Great Expectations

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CS109, Stanford University

Joint Random Variables

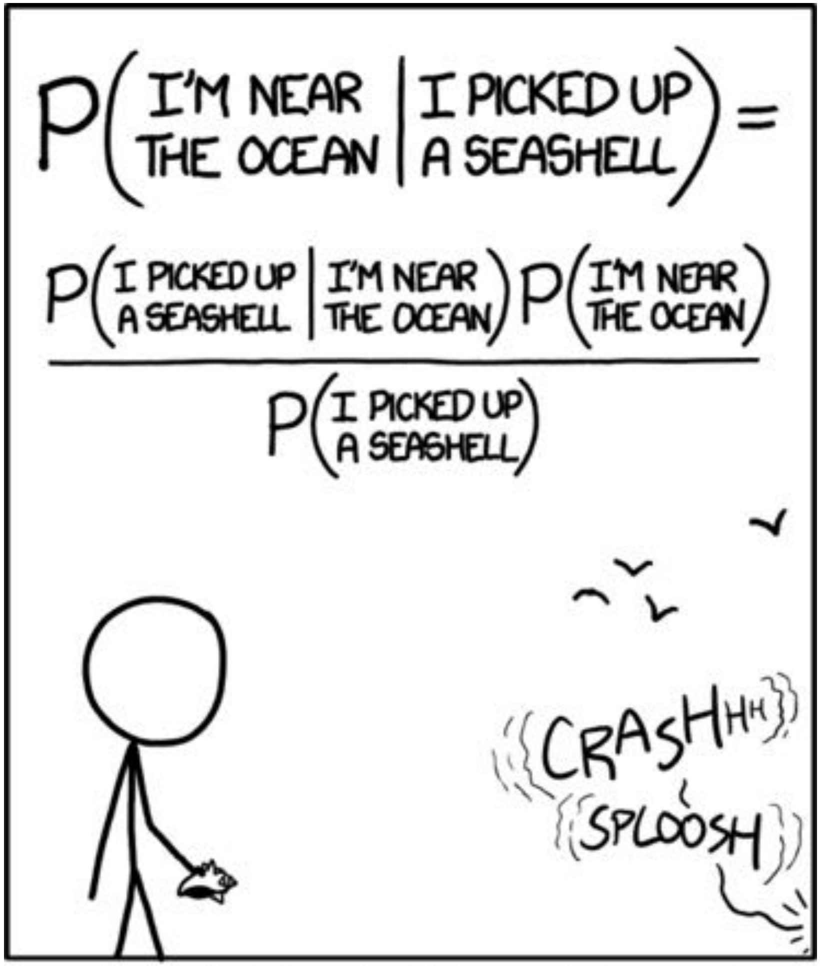
- Use a joint table, density function or CDF to solve probability question
- Think about **conditional** probabilities with joint variables (which might be continuous)
- Use and find **expectation** of multiple RVS
- Use and find **independence** of multiple RVS
- What happens when you **add** random variables?
- How do multiple variables **covary**?

Course Mean

$$E[\text{CS109}]$$

This is actual midpoint of course
(Just wanted you to know)

Sea side



STATISTICALLY SPEAKING, IF YOU PICK UP A SEASHELL AND *DON'T* HOLD IT TO YOUR EAR, YOU CAN PROBABLY HEAR THE OCEAN.

Review

Expected Values of Sums

E[X + Y] = E[X] + E[Y]

Generalized: $$E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i]$$

Holds regardless of dependency between X_i's

End Review

Bool Was Cool

- Let E_1, E_2, ... E_n be events with indicator RVs X_i
  - If event E_i occurs, then $X_i = 1$, else $X_i = 0$
  - Recall $E[X_i] = P(E_i)$
  - Why?

$$E[X_i] = 0 \cdot (1 - P(E_i)) + 1 \cdot P(E_i)$$

Bernoulli aka Indicator Random Variables were studied extensively by George Boole

