Independence

Learning Goals

1. Be able to recognize independent events
2. Use independence rules to calculate probabilities
3. Recognize and use *conditional* independencies

Summary

Two events A and B are called **independent** if:

$$P(AB) = P(A)P(B) \qquad P(A|B) = P(A)$$

Otherwise, they are called **dependent** events

Two events A and B are

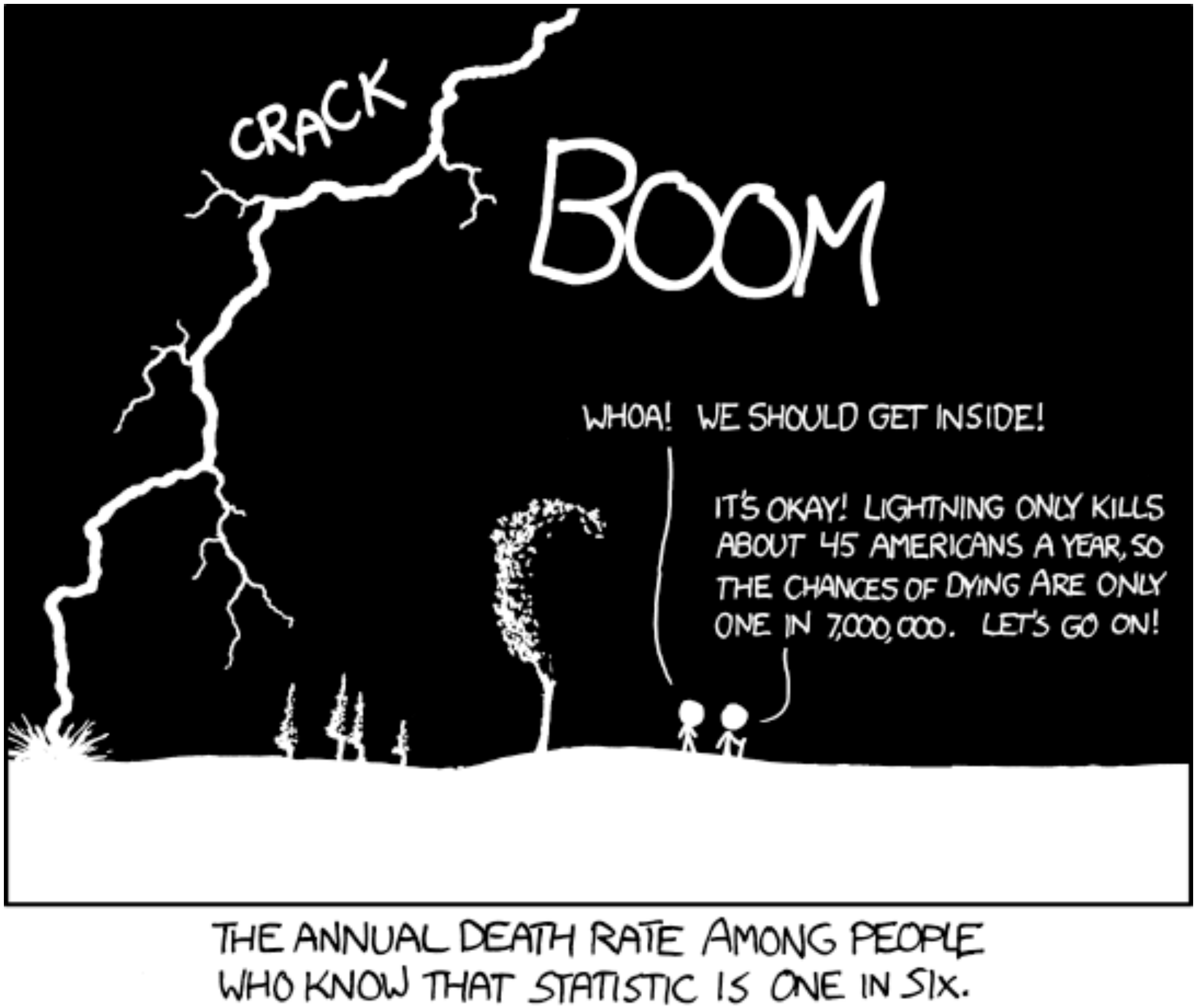
conditionally independent on C if:

$$P(AB|C) = P(A|C)P(B|C)$$

$$P(A|BC) = P(A|C)$$

Review

The Tragedy of Conditional Prob



Thanks xkcd! `http://xkcd.com/795/`

And vs Condition

$$P(AB) \text{ vs } P(A|B)$$

$$P(AB) = P(A|B)P(B)$$

A Few Useful Formulas

- For any events A and B:

$P(A\ B) = P(B\ A)$ (Commutativity)

$P(A\ B) = P(A \mid B)\, P(B)$ (Chain rule)

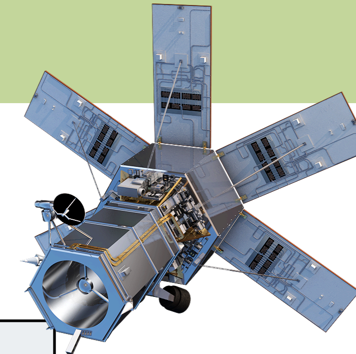
$= P(B \mid A)\, P(A)$

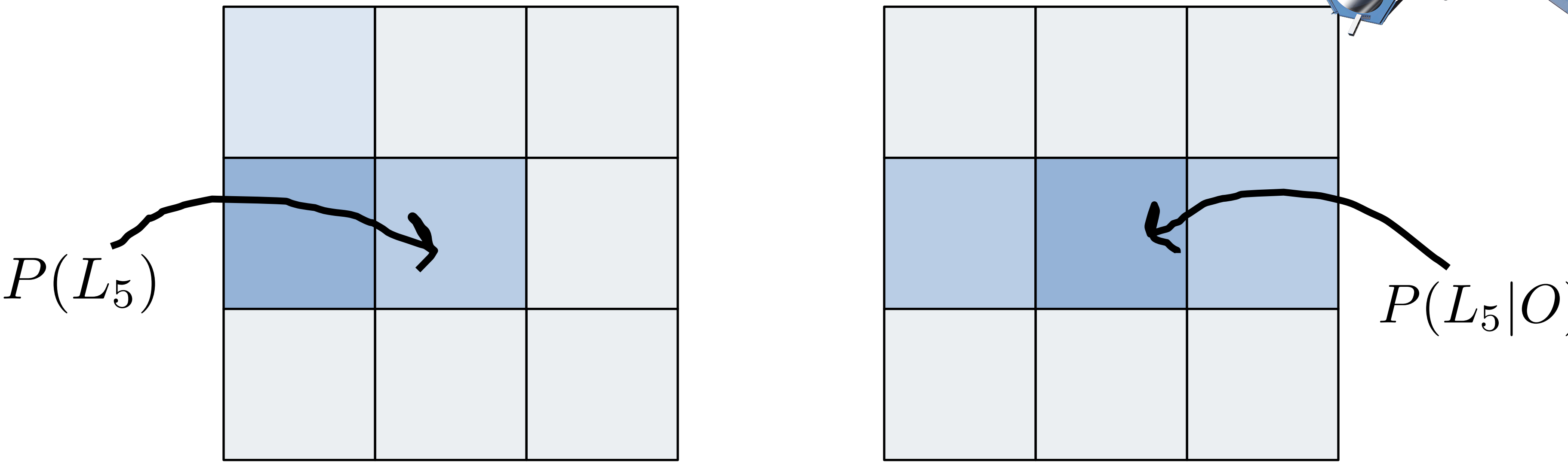
$P(A\ B^c) = P(A) - P(AB)$ (Intersection)

$P(A) + P(A^c) = 1$ (Total Probability)

$$P(A \mid B) = \frac{P(B \mid A)\, P(A)}{P(B)}$$ (Bayes Theorem)

Bayes: Update Belief





Before Observation

After Observation

$$P(L_5|O) = \frac{P(O|L_5)P(L_5)}{P(O)}$$

Generality of Conditional Probability

- For any events A, B, and E, you can condition consistently on E, and these formulas still hold:

$$P(A\,B \mid E) = P(B\,A \mid E)$$

$$P(A\,B \mid E) = P(A \mid B\,E)\,P(B \mid E)$$

$$P(A \mid B\,E) = \frac{P(B \mid A\,E)\,P(A \mid E)}{P(B \mid E)} \qquad \text{(Bayes' Thm.)}$$

- Can think of E as "everything you already know"
- Formally, P(• | E) satisfies 3 axioms of probability

BAE's Theorem?

$$P(A \mid B\ E) = \frac{P(B \mid A\ E)\ P(A \mid E)}{P(B \mid E)}$$

End Review

Today, start with a cool program


G_1
G_2
G_3
G_4
G_5
T

G_1 G_2 G_3 G_4 G_5 T

dna.txt — dna

```
False,True,False,False,True,False
True,True,False,True,True,False
True,True,False,True,True,True
False,True,False,True,True,False
False,True,False,False,True,False
True,True,False,True,True,True
False,False,True,False,False,False
False,False,True,False,True,False
True,False,False,True,False,False
False,True,False,True,True,False
True,False,False,True,False,False
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True,False,False,True,False,False
True,True,False,True,True,True
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False,False,False,True,False,False
```
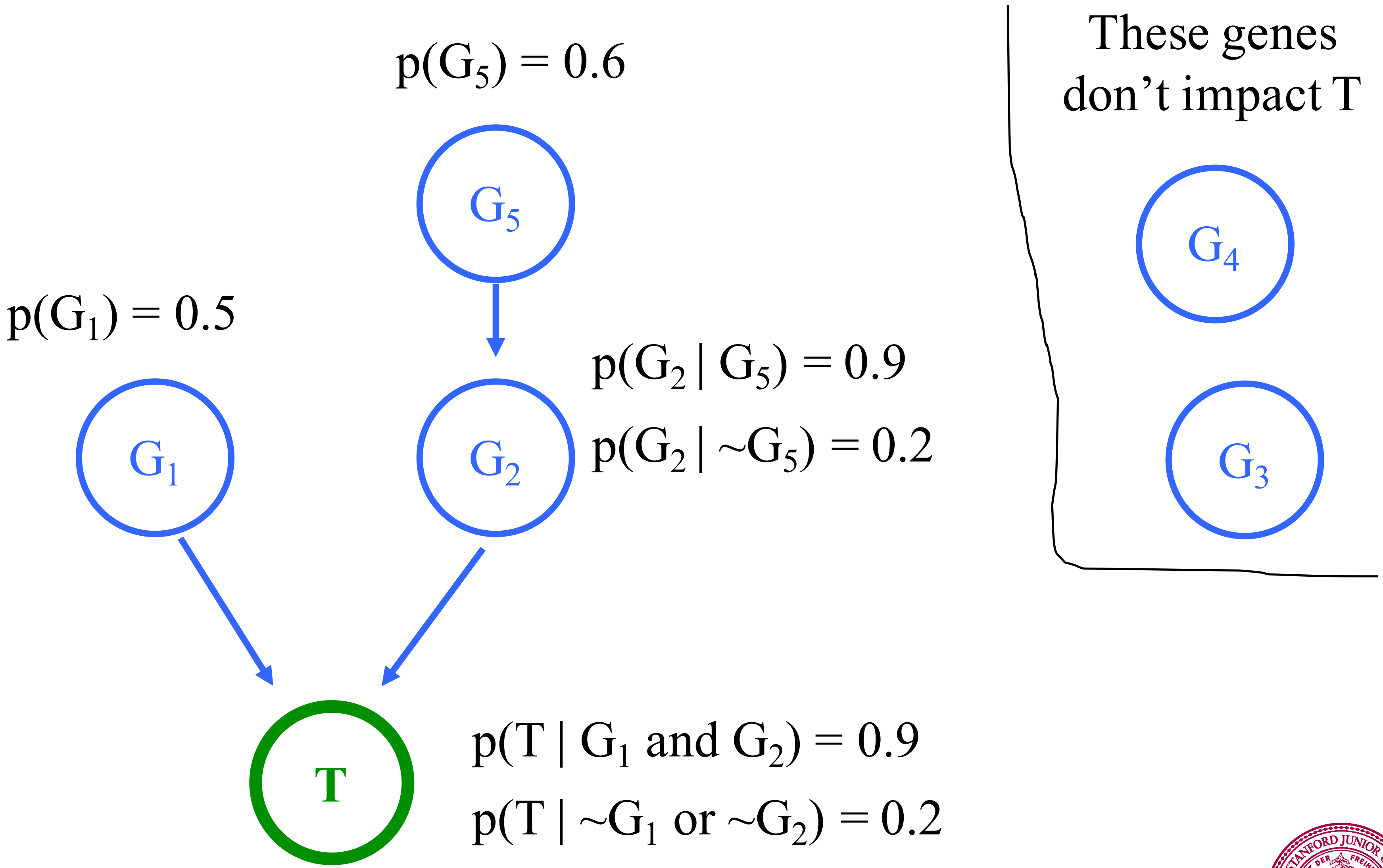

100,000 samples

6 observations per sample

Discovered Pattern



We’ve gotten ahead of ourselves

Source: The Ho

Start at the beggining

Source: The Ho

Independence

Two events A and B are called **independent** if:

$$P(AB) = P(A)P(B)$$

Or, equivalently:

$$P(A|B) = P(A)$$

Otherwise, they are called **dependent** events

Dice, Our Misunderstood Friends

- Roll two 6-sided dice, yielding values D_1 and D_2
  - Let E be event: $D_1 = 1$
  - Let F be event: $D_2 = 1$

- What is P(E), P(F), and P(EF)?
  - P(E) = 1/6, P(F) = 1/6, P(EF) = 1/36
  - P(EF) = P(E) P(F) → E and F *<u>independent</u>*

- Let G be event: $D_1 + D_2 = 5$ {(1, 4), (2, 3), (3, 2), (4, 1)}
- What is P(E), P(G), and P(EG)?
  - P(E) = 1/6, P(G) = 4/36 = 1/9, P(EG) = 1/36
  - P(EG) ≠ P(E) P(G) → E and G *<u>dependent</u>*

Intuition through proofs:

Independence with Proofs

Let A and B be independent

$$P(A|B) = \frac{P(AB)}{P(B)}$$

Definition of conditional probability

$$= \frac{P(A)P(B)}{P(B)}$$

Since A and B are independent

$$= P(A)$$

Taking the bus to cancel city

Knowing that event B happened, doesn't change our belief that A will happen.

