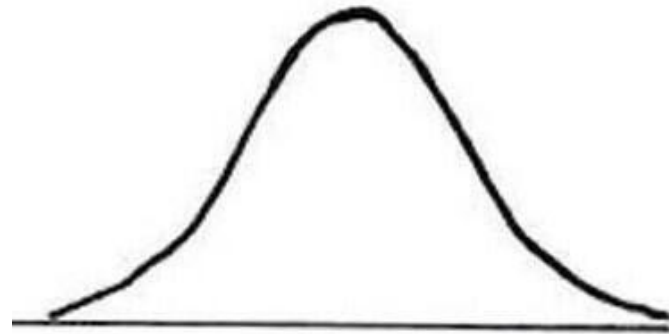


CS 109 Midterm Review!



NORMAL DISTRIBUTION



PARANORMAL DISTRIBUTION

Major Topics:

- Counting and Combinatorics
- Probability
- Conditional Probability
- Random Variables
- Discrete/Continuous Distributions
- Joint Distributions and Convolutions

Counting

- Sum Rule:
- If the outcome of an experiment can either be one of m outcomes or one of n outcomes, where none of the outcomes in the set of m outcomes is the same as the any of the outcomes in the set of n outcomes, then there are $m + n$ possible outcomes of the experiment.
- Example:
 - I could dress up as one of 10 characters from game of thrones or one of 50 disney characters for Halloween. How many possible costumes do I have to choose from?



Counting

- Inclusion Exclusion rule:
- If the outcome of an experiment can either be drawn from set A or set B, and sets A and B may potentially overlap (i.e., it is not guaranteed that $|A \cap B| = 0$), then the number of outcomes of the experiment is $|A \cup B| = |A| + |B| - |A \cap B|$.



Counting

- Product Rule:
- If an experiment has two parts, where the first part can result in one of m outcomes and the second part can result in one of n outcomes regardless of the outcome of the first part, then the total number of outcomes for the experiment is mn .
- Example:
 - We're planning our trick or treat route. We can start by going to any one of 6 neighborhoods, then go to any one of 7 neighborhoods. How many possible trick or treat routes are there?



Combinatorics

	Ordered	Unordered
Distinct	$n!$	$\binom{n}{k} = \frac{n!}{k!(n-k)!}$
Indistinct	$\frac{n!}{k_1! k_2! \dots k_n!}$	$\binom{n+r-1}{r-1}$

Probability

$P(E) = \frac{|E|}{|S|}$ if all outcomes are equally likely

$$0 \leq P(A) \leq 1$$

$$1 = P(S)$$

$$P(E) = 1 - P(E^c)$$

$P(A \cup B) = P(A) + P(B)$ if A and B are mutually exclusive

$P(A \cap B) = P(A)P(B)$ if A and B are independent

$$P(A \cap B) = 1 - P(A^c \cup B^c)$$

$$P(A \cup B) = 1 - P(A^c \cap B^c)$$

Conditional Probability

$$P(EF) = P(E|F)P(F)$$

$$P(E|F) = \frac{P(EF)}{P(F)}$$

$$P(F|E) = \frac{P(EF)}{P(E)}$$

$$P(F) = P(F|E)P(E) + P(F|E^c)P(E^c)$$

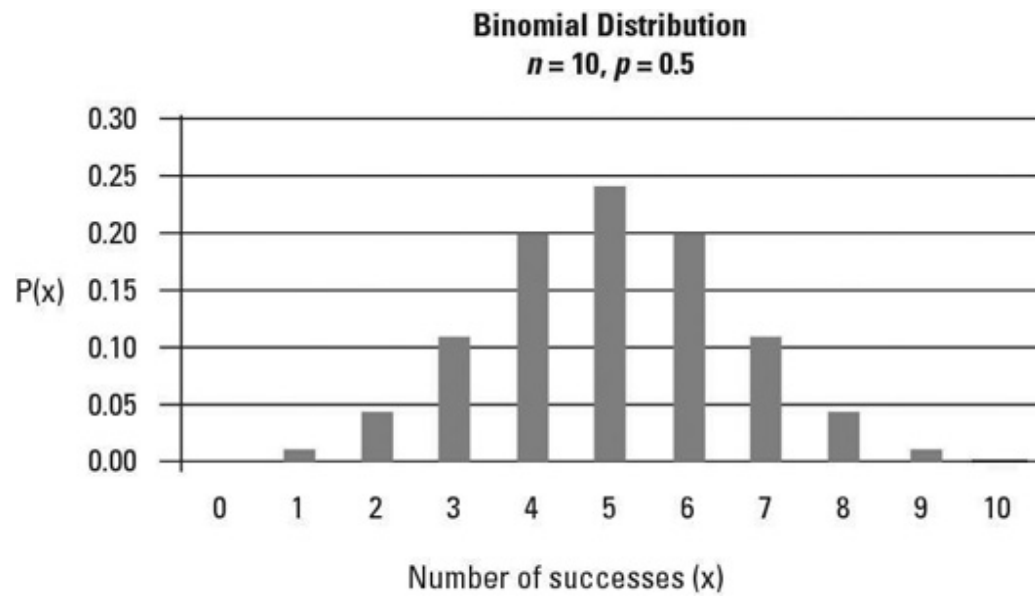
$$= \sum_i P(F|B_i)P(B_i) \text{ if } B_i \text{'s mutually exclusive and exhaustive}$$

