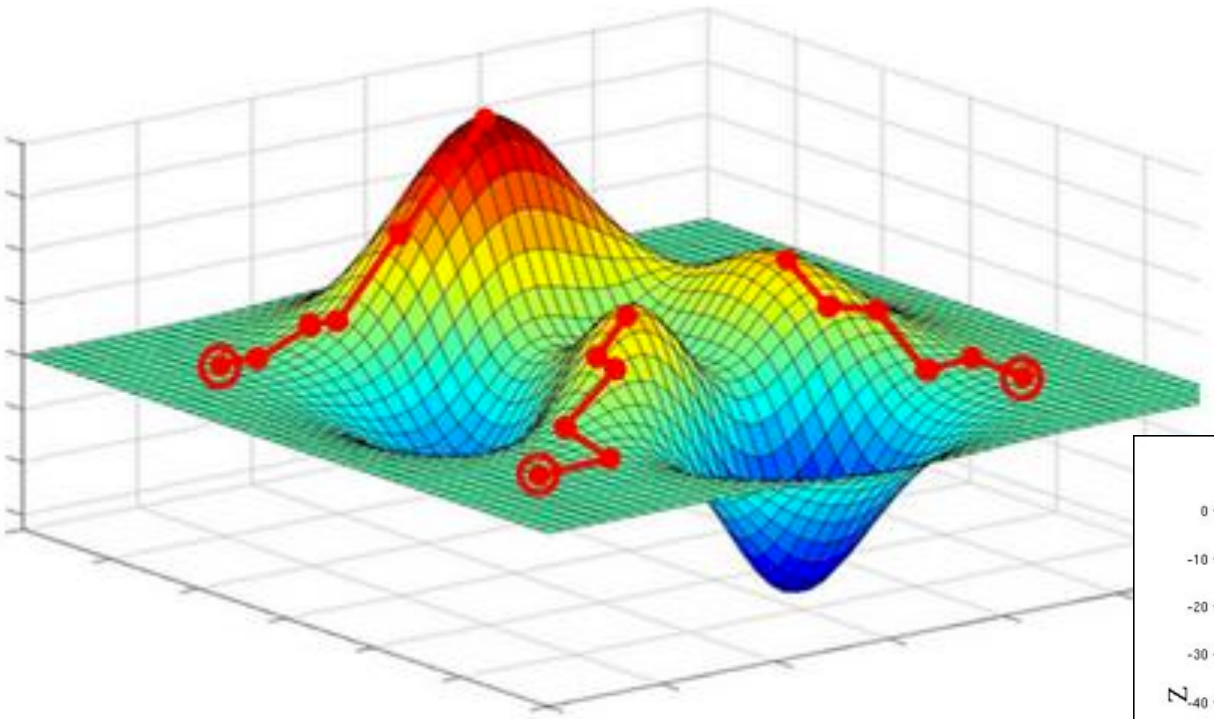
Recall:
Sigmoid function

$$\sigma(z) = \frac{1}{1 + e^{-z}}$$

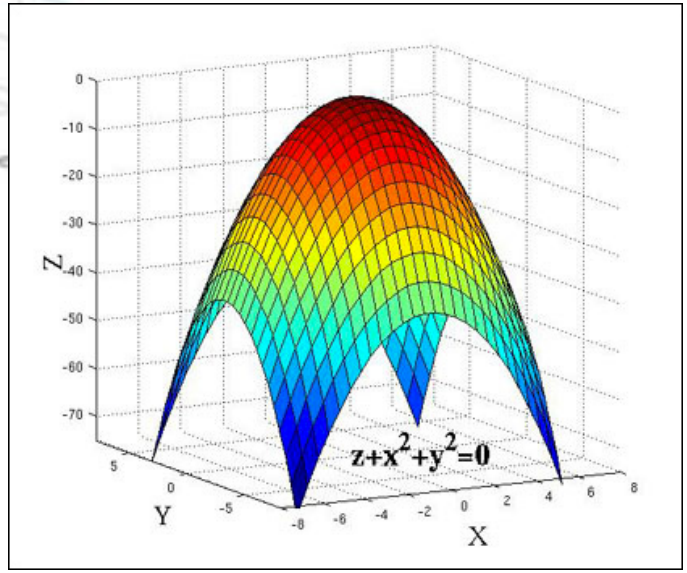
Logistic regression gets its ***intelligence*** from its thetas (aka its parameters)

The hard part is learning the thetas

Gradient Ascent



Logistic regression LL function is convex



Walk uphill and you will find a local maxima
(if your step size is small enough)

Math for Logistic Regression

1 Make logistic regression assumption

$$P(Y = 1|X = \mathbf{x}) = \sigma(\theta^T \mathbf{x})$$
$$P(Y = 0|X = \mathbf{x}) = 1 - \sigma(\theta^T \mathbf{x})$$

2 Calculate the log likelihood for all data

$$LL(\theta) = \sum_{i=0}^{n} y^{(i)} \log \sigma(\theta^T \mathbf{x}^{(i)}) + (1 - y^{(i)}) \log[1 - \sigma(\theta^T \mathbf{x}^{(i)})]$$

3 Get derivative of log likelihood with respect to thetas

$$\frac{\partial LL(\theta)}{\partial \theta_j} = \sum_{i=0}^{n} \left[y^{(i)} - \sigma(\theta^T \mathbf{x}^{(i)}) \right] x_j^{(i)}$$

Math for Logistic Regression

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$$\frac{\partial LL(\theta)}{\partial \theta_j} = \sum_{i=0}^{n} \left[y^{(i)} - \sigma(\theta^T \mathbf{x}^{(i)}) \right] x_j^{(i)}$$

Gradient Ascent Step

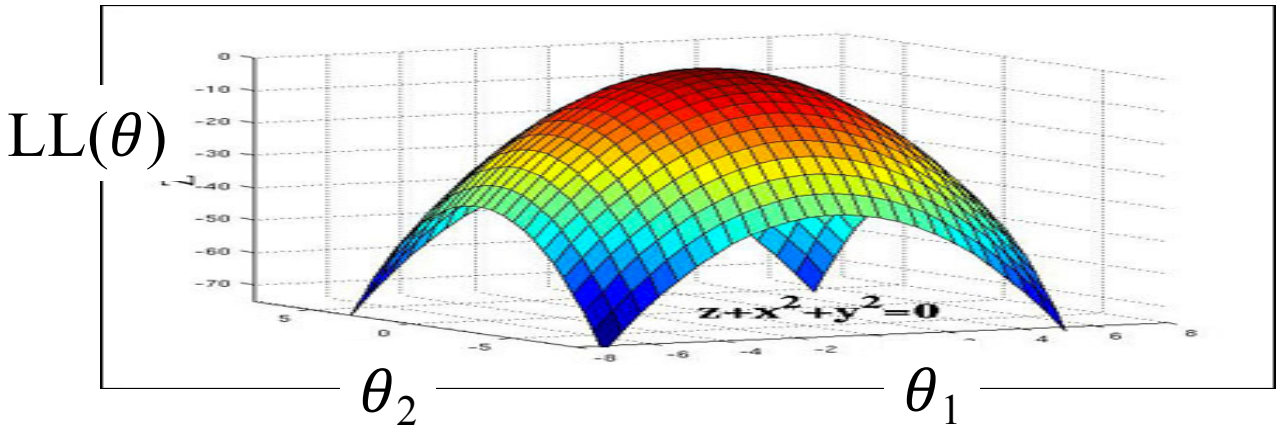
$$\frac{\partial LL(\theta)}{\partial \theta_j} = \sum_{i=0}^{n} \left[y^{(i)} - \sigma(\theta^T \mathbf{x}^{(i)}) \right] x_j^{(i)}$$

$$\theta_j^{\text{new}} = \theta_j^{\text{old}} + \eta \cdot \frac{\partial LL(\theta^{\text{old}})}{\partial \theta_j^{\text{old}}}$$

$$= \theta_j^{\text{old}} + \eta \cdot \sum_{i=0}^{n} \left[y^{(i)} - \sigma(\theta^T \mathbf{x}^{(i)}) \right] x_j^{(i)}$$

$$\eta = 0.0001$$

Do this for all thetas!



Logistic Regression Training

Initialize: $\theta_j = 0$ for all $0 \le j \le m$

Repeat many times:

gradient[j] = 0 for all $0 \le j \le m$

For each training example (x, y):

For each parameter j:

gradient[j] $+= \left[y - \sigma(\theta^T \mathbf{x})\right] \mathbf{x}_j$

θ_j += η * gradient[j] for all $0 \le j \le m$

Chapter 2: How Come?

Logistic Regression

1 Make logistic regression assumption

$$P(Y = 1|X = \mathbf{x}) = \sigma(\theta^T \mathbf{x})$$
$$P(Y = 0|X = \mathbf{x}) = 1 - \sigma(\theta^T \mathbf{x})$$

2 Calculate the log likelihood for all data

$$LL(\theta) = \sum_{i=0}^{n} y^{(i)} \log \sigma(\theta^T \mathbf{x}^{(i)}) + (1 - y^{(i)}) \log[1 - \sigma(\theta^T \mathbf{x}^{(i)})]$$

3 Get derivative of log likelihood with respect to thetas

$$\frac{\partial LL(\theta)}{\partial \theta_j} = \sum_{i=0}^{n} \left[y^{(i)} - \sigma(\theta^T \mathbf{x}^{(i)}) \right] x_j^{(i)}$$

Sanity Check

3 Get partial derivative of log likelihood with respect to each theta

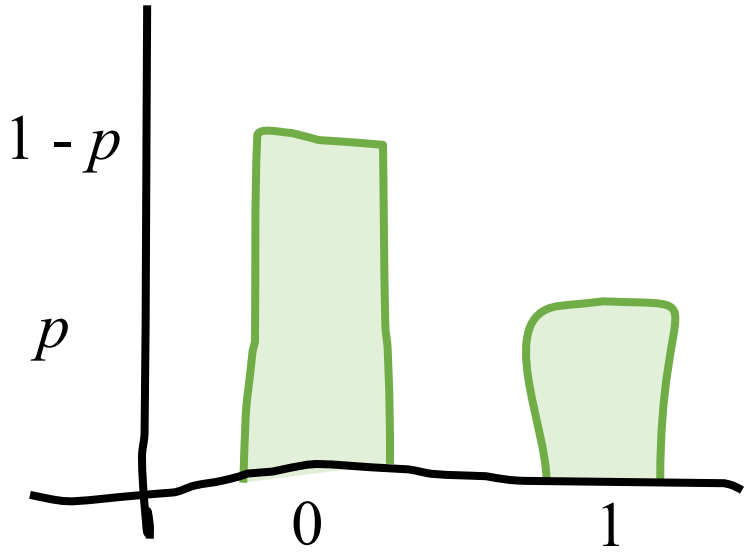
Why?

How did we get that LL function?

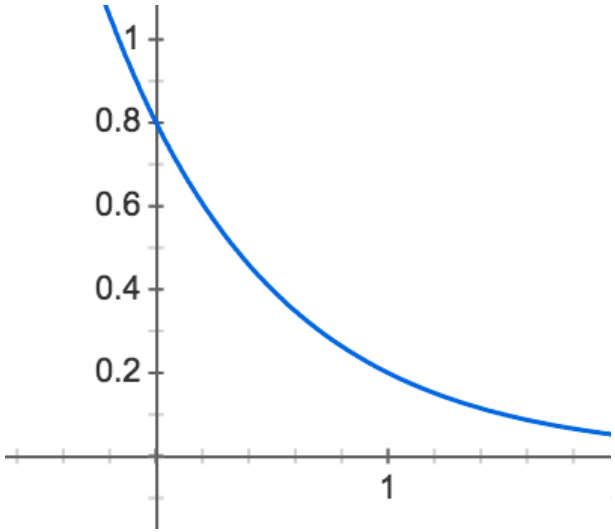
Recall: PMF of Bernoulli

- Y ~ Ber(*p*)
- Probability mass function: $P(Y = y)$

PMF of Bernoulli



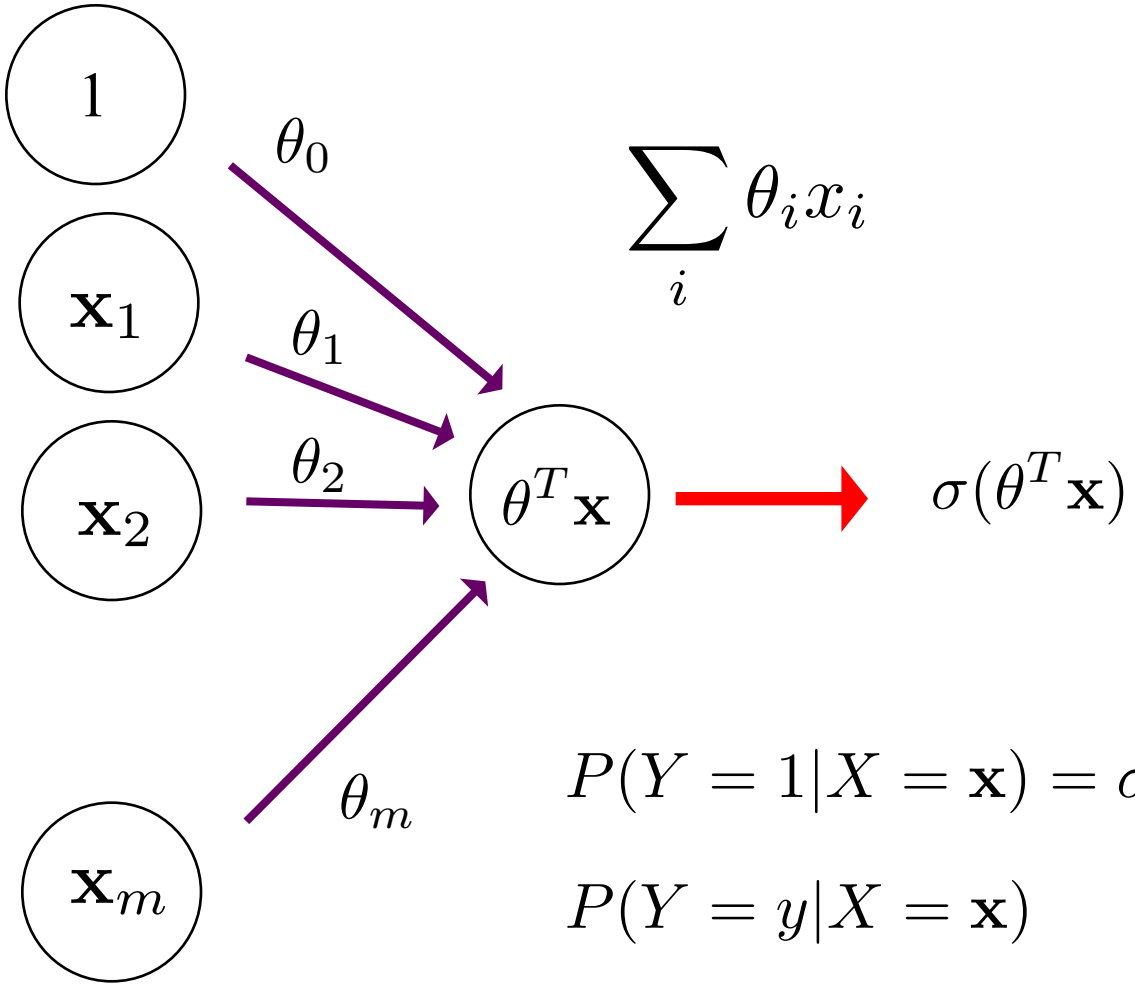
PMF of Bernoulli (*p* = 0.2)



$$P(Y = y) = p^y(1-p)^{1-y}$$

$$P(Y = y) = 0.2^y(0.8)^{1-y}$$

Logistic Regression



$$P(Y = 1|X = \mathbf{x}) = \sigma(\theta^T\mathbf{x})$$

$$P(Y = y|X = \mathbf{x})$$

$$= \sigma(\theta^T\mathbf{x})^y \cdot \left[1 - \sigma(\theta^T\mathbf{x})\right]^{(1-y)}$$