Boole died of being too cool

Expectation of Binomial

- Let $Y \sim \text{Bin}(n, p)$
  - n independent trials
  - Let X_i = 1 if i-th trial is "success", 0 otherwise
  - $X_i \sim \text{Ber}(p)$ $\quad E[X_i] = p$

$$Y = X_1 + X_2 + \cdots + X_n = \sum_{i=1}^{n} X_i$$

$$E[Y] = E[\sum_{i=1}^{n} X_i]$$

$$= \sum_{i=1}^{n} E[X_i]$$

$$= E[X_1] + E[X_2] + \ldots E[X_n]$$

$$= np$$

Expectation of Negative Binomial

- Let Y ~ NegBin(r, p)
  - Recall Y is number of trials until r "successes"
  - Let X_i = # of trials to get success after (i – 1)st success
  - X_i ~ Geo(p) (i.e., Geometric RV) $E[X_i] = \frac{1}{p}$

$$Y = X_1 + X_2 + \cdots + X_r = \sum_{i=1}^{r} X_i$$

$$E[Y] = E[\sum_{i=1}^{r} X_i]$$

$$= \sum_{i=1}^{r} E[X_i]$$

$$= E[X_1] + E[X_2] + \ldots E[X_r]$$

$$= \frac{r}{p}$$

Differential Privacy

Aims to provide means to **maximize the accuracy** of probabilistic queries while minimizing the **probability** of identifying its records.

Cynthia Dwork's celebrity lookalike is Cynthia Dwork.

Differential Privacy

100 independent values $X_1 \ldots X_{100}$ where $X_i \sim \text{Bern}(p)$

```
# Maximize accuracy, while preserving privacy.
def calculateYi(Xi):
    obfuscate = random()
    if obfuscate:
        return indicator(random())
    else:
        return Xi
```

`random()` returns `True` or `False` with equal likelihood

Differential Privacy

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random() returns True or False with equal likelihood

What is $E[Y_i]$?

$$E[Y_i] = P(Y_i = 1) = \frac{p}{2} + \frac{1}{4}$$

Differential Privacy

100 independent values $X_1 \dots X_{100}$ where $X_i \sim \text{Bern}(p)$

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# Maximize accuracy, while preserving privacy.
def calculateYi(Xi):
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```

random() returns True or False with equal likelihood

Let $Z = \sum_{i=1}^{100} Y_i$ What is the E[Z]?

$$E[Z] = E\left[\sum_{i=1}^{100} Y_i\right] = \sum_{i=1}^{100} E[Y_i] = \sum_{i=1}^{100} \left(\frac{p}{2} + \frac{1}{4}\right) = 50p + 25$$

Differential Privacy

100 independent values $X_1 \ldots X_{100}$ where $X_i \sim \text{Bern}(p)$

```
# Maximize accuracy, while preserving privacy.
def calculateYi(Xi):
    obfuscate = random()
    if obfuscate:
        return indicator(random())
    else:
        return Xi
```

random() returns True or False with equal likelihood

Let $Z = \sum_{i=1}^{100} Y_i$ $E[Z] = 50p + 25$ How do you estimate p?

$$p \approx \frac{Z - 25}{50}$$

Challenge: What is the probability that our estimate is good?

More Practice!

Computer Cluster Utilization

- Computer cluster with *k* servers
  - Requests independently go to server *i* with probability p_i
  - Let event A_i = server *i* receives no requests
  - Let Bernoulli B_i be an indicator for A_i
  - X = # of events A_1, A_2, … A_k that occur
  - Y = # servers that receive ≥ 1 request = k – X
  - E[Y] after first *n* requests?
  - Since requests independent: $P(A_i) = (1 - p_i)^n$

$$X = \sum_{i=1}^{k} B_i$$

$$E[X] = E\left[\sum_{I=1}^{k} B_i\right] = \sum_{i=1}^{k} E[B_i] = \sum_{i=1}^{k} P(A_i) = \sum_{i=1}^{k} (1 - p_i)^n$$

$$E[Y] = k - E[X] = k - \sum_{i=1}^{k} (1 - p_i)^n$$

amazon

amazon web services™

* 52% of Amazons Profits

**More profitable than Amazon’s North America commerce operations

http://www.zdnet.com/article/amazons-finds-its-profit-horse-in-aws-why-its-so-disruptive-to-its-old-guard/