Major Topics:

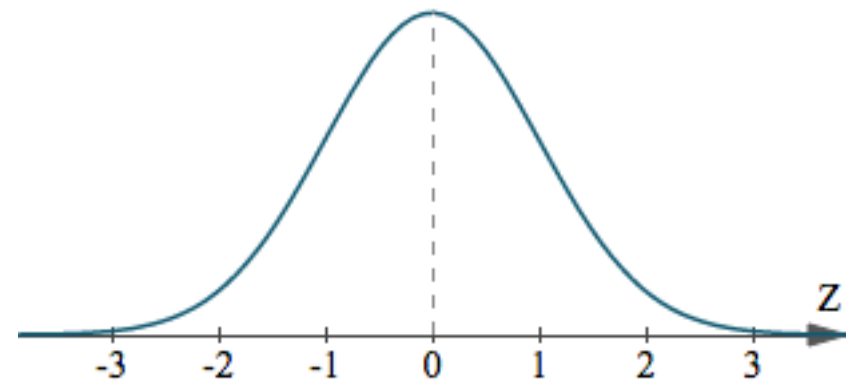
- ~~Counting and Combinatorics~~
- ~~Probability~~
- ~~Conditional Probability~~
- Random Variables
- Discrete/Continuous Distributions
- Joint Distributions and Convolutions