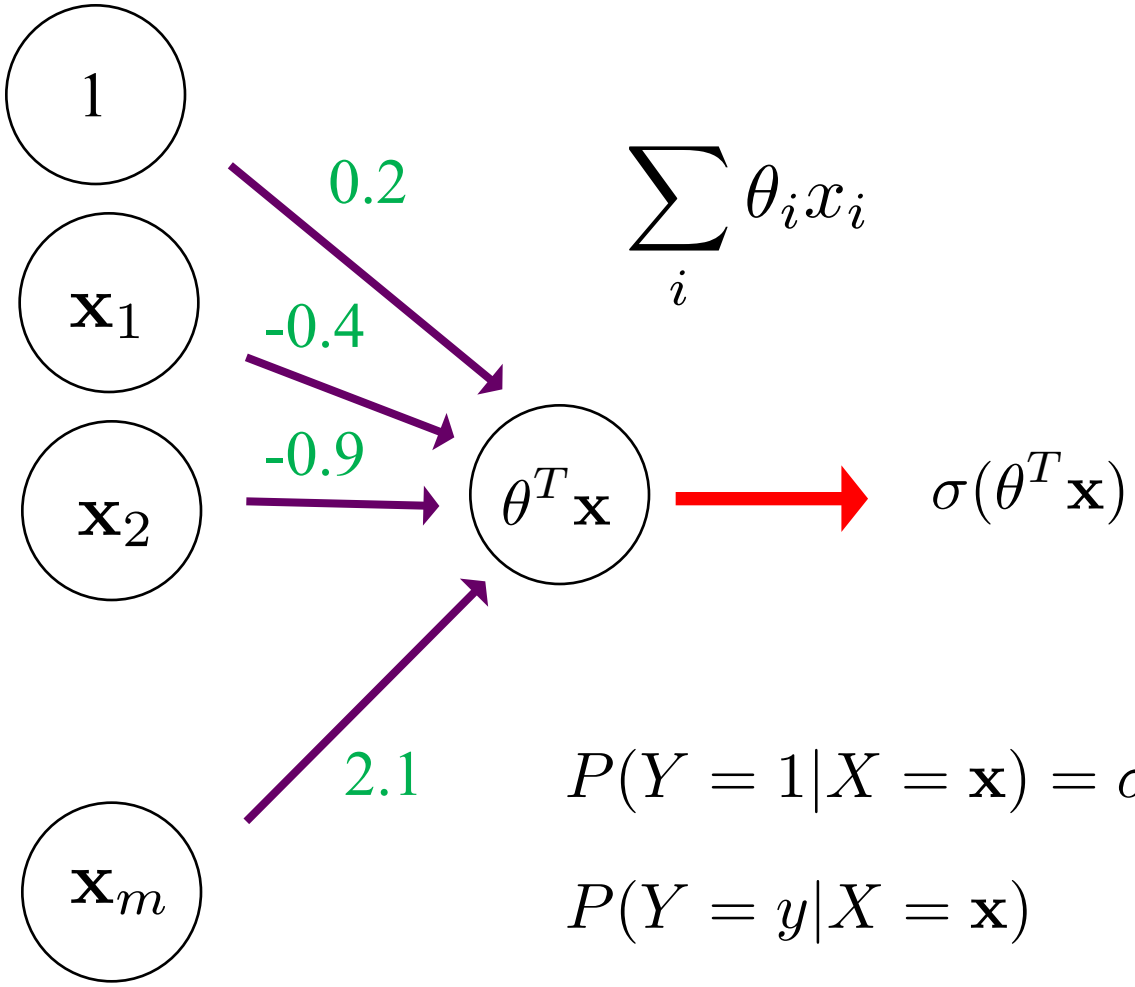
You can calculate likelihood of data, θ

Likelihood of a training example:

$$P(Y = y | X = \mathbf{x})$$

$$= \sigma(\theta^T \mathbf{x})^y \cdot \left[1 - \sigma(\theta^T \mathbf{x})\right]^{(1-y)}$$

Logistic Regression



$$P(Y = 1|X = \mathbf{x}) = \sigma(\theta^T\mathbf{x})$$

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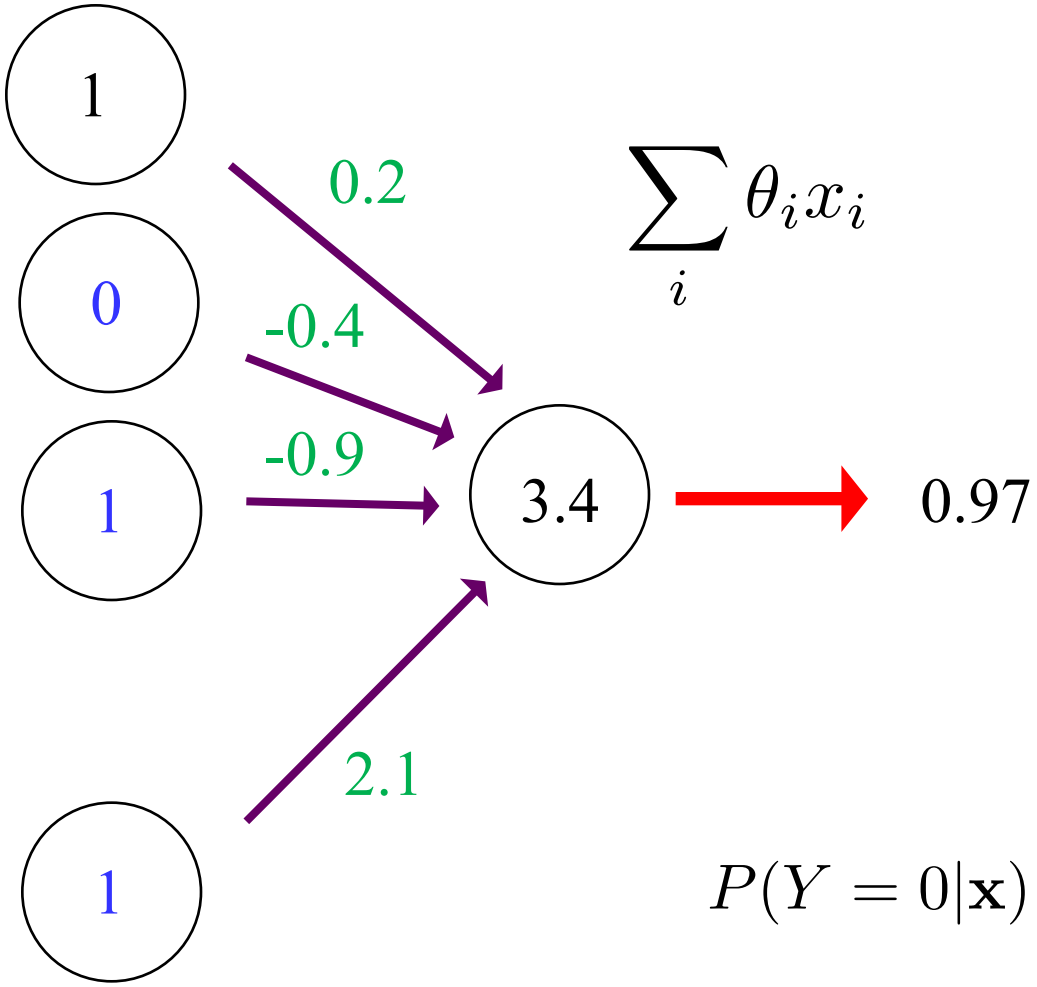
You can calculate likelihood of data, θ

Logistic Regression



Let's say that:

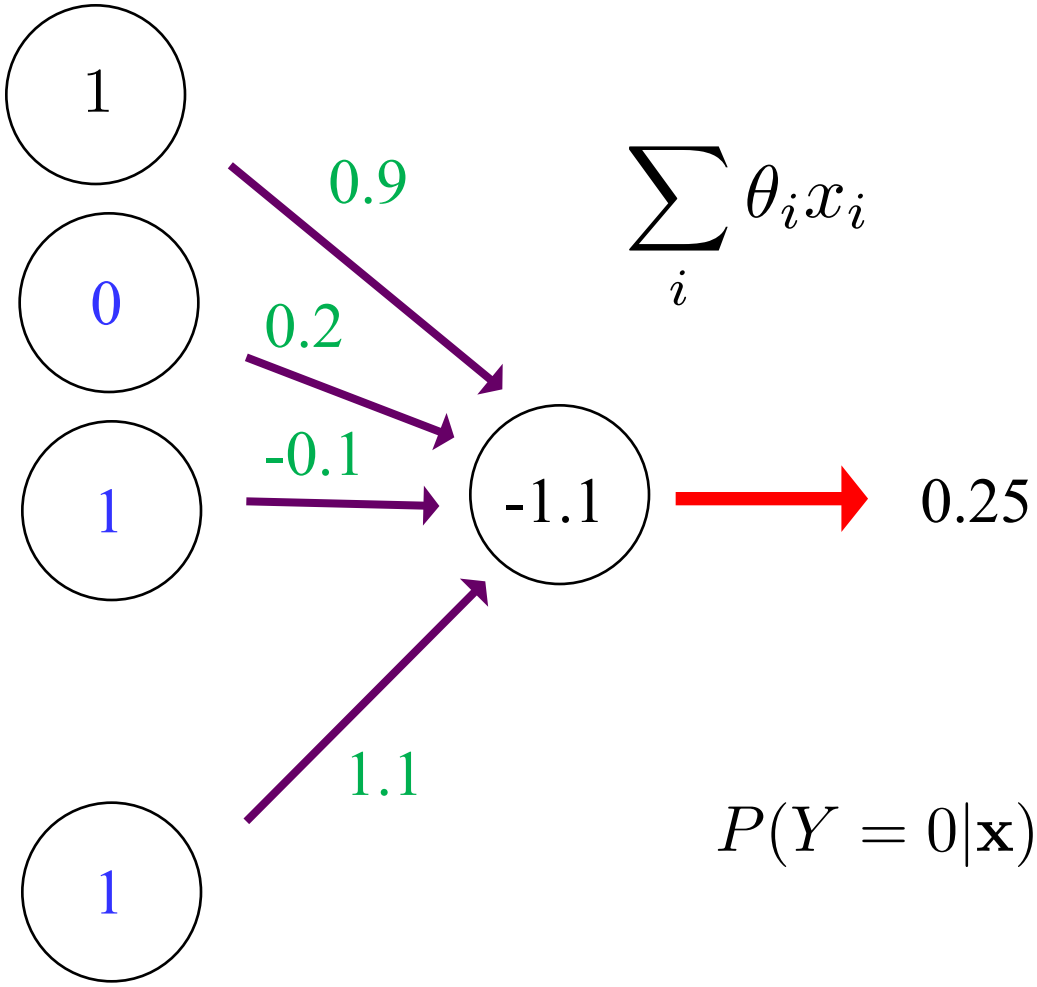
$$\mathbf{x} = [1, 0, 1, 1]$$
$$y = 0$$

$$\sum_i \theta_i x_i$$

$$P(Y = 0|\mathbf{x}) = (1 - \theta^T \mathbf{x})$$
$$= 0.03$$

We should change θ

These Parameters are Better



Let's say that:

$$\mathbf{x} = [1, 0, 1, 1]$$

$$y = 0$$

$$\sum_i \theta_i x_i$$

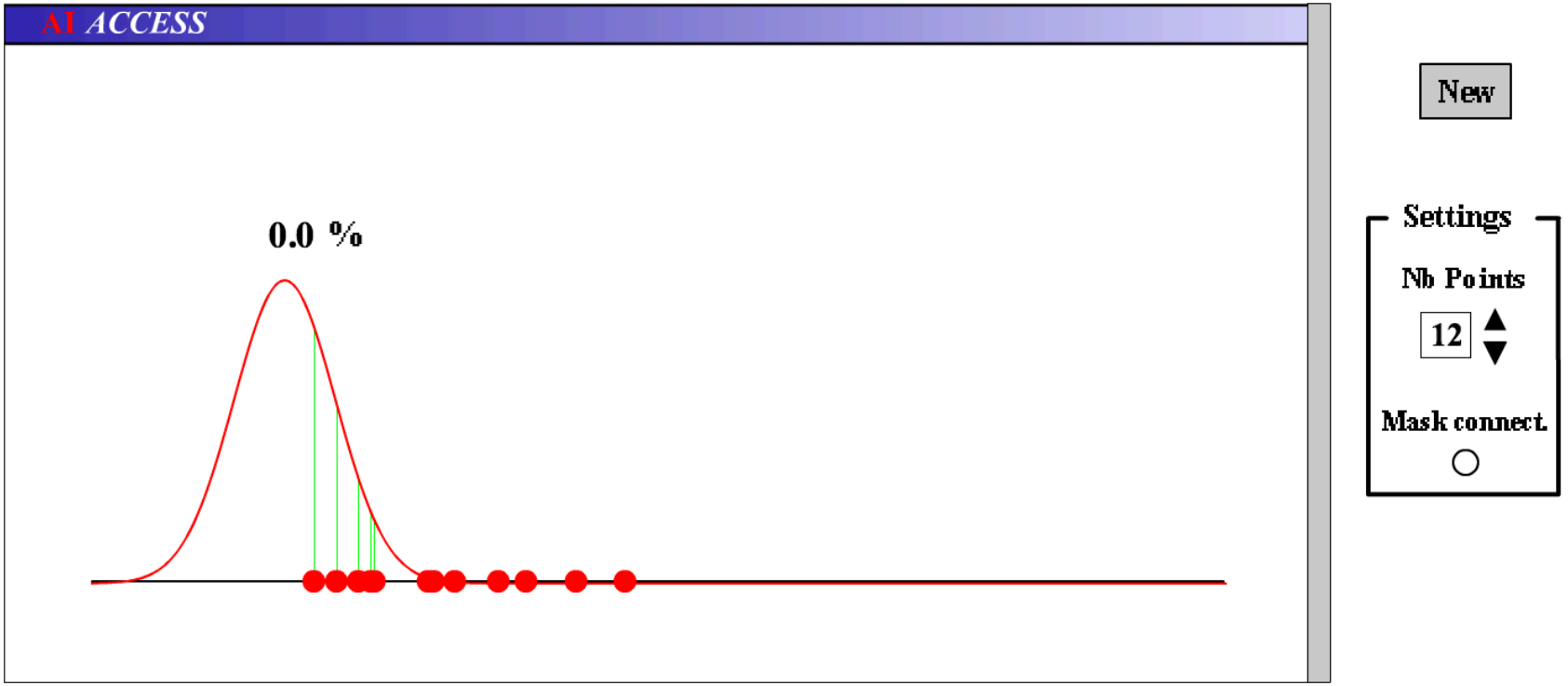
$$P(Y = 0|\mathbf{x}) = (1 - \theta^T \mathbf{x})$$

$$= 0.75$$

Better...

Maximum Likelihood Estimation

Likelihood of Data from a Normal



Remember this?

Log Probability of Data

$$P(Y = 1|X = \mathbf{x}) = \sigma(\theta^T \mathbf{x})$$
$$P(Y = 0|X = \mathbf{x}) = 1 - \sigma(\theta^T \mathbf{x})$$

Implies

$$P(Y = y|X = \mathbf{x}) = \sigma(\theta^T \mathbf{x})^y \cdot \left[1 - \sigma(\theta^T \mathbf{x})\right]^{(1-y)}$$

For IID data

$$L(\theta) = \prod_{i=1}^{n} P(Y = y^{(i)}|X = \mathbf{x}^{(i)})$$
$$= \prod_{i=1}^{n} \sigma(\theta^T \mathbf{x}^{(i)})^{y^{(i)}} \cdot \left[1 - \sigma(\theta^T \mathbf{x}^{(i)})\right]^{(1-y^{(i)})}$$

Take the log

$$LL(\theta) = \sum_{i=0}^{n} y^{(i)} \log \sigma(\theta^T \mathbf{x}^{(i)}) + (1 - y^{(i)}) \log[1 - \sigma(\theta^T \mathbf{x}^{(i)})]$$

Logistic Regression

1 Make logistic regression assumption

$$P(Y = 1|X = \mathbf{x}) = \sigma(\theta^T \mathbf{x})$$

$$P(Y = 0|X = \mathbf{x}) = 1 - \sigma(\theta^T \mathbf{x})$$

2 Write the log likelihood for all data

$$LL(\theta) = \sum_{i=0}^{n} y^{(i)} \log \sigma(\theta^T \mathbf{x}^{(i)}) + (1 - y^{(i)}) \log[1 - \sigma(\theta^T \mathbf{x}^{(i)})]$$

3 Optimize likelihood of data given theta

$$\frac{\partial LL(\theta)}{\partial \theta_j} = \sum_{i=0}^{n} \left[y^{(i)} - \sigma(\theta^T \mathbf{x}^{(i)}) \right] x_j^{(i)}$$

How did we get that gradient?

Big Idea

It's called chain rule

$$\frac{\partial f(x)}{\partial x} = \frac{\partial f(z)}{\partial z} \cdot \frac{\partial z}{\partial x}$$

Sigmoid has a Beautiful Slope

$$\frac{\partial}{\partial \theta_j}\sigma(\theta^T x)?$$

$$\frac{\partial}{\partial z}\sigma(z) = \sigma(z)[1 - z]$$

True fact about sigmoid functions

$$\frac{\partial}{\partial \theta_j}\sigma(\theta^T x) = \frac{\partial}{\partial z}\sigma(z) \cdot \frac{\partial z}{\partial \theta_j}$$

Chain rule!

$$\frac{\partial}{\partial \theta_j}\sigma(\theta^T x) = \sigma(\theta^T x)[1 - \sigma(\theta^T x)]x_j$$

Plug and chug

Sigmoid, you should be a ski hill

Gradient Update

$$LL(\theta) = \sum_{i=0}^{n} y^{(i)} \log \sigma(\theta^T \mathbf{x}^{(i)}) + (1 - y^{(i)}) \log[1 - \sigma(\theta^T \mathbf{x}^{(i)})]$$

$$\frac{\partial LL(\theta)}{\partial \theta_j} = \frac{\partial}{\partial \theta_j} y \log \sigma(\theta^T \mathbf{x}) + \frac{\partial}{\partial \theta_j} (1 - y) \log[1 - \sigma(\theta^T \mathbf{x}]$$

Imagine only one data point

$$= \left[\frac{y}{\sigma(\theta^T x)} - \frac{1 - y}{1 - \sigma(\theta^T x)} \right] \frac{\partial}{\partial \theta_j} \sigma(\theta^T x)$$

$$= \left[\frac{y}{\sigma(\theta^T x)} - \frac{1 - y}{1 - \sigma(\theta^T x)} \right] \frac{\partial}{\partial \theta_j} \sigma(\theta^T x)$$

$$= \left[\frac{y - \sigma(\theta^T x)}{\sigma(\theta^T x)[1 - \sigma(\theta^T x)]} \right] \sigma(\theta^T x)[1 - \sigma(\theta^T x)] x_j$$

$$= \left[y - \sigma(\theta^T x) \right] x_j$$

$$\frac{\partial LL(\theta)}{\partial \theta_j} = \sum_{i=0}^{n} \left[y^{(i)} - \sigma(\theta^T \mathbf{x}^{(i)}) \right] x_j^{(i)}$$

For many data points

Logistic Regression

1 Make logistic regression assumption

$$P(Y = 1|X = \mathbf{x}) = \sigma(\theta^T \mathbf{x})$$
$$P(Y = 0|X = \mathbf{x}) = 1 - \sigma(\theta^T \mathbf{x})$$