Independence

Given independent events A and B, prove that A and B^C are independent

We want to show that $P(AB^C) = P(A)P(B^C)$

$$
\begin{aligned}
P(AB^C) &= P(A) - P(AB) && \text{By Intersection Rule} \\
&= P(A) - P(A)P(B) && \text{By independence} \\
&= P(A)[1 - P(B)] && \text{Factoring} \\
&= P(A)P(B^C) && \text{Since } P(B) + P(B^C) = 1
\end{aligned}
$$

So if A and B are independent A and B^C are also independent

Independence

Let A and B be independent

$$P(A|B) = P(A)$$

From our first proof

A and B^C are independent

From our second proof

And thus:

$$P(A|B^C) = P(A)$$

Since A and BC are independent

$$P(A|B) = P(A) = P(A|B^C)$$

Put it all together

Intuitively, if A and B are independent, knowing whether B holds gives us no information about A

Generalization

Generalized Independence

- General definition of Independence:

  Events E_1 , E_2 , ..., E_n are independent if for every subset with r elements (where *r* ≤ *n*) it holds that:

$$P(E_{s_1}, E_{s_2}, E_{s_3}, \ldots E_{s_r})$$
$$= P(E_{s_1})P(E_{s_2})P(E_{s_3})\ldots P(E_{s_r})$$

- <u>Example</u>: outcomes of *n* separate flips of a coin are all independent of one another
  - Each flip in this case is called a “trial” of the experiment

Math > Intuition

Two Dice

- Roll two 6-sided dice, yielding values D_1 and D_2
  - Let E be event: $D_1 = 1$
  - Let F be event: $D_2 = 6$
  - Are E and F independent? Yes!
- Let G be event: $D_1 + D_2 = 7$
  - Are E and G independent? Yes!
  - P(E) = 1/6, P(G) = 1/6, P(E G) = 1/36 [roll (1, 6)]
  - Are F and G independent? Yes!
  - P(F) = 1/6, P(G) = 1/6, P(F G) = 1/36 [roll (1, 6)]
  - Are E, F and G independent? No!
  - $P(EFG) = 1/36 \neq 1/216 = (1/6)(1/6)(1/6)$

New Ability

Generating Random Bits

- A computer produces a series of random bits, with probability *p* of producing a 1.
  - Each bit generated is an independent trial
  - E = first *n* bits are 1's, followed by a single 0
  - What is P(E)?
- Solution
  - P(first *n* 1's) = P(1st bit=1) P(2nd bit=1) ... P(nth bit=1)
    $= p^n$
  - P(*n+1* bit=0) = (1 – p)
  - P(E) = P(first *n* 1's) P(*n+1* bit=0) = p^n (1 – p)

Coin Flips

- Say a coin comes up heads with probability p
  - Each coin flip is an independent trial

- P(*n* heads on *n* coin flips) = p^n
- P(*n* tails on *n* coin flips) = $(1 - p)^n$

- P(first *k* heads, then *n* – *k* tails) = $p^k(1-p)^{n-k}$

- P(exactly *k* heads on *n* coin flips) =?

Important Result

P(exactly *k* heads on *n* coin flips)? $\binom{n}{k} p^k (1-p)^{n-k}$

Think of the flips as ordered:

Ordering 1: T, H, H, T, T, T….

Ordering 2: H, T, H, T, T, T….

And so on…

The coin flips are independent!

$$P(F_i) = p^k (1-p)^{n-k}$$

Let's make each ordering with *k* heads an event… F_i

P(exactly *k* heads on *n* coin flips) = P(any one of the events)

P(exactly *k* heads on *n* coin flips) = P(F_1 or F_2 or F_3…)

Those events are mutually exclusive!

Moment of Crystallization

Add vs Multiply?


Batman vs Superman
COMING SOON
#BATMAN v SUPERMAN
SEE IT IN 3D

Add vs Multiply

+ vs ×

Add vs Multiply

Probabilities

multiply

P(AB)	Generally:	P(A)P(B|A)
	Independent:	P(A)P(B)
P(A∪B)	Generally:	P(A) + P(B) – P(AB)
	Mutually Exclusive:	P(A) + P(B)

add

Add vs Multiply

Counting

multiply

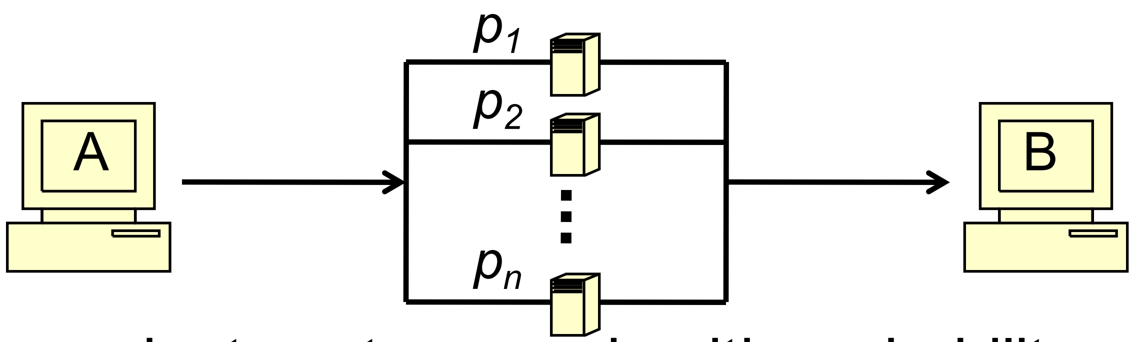
A then B	Generally:	*Not handled*
	Independent Counts:	$\lvert A \rvert \lvert B \rvert$
$A \cup B$	Generally:	$\lvert A \rvert + \lvert B \rvert - \lvert AB \rvert$
	Mutually Exclusive:	$\lvert A \rvert + \lvert B \rvert$

add

Combining with Previous Skills

Sending a Message Through Network

- Consider the following parallel network:



  - n independent routers, each with probability p_i of functioning (where $1 \leq i \leq n$)
  - E = functional path from A to B exists. What is P(E)?

- Solution:
  - P(E) = 1 – P(all routers fail)

$$= 1 - (1 - p_1)(1 - p_2)\ldots(1 - p_n)$$

$$= 1 - \prod_{i=1}^{n} (1 - p_i)$$

Yet More Hash Tables

- m strings are hashed (unequally) into a hash table with n buckets
  - Each string hashed is an independent trial, with probability p_i of getting hashed to bucket i
  - E = At least 1 of buckets 1 to k has ≥ 1 string hashed to it
- Solution
  - F_i = at least one string hashed into i-th bucket
  - $P(E) = P(F_1 \cup F_2 \cup \ldots \cup F_k) = 1 - P((F_1 \cup F_2 \cup \ldots \cup F_k)^c)$
    $= 1 - P(F_1^c F_2^c \ldots F_k^c)$ (DeMorgan's Law)
  - $P(F_1^c F_2^c \ldots F_k^c) = P(\text{no strings hashed to buckets 1 to } k)$
    $= (1 - p_1 - p_2 - \ldots - p_k)^m$
  - $P(E) = 1 - (1 - p_1 - p_2 - \ldots - p_k)^m$

The Hardest Example

- *m* strings are hashed (unequally) into a hash table with *n* buckets
  - Each string hashed is an independent trial, with probability p_i of getting hashed to bucket *i*
  - E = Each of buckets 1 to *k* has ≥ 1 string hashed to it
- Solution
  - F_i = at least one string hashed into *i*-th bucket
  - P(E) $= P(F_1 F_2 \ldots F_k) = 1 - P((F_1 F_2 \ldots F_k)^c)$

$$= 1 - P(F_1^c \cup F_2^c \cup \ldots \cup F_k^c) \quad \text{(DeMorgan's Law)}$$

$$= 1 - P\left(\bigcup_{i=1}^{k} F_i^c\right) = 1 - \sum_{r=1}^{k} (-1)^{(r+1)} \sum_{i_1 < \ldots < i_r} P(F_{i_1}^{\ c} F_{i_2}^{\ c} \ldots F_{i_r}^{\ c})$$

where $P(F_{i_1}^{\ c} F_{i_2}^{\ c} \ldots F_{i_r}^{\ c}) = (1 - p_{i_1} - p_{i_2} - \ldots - p_{i_r})^m$

Phew…

Conditional Independence

Recall, two events A and B are **independent** if:

$$P(A) = P(A)P(B)$$

$$P(A|B) = P(A)$$

Two events E and F are **conditionally independent** on C if:

$$P(AB|C) = P(A|C)P(B|C)$$

$$P(A|BC) = P(A|C)$$

NETFLIX

And Learn

Netflix and Learn

What is the probability that a user will watch Life is Beautiful?

$P(E)$

$$P(E) = \lim_{n \to \infty} \frac{n(E)}{n} \approx \frac{\#\text{people who watched movie}}{\#\text{people on Netflix}}$$

$P(E) = 10{,}234{,}231 \;/\; 50{,}923{,}123 = 0.20$

Netflix and Learn

What is the probability that a user will watch Life is Beautiful, given they watched Amelie?

$P(E|F)$

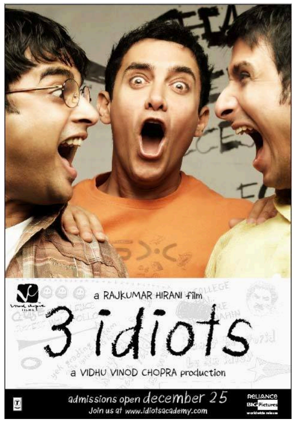
$$P(E|F) = \frac{P(EF)}{P(F)} = \frac{\#\text{people who watched both}}{\#\text{people who watched } F}$$

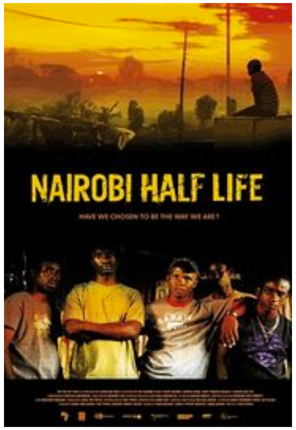
$P(E|F) = 0.42$

Conditioned on watching a set of movies?

Netflix and Learn

Each event corresponds to watching a particular movie





E_1 E_2 E_3 E_4

$$P(E_4 | E_1, E_2, E_3)?$$

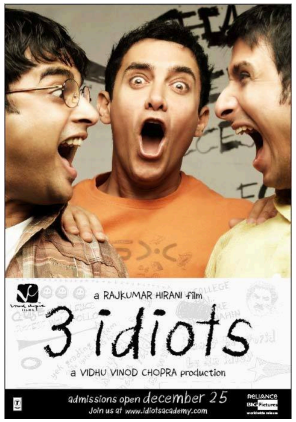
Is E_4 independent of E_1,E_2,E_3?

Netflix and Learn

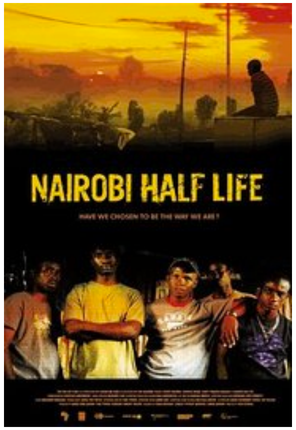
Is E_4 independent of E_1, E_2, E_3?