Random Variables

Discrete



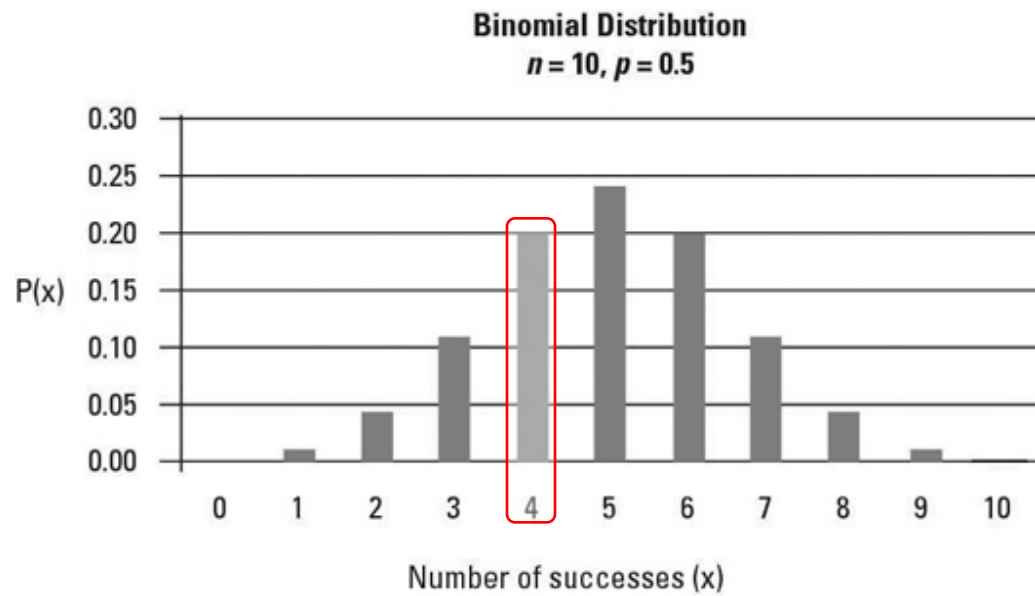
Continuous



Random Variables

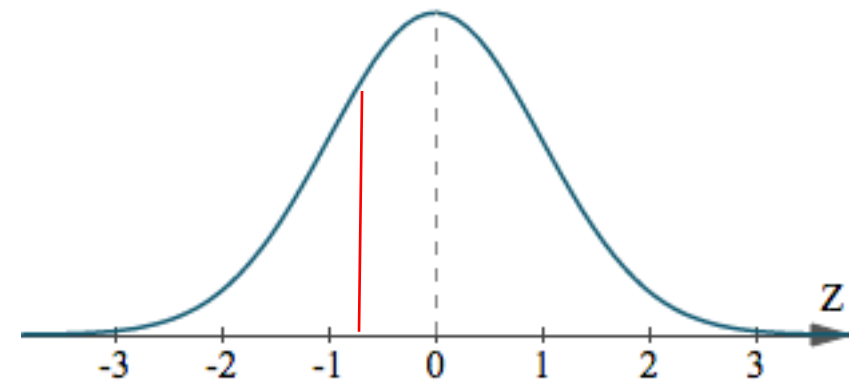
Discrete

PMF:



Continuous

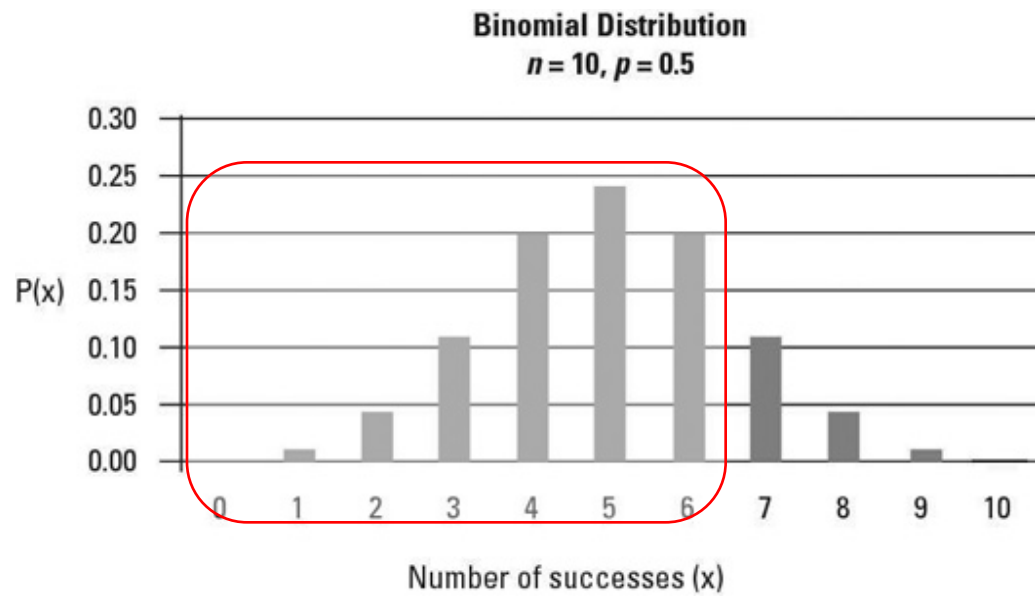
PDF:



Random Variables

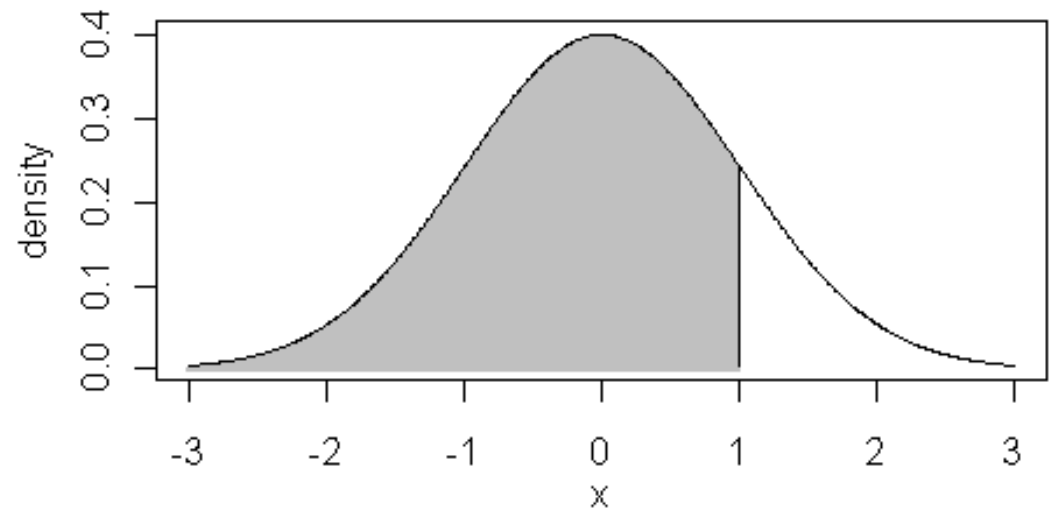
Discrete

CDF:



Continuous

CDF:



Random Variables

Discrete

$$E[X] = \sum_{x:P(x)>0} x * P(x)$$

Continuous

$$E[X] = \int_x x * p(X) dx$$

$$E[X + Y] = E[X] + E[Y]$$

$$E[aX + b] = aE[X] + b$$

$$\text{Var}(X) = E[X^2] - E[X]^2$$

$$\text{Var}[aX + b] = a^2 \text{Var}[X]$$

Distributions

Discrete	Continuous
Bin(n,p)	
Poi(λ)	
Bernoulli(p)	
Geo(p)	Exp(λ)
	Uniform(a,b)

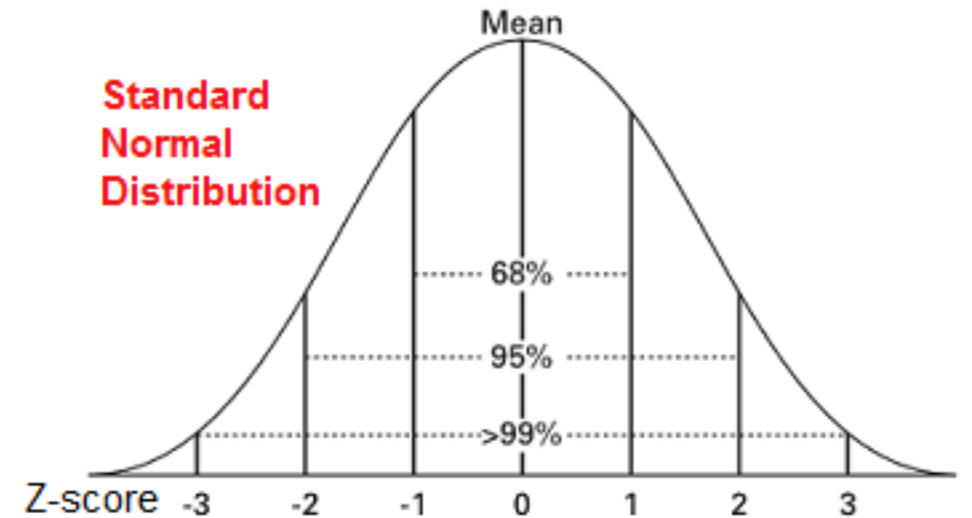
Normal Distribution

$$X \sim N(\mu, \sigma^2)$$

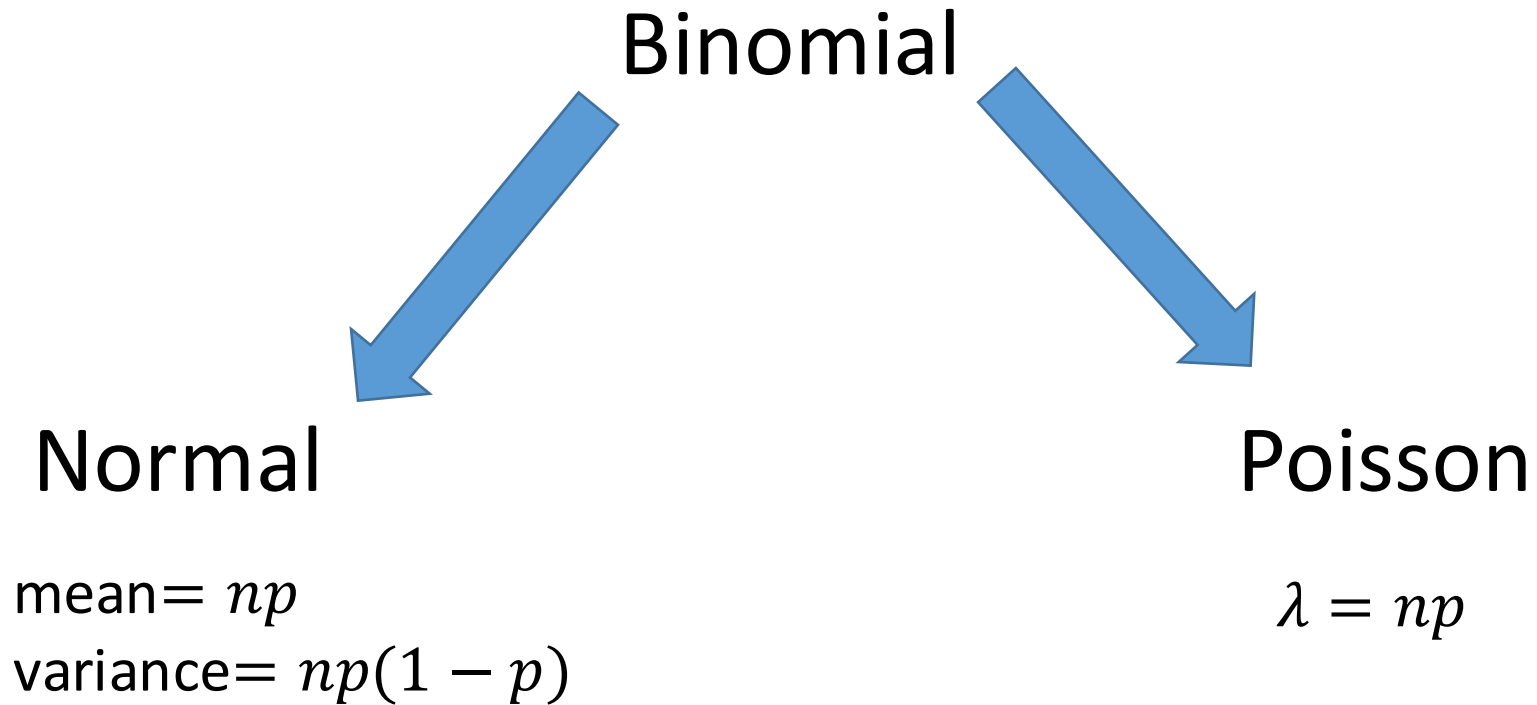
$$f_x(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$F_x(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

