

## Practice Midterm Solutions

With solutions by Mehran Sahami

1. a. The answer to this question is simply a multinomial coefficient, which can be written/computed in numerous ways:

$$\binom{12}{5, 4, 3} = \frac{12!}{5!4!3!} = \binom{12}{5} \binom{7}{4} \binom{3}{3} = \binom{12}{5} \binom{7}{4}$$

b.  $\binom{10}{3,4,3} + \binom{10}{5,2,3} + \binom{10}{5,4,1}$

We select (remove) two drinks of the same type to give to Larry and Sergey (there is only 1 way to do this for each type of drink). The remaining 10 drinks are then distributed to the remaining 10 students. The three terms above correspond respectively to CapriSuns, Cokes, and Otter Pops being given to Larry and Sergey.

Note that each of the multinomial coefficients could have been written in different ways (analogously to what was shown in part (a)).

c.  $\binom{10}{3,4,3} + \binom{10}{5,2,3} + \binom{10}{5,4,1} + \binom{10}{4,3,3} + \binom{10}{4,4,2} + \binom{10}{5,3,2}$

We select two drinks to remain in the bag and the remaining 10 drinks are then distributed to the 10 students. The six terms above correspond respectively to the cases where the two drinks left in the cooler are: (a) 2 CapriSuns, (b) 2 Cokes, (c) 2 Otter Pops, (d) 1 CapriSun and 1 Coke, (e) 1 CapriSun and 1 Otter Pops, and (f) 1 Coke and 1 Otter Pops.

Note that each of the multinomial coefficients could have been written in different ways (analogously to what was shown in part (a)).

2. There are multiple ways to obtain this answer; here are two:

The first (common) method is to let  $X$  = number of slices of pizza eaten immediately after last slice of cheese pizza is eaten. Note that  $X \sim \text{NegBin}(12, 0.5)$  since there are 12 slices of cheese pizza and slices of the two pizzas are equally likely to get eaten.

Now, we want to consider all cases where  $12 \leq X \leq 21$ , since at least 12 slices of pizza must be eaten in order for there to be a chance that the last cheese slice was eaten, and if no more than 21 (out of 24) slices are eaten when the last cheese slice is eaten, then at least 3 slices of pepperoni must remain. Thus, the probability we want is given by the expression:

$$\sum_{i=12}^{21} P(X = i) = \sum_{i=12}^{21} \binom{i-1}{11} \left(\frac{1}{2}\right)^{12} \left(\frac{1}{2}\right)^{i-12} = \sum_{i=12}^{21} \binom{i-1}{11} \left(\frac{1}{2}\right)^i$$

A second method to compute the answer is to use a set of Binomial variables defined as:  $Y_i$  = number of cheese slices eaten at time when  $i$  total slices have been eaten. We have  $Y_i \sim \text{Bin}(i, 0.5)$ , since we have  $i$  trials (slices of pizza eaten), where there is a 50% chance that each slice eaten is cheese. Here, we want to compute:  $\frac{1}{2} \sum_{i=11}^{20} P(Y_i = 11)$ , since we want to

find the probability that when 11 slices of cheese pizza have been eaten (i.e., only one cheese slice remains), a total of 11 to 20 slices of pizza have been eaten. We then multiply by  $1/2$  to denote the chance that the next slice eaten is in fact the 12th (last) slice of cheese. At that time a total of 12 to 21 slices of pizza will have been eaten, with 12 of those slices having been cheese, which means there are at least 3 slices of pepperoni remaining. Solving yields:

$$\frac{1}{2} \sum_{i=11}^{20} P(Y_i = 11) = \frac{1}{2} \sum_{i=11}^{20} \binom{i}{11} \left(\frac{1}{2}\right)^{11} \left(\frac{1}{2}\right)^{i-11} = \sum_{i=11}^{20} \binom{i}{11} \left(\frac{1}{2}\right)^{i+1}$$

And just to show the equivalence of this result, if we let  $j = i + 1$ , we can rewrite the expression immediately above in the same way we computed it in the first method:

$$\sum_{i=11}^{20} \binom{i}{11} \left(\frac{1}{2}\right)^{i+1} = \sum_{j=12}^{21} \binom{j-1}{11} \left(\frac{1}{2}\right)^j$$

3. a. Let  $X$  = the number of times the randomly chosen song is played.