When stuck, brainstorm about random variables

Hash Tables (aka Toy Collecting)

- Consider a hash table with *n* buckets
  - Each string equally likely to get hashed into any bucket
  - Let X = # strings to hash until each bucket ≥ 1 string
  - What is E[X]?
  - Let X_i = # of trials to get success after *i*-th success
    - where "success" is hashing string to previously empty bucket
    - After *i* buckets have ≥ 1 string, probability of hashing a string to an empty bucket is $p = (n - i) / n$
    - $P(X_i = k) = \frac{n-i}{n}\left(\frac{i}{n}\right)^{k-1}$ equivalently: $X_i \sim \text{Geo}((n - i) / n)$
    - $E[X_i] = 1 / p = n / (n - i)$
  - $X = X_0 + X_1 + \ldots + X_{n-1} \Rightarrow E[X] = E[X_0] + E[X_1] + \ldots + E[X_{n-1}]$

$$E[X] = \frac{n}{n} + \frac{n}{n-1} + \frac{n}{n-2} + \ldots + \frac{n}{1} = n\left[\frac{1}{n} + \frac{1}{n-1} + \ldots + 1\right] = O(n \log n)$$

This is your final answer

Break

Conditional Expectation

Conditional Expectation

- X and Y are jointly discrete random variables
  - Recall conditional PMF of X given Y = y:

$$p_{X|Y}(x \mid y) = P(X = x \mid Y = y) = \frac{p_{X,Y}(x, y)}{p_Y(y)}$$

- Define conditional expectation of X given Y = y:

$$E[X \mid Y = y] = \sum_x x P(X = x \mid Y = y) = \sum_x x\, p_{X|Y}(x \mid y)$$

- Analogously, jointly continuous random variables:

$$f_{X|Y}(x \mid y) = \frac{f_{X,Y}(x, y)}{f_Y(y)} \qquad E[X \mid Y = y] = \int_{-\infty}^{\infty} x\, f_{X|Y}(x \mid y)\, dx$$

Rolling Dice

- Roll two 6-sided dice D_1 and D_2
  - X = value of $D_1 + D_2$ Y = value of D_2
  - What is E[X | Y = 6]?

$$E[X \mid Y = 6] = \sum_x x P(X = x \mid Y = 6)$$

$$= \left(\frac{1}{6}\right)(7 + 8 + 9 + 10 + 11 + 12) = \frac{57}{6} = 9.5$$

  - Intuitively makes sense: 6 + E[value of D_1] = 6 + 3.5

Properties of Conditional Expectation

- X and Y are jointly distributed random variables

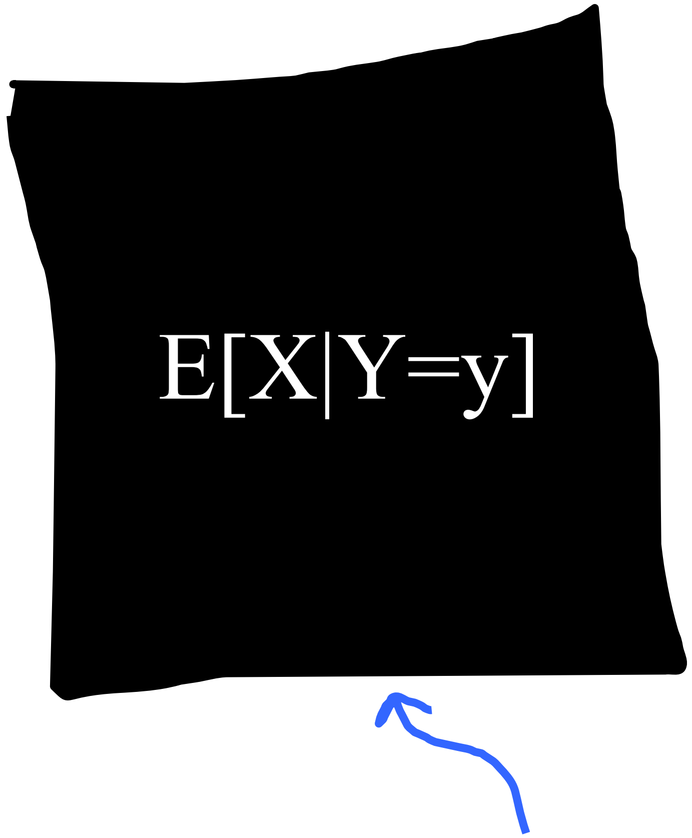
$$E[g(X) \mid Y = y] = \sum_x g(x) p_{X|Y}(x \mid y) \quad \text{or} \quad \int_{-\infty}^{\infty} g(x) f_{X|Y}(x \mid y)\, dx$$