2 Calculate the log probability for all data

$$LL(\theta) = \sum_{i=0}^{n} y^{(i)} \log \sigma(\theta^T \mathbf{x}^{(i)}) + (1 - y^{(i)}) \log[1 - \sigma(\theta^T \mathbf{x}^{(i)})]$$

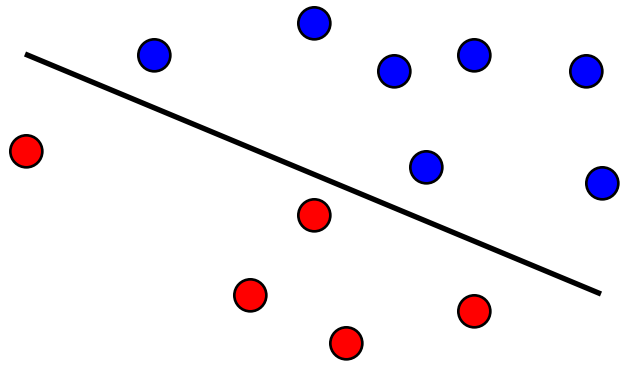
3 Get derivative of log probability with respect to thetas

$$\frac{\partial LL(\theta)}{\partial \theta_j} = \sum_{i=0}^{n} \left[y^{(i)} - \sigma(\theta^T \mathbf{x}^{(i)}) \right] x_j^{(i)}$$

Chapter 3: Philosophy

Discrimination Intuition

- Logistic regression is trying to fit a **line** that separates data instances where $y = 1$ from those where $y = 0$



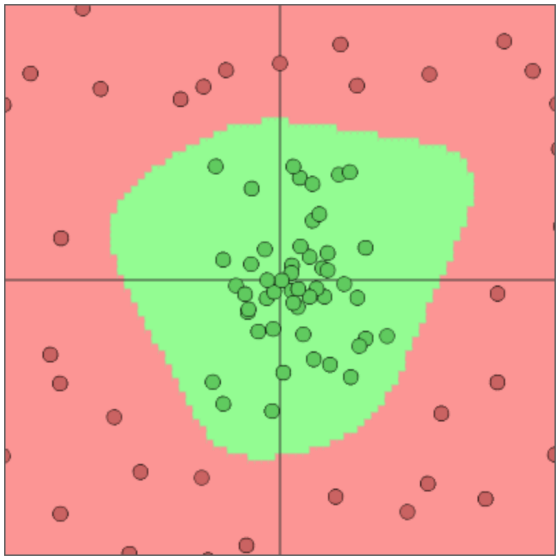
$$\theta^T \mathbf{x} = 0$$

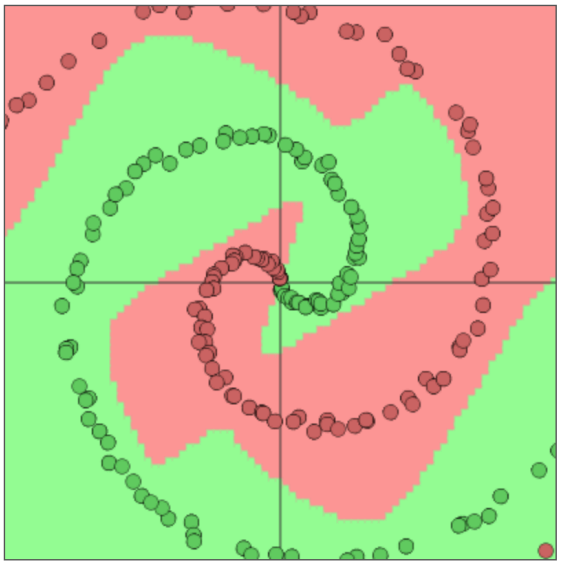
$$\theta_0 x_0 + \theta_1 x_1 + \cdots + \theta_m x_m = 0$$

- We call such data (or the functions generating the data) "**linearly separable**"
- Naïve bayes is linear too as there is no interaction between different features.

Some Data Not Linearly Seperable

- Some data sets/functions are not separable





  - Not possible to draw a line that successfully separates all the $y = 1$ points (green) from the $y = 0$ points (red)
  - Despite this fact, logistic regression and Naive Bayes still often work well in practice

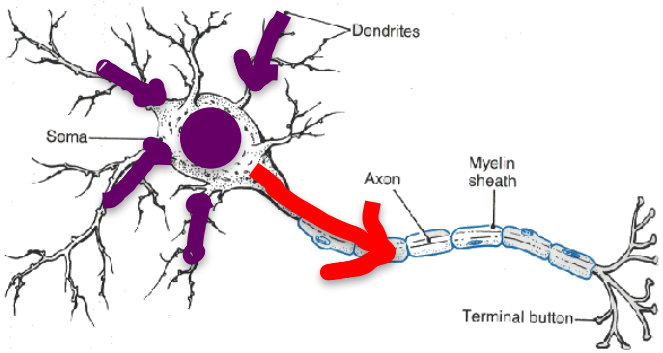
Logistic Regression vs Naïve Bayes

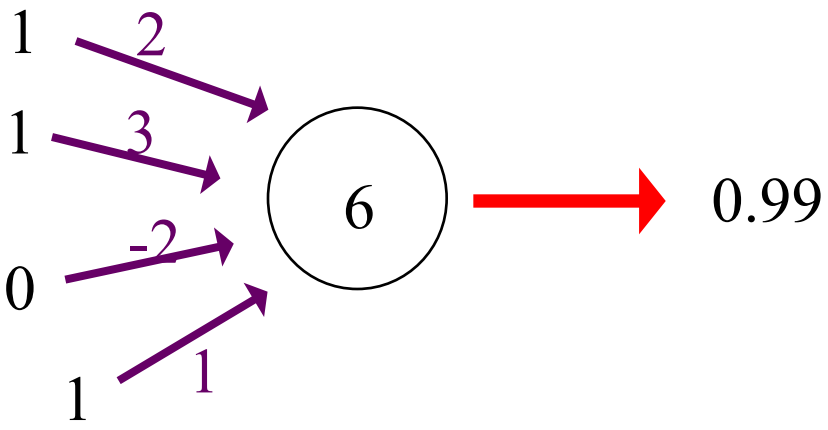
- Compare Naive Bayes and Logistic Regression
  - Recall that Naive Bayes models P(**X**, Y) = P(**X** | Y) P(Y)
  - Logistic Regression directly models P(Y | **X**)
  - We call Naive Bayes a "generative model"
    - Tries to model joint distribution of how data is "generated"
    - I.e., could use P(X, Y) to generate new data points if we wanted
    - But lots of effort to model something that may not be needed
  - We call Logistic Regression a "discriminative model"
    - Just tries to model way to discriminate y = 0 vs. y = 1 cases
    - Cannot use model to generate new data points (no P(X, Y))
    - Note: Logistic Regression can be generalized to more than two output values for y (have multiple sets of parameters β_j)

Choosing an Algorithm?

- Many trade-offs in choosing learning algorithm
  - Continuous input variables
    - Logistic Regression easily deals with continuous inputs
    - Naive Bayes needs to use some parametric form for continuous inputs (e.g., Gaussian) or "discretize" continuous values into ranges (e.g., temperature in range: <50, 50-60, 60-70, >70)
  - Discrete input variables
    - Naive Bayes naturally handles multi-valued discrete data by using multinomial distribution for $P(X_i \mid Y)$
    - Logistic Regression requires some sort of representation of multi-valued discrete data (e.g., one hot vector)
    - Say $X_i \in \{A, B, C\}$. Not necessarily a good idea to encode X_i as taking on input values 1, 2, or 3 corresponding to A, B, or C.

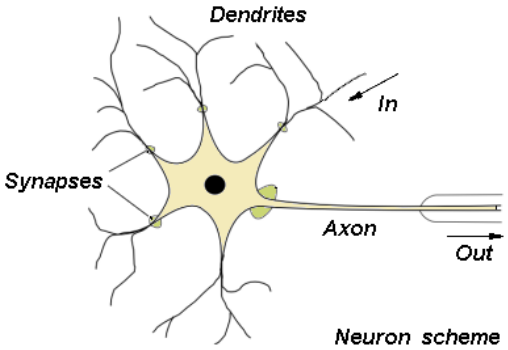
Artificial Neuron

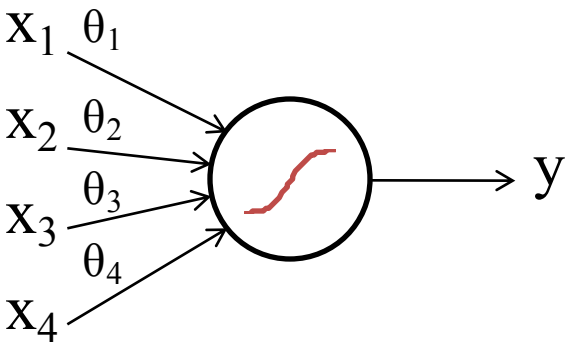




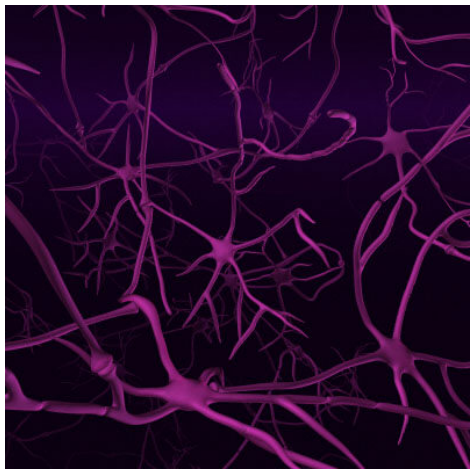
Biological Basis for Neural Networks

- A neuron

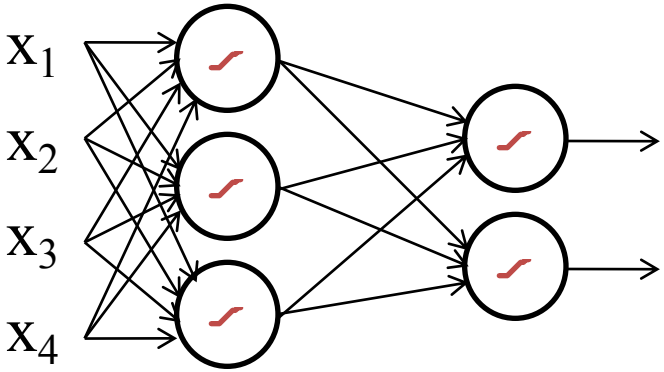




- Your brain



Actually, it's probably someone else's brain

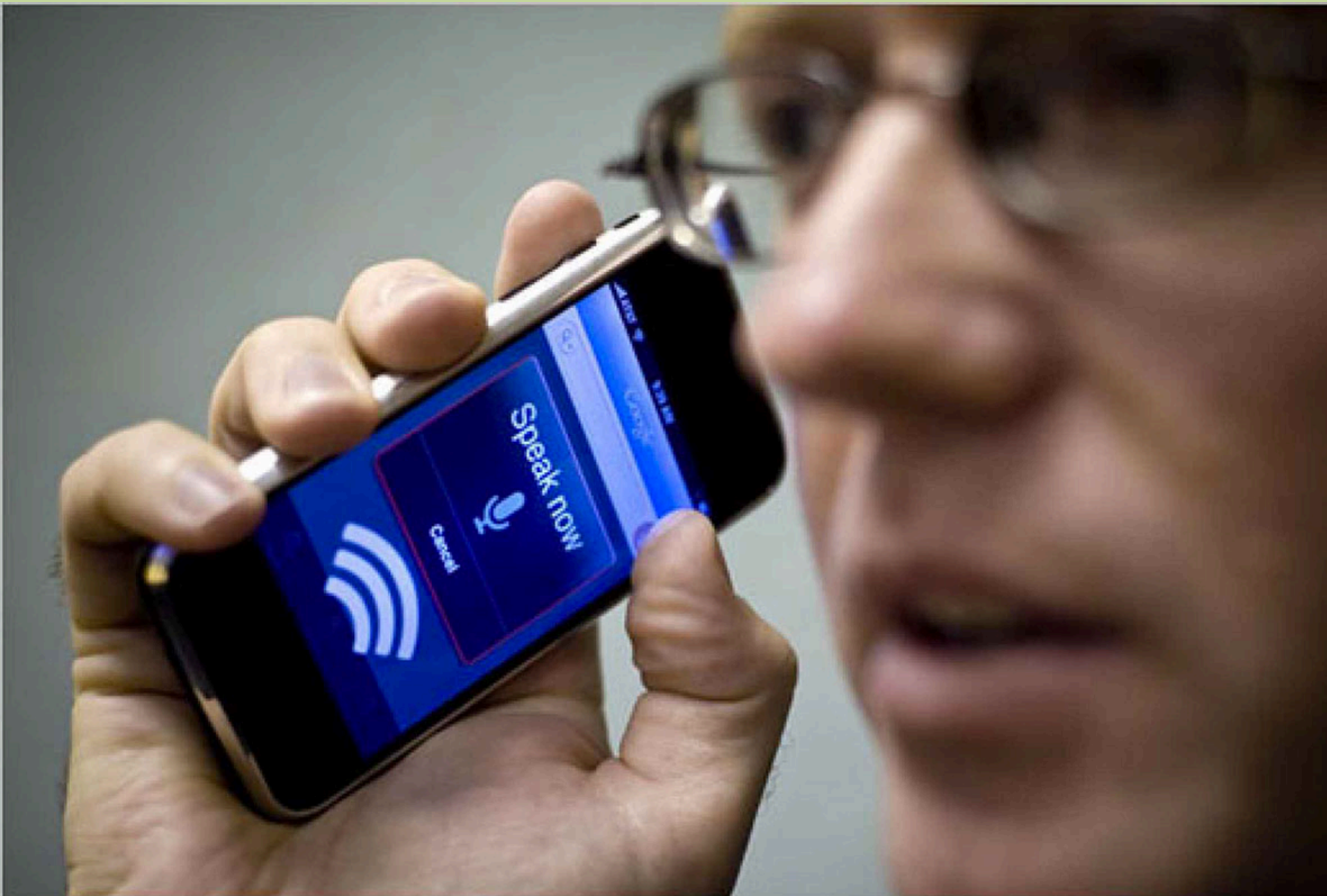


Next up: Deep Learning!

Alpha GO

Computer Vision

Revolution in AI



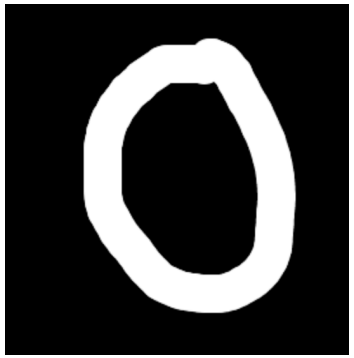
Computers Making Art

(aka Neural Networks)

Deep learning is (at its core) many logistic regression pieces stacked on top of each other.

Digit Recognition Example

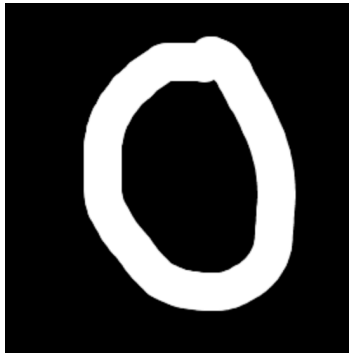
Let's make feature vectors from pictures of numbers

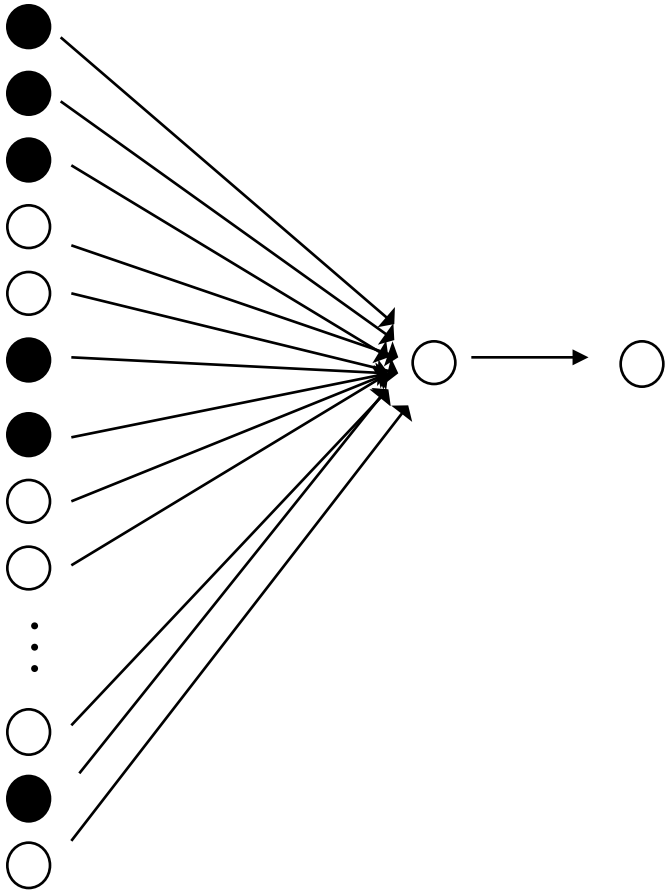


$$\mathbf{x}^{(i)} = [0, 0, 0, 0, \ldots, 1, 0, 0, 1, \ldots 0, 0, 1, 0]$$
$$y^{(i)} = 0$$

$$\mathbf{x}^{(i)} = [0, 0, 1, 1, \ldots, 0, 1, 1, 0, \ldots 0, 1, 0, 0]$$
$$y^{(i)} = 1$$