E_1

E_2



E_3



E_4

$$P(E_4 | E_1, E_2, E_3) \stackrel{?}{=} P(E_4)$$

Netflix and Learn

- What is the probability that a user watched four particular movies?
  - There are 13,000 titles on Netflix
  - The user watches 30 random titles
  - E = movies watched include the given four.
- Solution:

Watch those four

Choose 24 movies not in the set

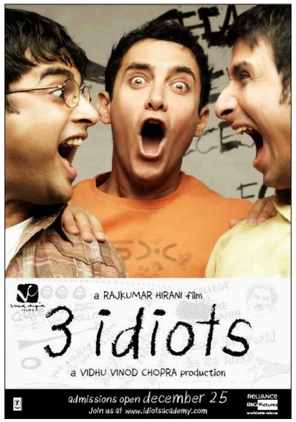
$$P(E) = \frac{\binom{4}{4}\binom{12996}{24}}{\binom{13000}{30}} = 10^{-11}$$

Choose 30 movies from netflix

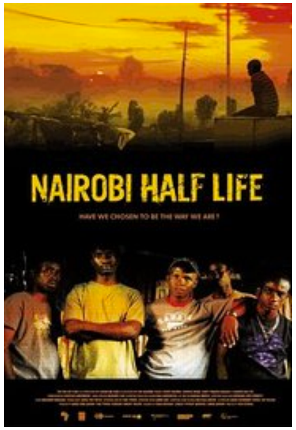
Netflix and Learn

E_1

E_2



E_3



E_4

Netflix and Learn

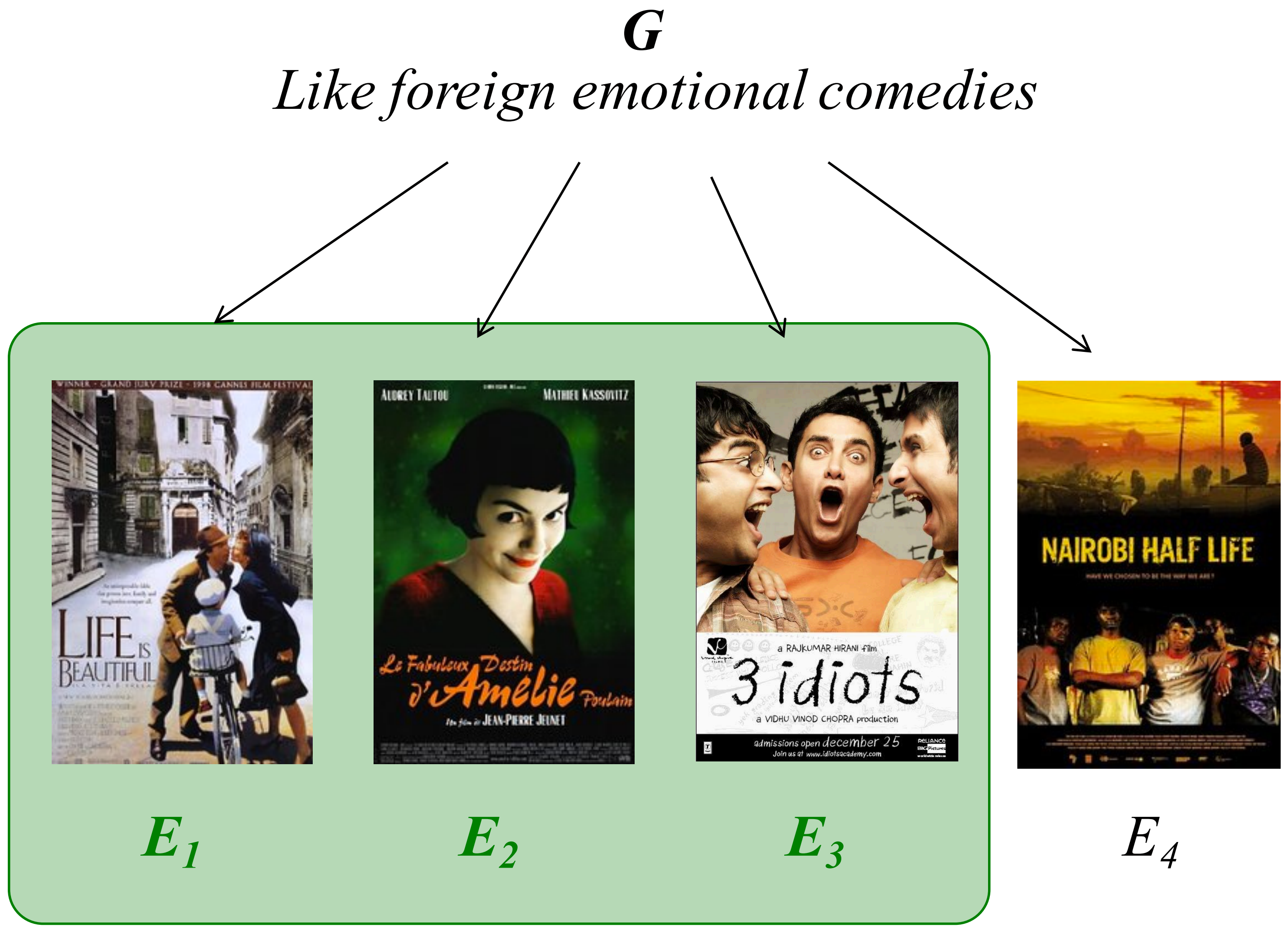
G

Like foreign emotional comedies

E_1 E_2 E_3 E_4

Assume E_1, E_2, E_3 and E_4 are conditionally independent given G

Netflix and Learn

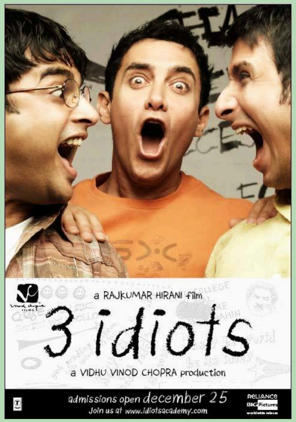


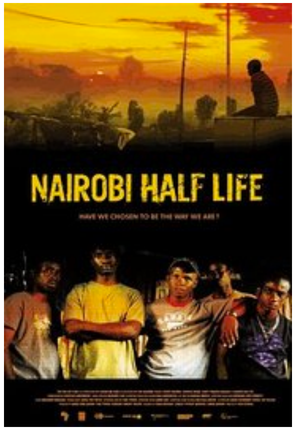
Assume E_1, E_2, E_3 and E_4 are conditionally independent given G

Netflix and Learn

G

Like foreign emotional comedies

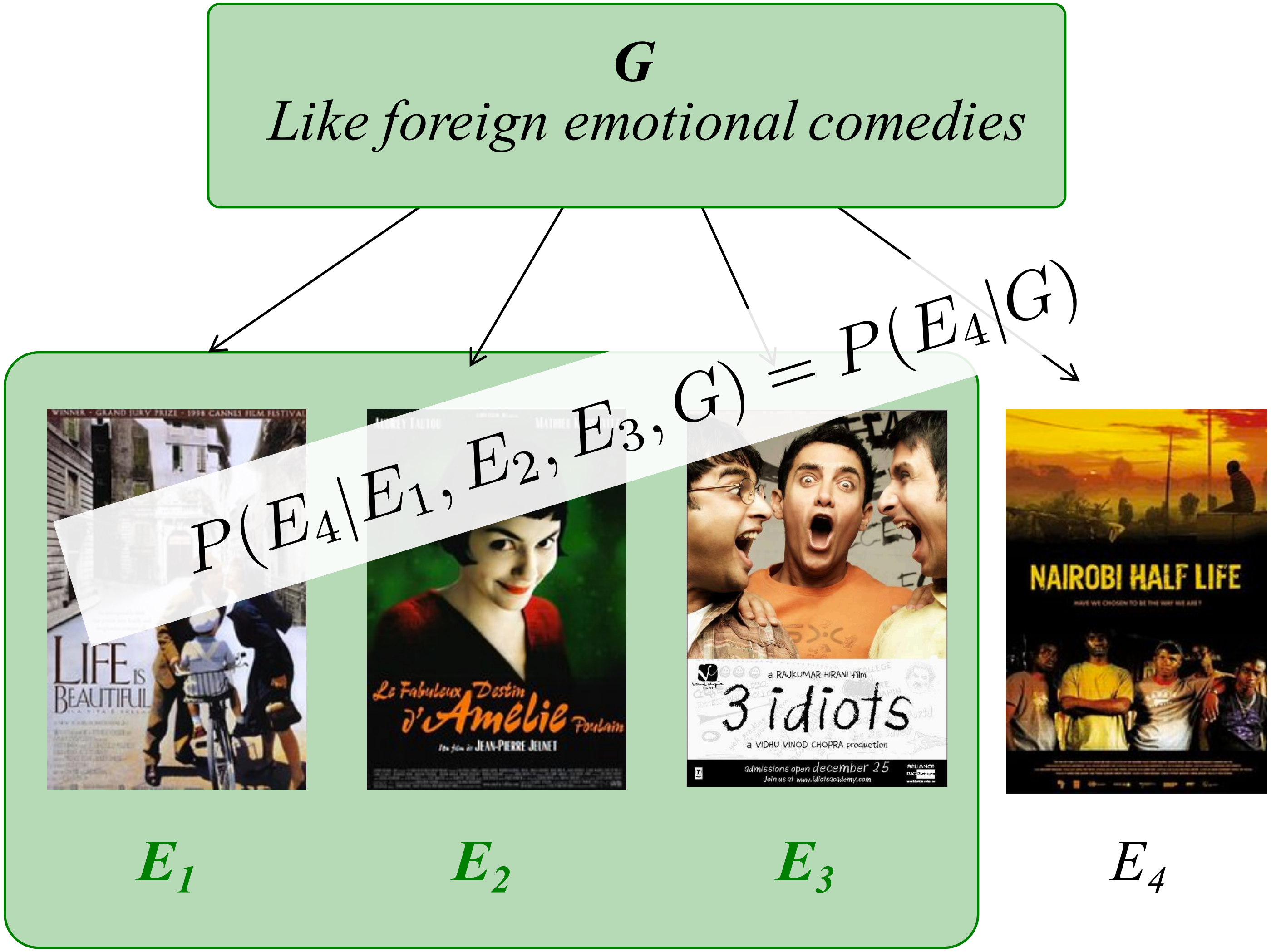




E_1 ***E_2*** ***E_3*** *E_4*

Assume E_1, E_2, E_3 and E_4 are conditionally independent given G

Netflix and Learn



$$P(E_4 | E_1, E_2, E_3, G) = P(E_4 | G)$$

Assume E_1, E_2, E_3 and E_4 are conditionally independent given G

Conditional independence is a practical, real world way of decomposing hard probability questions.

Big Deal

"Exploiting conditional independence to generate fast probabilistic computations is one of the main contributions CS has made to probability theory"

-Judea Pearl wins 2011 Turing Award, *"For fundamental contributions to artificial intelligence through the development of a calculus for probabilistic and causal reasoning"*

When we introduced conditions

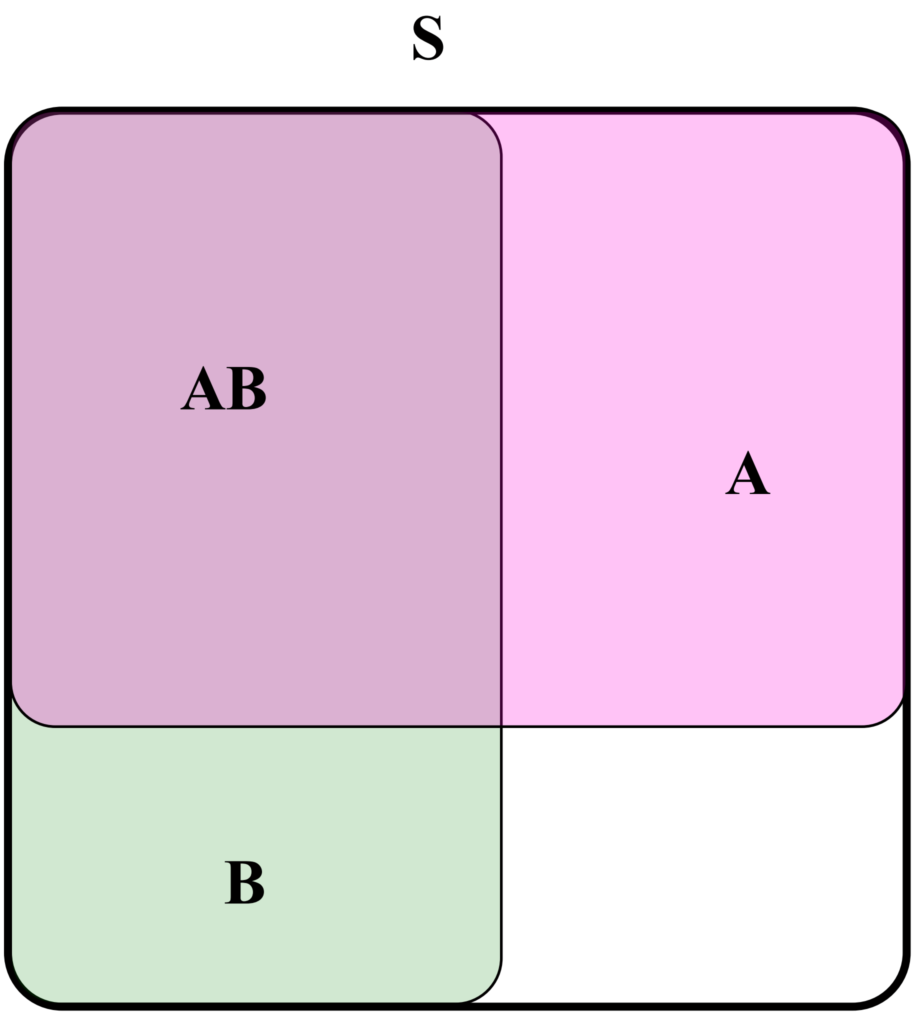
Identities of probability remain the same

But sometimes independence / dependence relationships change

What the fish?

What does independence look like?