Here the probability  $p$  of selecting the particular song =  $1/500$  and the number of independent trials (song selections)  $n = 200$ . So, we have  $X \sim \text{Bin}(200, 1/500)$ . We want to compute:

$$P(X > 4) = 1 - P(X \leq 4) = 1 - \sum_{i=0}^4 P(X = i) = 1 - \sum_{i=0}^4 \binom{200}{i} \left(\frac{1}{500}\right)^i \left(\frac{499}{500}\right)^{200-i}$$

- b. Let  $p$  = probability that a randomly chosen song is played more than 4 times. As determined in part (a):  $p = 1 - \sum_{i=0}^4 \binom{200}{i} \left(\frac{1}{500}\right)^i \left(\frac{499}{500}\right)^{200-i}$

Now, let  $Y$  = the number of songs that have been heard more than 4 times. Here, this problem set-up fits the Poisson paradigm (it is really the same as computing if 3 buckets in a hash table each have more the 4 strings hashed to them). Thus, we have:  $Y \sim \text{Poi}(\lambda)$  where  $\lambda = 500p$ , and  $p$  is defined as above.

$$P(Y = 3) = \frac{e^{-\lambda} \lambda^3}{3!} \text{ where } \lambda = 500 \left(1 - \sum_{i=0}^4 \binom{200}{i} \left(\frac{1}{500}\right)^i \left(\frac{499}{500}\right)^{200-i}\right).$$

Note that a normal approximation is not as appropriate as a Poisson approximation here since  $p$  is a very small value.

4. a. Let  $X_i$  = the value rolled on die  $i$ , where  $1 \leq i \leq 4$ .  $P(X \geq k) = P(X_1 \geq k, X_2 \geq k, X_3 \geq k, X_4 \geq k) = \left(\frac{6-k+1}{6}\right)^4$ , since all four rolls must be greater than or equal to  $k$ .

- b. Using the definition of expectation:

$$\begin{aligned} E[X] &= \sum_{x=1}^6 x \cdot P(X = x) = \sum_{x=1}^6 x \cdot [P(X \geq x) - P(X \geq x + 1)] \\ &= \sum_{x=1}^6 x \cdot \left[ \left(\frac{6-x+1}{6}\right)^4 - \left(\frac{6-x}{6}\right)^4 \right] \end{aligned}$$

Alternatively, one can use a property covered in Lecture 12 (and therefore not required knowledge for the midterm), which is that if  $X$  is non-negative, then:

$$E[X] = \sum_{x=1}^6 P(X \geq x) = \sum_{x=1}^6 \left(\frac{6-x+1}{6}\right)^4 = \left(\frac{6}{6}\right)^4 + \left(\frac{5}{6}\right)^4 + \left(\frac{4}{6}\right)^4 + \left(\frac{3}{6}\right)^4 + \left(\frac{2}{6}\right)^4 + \left(\frac{1}{6}\right)^4$$

The two expressions to compute  $E[X]$  above are, indeed, equivalent.

c.  $E[S] = E[T - X] = E[T] - E[X]$

Let  $X_i$  = the value rolled on die  $i$ , where  $1 \leq i \leq 4$ . As computed in class, we know that  $E[X_i] = 3.5$  for all  $1 \leq i \leq 4$ .

$$E[T] = E[X_1 + X_2 + X_3 + X_4] = E[X_1] + E[X_2] + E[X_3] + E[X_4] = 4(3.5) = 14$$

So,  $E[S] = 14 - E[X]$ , where  $E[X]$  is as computed in part (b).