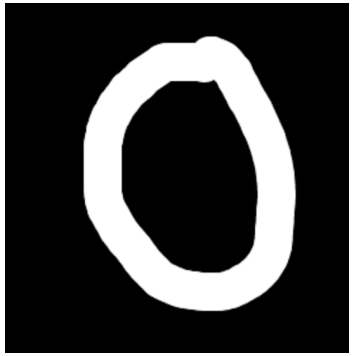
Logistic Regression

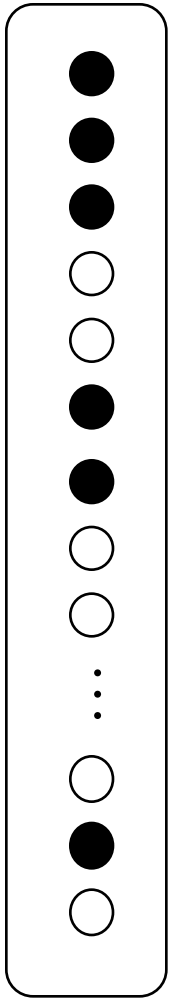




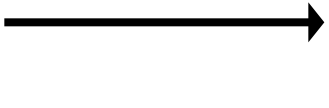
This means it predicts a 0

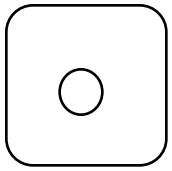
Logistic Regression





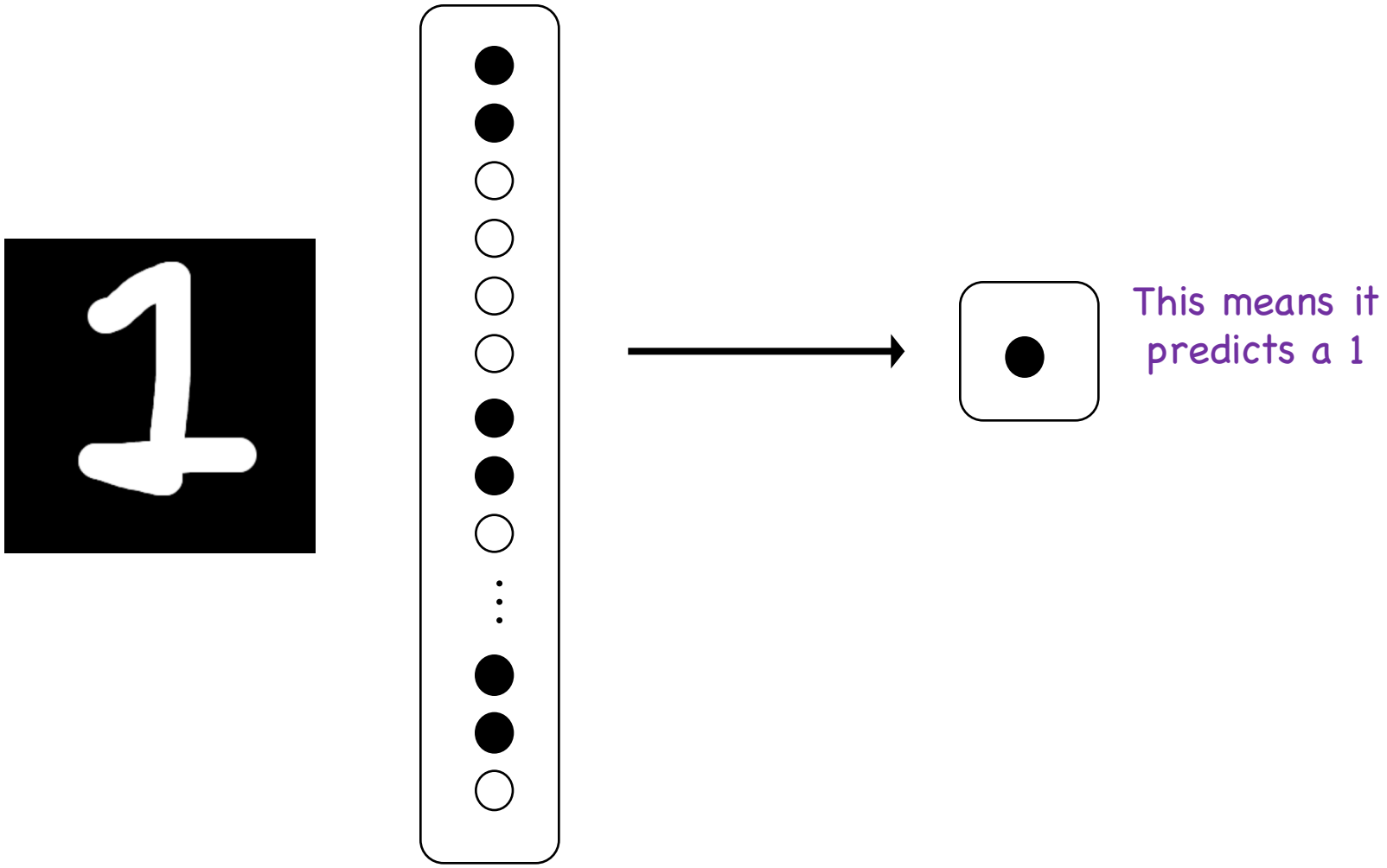
Indicates logistic regression connection



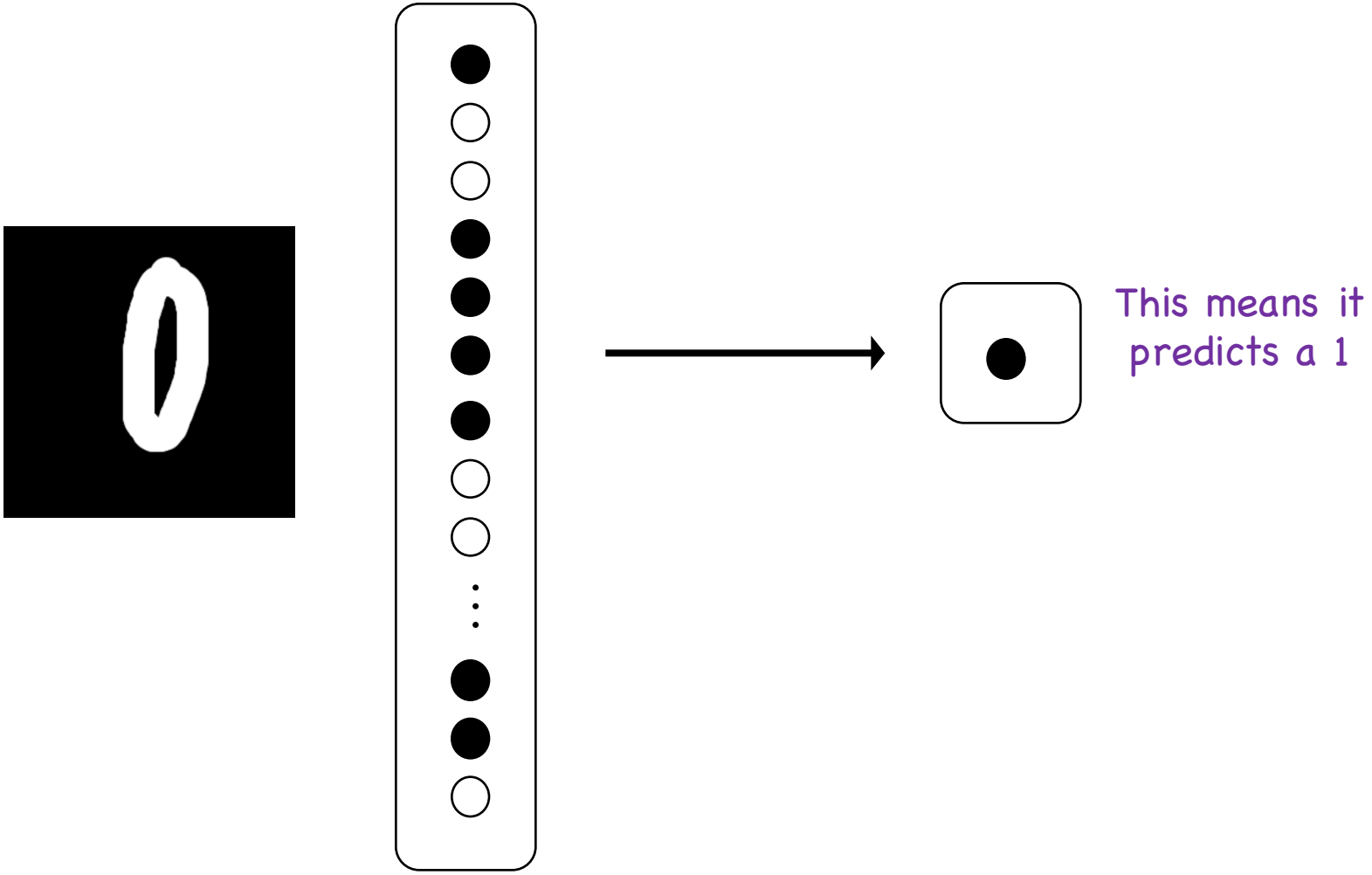


This means it predicts a 0

Logistic Regression

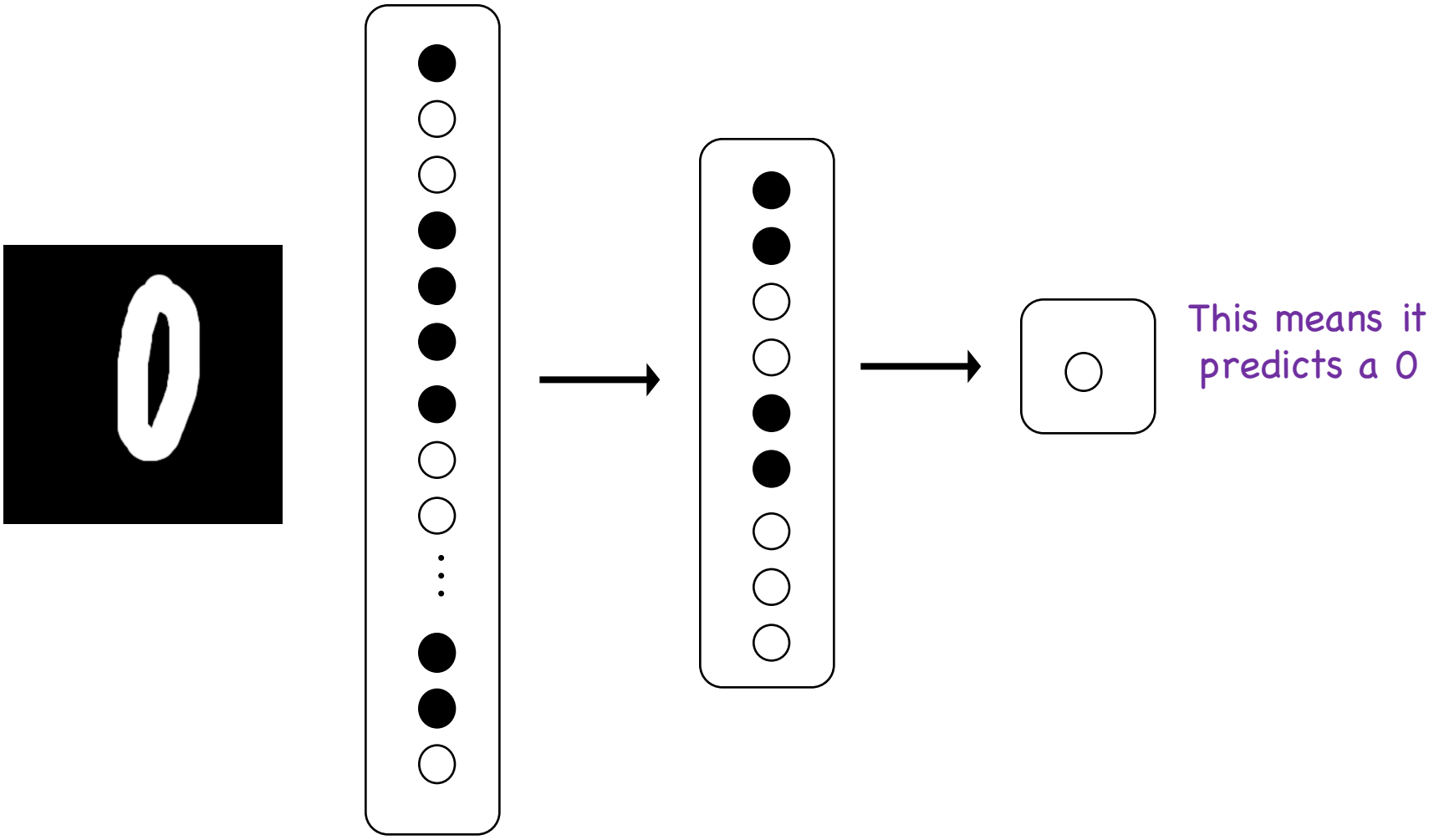


Not So Good

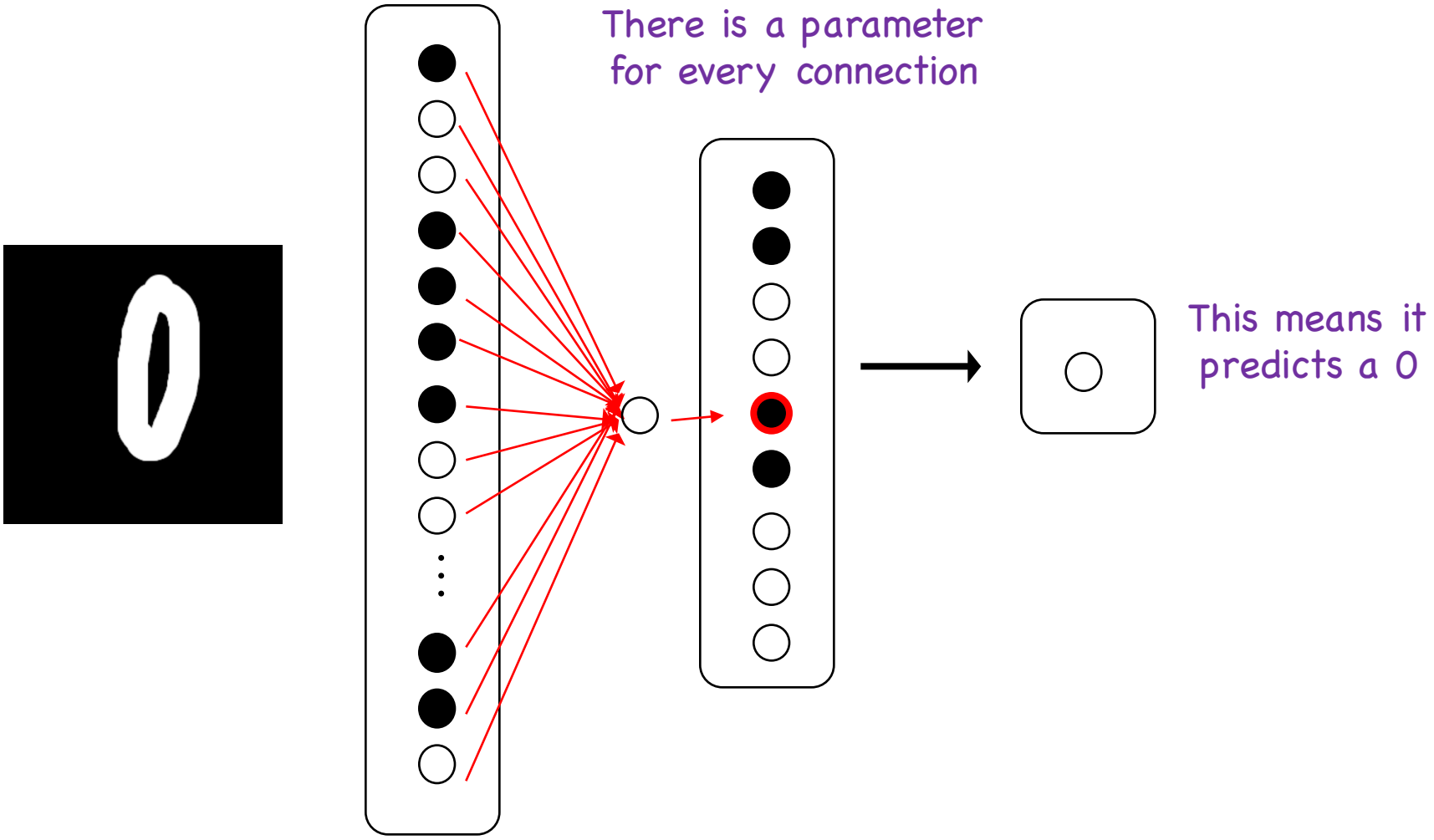


What can we do?