- Expectation of conditional sum:

$$E\left[\sum_{i=1}^{n} X_i \mid Y = y\right] = \sum_{i=1}^{n} E[X_i \mid Y = y]$$

Conditional Expectation Functions

- Define $g(Y) = E[X \mid Y]$
- This is just function of Y



This is a function with Y as input

Conditional Expectation Functions

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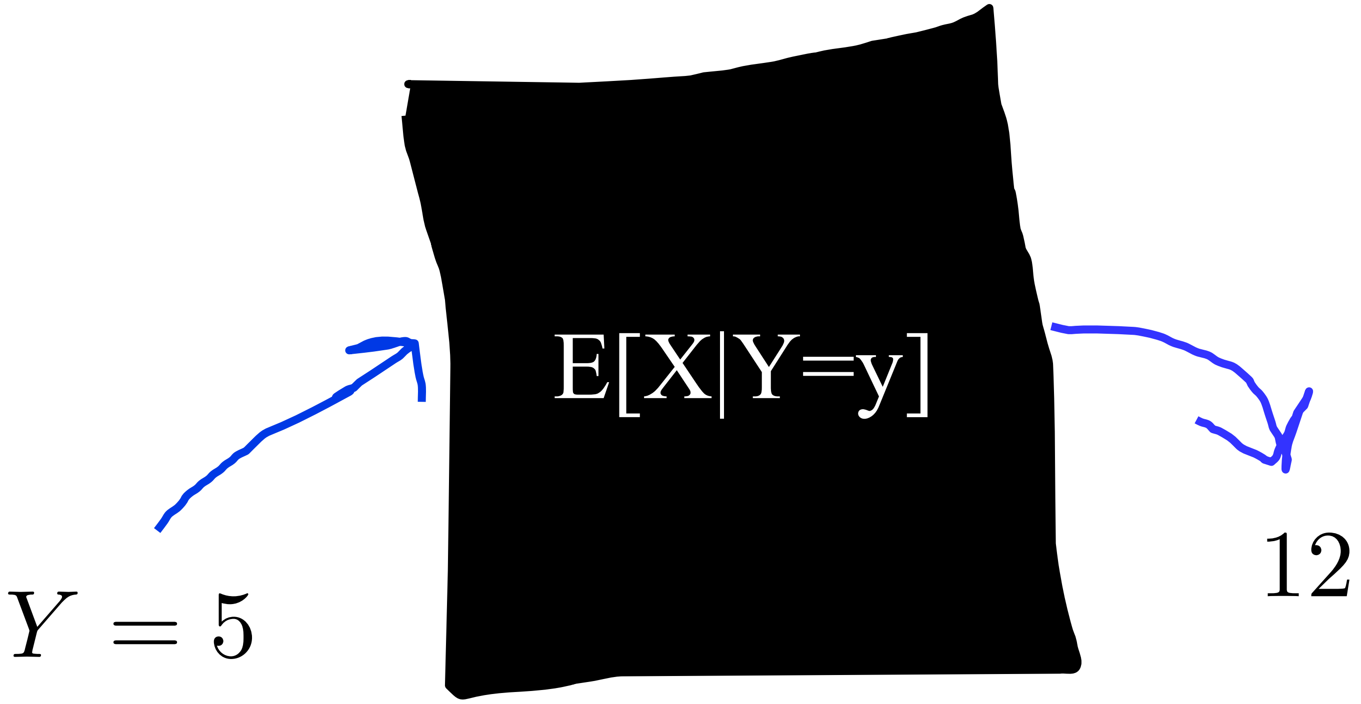
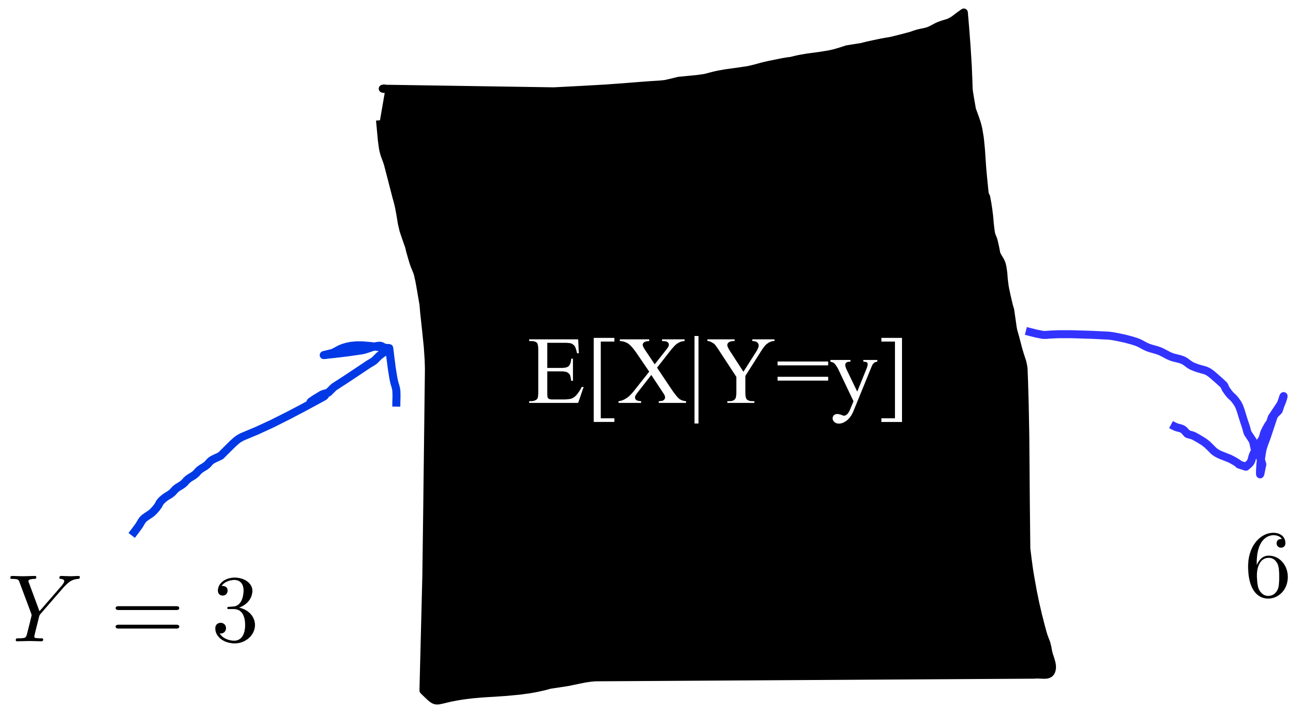


Conditional Expectation Functions

- Define $g(\mathrm{Y}) = \mathrm{E}[\mathrm{X} \mid \mathrm{Y}]$
- This is just function of Y



Conditional Expectation Functions

This is a number:

$$E[X]$$

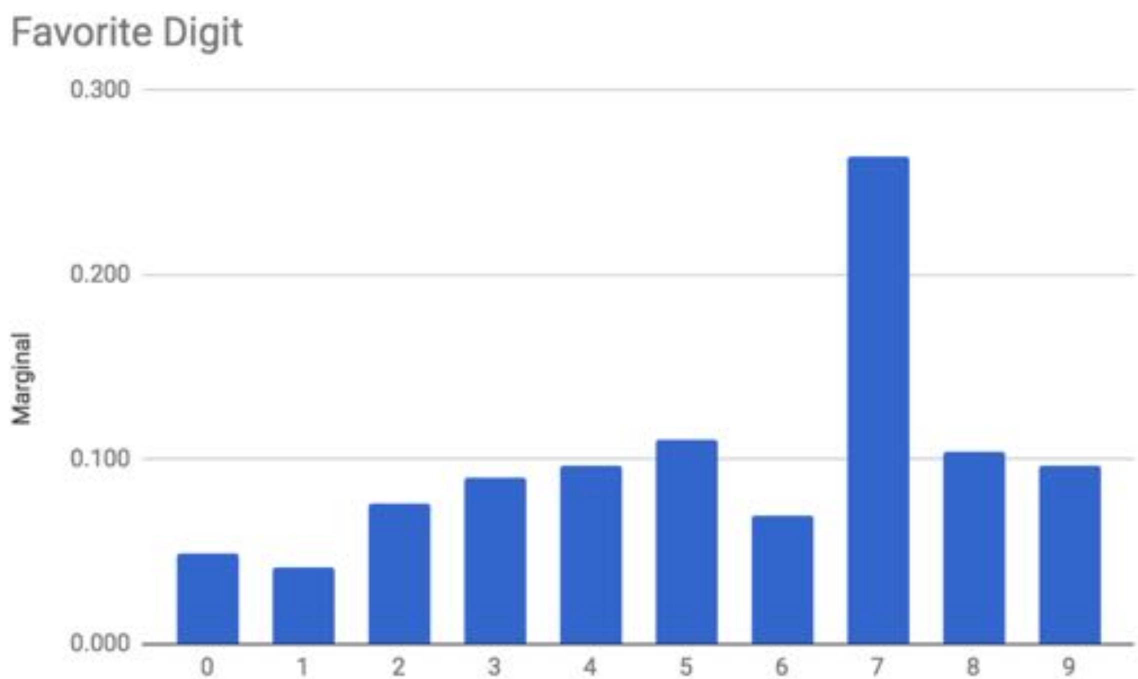
This is a function of y:

$$E[X|Y = y]$$

$E[X = 5]$ Doesn't make sense. Take expectation of random variables, not events

Conditional Expectation Functions

X = favorite number
Y = year in school



E[X] = 0 * 0.05 + … + 9 * 0.10 = 5.38

Conditional Expectation Functions

X = favorite number
Y = year in school

E[X | Y] ?