5. a. We are given the PMF for the random variable  $X$ , which is the popularity rank of the song for a random play. So we can plug in  $i = 10$ :

$$P(X = 10) = \frac{\frac{1}{10}}{\sum_{n=1}^{3 \cdot 10^7} \frac{1}{n}}$$

- b. Let  $Y$  be a random variable equal to the number of times the most popular song is listened to over the course of the day. If we consider each play to be a trial which succeeds if the song is the most popular, then  $Y \sim \text{Bin}(n, p)$ , where  $n$  is the number of plays (1 billion =  $10^9$ ) and  $p$  is the probability that the song is the most popular. From the PMF, the probability that the song is the most popular is

$$p = P(X = 1) = \frac{\frac{1}{1}}{\sum_{n=1}^{3 \cdot 10^7} \frac{1}{n}}$$

Here,  $n$  is very large, and  $p$  is fairly small (using the fact that  $\sum_{n=1}^{3 \cdot 10^7} \frac{1}{n} \approx 17.8$ , we can figure out that  $p = \frac{1}{17.8} \approx 0.056$ ). So a Poisson approximation is a good choice here. We can approximate  $Y \approx W \sim \text{Poi}(\lambda = np)$ .

$$P(Y > 10^8) \approx P(W > 10^8) = 1 - \sum_{i=0}^{10^8} e^{-np} \frac{(np)^i}{i!} = 1 - \sum_{i=0}^{10^8} e^{-\frac{10^9}{17.8}} \frac{\left(\frac{10^9}{17.8}\right)^i}{i!}$$

(The last step, plugging in the values we already defined for  $n$  and  $p$ , is not necessary for full credit.)

6. Let  $X$  = lifetime of screen in our laptop.

Let event  $A$  = manufacturer A produced the screen.

Let event  $B$  = manufacturer B produced the screen.

a. We want to compute  $P(A | X > 18)$ . Using Bayes Theorem, we have:

$$P(A | X > 18) = \frac{P(X > 18 | A)P(A)}{P(X > 18)} = \frac{(1 - P(X \leq 18 | A)) \cdot 0.5}{P(X > 18)}$$

Noting that  $(X | A) \sim N(20, 4)$ , we have:

$$\begin{aligned} P(A | X > 18) &= \frac{(0.5) \left(1 - P\left(\frac{X-20}{2} \leq \frac{18-20}{2}\right)\right)}{P(X > 18)} \\ &= \frac{(0.5)\Phi(1)}{P(X > 18)} = \frac{(0.5)(0.8413)}{P(X > 18)} \end{aligned}$$

Now, we need to compute  $P(X > 18)$ :

$$\begin{aligned} P(X > 18) &= P(X > 18 | A)P(A) + P(X > 18 | B)P(B) \\ &= P(X > 18 | A)(0.5) + P(X > 18 | B)(0.5) \\ &= 0.5 \cdot \left(1 - P\left(\frac{X-20}{2} \leq \frac{18-20}{2}\right)\right) + 0.5 \left[1 - \left(1 - e^{-\frac{18}{20}}\right)\right] \\ &= 0.5 \cdot (1 - P(Z \leq -1)) + 0.5e^{-\frac{9}{10}} \\ &= 0.5 \cdot (1 - (1 - P(Z \leq 1))) + 0.5e^{-\frac{9}{10}} \\ &= 0.5\Phi(1) + 0.5e^{-\frac{9}{10}} \\ &= 0.5 \cdot 0.8413 + 0.5e^{-\frac{9}{10}} \end{aligned}$$

Substituting  $P(X > 18)$  into the expression for  $P(A | X > 18)$ , yields the answer:

$$P(A | X > 18) = \frac{(0.5)(0.8413)}{P(X > 18)} = \frac{0.8413}{0.8413 + e^{-\frac{9}{10}}}$$

b. Here, we want to compute  $P(B | X > 18)$ . Using Bayes Theorem, we have:

$$P(B | X > 18) = \frac{P(X > 18 | B)P(B)}{P(X > 18)} = \frac{(1 - P(X \leq 18 | B)) \cdot 0.5}{P(X > 18)}$$

Noting that  $(X | B) \sim \text{Exp}(1/20)$ , we have:

$$P(B | X > 18) = \frac{0.5 \left(1 - \left(1 - e^{-\frac{18}{20}}\right)\right)}{P(X > 18)} = \frac{0.5e^{-\frac{9}{10}}}{P(X > 18)}$$

Substituting the previously computed value for  $P(X > 18)$  into the expression for  $P(B | X > 18)$ , yields the final answer:

$$P(B | X > 18) = \frac{0.5e^{-\frac{9}{10}}}{P(X > 18)} = \frac{e^{-\frac{9}{10}}}{0.8413 + e^{-\frac{9}{10}}}$$