Independence



Independence Definition 1:

$$P(AB) = P(A)P(B)$$

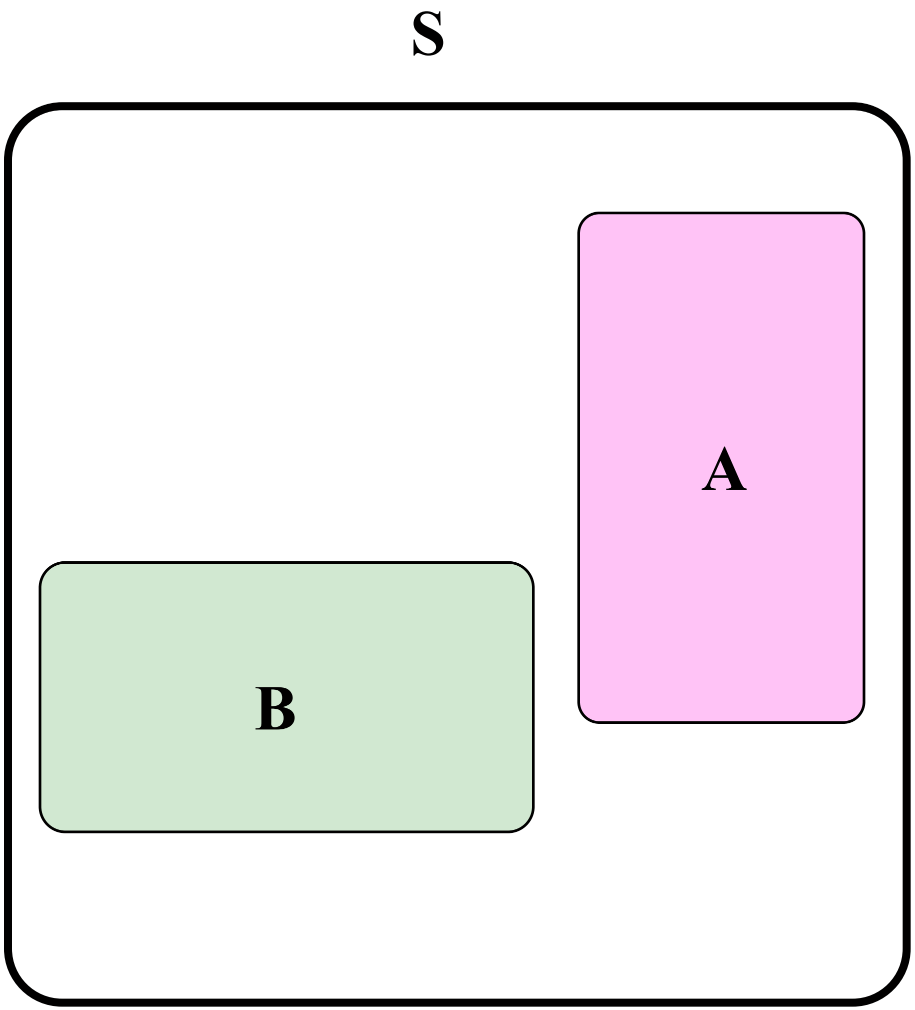
$$\frac{|AB|}{|S|} = \frac{|A|}{|S|} \times \frac{|B|}{|S|}$$

Independence Definition 2:

$$P(A|B) = P(A)$$

$$\frac{|AB|}{|B|} = \frac{|A|}{|S|}$$

Independence?

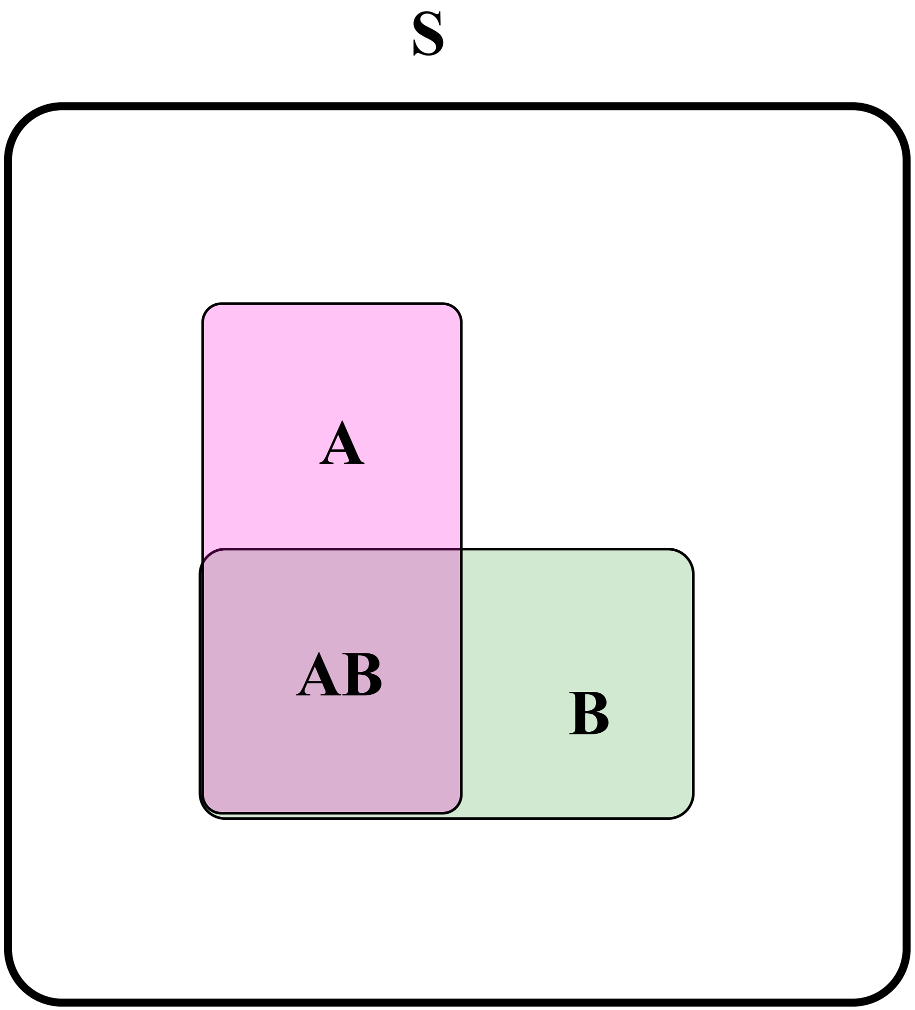


Independence Definition 1:

$$P(AB) = P(A)P(B)$$

$$\frac{|AB|}{|S|} = \frac{|A|}{|S|} \times \frac{|B|}{|S|}$$

Independence?



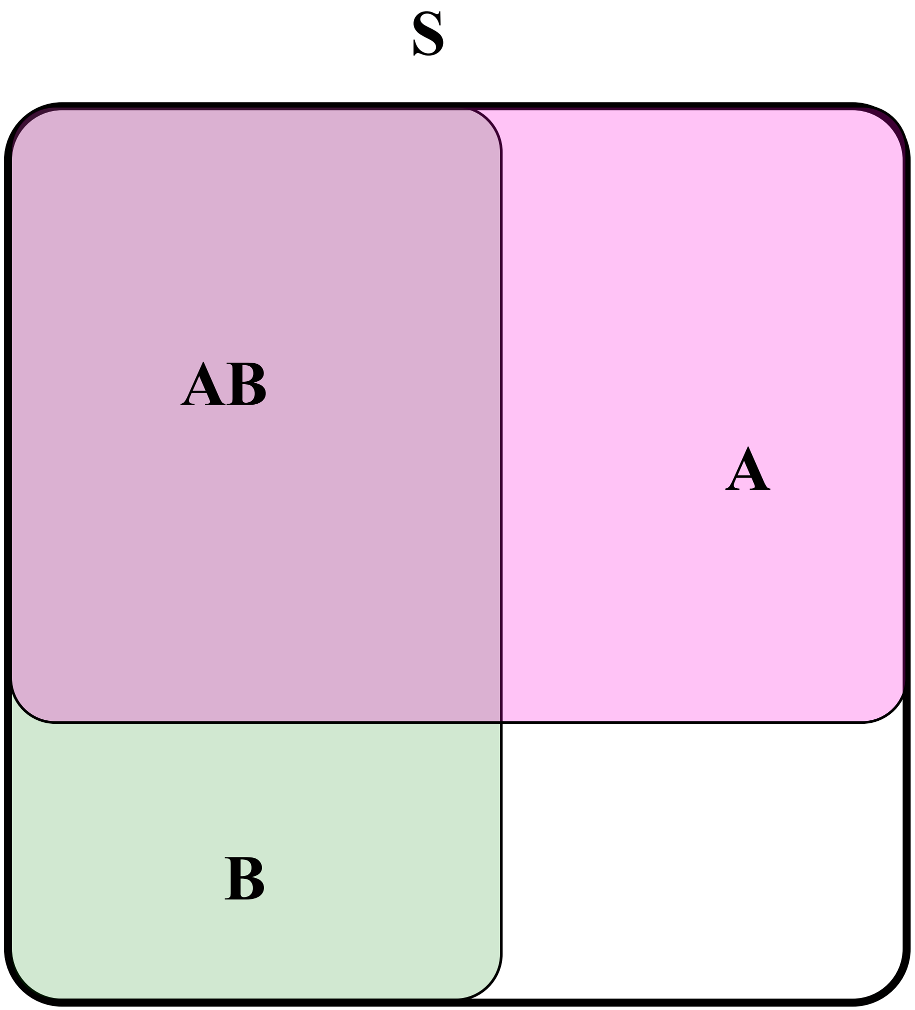
Independence Definition 2:

$$P(A|B) \stackrel{?}{=} P(A)$$

$$\frac{|AB|}{|B|} \stackrel{?}{=} \frac{|A|}{|S|}$$

$$\frac{1}{2} \neq \frac{2}{16}$$

Independence



Independence Definition 1:

$$P(AB) = P(A)P(B)$$

$$\frac{|AB|}{|S|} = \frac{|A|}{|S|} \times \frac{|B|}{|S|}$$

Independence Definition 2:

$$P(A|B) = P(A)$$

$$\frac{|AB|}{|B|} = \frac{|A|}{|S|}$$

Friday Night Fever

- Population of 10,000 people.
  - Of those, 300 have Malaria (event M) and 200 have Bacterial Infection (event B). 6 people have both.
  - Have Fever if and only if you have Malaria or Bacteria.
  - Are M and B independent?
- Solution:
  - $P(M) = 300 / 10{,}000 = 0.03$
  - $P(B) = 200 / 10{,}000 = 0.02$
  - $P(MB) = 6 / 10{,}000 = 0.0006$
  - $P(M)P(B) = 0.0006$
  - $P(M)P(B) = P(MB)$
  - Independent