We Can Put Neurons Together

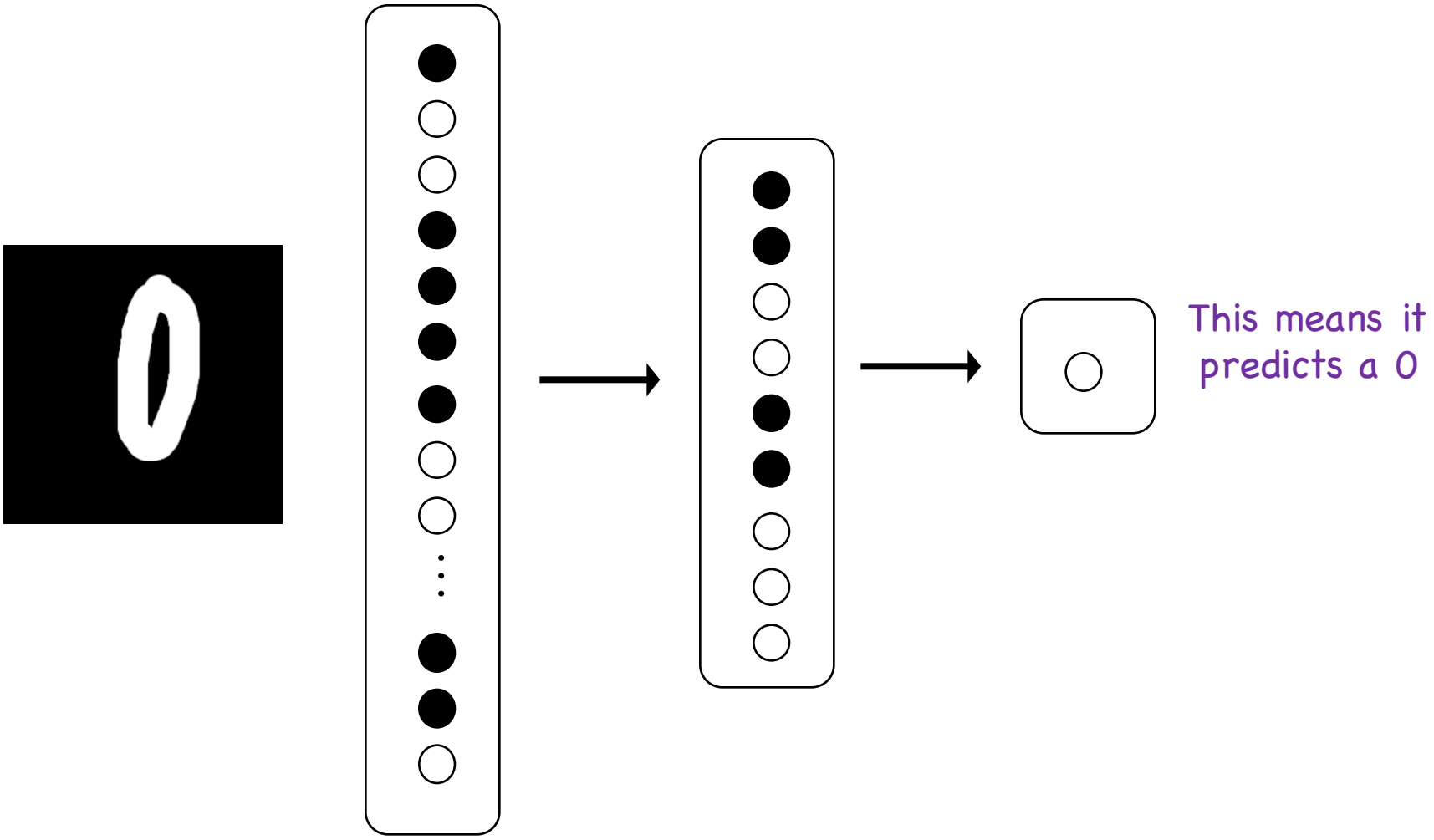


We Can Put Neurons Together

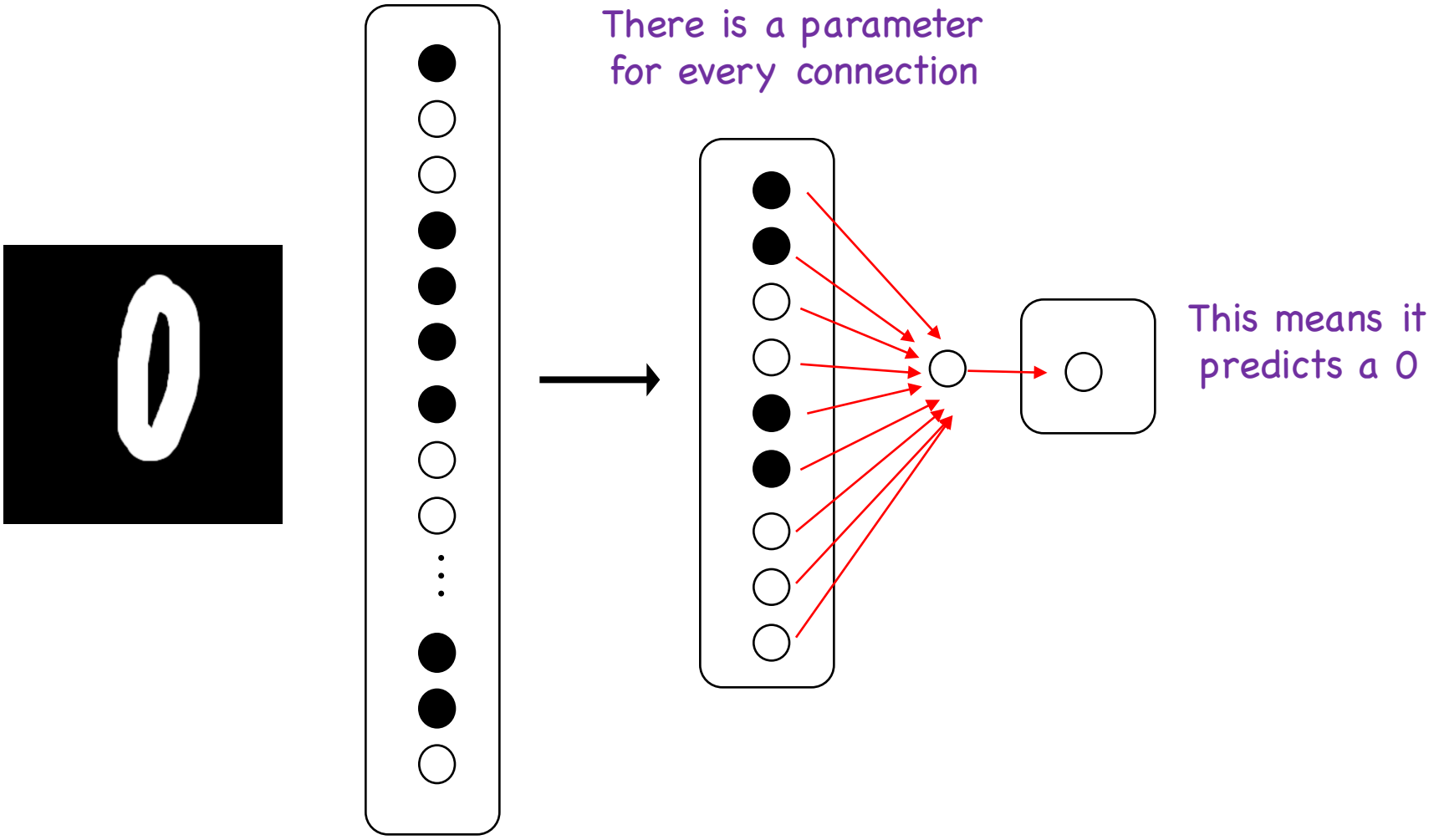


Look at a single "hidden" neuron

We Can Put Neurons Together

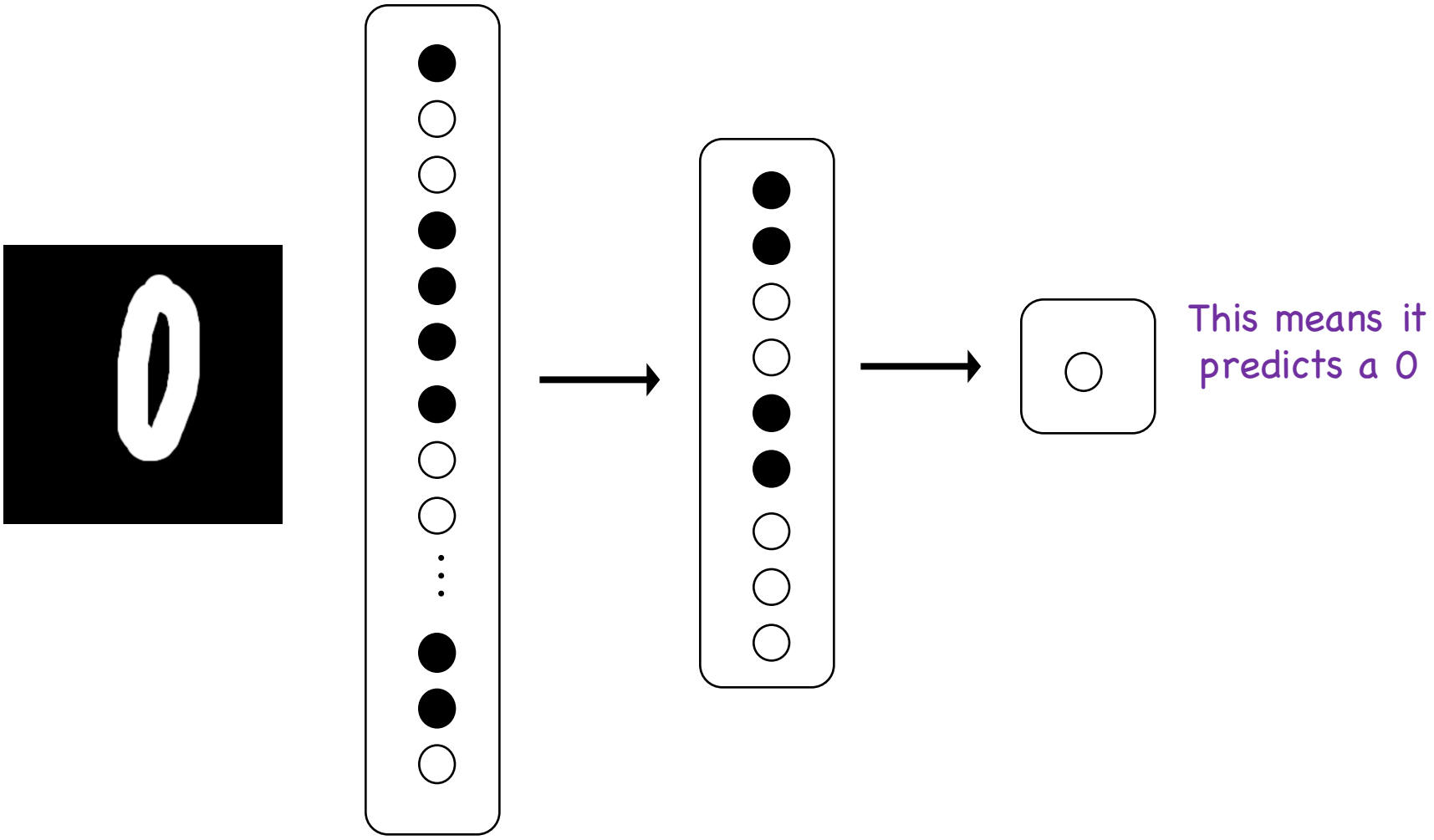


We Can Put Neurons Together



Look at another neuron

We Can Put Neurons Together

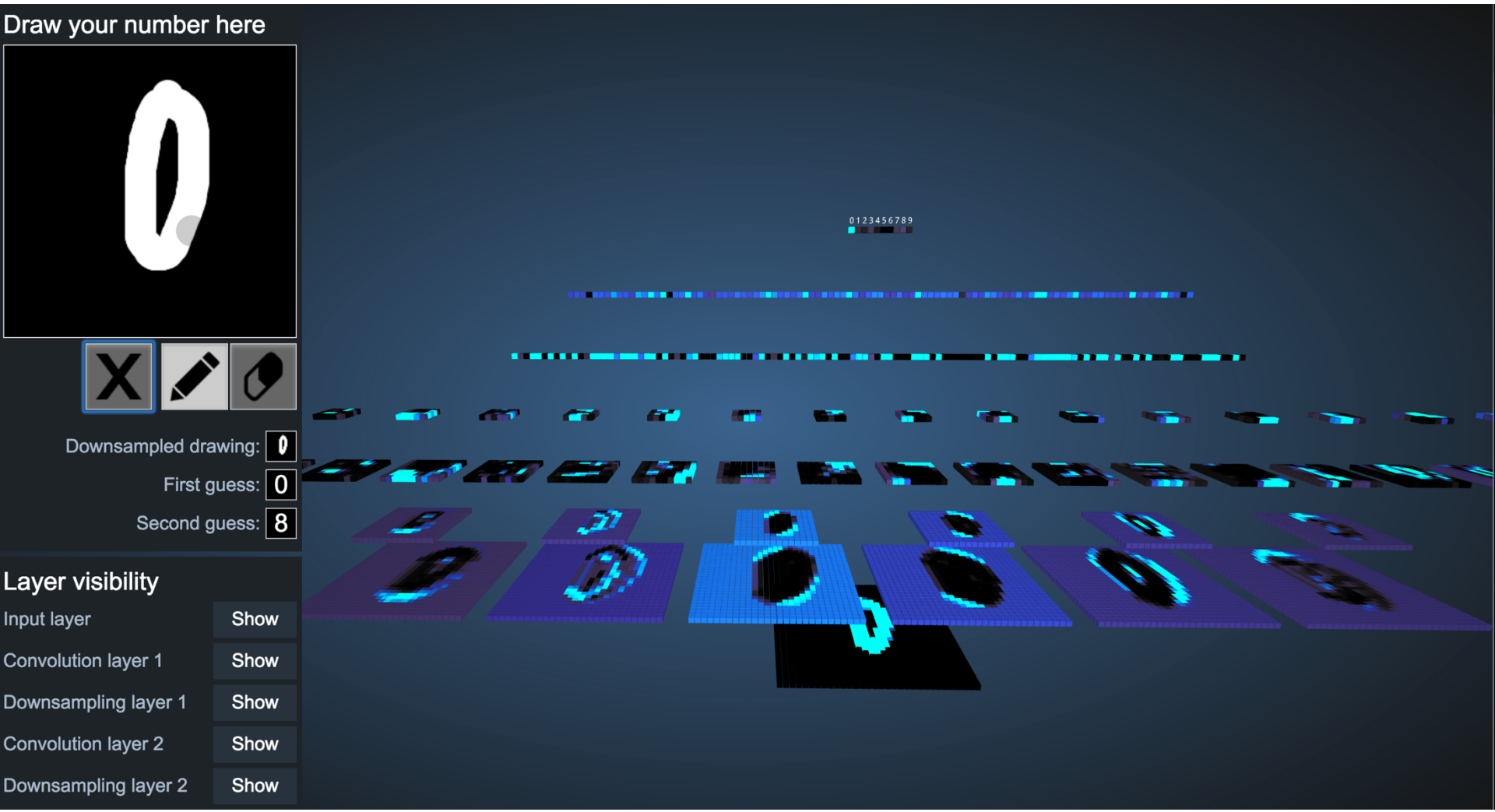


Deep Learning Assumption

- Model *conditional* likelihood P(Y | **X**) directly

$$P(Y = 1|\mathbf{X}) =$$ The output of a chain of logistic regressions

Demonstration



http://scs.ryerson.ca/~aharley/vis/conv/

Deep learning gets its ***intelligence*** from its thetas (aka its parameters)

How do we train?

MLE of Thetas!

2 Min Pause

Summarize what we have learned

MLE of Thetas!

First: Learning Goals…

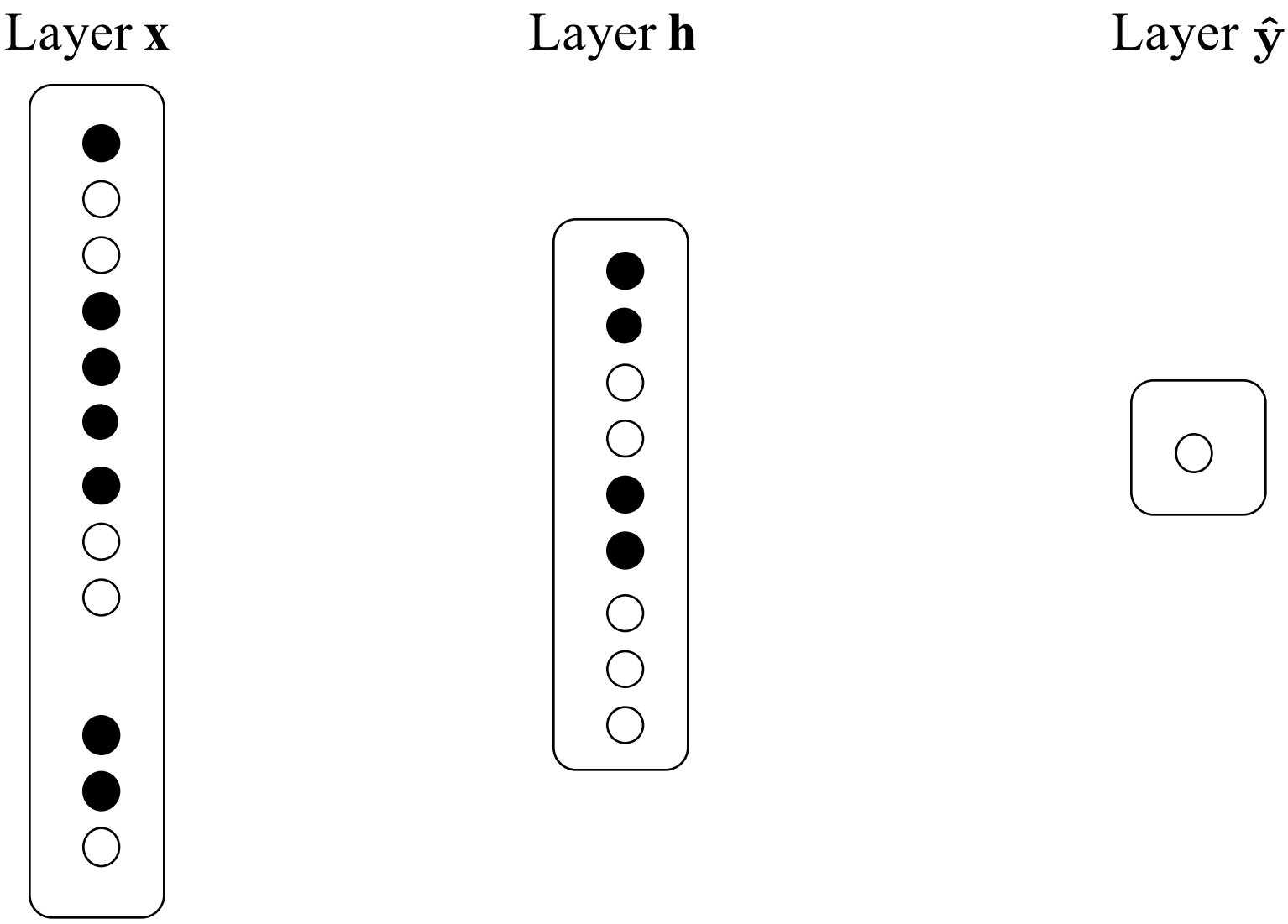
1. Understand Chain Rule as ♡ of Deep Learning