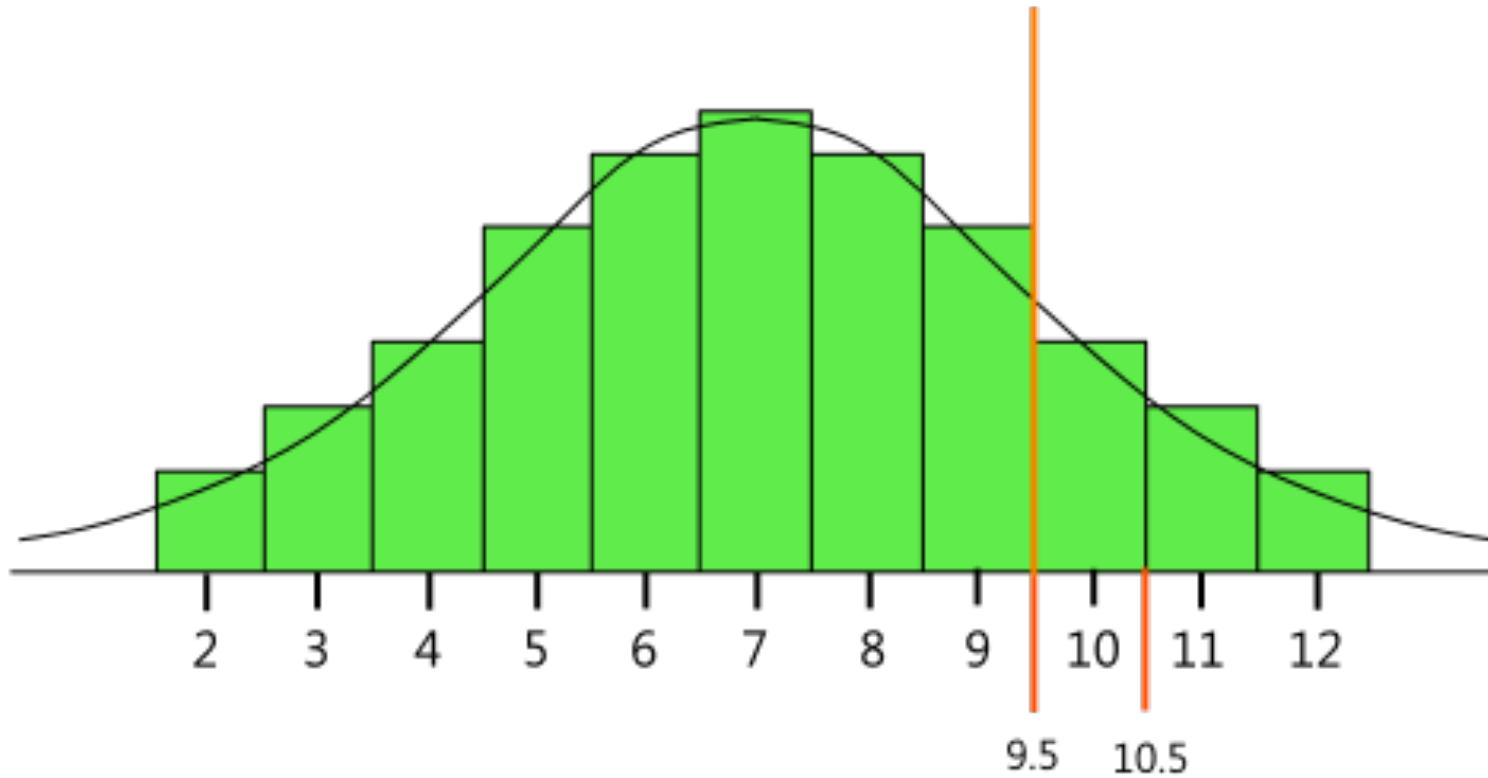
If $Y = aX + b$, and $B \sim N(\mu, \sigma^2)$, then $Y \sim N(a\mu + b, a^2\sigma^2)$