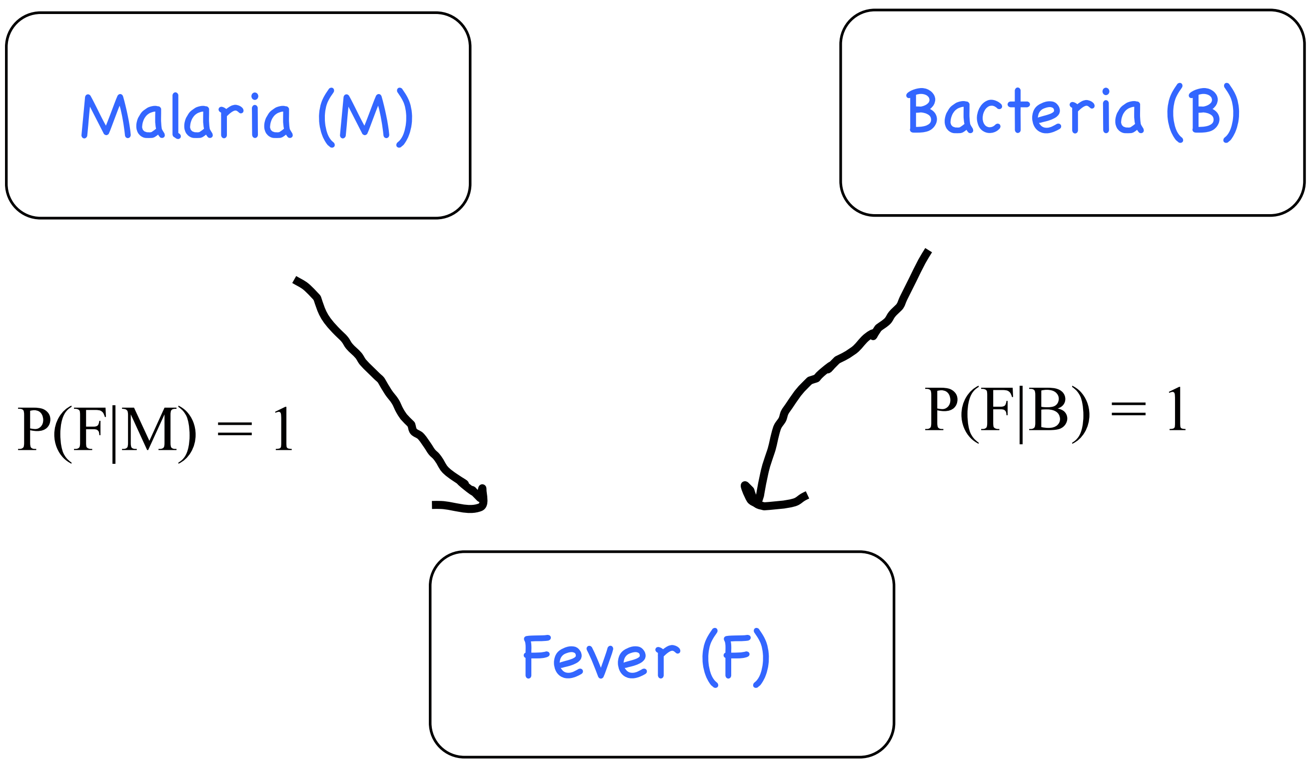
Causality

Malaria does not cause Bacteria and
Bacteria does not cause Malaria

This is 9/10 important

*This is a “causal” diagram. It helps explain why things are independent

Causality

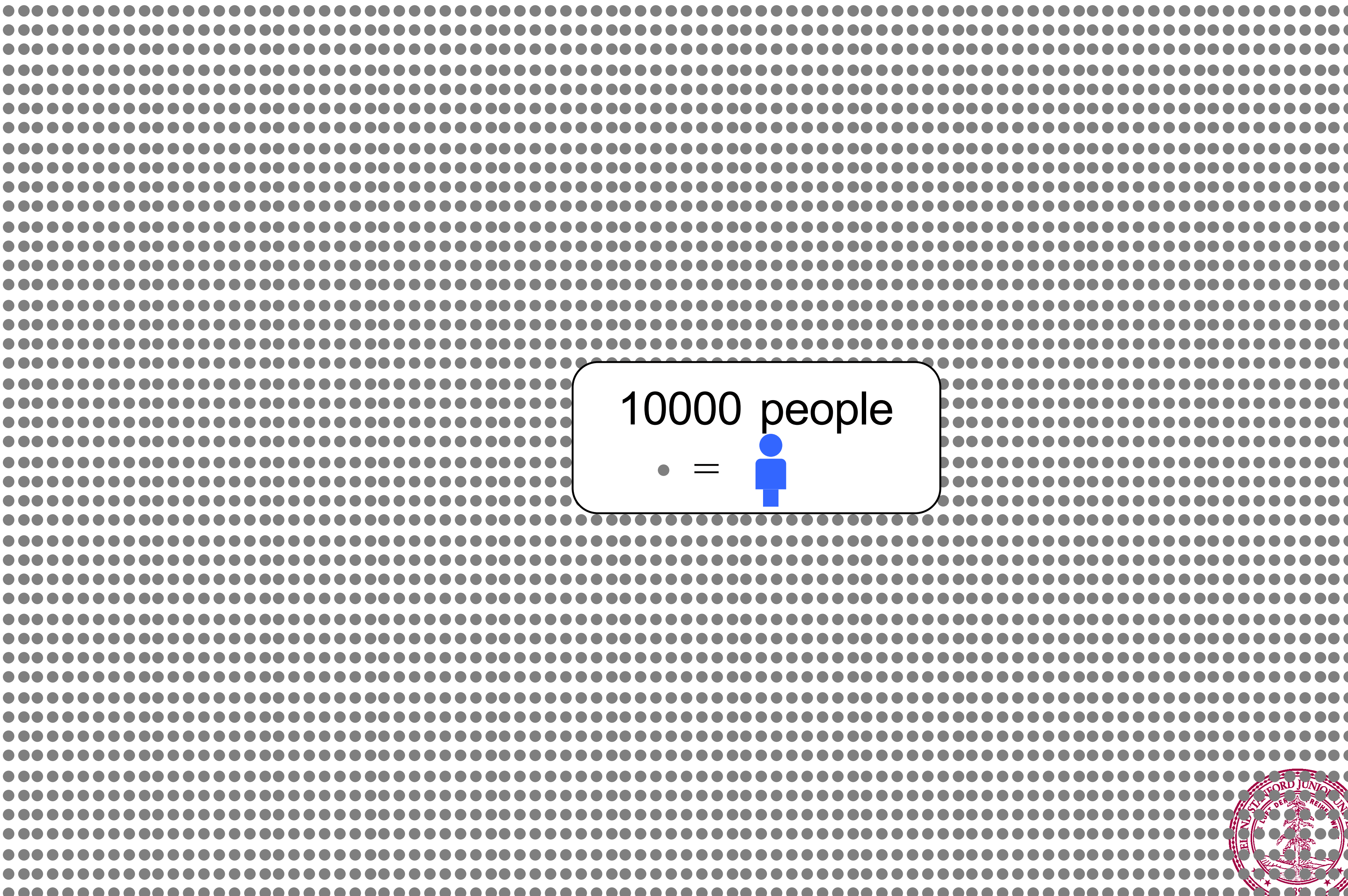


*This is a "causal" diagram. It helps explain why things are independent

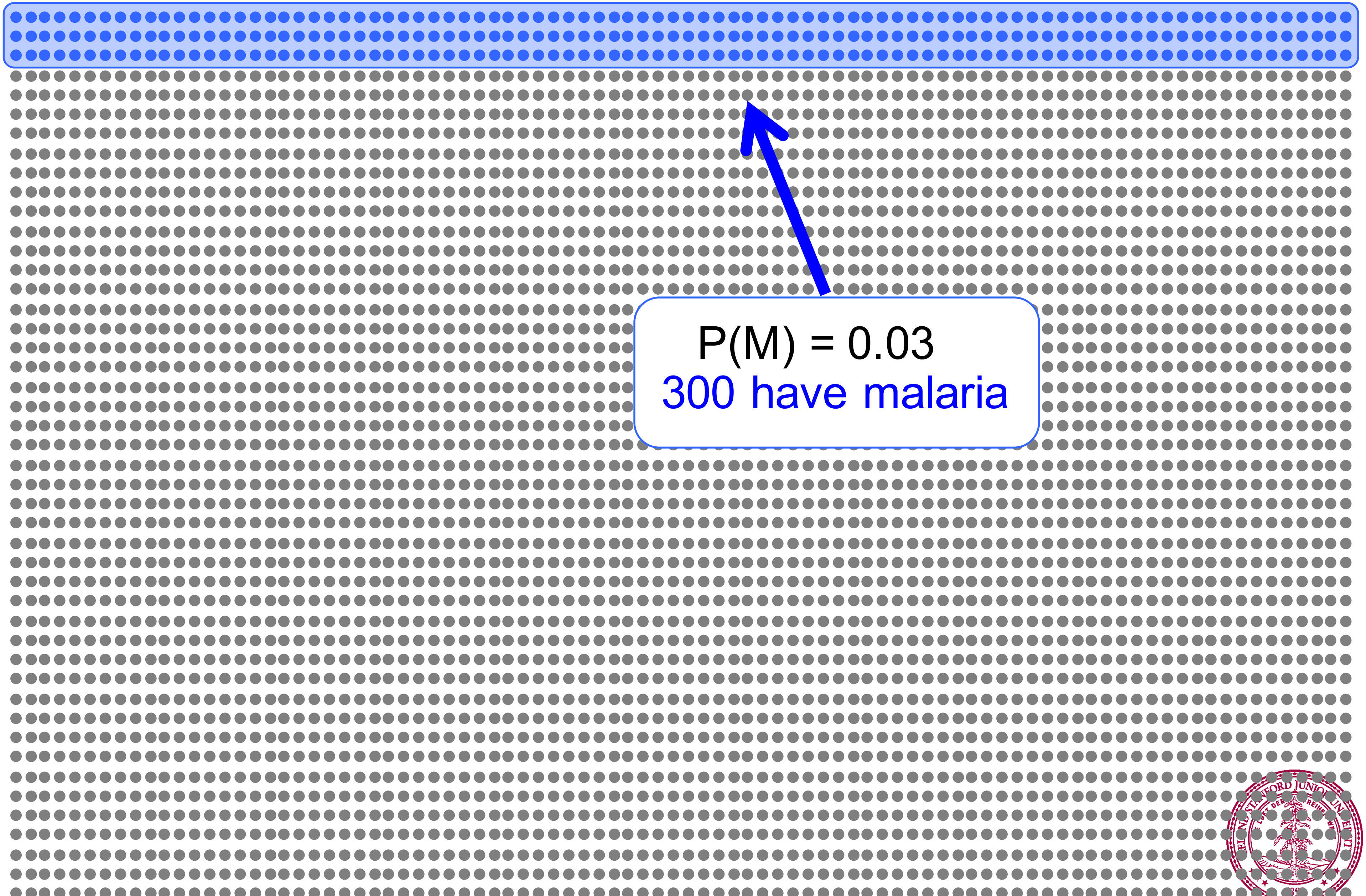
Friday Night Fever

- Population of 10,000 people.
  - Of those, 300 have Malaria (event M) and 200 have Bacterial Infection (event B). 6 people have both.
  - Have Fever if and only if you have Malaria or Bacteria.
  - Are M and B independent given F?
- Solution:
  - Total people with Fever = 200+300 – 6 = 494
  - $P(M|F) = 300 / 494 = 0.61$
  - $P(B|F) = 200 / 494 = 0.40$
  - $P(MB|F) = 6 / 494 = 0.012$
  - $P(M|F)P(B|F) = 0.224$
  - $P(M|F)P(B|F) \neq P(MB|F)$
  - Conditionally dependent