2. Everyone should be able to do simple derivations

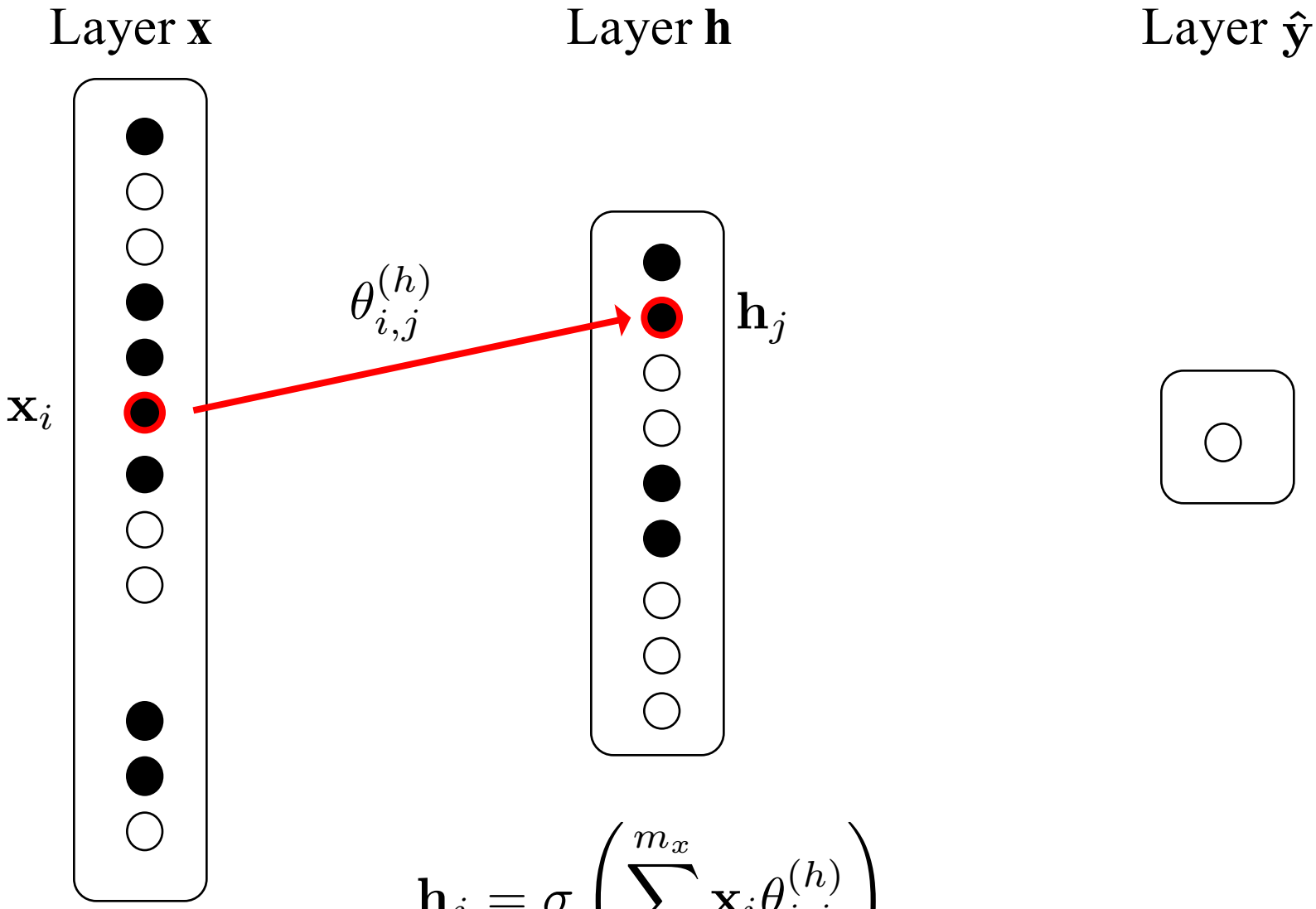
3. Be ready to rock
the socks off of CS221 and CS224N

Math worth knowing:

New Notation

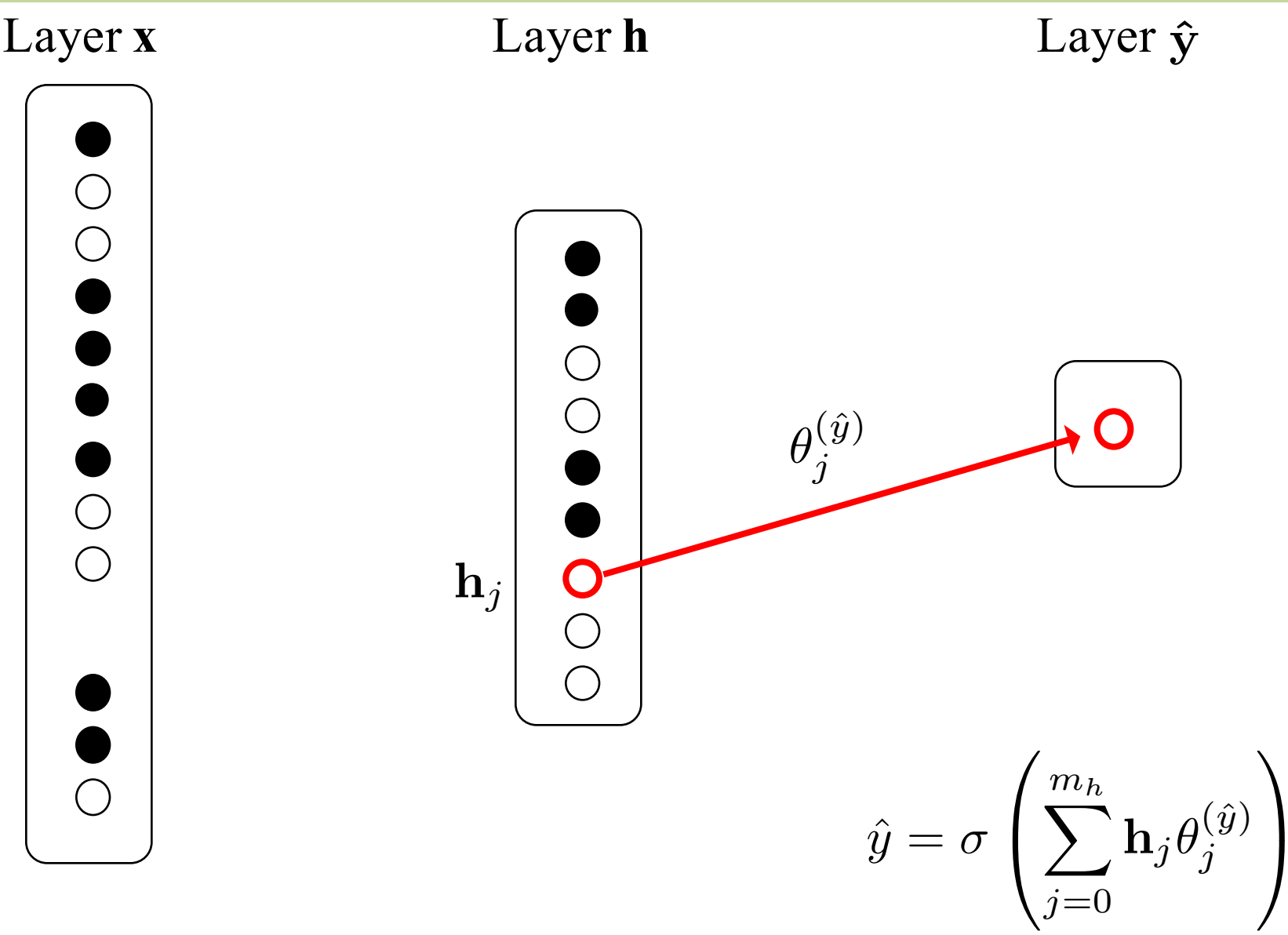


New Notation



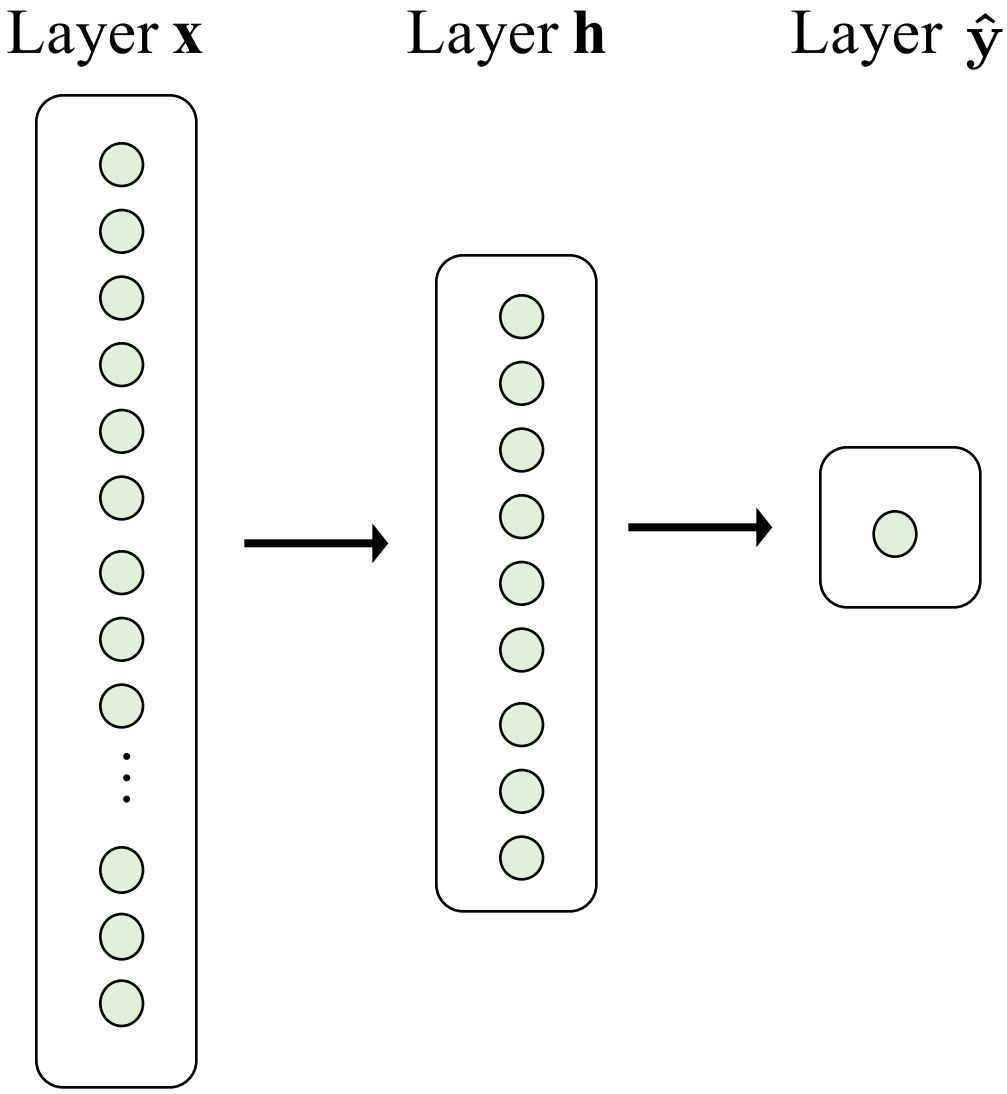
$$\mathbf{h}_j = \sigma\left(\sum_{i=0}^{m_x} \mathbf{x}_i \theta_{i,j}^{(h)}\right)$$

New Notation

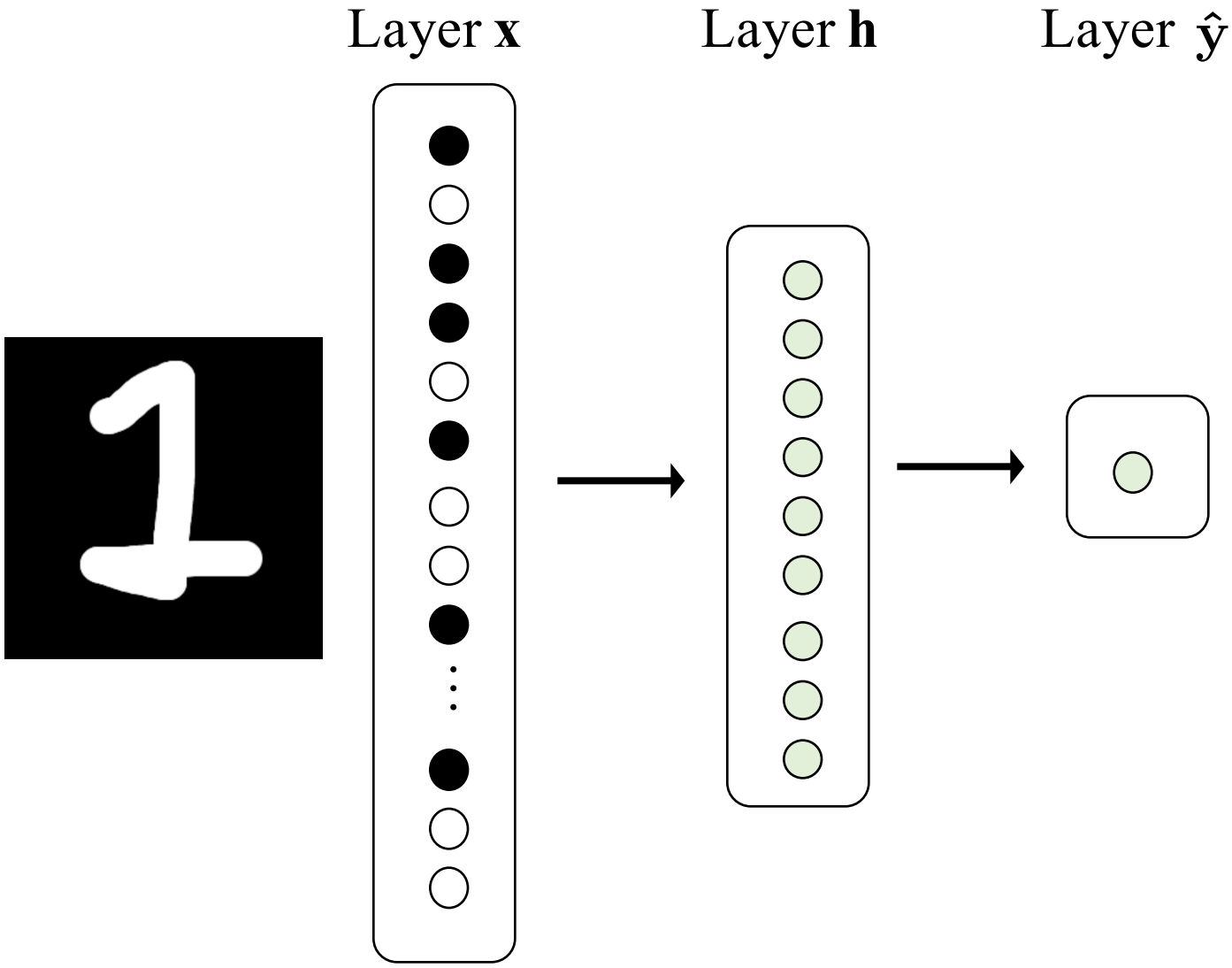


$$\hat{y} = \sigma\left(\sum_{j=0}^{m_h} \mathbf{h}_j \theta_j^{(\hat{y})}\right)$$