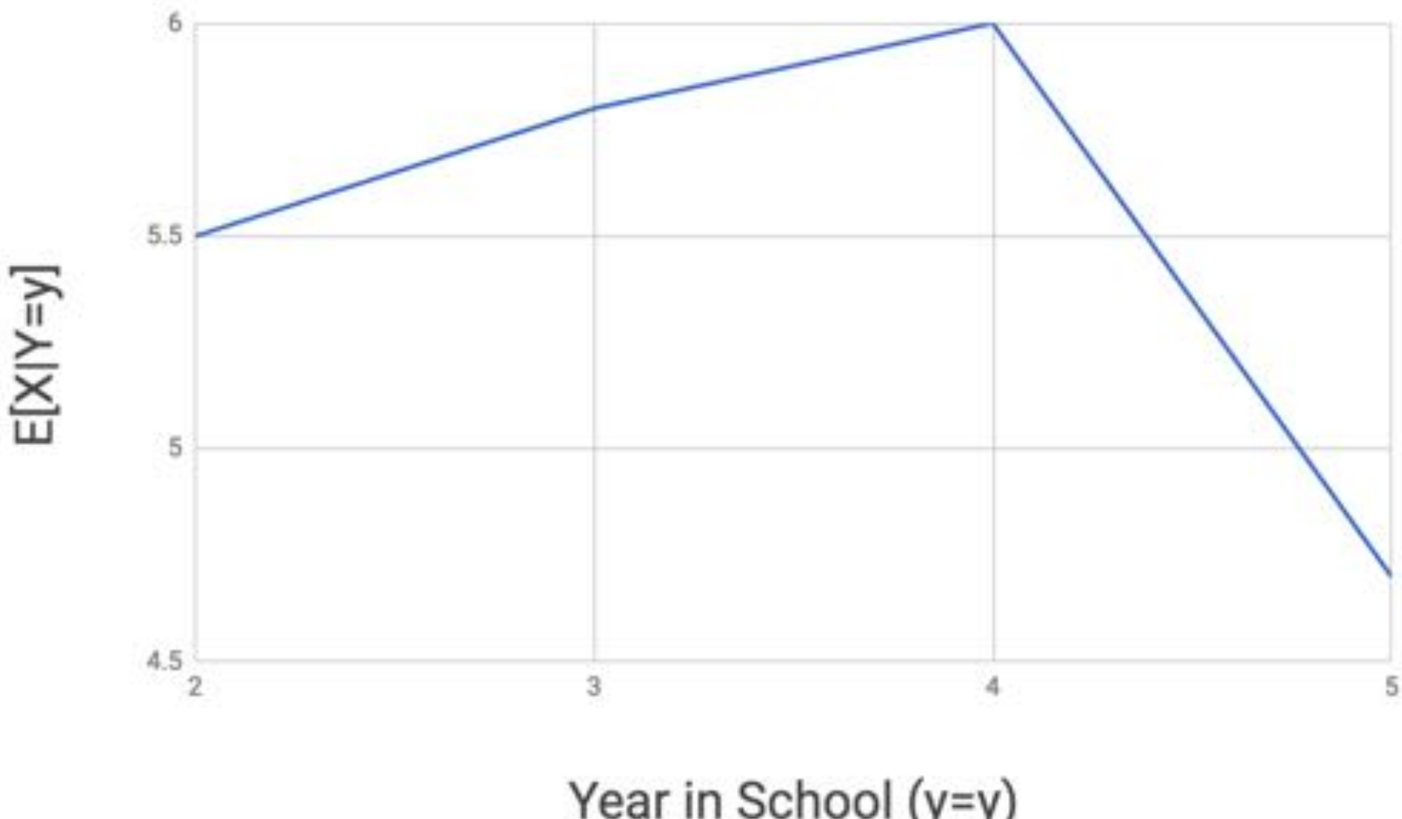
Year in school, Y = y	E[X | Y = y]
2	5.5
3	5.8
4	6.0
5	4.7

Conditional Expectation Functions

X = favorite number

Y = year in school

E[X | Y] ?