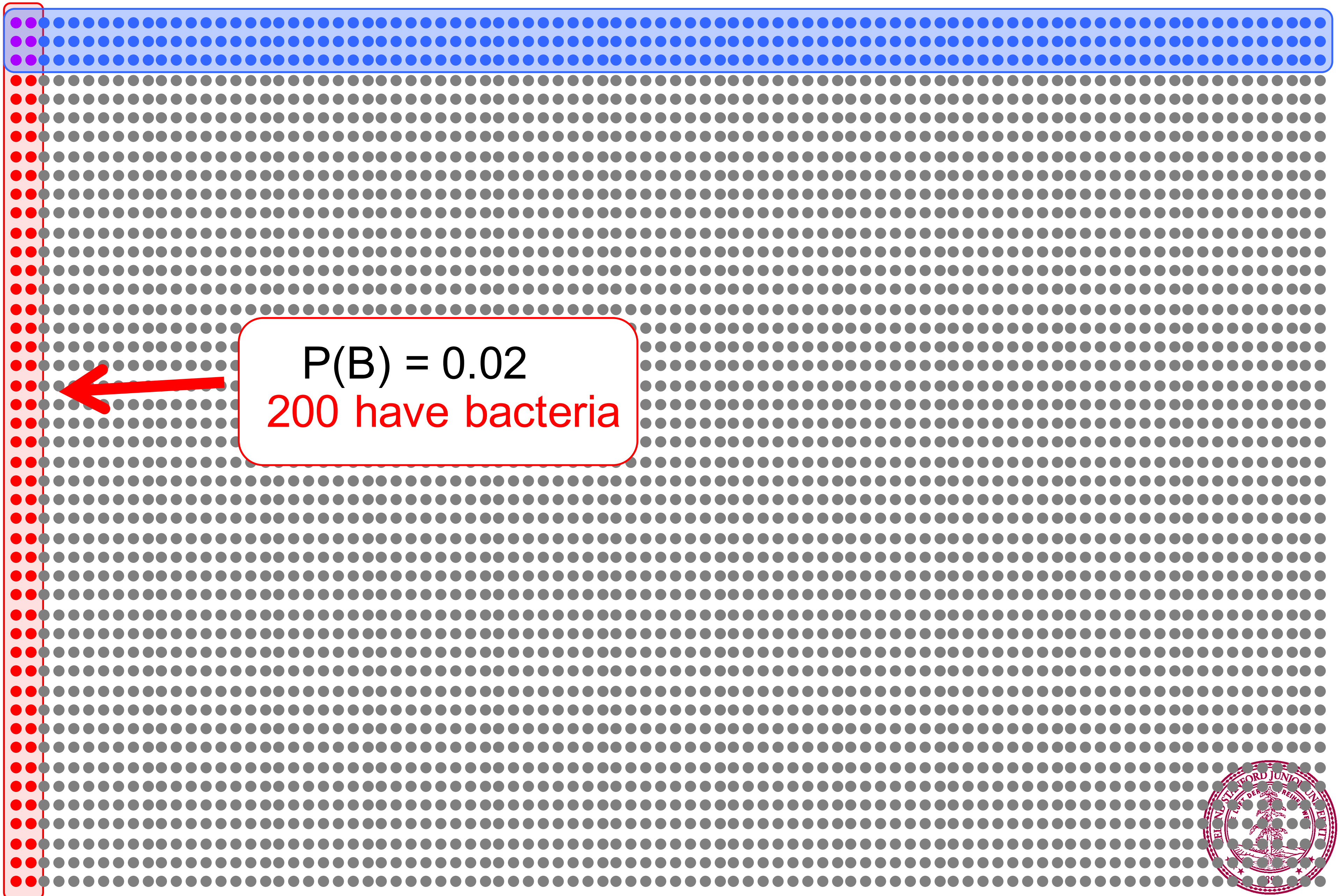
Conditional Dependence



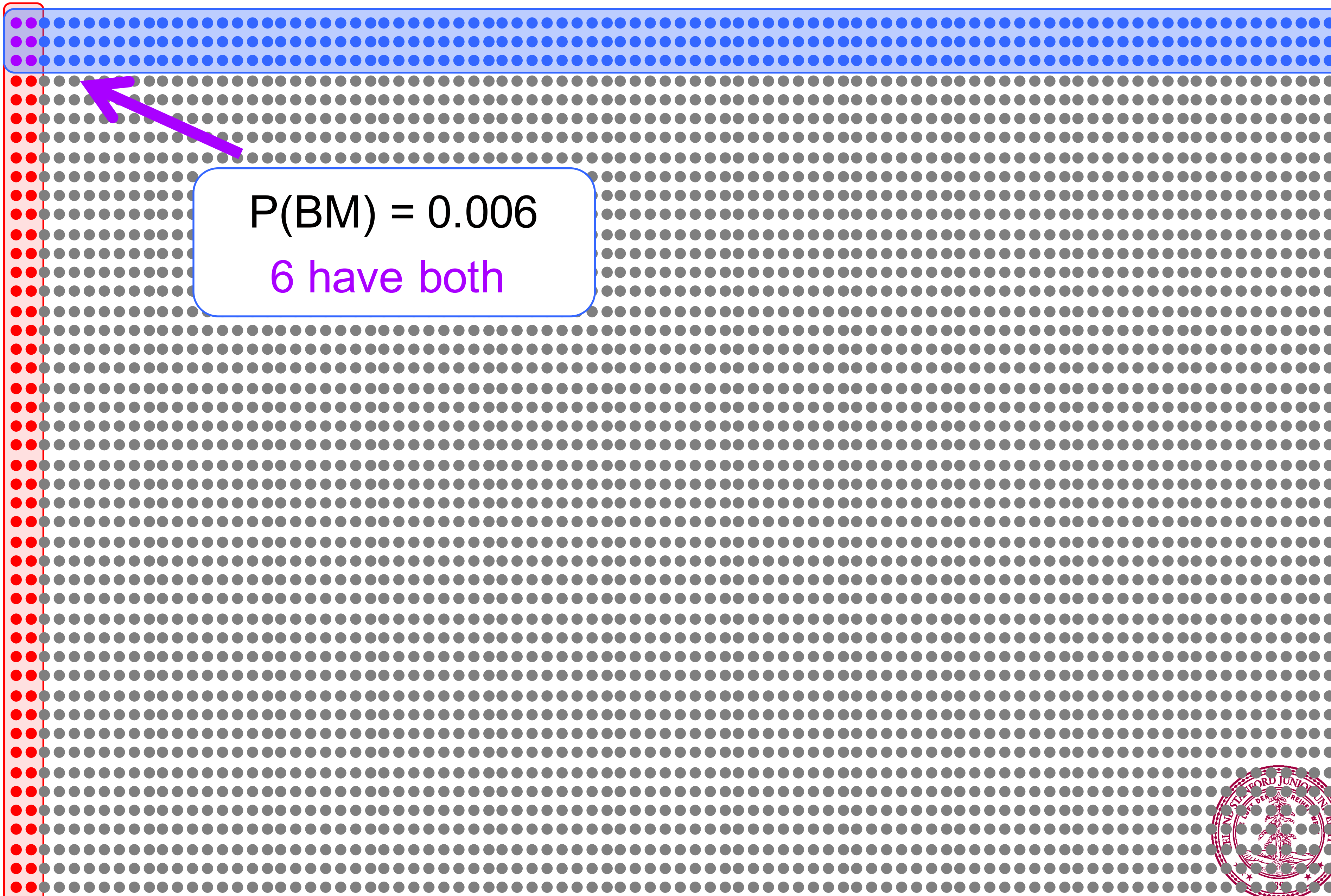
Conditional Dependence



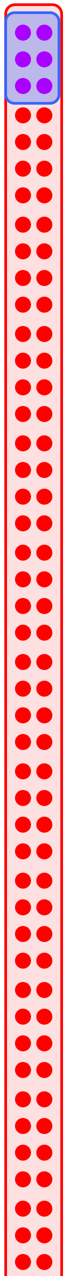
Conditional Dependence



Conditional Dependence



Conditional Dependence

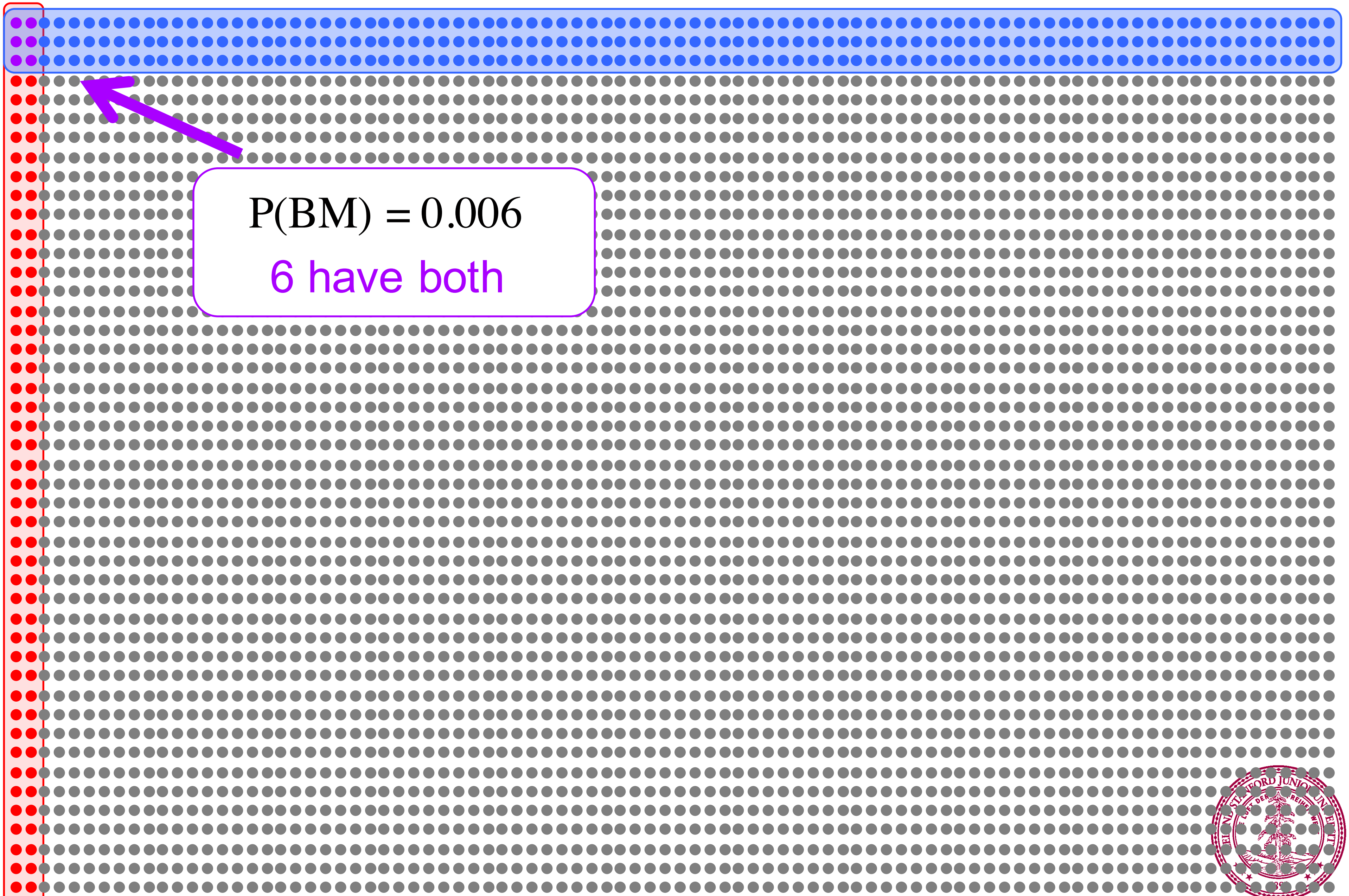


If we condition on B, the same ratio of people have malaria

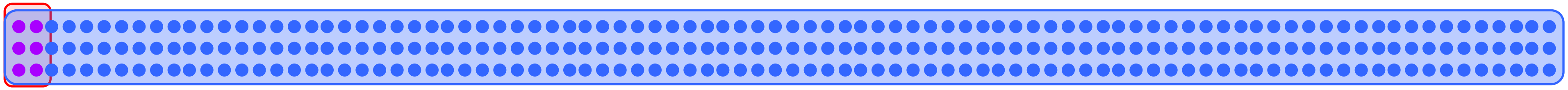
$$P(M|B) = 6/200 = 0.03$$
$$P(M) = 300/10000 = 0.03$$
$$P(M) = P(M|B)$$

That's the math definition of independence

Conditional Dependence



Conditional Dependence

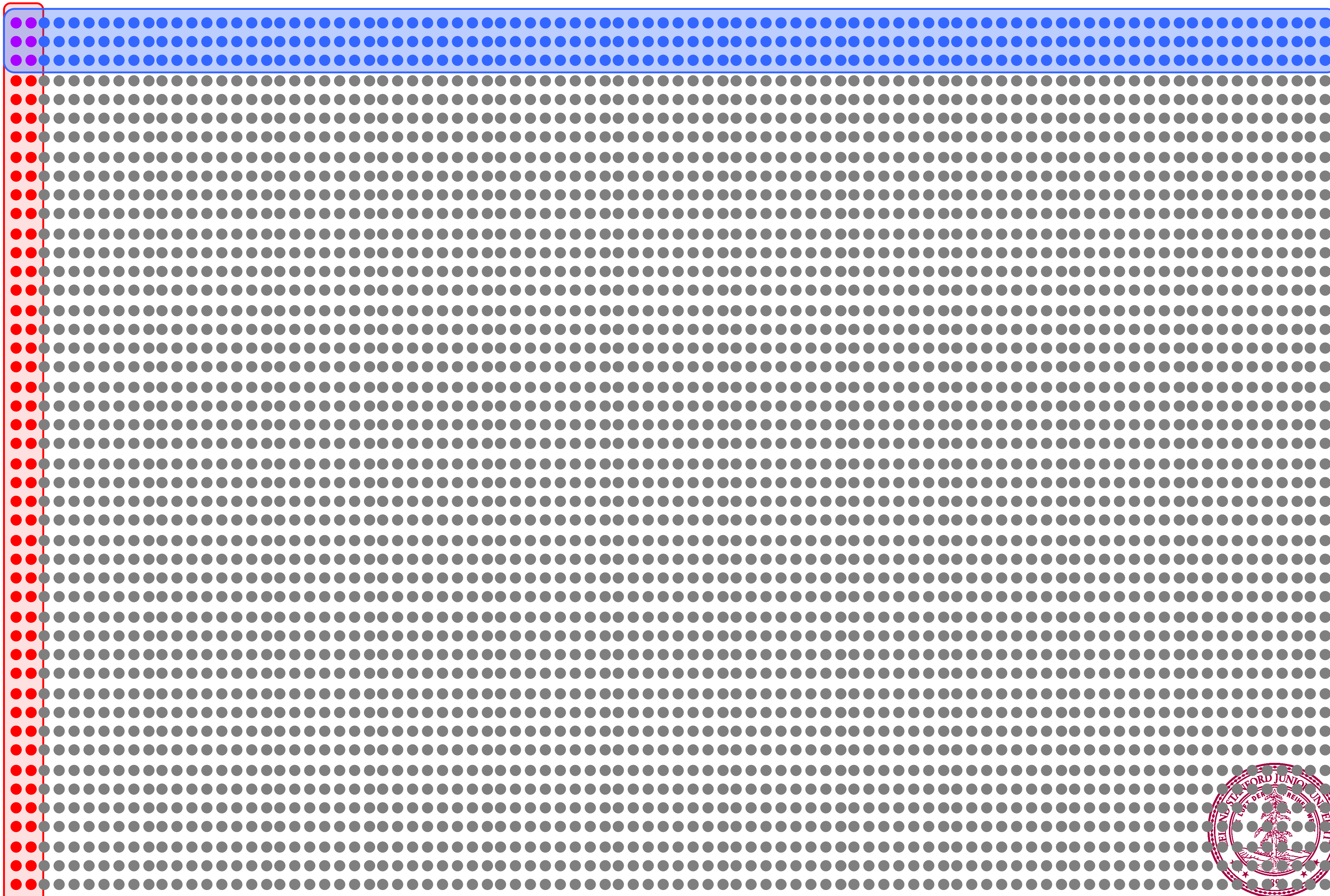


If we condition on M, the same ratio of people have bacteria

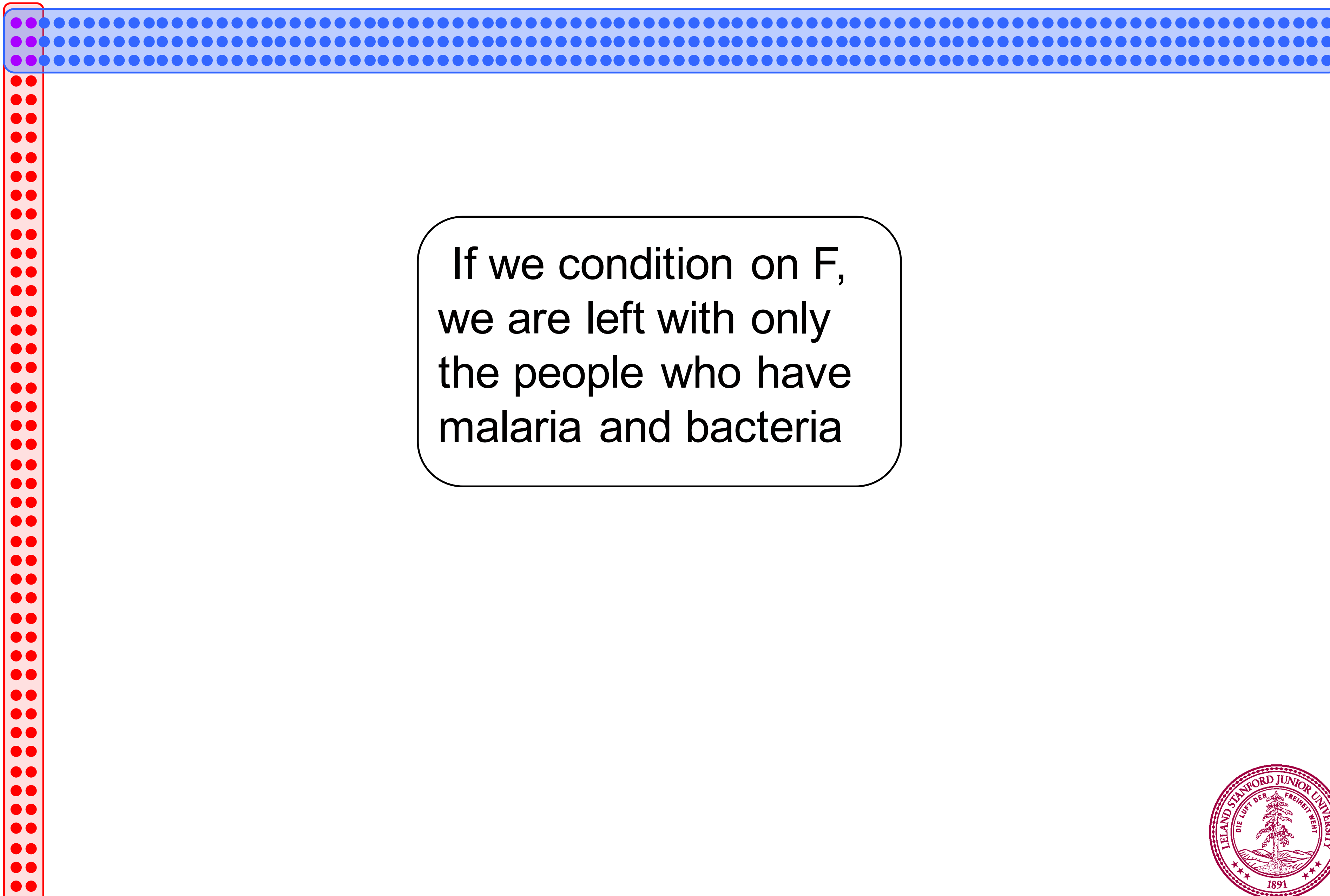
There it is again!