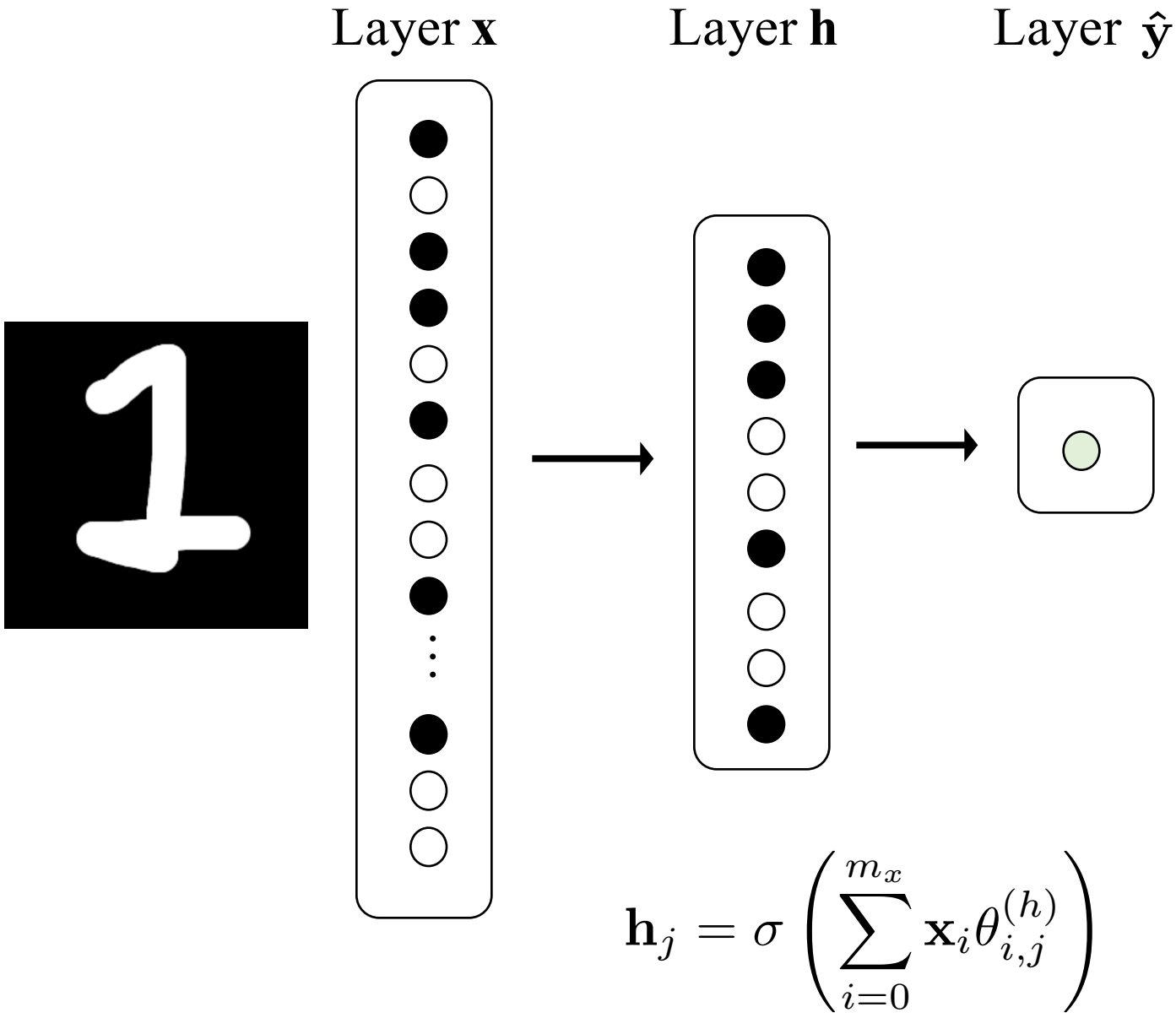
Forward Pass



Forward Pass

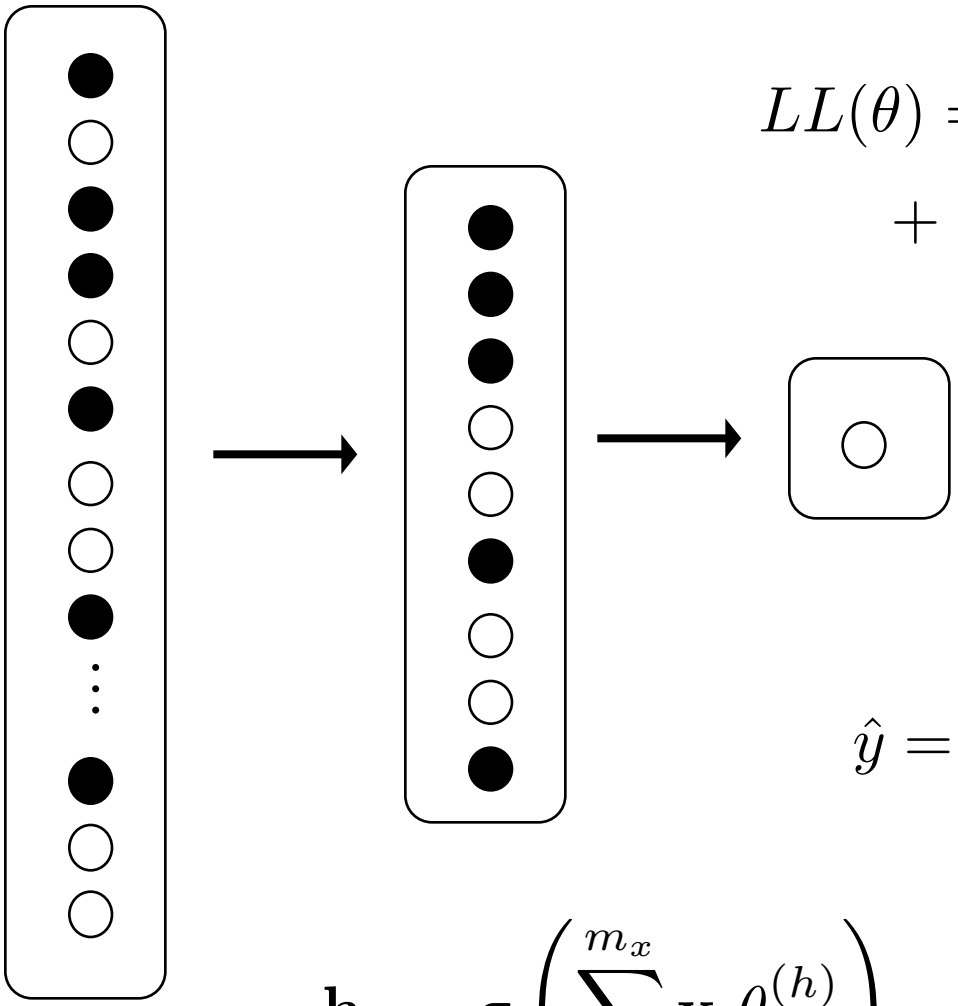


Forward Pass



$$\mathbf{h}_j = \sigma\left(\sum_{i=0}^{m_x} \mathbf{x}_i \theta_{i,j}^{(h)}\right)$$

Forward Pass

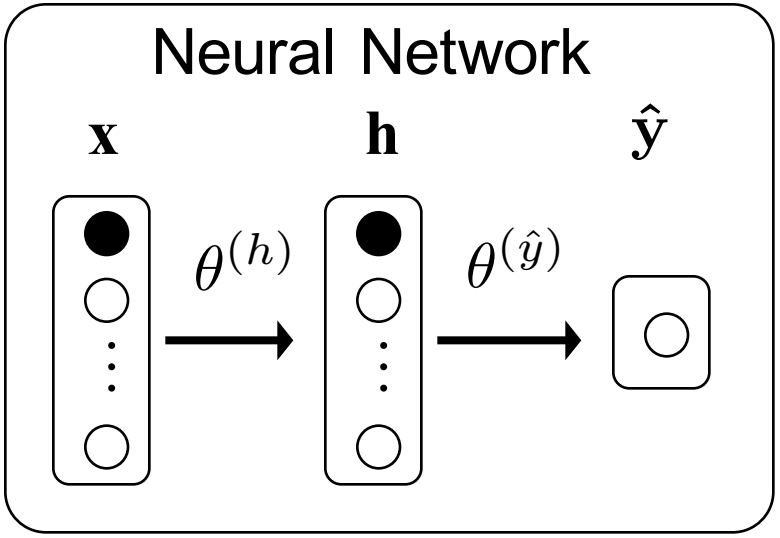


$$LL(\theta) = y \log \hat{y} + (1 - y) \log[1 - \hat{y}]$$

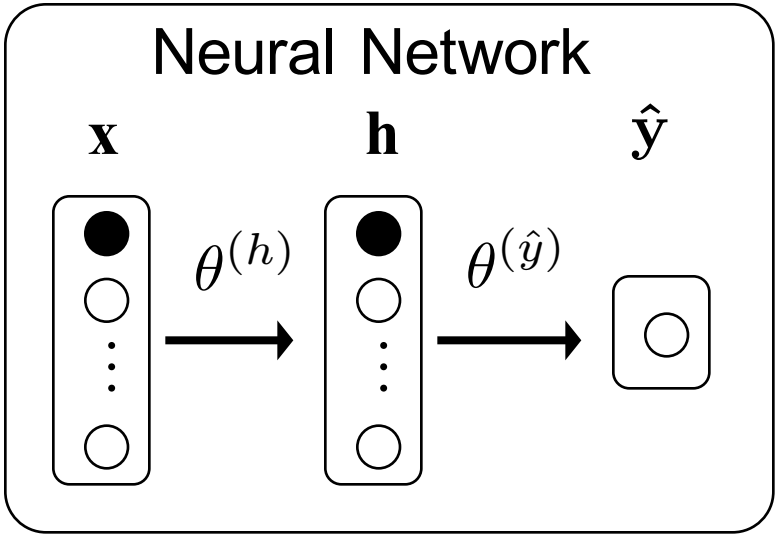
$$\hat{y} = \sigma\left(\sum_{j=0}^{m_h} \mathbf{h}_j \theta_j^{(\hat{y})}\right)$$

$$\mathbf{h}_j = \sigma\left(\sum_{i=0}^{m_x} \mathbf{x}_i \theta_{i,j}^{(h)}\right)$$

All Together



Sanity Check



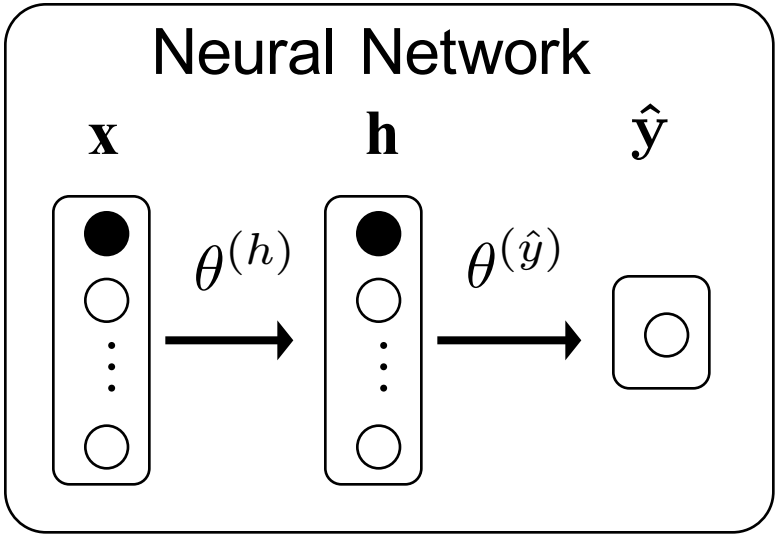
$$|\mathbf{x}| = 40$$

$$|\mathbf{h}| = 20$$

How many parameters in $\theta^{(h)}$?

c) 2 a) 20 b) 40 c) 800

Sanity Check 2



$$|\mathbf{x}| = 40$$

$$|\mathbf{h}| = 20$$

How many parameters in $\theta^{(\hat{y})}$?

c) 2 a) 20 b) 40 c) 800

Today: Do Something Brave

Only Have to Do Three Things

1. Make deep learning assumption

2. Calculate the log probability for all data

3. Get partial derivative of log likelihood with respect to each theta

Sanity Check

3 Get partial derivative of log likelihood with respect to each theta

Why?

Same Assumption, Same LL

$$P(Y = 1|X = \mathbf{x}) = \hat{\mathbf{y}}$$

For one datum

$$P(Y = y|X = \mathbf{X}) = (\hat{y})^y(1 - \hat{y})^{1-y}$$

For IID data

$$L(\theta) = \prod_{i=1}^{n} P(Y = y^{(i)}|X = \mathbf{x}^{(i)})$$

$$= \prod_{i=1}^{n} (\hat{y}^{(i)})^{y^{(i)}} \cdot \left[1 - (\hat{y}^{(i)})\right]^{(1-y^{(i)})}$$

Take the log

$$LL(\theta) = \sum_{i=0}^{n} y^{(i)} \log \hat{y}^{(i)} + (1 - y^{(i)}) \log[1 - \hat{y}^{(i)}]$$

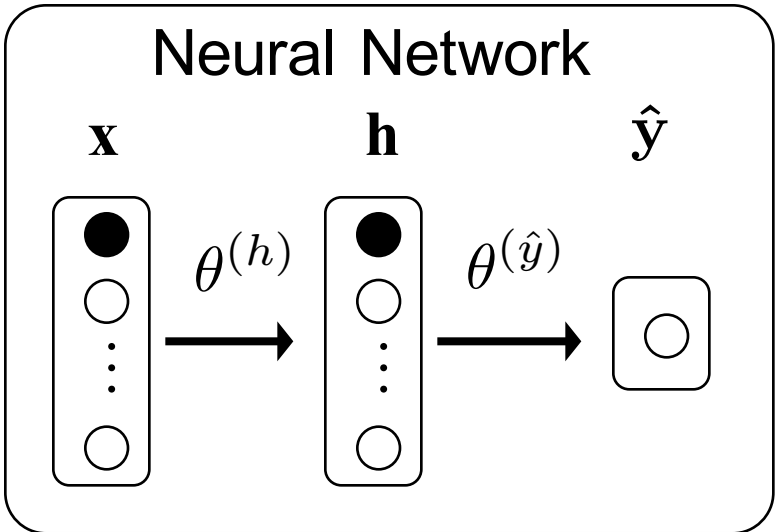
Derivative Goals

Loss with respect to output layer params

$$\frac{\partial LL(\theta)}{\partial \theta_i^{(\hat{y})}}$$

Loss with respect to hidden layer params

$$\frac{\partial LL(\theta)}{\partial \theta_{i,j}^{(h)}}$$



Think About Only One Training Instance

$$LL(\theta) = \sum_{i=0}^{n} y^{(i)} \log \hat{y}^{(i)} + (1 - y^{(i)}) \log[1 - \hat{y}^{(i)}]$$

We only need to calculate the gradient for one training example!

$$\frac{\partial}{\partial x} \sum f(x) = \sum \frac{\partial}{\partial x} f(x)$$

We will pretend we only have one example

$$LL(\theta) = y \log \hat{y} + (1 - y) \log[1 - \hat{y}]$$

We can sum up the gradients of each example to get the correct answer

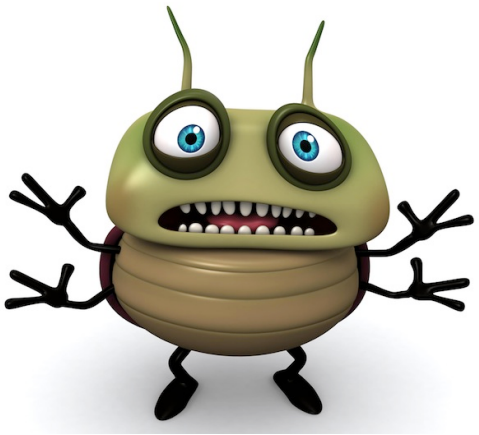
Bad Approach

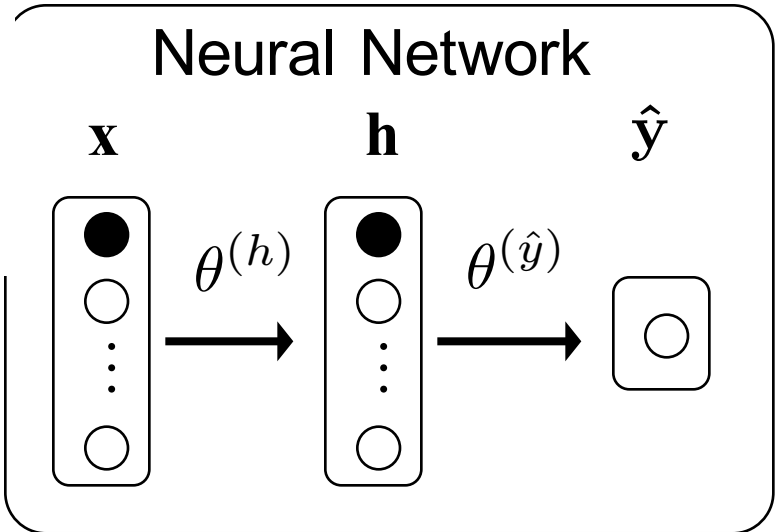
$$LL(\theta) = y \log \hat{y} + (1 - y) \log[1 - \hat{y}]$$

$$\hat{\mathbf{y}} = \sigma\left(\sum_{i=0}^{m_h} \mathbf{h}_i \theta_i^{(\hat{\mathbf{y}})}\right)$$

Math bug

$$= \sigma\left(\sum_{i=0}^{m_h}\left[\sigma\left(\sum_{i=0}^{m_x} \mathbf{x}_i \theta_{i,j}^{(\mathbf{h})}\right)\right]\theta_i^{(\hat{\mathbf{y}})}\right)$$





Big Idea

Big Idea

It's called chain rule

$$\frac{\partial f(x)}{\partial x} = \frac{\partial f(z)}{\partial z} \cdot \frac{\partial z}{\partial x}$$

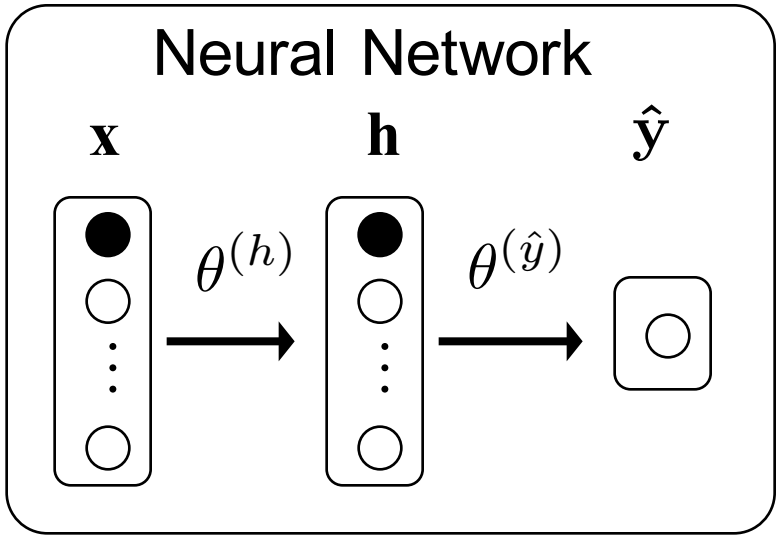
First use:

$$\frac{\partial LL(\theta)}{\partial \theta_i^{(\hat{y})}} = \frac{\partial LL}{\partial \hat{y}} \cdot \frac{\partial \hat{y}}{\partial \theta_i^{(\hat{y})}}$$