Approximations



Continuity Correction



Joint Distributions

$$P_{X,Y}(a, b) = P(X = a, Y = b)$$

$$\textit{Discrete: } P_X(a) = P(X = a) = \sum_y p_{X,Y}(a, y)$$

$$\textit{Continuous: } f_X(a) = \int_{-\infty}^{\infty} f_{X,Y}(a, y) dy$$

$$\begin{aligned} P(a_1 \leq X \leq a_2, b_1 \leq Y \leq b_2) &= \int_{a_1}^{a_2} \int_{b_1}^{b_2} f_{X,Y}(x, y) \\ &= F(a_2, b_2) - F(a_2, b_1) - F(a_1, b_2) + F(a_1, b_1) \end{aligned}$$

Independent if $P(X = x, Y = y) = P(X = x)P(Y = y)$ for all x, y

Convolutions

$$X \sim \text{Bin}(n_1, p), Y \sim \text{Bin}(n_2, p) \Rightarrow X + Y \sim \text{Bin}(n_1 + n_2, p)$$

$$X \sim \text{Poi}(\lambda_1), Y \sim \text{Poi}(\lambda_2) \Rightarrow X + Y \sim \text{Poi}(\lambda_1 + \lambda_2)$$

$$X \sim \text{N}(\mu_1, \sigma_1^2), Y \sim \text{N}(\mu_2, \sigma_2^2) \Rightarrow X + Y \sim \text{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$$

$$f_{X+Y}(a) = \int_{y=-\infty}^{\infty} f_X(a-y)f_Y(y)dy$$

Major Topics:

- Counting and Combinatorics
- Probability
- Conditional Probability
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Practice Problems!

Jeopardy

Counting/
Combinatorics

100

200

300

(Conditional)
Probability

100

200

300

Random Variables
and Distributions

100

200

300

Joint Distributions
and Convolutions

100

200

300

How many ways are there to arrange the letters in the word “Statistics”?

How many ways are there to arrange the letters in the word “Statistics”?

There are 10 letters: 3 s's, 3 t's, 2 i's, 1 a, and 1 c.

There are $\frac{10!}{3!3!2!1!1!}$ ways.



How many ways can I rearrange the letters of the alphabet such that none of the 5 vowels are next to each other?

Think of the 5 vowels as dividers for buckets, and imagine at least one consonant must go in the middle four buckets. That means that there are $\binom{17 + 6 - 1}{6 - 1}$ arrangements of vowels and consonants (17 consonants into 6 buckets). Then there are $21!$ ways to arrange the consonants and $5!$ ways to arrange the vowels, so our final answer is

$$21!5! \binom{17 + 6 - 1}{6 - 1}$$



Pretend it normally rains 300 days a year in Portland, and tomorrow, the newspaper says the cloud patterns mean there will be rain.

- When there is ***no*** rain, the newspapers have reported that there will be rain 10% of the time
- 70% of the days, there is rain ***and*** the newspaper predicts rain

What is the probability there is rain tomorrow?

$$P(\text{Rain}|\text{Prediction}) = 1 - P(\text{Rain}^c|\text{Prediction})$$

$$P(\text{Rain}^c|\text{Prediction}) = \frac{P(\text{Prediction}|\text{Rain}^c)P(\text{Rain}^c)}{P(\text{Prediction}|\text{Rain}^c)P(\text{Rain}^c) + P(\text{Prediction}|\text{Rain})P(\text{Rain})}$$

$$= \frac{P(\text{Prediction}|\text{Rain}^c)P(\text{Rain}^c)}{P(\text{Prediction}|\text{Rain}^c)P(\text{Rain}^c) + P(\text{Prediction}, \text{Rain})}$$

$$= \frac{.1 * \frac{65}{365}}{.1 * \frac{300}{365} + .7}$$

$$P(\text{Rain}|\text{Prediction}) = 1 - \frac{.1 * \frac{65}{365}}{.1 * \frac{300}{365} + .7}$$