$$P(B|M) = 6/300 = 0.02$$
$$P(B) = 200/10000 = 0.02$$
$$P(B|M) = P(B)$$

Conditional Dependence



Conditioned on Fever

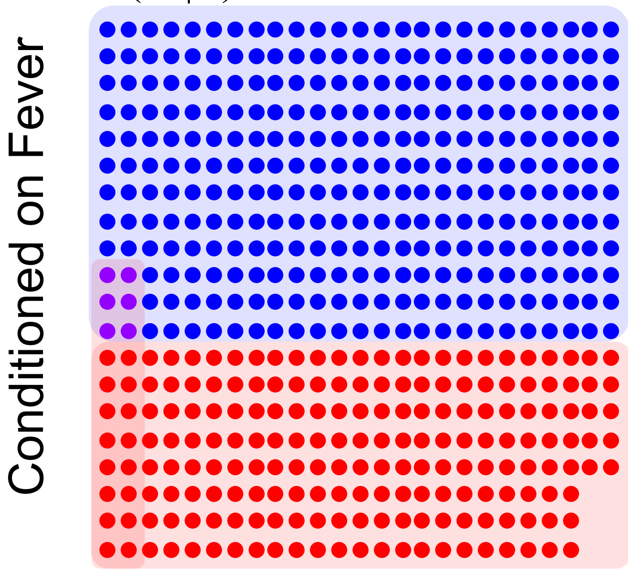


If we condition on F, we are left with only the people who have malaria and bacteria

Conditioned on Fever

P(B|F) = 200/494 = 0.40

P(M|F) = 300/494 = 0.61

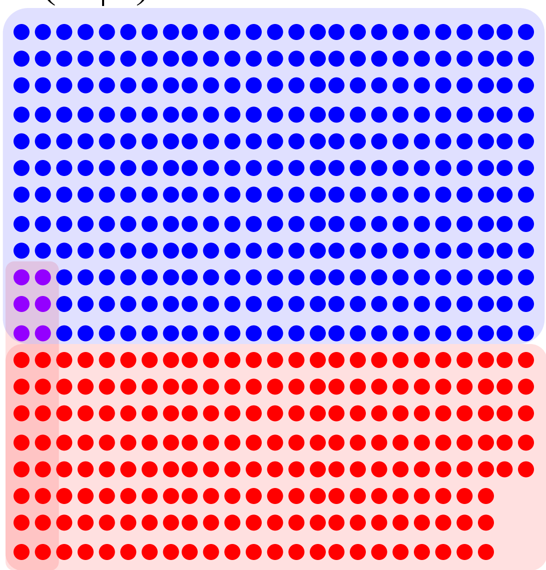


Conditioned on Fever

$P(B|F) = 200/494 = 0.40$

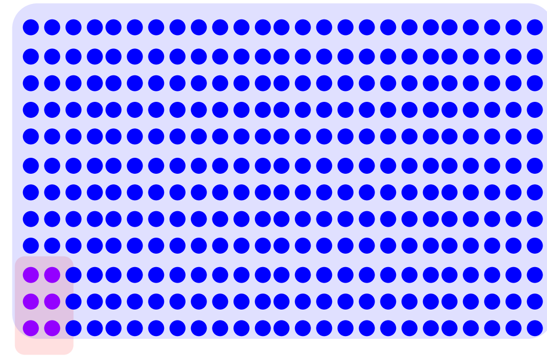
$P(M|F) = 300/494 = 0.61$

Conditioned on Fever



Test shows Malaria →

Conditioned on Fever + Malaria



$P(B|MF) = 6/300 = 0.02$

$P(B|F) \neq P(B|MF)$

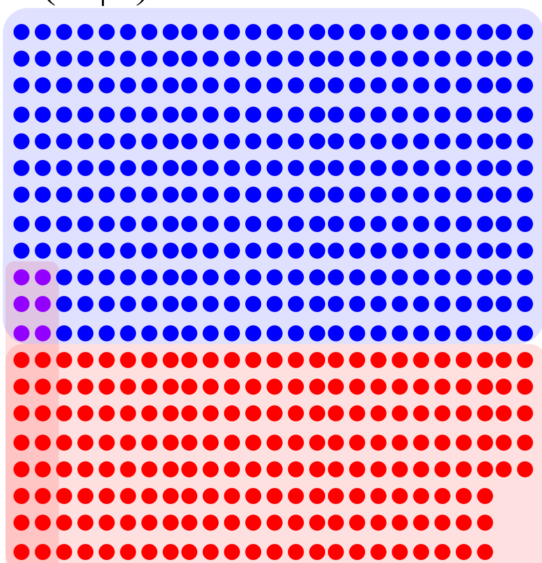
That's the math definition of conditional dependence

Conditioned on Fever

$P(B|F) = 200/494 = 0.40$

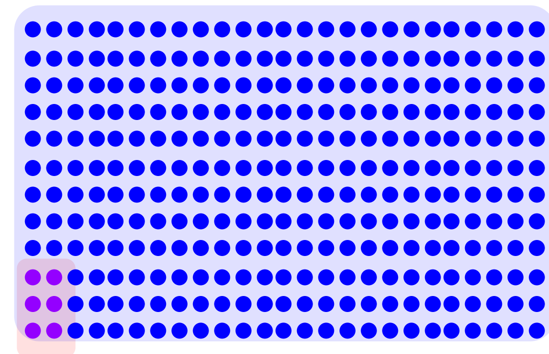
$P(M|F) = 300/494 = 0.61$

Conditioned on Fever



Test shows Malaria

Conditioned on Fever + Malaria



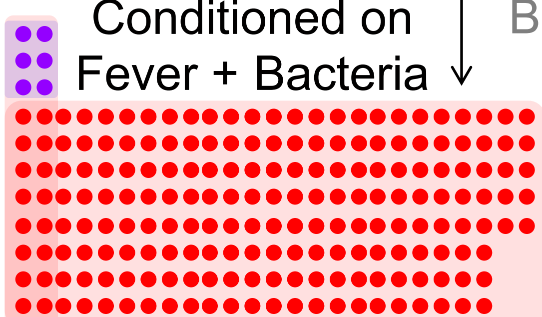
$P(B|MF) = 6/300 = 0.02$

$P(B|F) \ != P(B|MF)$

That's the math definition of conditional dependence

Test shows Bacteria

Conditioned on Fever + Bacteria

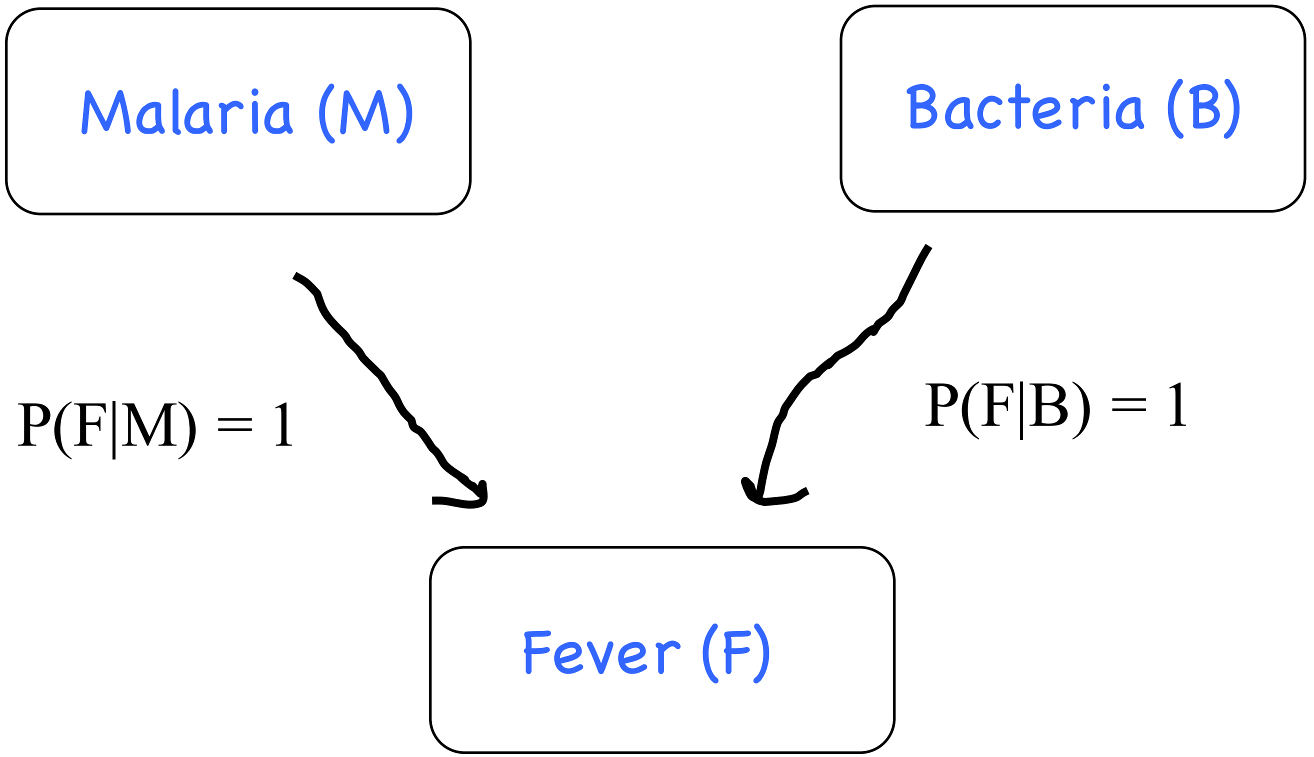


$P(M|BF) = 6/200 = 0.03$

$P(M|F) \neq P(M|BF)$

If we condition on F, the events bacteria and malaria become dependent

Conditional Dependence



*This is a “causal” diagram. It helps explain why things are independent

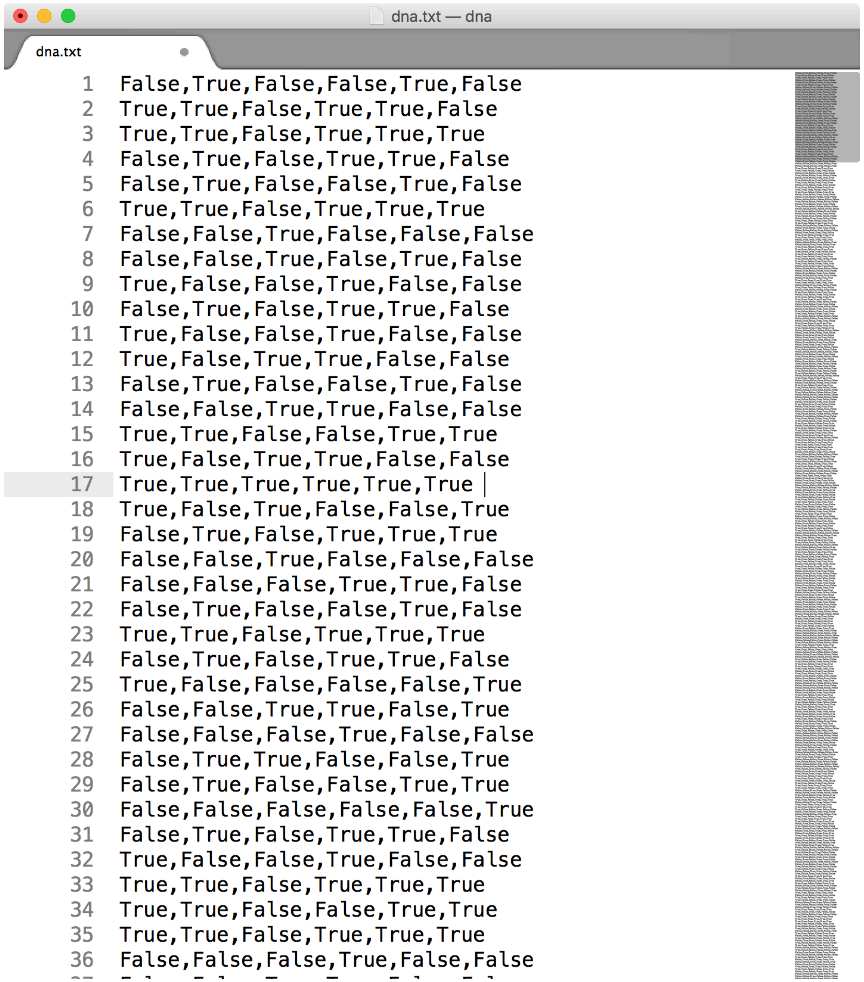
Parents With a Common Child

Say two independent parents have a common child:
When conditioned on the child they are no longer independent