Chain Rule Example 1

$$\frac{\partial LL(\theta)}{\partial \theta_i^{(\hat{y})}}$$

Goal



Network

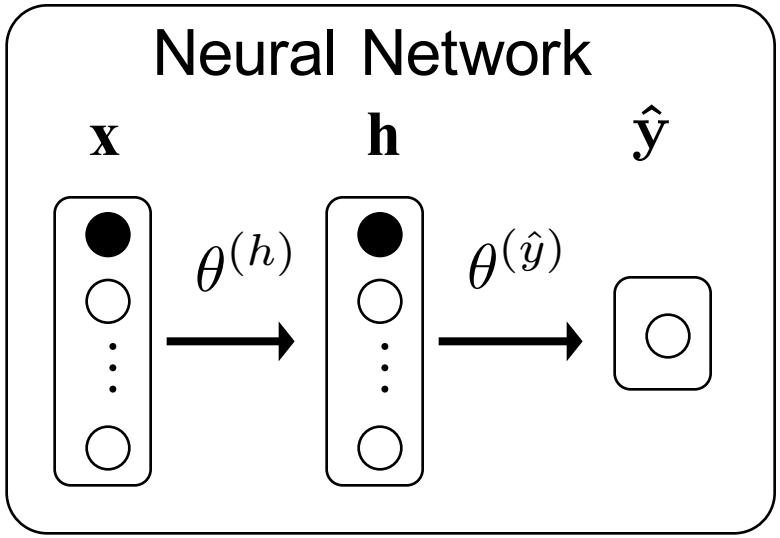
$$\frac{\partial LL(\theta)}{\partial \theta_i^{(\hat{y})}} = \frac{\partial LL}{\partial \hat{y}} \cdot \frac{\partial \hat{y}}{\partial \theta_i^{(\hat{y})}}$$

Decomposition

Chain Rule Example 2

$$\frac{\partial LL(\theta)}{\partial \theta_{i,j}^{(h)}}$$

Goal



Network

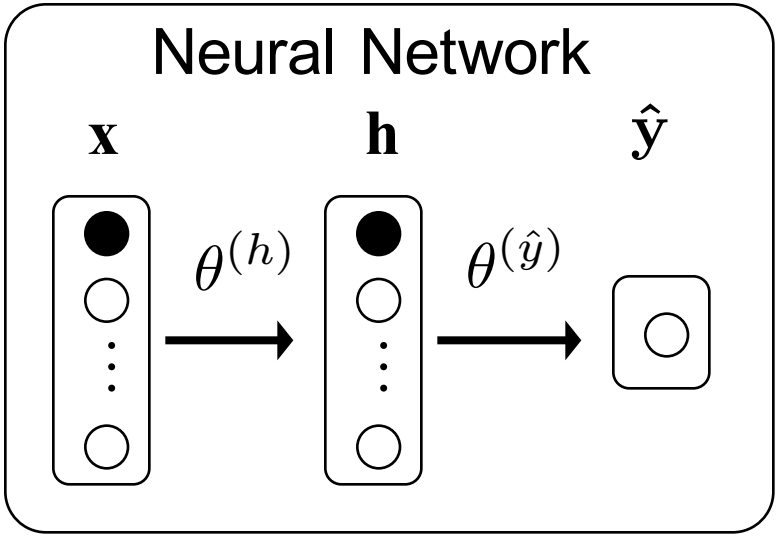
$$\frac{\partial LL(\theta)}{\partial \theta_{i,j}^{(h)}} = \frac{\partial LL}{\partial \hat{y}} \cdot \frac{\partial \hat{y}}{\partial \mathbf{h}_j} \cdot \frac{\partial \mathbf{h}_j}{\partial \theta_{i,j}^{(h)}}$$

Decomposition

Decomposition

Gradient of output layer params

$$\frac{\partial LL(\theta)}{\partial \theta_i^{(\hat{y})}} = \frac{\partial LL}{\partial \hat{y}} \cdot \frac{\partial \hat{y}}{\partial \theta_i^{(\hat{y})}}$$



Gradient of output layer params

$$\frac{\partial LL(\theta)}{\partial \theta_i^{(\hat{y})}} = \frac{\partial LL}{\partial \hat{y}} \cdot \frac{\partial \hat{y}}{\partial \theta_i^{(\hat{y})}}$$

$$LL(\theta) = y \log \hat{y} + (1 - y) \log[1 - \hat{y}]$$

$$\frac{\partial LL(\theta)}{\partial \hat{y}} = \frac{y}{\hat{y}} + \frac{(1 - y)}{(1 - \hat{y})} \cdot \frac{\partial (1 - \hat{y})}{\partial \hat{y}}$$

$$\frac{\partial LL(\theta)}{\partial \hat{y}} = \frac{y}{\hat{y}} - \frac{(1 - y)}{(1 - \hat{y})}$$

Recall: Sigmoid has a Beautiful Slope

True fact about sigmoid functions

$$\frac{\partial}{\partial z}\sigma(z) = \sigma(z)[1 - z]$$

Sigmoid, you should be a ski hill

Gradient of output layer params

$$\frac{\partial LL(\theta)}{\partial \theta_i^{(\hat{y})}} = \frac{\partial LL}{\partial \hat{y}} \cdot \frac{\partial \hat{y}}{\partial \theta_i^{(\hat{y})}}$$

$$\hat{y} = \sigma\left(\sum_{j=0}^{m_h} \mathbf{h}_j \theta_j^{(\hat{y})}\right)$$

$$\frac{\partial \hat{y}}{\partial \theta_i^{(\hat{y})}} = \sigma\left(\sum_{j=0}^{m_h} \mathbf{h}_j \theta_j^{(\hat{y})}\right)\left[1 - \sigma\left(\sum_{j=0}^{m_h} \mathbf{h}_j \theta_j^{(\hat{y})}\right)\right] \cdot \frac{\partial}{\partial \theta_i^{(\hat{y})}} \sum_{j=0}^{m_h} \mathbf{h}_j \theta_j^{(\hat{y})}$$

$$= \hat{y}[1 - \hat{y}] \cdot \frac{\partial}{\partial \theta_i^{(\hat{y})}} \sum_{j=0}^{m_h} \mathbf{h}_j \theta_j^{(\hat{y})}$$

$$= \hat{y}[1 - \hat{y}] \cdot h_i$$

What! That's not scary!

Make it Simple

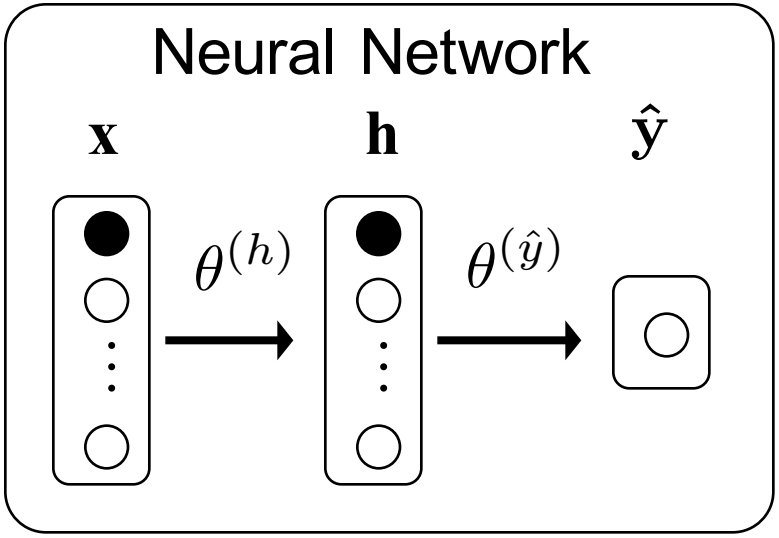
$$\frac{\partial LL(\theta)}{\partial \theta_i^{(\hat{y})}} = \text{  } \cdot \text{  }$$

$$\text{  } = \frac{y}{\hat{y}} - \frac{(1-y)}{(1-\hat{y})}$$

$$\text{  } = \hat{y}[1-\hat{y}] \cdot h_i$$

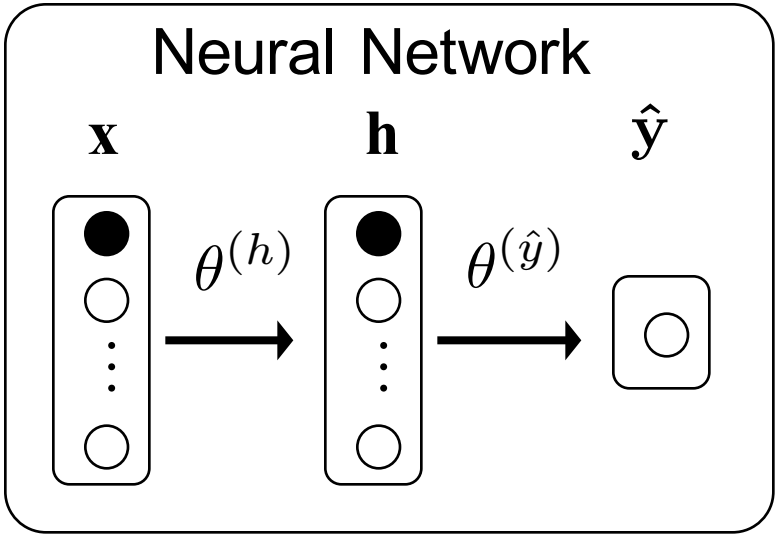
Boom!

$$\frac{\partial LL(\theta)}{\partial \theta_{i,j}^{(h)}}$$



Gradient of hidden layer params

$$\frac{\partial LL(\theta)}{\partial \theta_{i,j}^{(h)}} = \frac{\partial LL}{\partial \hat{y}} \cdot \frac{\partial \hat{y}}{\partial \mathbf{h}_j} \cdot \frac{\partial \mathbf{h}_j}{\partial \theta_{i,j}^{(h)}}$$



Gradient of hidden layer params

$$\frac{\partial LL(\theta)}{\partial \theta_{i,j}^{(h)}} = \frac{\partial LL}{\partial \hat{y}} \cdot \frac{\partial \hat{y}}{\partial \mathbf{h}_j} \cdot \frac{\partial \mathbf{h}_j}{\partial \theta_{i,j}^{(h)}}$$

$$\hat{y} = \sigma\left(\sum_{i=0}^{m_h} \mathbf{h}_i \theta_i^{(\hat{y})}\right)$$

$$\frac{\partial \hat{y}}{\partial \mathbf{h}_j} = \hat{y}[1-\hat{y}]\theta_j^{(\hat{y})}$$

Wait is it over?

Gradient of hidden layer params

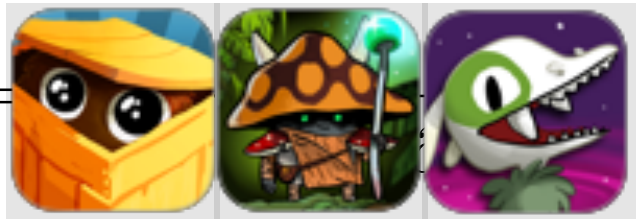
$$\frac{\partial LL(\theta)}{\partial \theta_{i,j}^{(h)}} = \frac{\partial LL}{\partial \hat{y}} \cdot \frac{\partial \hat{y}}{\partial \mathbf{h}_j} \cdot \frac{\partial \mathbf{h}_j}{\partial \theta_{i,j}^{(h)}}$$

$$\mathbf{h}_j = \sigma\left(\sum_{k=0}^{m_x} \mathbf{x}_k \theta_{k,j}\right)$$

$$\frac{\partial \mathbf{h}_j}{\partial \theta_{i,j}^{(h)}} = \mathbf{h}_j[1 - \mathbf{h}_j]\mathbf{x}_j$$

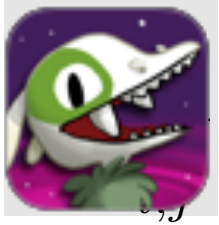
That one too?

Make it Simple

$$\frac{\partial LL(\theta)}{\partial \theta_{i,j}^{(h)}} =  $$

$$= \frac{y}{\hat{y}} - \frac{(1-y)}{(1-\hat{y})}$$

$$= \hat{y}[1-\hat{y}]\theta_j^{(\hat{y})}$$



$$= \mathbf{h}_j[1-\mathbf{h}_j]\mathbf{x}_j$$

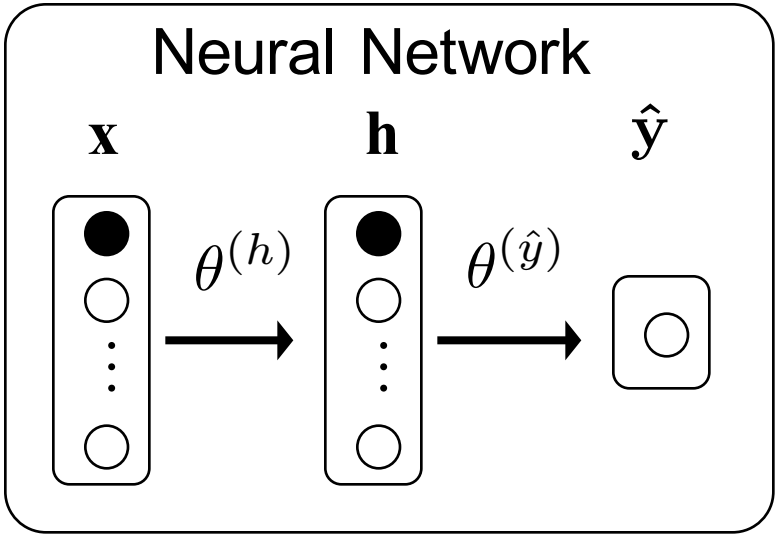
Summary: Simple Calculations For

Loss with respect to output layer params

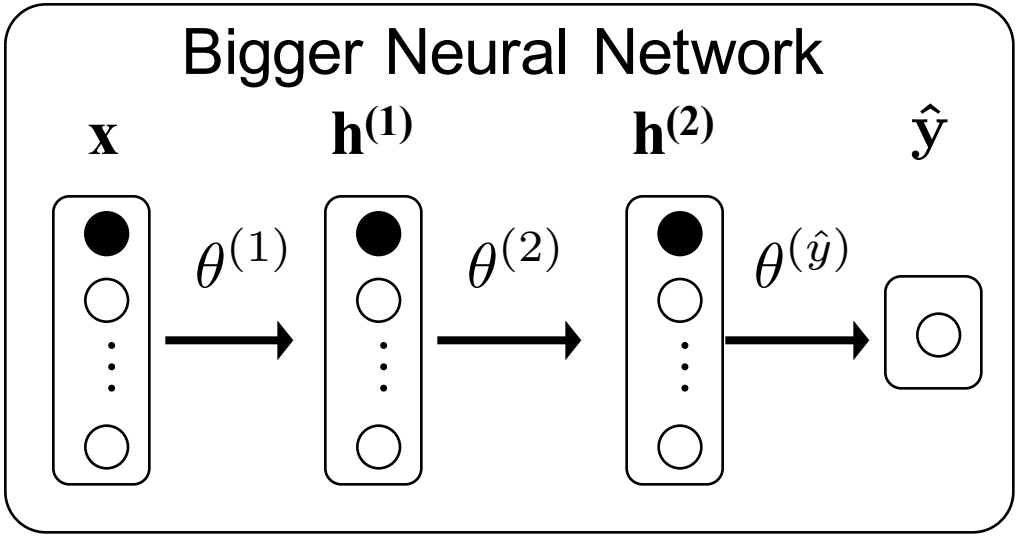
$$\frac{\partial LL(\theta)}{\partial \theta_i^{(\hat{y})}}$$

Loss with respect to hidden layer params

$$\frac{\partial LL(\theta)}{\partial \theta_{i,j}^{(h)}}$$



What Would You Do Here?

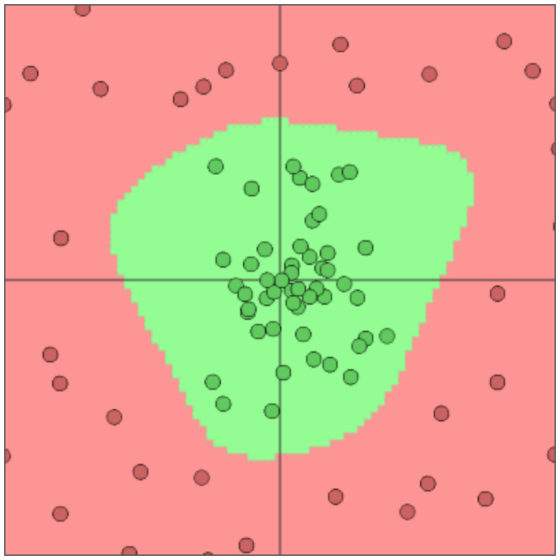


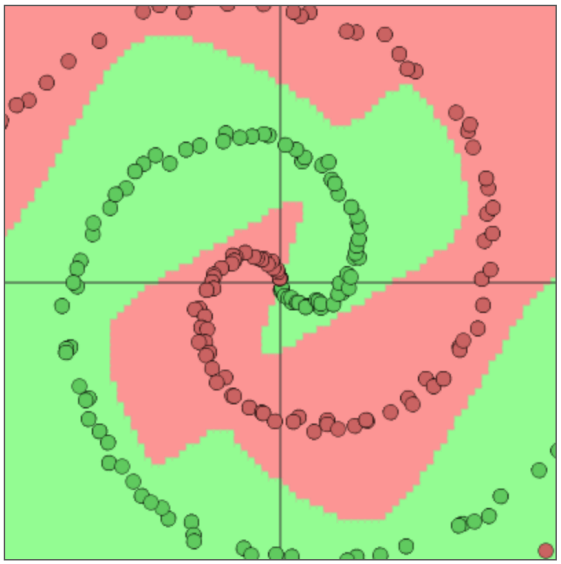
Chain rule:
Game changer for
artificial intelligence

Congrats. You now know backpropagation

Neural Networks Can Learn Complex Functions

- Some data sets/functions are not separable





  - These are classifiers learned by neural networks

http://cs.stanford.edu/people/karpathy/convnetjs/demo/classify2d.html