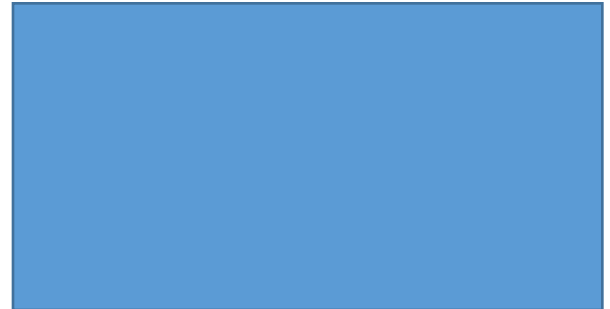
A gambler goes to bet. The dealer has 3 fair dice.

The dealer says: "You can choose your bet on a number, any number from 1 to 6. Then I'll roll the 3 dice. If none show the number you bet, you'll lose \$1. If one shows the number you bet, you'll win \$1. If two or three dice show the number you bet, you'll win \$3 or \$5, respectively."

Is it a fair game?

$$E[Game] = -1 * P(No Dice) + 1 * P(One Dice) + 3 * P(Two Dice) + 5 * P(Three Dice)$$

$$E[Game] = -1 * \left(\frac{5}{6}\right)^3 + 1 * \binom{3}{1} \frac{1}{6} \left(\frac{5}{6}\right)^2 + 3 * \binom{3}{2} \frac{5}{6} \left(\frac{1}{6}\right)^2 + 5 * \left(\frac{1}{6}\right)^3$$
$$= 0$$

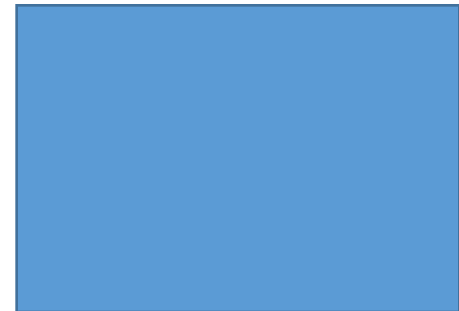


King Arthur, Merlin, Sir Lancelot, Sir Gawain, and Guinevere decide to go to their favorite restaurant. They sit down at a round table for five, and as soon as they do, Lancelot notes, "We sat down around the table in age order! What is the probability of that?"

Merlin smiles broadly. "This is easily solved without any magic." He then shared the answer. What did he say the probability was?

Imagine they sat down in age order, with each person randomly picking a seat. The first person is guaranteed to pick a seat that "works". The second oldest can sit to his right or left, since these five can sit either clockwise or counterclockwise. The probability of picking a seat that works is thus $2/4$, or $1/2$. The third oldest now has three chairs to choose from, one of which continues the progression in the order determined by the second person, for a probability of $1/3$. This leaves two seats for the fourth oldest, or a $1/2$ chance. The youngest would thus be guaranteed to sit in the right seat, since there is only one seat left. This gives $1 * 1/2 * 1/3 * 1/2 * 1 = 1/12$ for the probability.

Alternatively, we can think event space over sample space: $\frac{2*5}{5!}$



Based on a historical averages we expect a year until the next earthquake. One of your friends suggests using the cdf of a uniform distribution with parameters $\alpha=0$ and $\beta=730$ to measure time until there's an earthquake. When does this model begin to overestimate the probability of the next earthquake, when compared with the cdf of an exponential?

CDF of a uniform: $F(x) = \frac{x - a}{b - a}$

Exponential distribution: $F(x) = 1 - e^{-\lambda x}$

Expectation of Exponential Distribution is λ^{-1}

CDF of a normal: $F(x) = \frac{x - a}{b - a}$

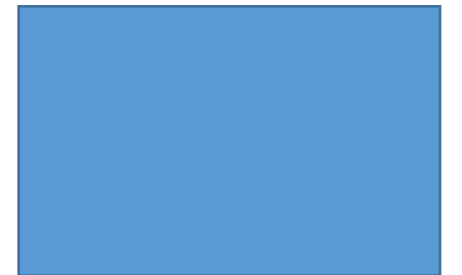
Exponential distribution: $F(x) = 1 - e^{-\lambda x}$

Expectation of Exponential Distribution is λ^{-1}

When is $\frac{\frac{x - a}{b - a}}{1 - e^{-\lambda x}} > 1$? Lets use days as units

$$\frac{\frac{x - 0}{730 - 0}}{1 - e^{-\frac{1}{365}x}} > 1$$

$x > 581.67$, so on day 582



A jar has 7 marbles: 1 yellow, 1 blue, 1 green, and 4 red. What is the probability of drawing two red marbles?