And Here We Are


G_1
G_2
G_3
G_4
G_5
T

G_1 G_2 G_3 G_4 G_5 T




```
False,True,False,False,True,False
True,True,False,True,True,False
True,True,False,True,True,True
False,True,False,True,True,False
False,True,False,False,True,False
True,True,False,True,True,True
False,False,True,False,False,False
False,False,True,False,True,False
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True,True,False,False,True,True
True,True,False,True,True,True
False,False,False,True,False,False
```

100,000 samples

6 observations per sample

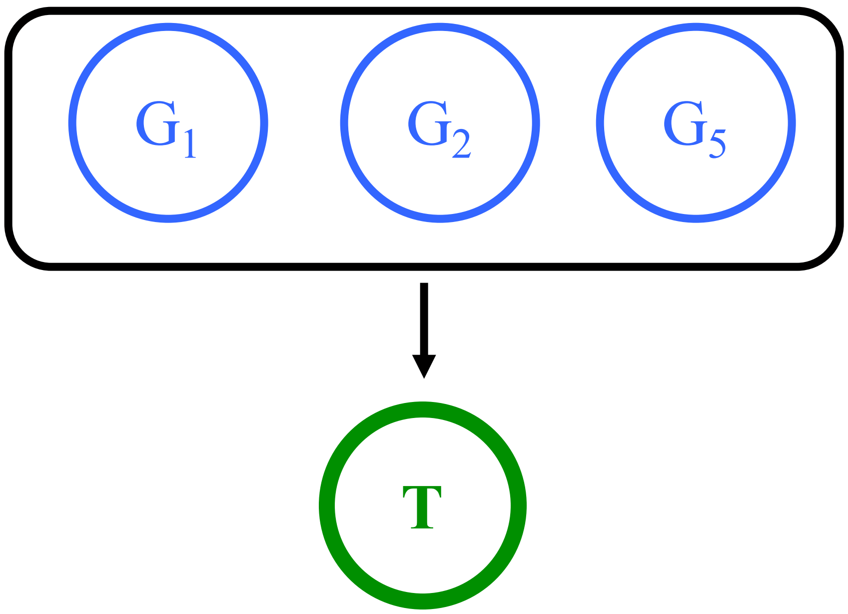
Correlation does not imply causation

Independence implies lack of causation

Model Discovery

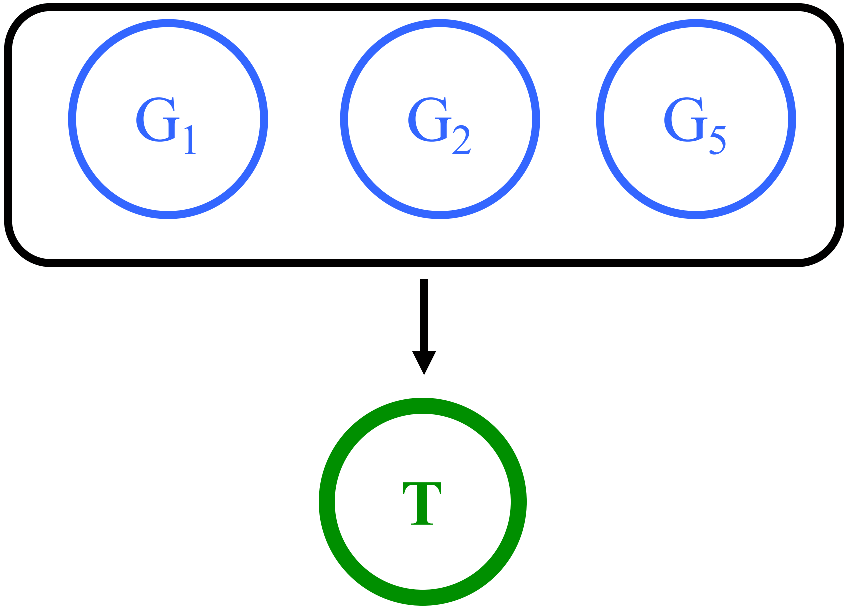
```
p(G1) = 0.500
p(G2) = 0.545
p(G3) = 0.299
p(G4) = 0.701
p(G5) = 0.600
p(T) = 0.390
```

```
p(T and G1) = 0.291 , P(T)p(G1) = 0.195
p(T and G2) = 0.300 , P(T)p(G2) = 0.213
p(T and G3) = 0.116 , P(T)p(G3) = 0.117
p(T and G4) = 0.273 , P(T)p(G4) = 0.273
p(T and G5) = 0.309 , P(T)p(G5) = 0.234
```



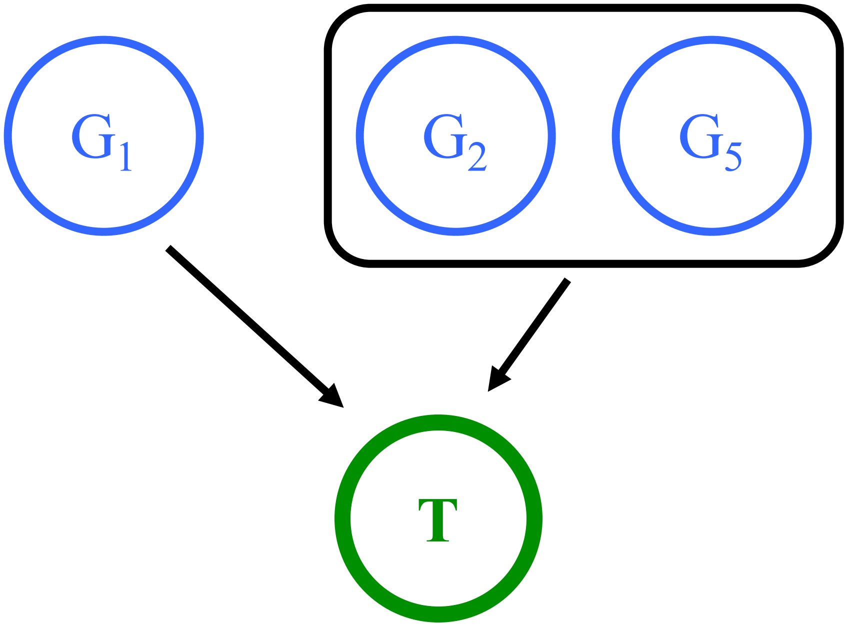
Model Discovery

```
T is independent of G3
T is independent of G4
G1 is independent of G2
G1 is independent of G5
T is independent of G5 | G2
```



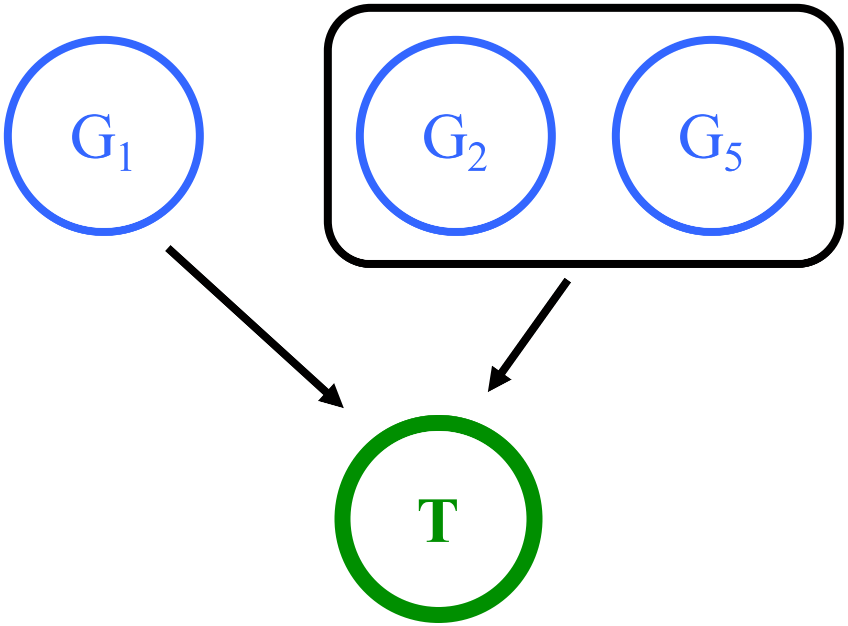
Model Discovery

```
T is independent of G3
T is independent of G4
G1 is independent of G2
G1 is independent of G5
T is independent of G5 | G2
```



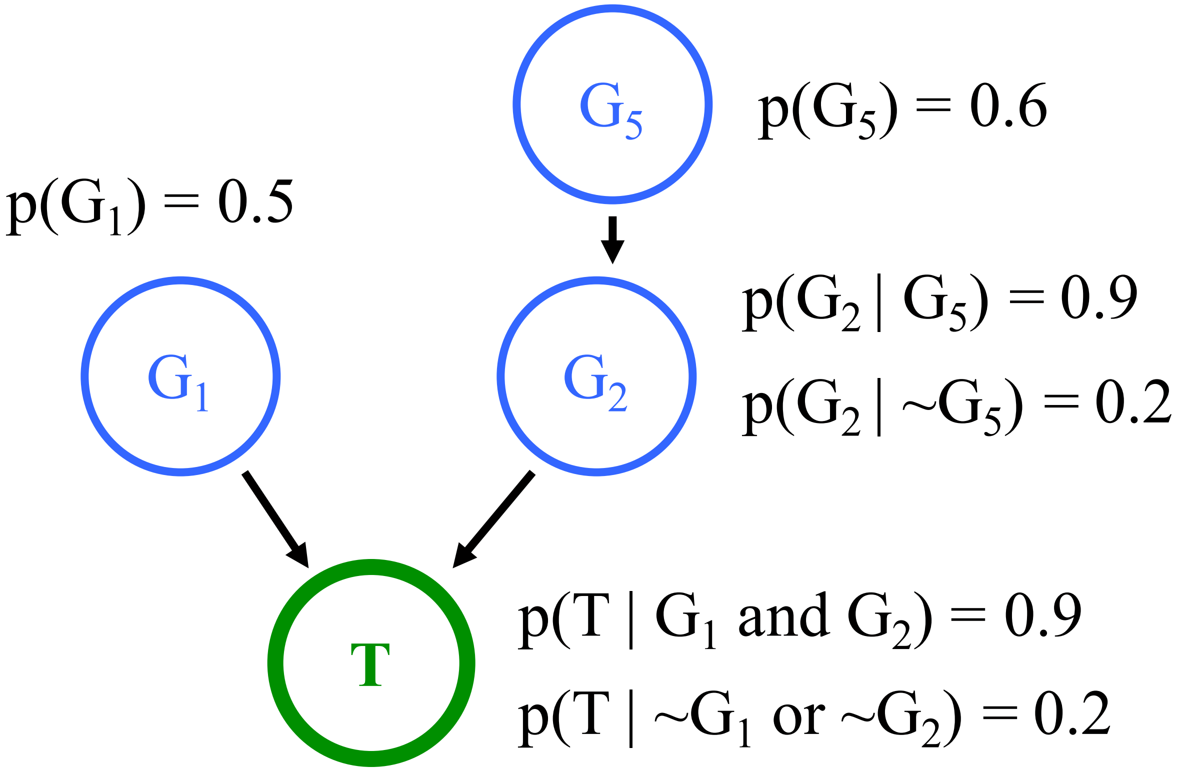
Model Discovery

```
T is independent of G3
T is independent of G4
G1 is independent of G2
G1 is independent of G5
T is independent of G5 | G2
```



Model Discovery

```
T is independent of G3
T is independent of G4
G1 is independent of G2
G1 is independent of G5
T is independent of G5 | G2
```



Summary

Two events A and B are called **<u>independent</u>** if:

$$P(AB) = P(A)P(B) \qquad P(A|B) = P(A)$$

Otherwise, they are called **<u>dependent</u>** events

Two events A and B are **<u>conditionally independent</u>** on C if:

$$P(AB|C) = P(A|C)P(B|C)$$

$$P(A|BC) = P(A|C)$$

Advanced Reading

Chow–Liu tree - Wikipedia

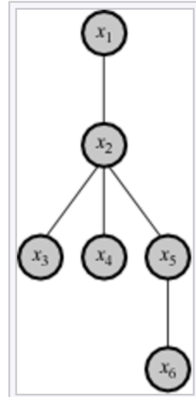
Chris Piech

Secure https://en.wikipedia.org/wiki/Chow–Liu_tree

Chow–Liu tree

From Wikipedia, the free encyclopedia

In probability theory and statistics **Chow–Liu tree** is an efficient method for constructing a second-order product approximation of a joint probability distribution, first described in a paper by Chow & Liu (1968). The goals of such a decomposition, as with such Bayesian networks in general, may be either data compression or inference.



A first-order dependency tree representing the product on the left.

Contents [hide]

The Chow–Liu representation [edit]

The Chow–Liu method describes a joint probability distribution $P(X_1, X_2, \ldots, X_n)$ as a product of second-order conditional and marginal distributions. For example, the six-dimensional distribution $P(X_1, X_2, X_3, X_4, X_5, X_6)$ might be approximated as

$$P'(X_1, X_2, X_3, X_4, X_5, X_6) = P(X_6|X_5)P(X_5|X_4)P(X_4|X_3)P(X_3|X_2)P(X_2|X_1)P(X_1)$$

where each new term in the product introduces just one new variable, and the product can be represented as a first-order