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CS 109

Supplementary Handout #2
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Calculation Reference

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Useful identities related to summations

Since it may have been a while since some folks have worked with summations, I just wanted to provide a reference on them that you may find useful in your future work. Here are some useful identities and rules related to working with summations. In the rules below, f and g are arbitrary real-valued functions.

Pulling a constant out of a summation:

$$\sum_{n=s}^{t} C \cdot f(n) = C \cdot \sum_{n=s}^{t} f(n), \text{ where } C \text{ is a constant.}$$

Eliminating the summation by summing over the elements:

$$\sum_{i=1}^{n} x = nx$$

$$\sum_{i=m}^{n} x = (n - m + 1)x$$

$$\sum_{i=s}^{n} f(C) = (n - s + 1) f(C), \text{ where } C \text{ is a constant.}$$

Combining related summations:

$$\sum_{n=s}^{j} f(n) + \sum_{n=j+1}^{t} f(n) = \sum_{n=s}^{t} f(n)$$

$$\sum_{n=s}^{t} f(n) + \sum_{n=s}^{t} g(n) = \sum_{n=s}^{t} [f(n) + g(n)]$$

Changing the bounds on the summation:

$$\sum_{n=s}^{t} f(n) = \sum_{n=s+p}^{t+p} f(n - p)$$

"Reversing" the order of the summation:

$$\sum_{n=a}^{b} f(n) = \sum_{n=b}^{a} f(n)$$

Arithmetic series:

$$\sum_{i=0}^{n} i = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$ (with a moment of silence for C. F. Gauss.)

$$\sum_{i=m}^{n} i = \frac{(n-m+1)(n+m)}{2}$$

Arithmetic series involving higher order polynomials:

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$$

$$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2 = \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4} = \left[\sum_{i=1}^{n}\right]^2$$

Geometric series:

$$\sum_{i=0}^{n} x^i = \frac{1 - x^{n+1}}{1 - x}$$

$$\sum_{i=m}^{n} x^i = \frac{x^{n+1} - x^m}{x - 1}$$

More exotic geometric series:

$$\sum_{i=0}^{n} i2^i = 2 + 2^{n+1}(n-1)$$

$$\sum_{i=0}^{n} \frac{i}{2^i} = \frac{2^{n+1} - n - 2}{2^n}$$

Taylor expansion of exponential function:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

Binomial coefficient:

$$\sum_{i=0}^{n} \binom{n}{i} = 2^n$$

Much more information on binomial coefficients is available in the Ross textbook.

Growth rates of summations

Besides solving a summation explicitly, it is also worthwhile to know some general growth rates on sums, so you can (tightly) bound a sum if you are trying to prove something in the big-Oh/Theta world. If you're not familiar with big-Theta (Θ) notation, you can think of it like big-Oh notation, but it actually provides a "tight" bound. Namely, big-Theta means that the function grows no more quickly and no more slowly than the function specified, up to constant factors, so it's actually more informative than big-Oh.

Here are some useful bounds:

$$\sum_{i=1}^{n} i^c = \Theta(n^{c+1}), \text{ for } c \geq 0.$$

$$\sum_{i=1}^{n} \frac{1}{i} = \Theta(\log n)$$

$$\sum_{i=1}^{n} c^i = \Theta(c^n), \text{ for } c \geq 2.$$

A few identities related to products

Recall that the mathematical symbol Π represents a product of terms (analogous to Σ representing a sum of terms). Below, we give some useful identities related to products.

Definition of factorial:

$$\prod_{i=1}^{n} i = n!$$

Note that $0! = 1$ by definition.

Stirling's approximation for $n!$ is given below. This approximation is useful when computing $n!$ for large values of n (particularly when $n > 30$).

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n, \text{ or equivalently } n! \approx \sqrt{2\pi}n^{\left(n+\frac{1}{2}\right)}e^{-n}$$

Eliminating the product by multiplying over the elements:

$$\prod_{i=1}^{n} C = C^n$$, where C is a constant.

Combining products:

$$\prod_{i=1}^{n} f(i) \prod_{i=1}^{n} g(i) = \prod_{i=1}^{n} f(i) \cdot g(i)$$

Turning products into summations (by taking logarithms, assuming $f(i) > 0$ for all i):

$$\log\left(\prod_{i=1}^{n} f(i)\right) = \sum_{i=1}^{n} \log f(i)$$

Suggestions for computing permutations and combinations

For your problem set solutions it is fine for your answers to include factorials, exponentials, or combinations; you don't need to calculate those all out to get a single numeric answer. However, if you'd like to work with those in Python, R, or Microsoft Excel, here are a few functions you may find useful.

In Python:

`math.factorial(n)`	computes $n!$	
`scipy.special.binom(n, m)`	computes $\binom{n}{m}$	(as a `float`)
`math.exp(n)`	computes e^n	
`n ** m`	computes n^m	

Names to the left of the dots (`.`) are modules that need to be imported before being used: `import math, scipy.special`.

In R:

`factorial(n)`	computes $n!$
`choose(n, m)`	computes $\binom{n}{m}$
`exp(n)`	computes e^n
`n^m`	computes n^m

In Microsoft Excel:

`FACT(n)`	computes $n!$
`COMBIN(n, m)`	$\binom{n}{m}$
`EXP(n)`	computes e^n
`POWER(n, m)`	computes computes n^m

To use functions in Excel, you need to set a cell to equal a function value. For example, to compute $3! \cdot \binom{5}{2}$, you would put the following in a cell:

```
= FACT(3) * COMBIN(5, 2)
```

Note the equals sign (=) at the beginning of the expression.

A little review of calculus

Since it may have been a while since you did calculus, here are a few rules that you might find useful.

Product Rule for derivatives:

$$d(u \cdot v) = du \cdot v + u \cdot dv$$

Derivative of exponential function:

$$\frac{d}{dx} e^u = e^u \frac{du}{dx}$$

Integral of exponential function:

$$\int du\, e^u = e^u$$

Derivative of natural logarithm:

$$\frac{d}{dx} \ln(x) = \frac{1}{x}$$

Integral of 1/x:

$$\int dx\, \frac{1}{x} = \ln(x)$$

Integration by parts (everyone's favorite!):

> Choose a suitable u and dv to decompose the integral of interest:
>
> $$\int u \cdot dv = u \cdot v - \int v \cdot du$$
>
> Here's the underlying rule that integration by parts is derived from:
>
> $$\int d(u \cdot v) = u \cdot v = \int du \cdot v + \int u \cdot dv$$

Bibliography

Additional information on sums and products can generally be found in a good calculus or discrete mathematics book. The discussion of summations above is based on Wikipedia (`http://en.wikipedia.org/wiki/Summation`).