There are $\frac{\binom{4}{2}}{\binom{7}{2}}$ ways. 4 choose 2 ways to draw red marbles, and 7 choose 2 ways total to draw marbles.



Say the SAT is normally distributed with mean 500 and variance 1000. If two students take the exam, what is the probability that their combined score is greater than 1020?

Score of student 1 $\sim N(500, 1000)$

Score of student 2 $\sim N(500, 1000)$

Score of both students varies as $N(500, 1000) + N(500, 1000) = N(1000, 2000)$.

$$P(N(1000, 2000) > 1020) = 1 - P(N(1000, 2000) < 1020.5)$$

$$P(N(1000, 2000) < 1020.5) = P\left(Z < \frac{1020.5 - 1000}{\sqrt{2000}}\right)$$
$$= P(Z < .458) = .677$$

So our final probability is $1 - .677$ or $.323$



A pirate is digging up buried treasure on a perfectly square island of size 3 miles by 3 miles. The pirate only has an hour until his ship arrives to take him back, and in that time he'll cover the easternmost third of the island. The treasure is buried at a location x, y with probability $P(X = x, Y = y) = cx^2y$. What is the probability that he finds the treasure?

The first step is to find c . We need $1 = \int_0^3 \int_0^3 cx^2y = c \int_0^3 \int_0^3 x^2y$

$$\int_0^3 \int_0^3 x^2y = 40.5, \text{ so } c = \frac{1}{40.5}$$

The probability that he finds the treasure is:

$$\int_2^3 \int_0^3 \frac{1}{40.5} * x^2y dy dx = 70.4\%$$



Table of House in Hogwarts vs. favorite pet. Are they independent?

	Dog	Cat	Fish
Gryffindor	.12	.12	.06
Slytherin	.04	.04	.02
Ravenclaw	.16	.16	.08
Hufflepuff	.08	.08	.04

Table of House in Hogwarts vs. favorite pet. Are they independent? YES

	Dog .4	Cat .4	Fish .2
Gryffindor .3	.12	.12	.06
Slytherin .1	.04	.04	.02
Ravenclaw .4	.16	.16	.08
Hufflepuff .2	.08	.08	.04



You're head of a store that sells building blocks for kids, and you just received a shipment of 10,000 lego kits. You look through the kits, and you estimate that more than 2,000 and less than 3,000 of them are defective. You know historically, the shipping company messes up and with probability .1 sends you a ripoff brand and probability .3 sends you megablocks. If normal lego kits and megablocks have a probability .1 of being defective and the ripoff brand has probability .5 of being defective, what is the probability that you actually received the ripoff brand instead of legos? ***Approximate*** your answer or provide an approximate formula that you could plug into a calculator.

$$P(\text{Ripoff}|\text{defects})$$

$$= \frac{P(\text{defects}|\text{Ripoff})P(\text{Ripoff})}{P(\text{defects}|\text{Ripoff})P(\text{Ripoff}) + P(\text{defects}|\text{Legos})P(\text{Legos}) + P(\text{defects}|\text{MegaBlocks})P(\text{Megablocks})}$$

$$= \frac{P(\text{defects}|\text{Ripoff}).1}{P(\text{defects}|\text{Ripoff}).1 + P(\text{defects}|\text{Legos}).6 + P(\text{defects}|\text{MegaBlocks}).3}$$

$$= \frac{P(\text{defects}|\text{Ripoff}).1}{P(\text{defects}|\text{Ripoff}).1 + P(\text{defects}|\text{.1 failure rate}).9}$$

Approximate ripoff brand with a normal distribution and brands with a .1 failure rate as a poisson distribution

$$R \sim \text{Normal}(5000, 2500)$$

$$LM \sim \text{Poi}(1000)$$

$$P(\text{defects}|\text{Ripoff}) = P(2000 < R < 3000) = P(7000 < R < 8000) = \Phi\left(\frac{7999.5 - 5000}{50}\right) - \Phi\left(\frac{7000.5 - 5000}{50}\right)$$

$$P(\text{defects}|\text{LM}) = P(2000 < LM < 3000) = \sum_{i=2000}^{3000} \frac{1000^i e^{-1000}}{i!}$$