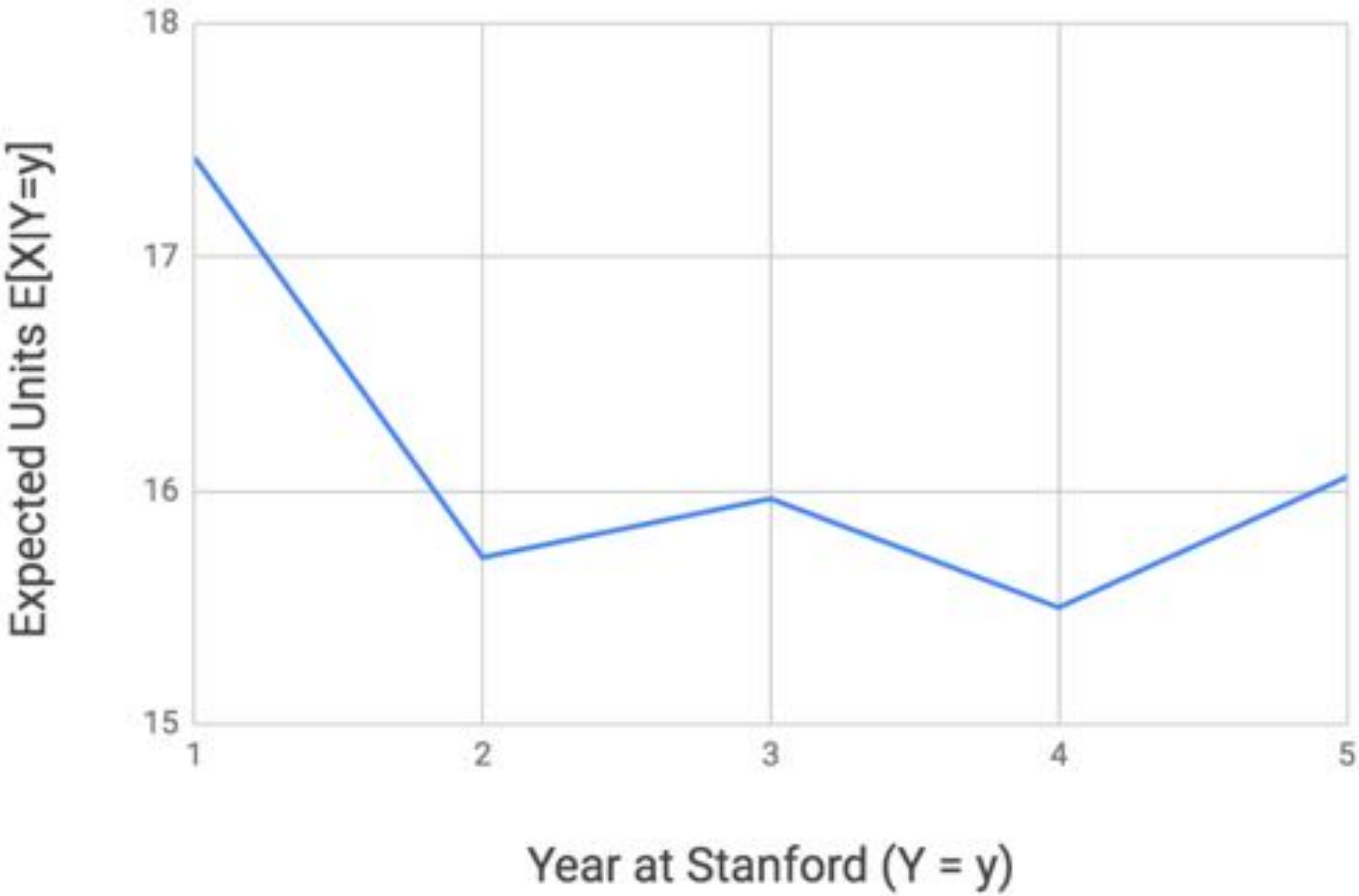


Conditional Expectation Functions

X = units in fall quarter

Y = year in school

E[X | Y] ?



Law of Total Expectation

$$E[E[X|Y]] = E[X]$$

$$E[E[X|Y]] = \sum_y E[X|Y=y]P(Y=y)$$ g(Y) = E[X|Y]

$$= \sum_y \sum_x xP(X=x|Y=y)P(Y=y)$$ Def of E[X|Y]

$$= \sum_y \sum_x xP(X=x, Y=y)$$ Chain rule!

$$= \sum_x \sum_y xP(X=x, Y=y)$$ I switch the order of the sums

$$= \sum_x x \sum_y P(X=x, Y=y)$$ Move that x outside the y sum

$$= \sum_x xP(X=x)$$ Marginalization

$$= E[X]$$ Def of E[X]

Law of Total Expectation

For any random variable X and any discrete random variable Y

$$E[X] = \sum_y E[X|Y = y]P(Y = y)$$

Analyzing Recursive Code

```
int Recurse() {
    int x = randomInt(1, 3);  // Equally likely values

    if (x == 1) return 3;
    else if (x == 2) return (5 + Recurse());
    else return (7 + Recurse());
}
```

- Let Y = value returned by `Recurse()`. What is E[Y]?

$$E[Y] = E[Y \mid X=1]P(X=1) + E[Y \mid X=2]P(X=2) + E[Y \mid X=3]P(X=3)$$

$$E[Y \mid X=1] = 3$$

$$E[Y \mid X=2] = E[5+Y] = 5 + E[Y]$$

$$E[Y \mid X=3] = E[7+Y] = 7 + E[Y]$$

$$E[Y] = 3(1/3) + (5 + E[Y])(1/3) + (7 + E[Y])(1/3) = (1/3)(15 + 2E[Y])$$

$$E[Y] = 15$$

Protip: do this in CS161

If we have time…

Hiring and Engineer

Your company has one job opening for a software engineer.

You have *n* candidates. But you have to say yes/no **immediately** after each interview!

Proposed algorithm: reject the first *k* and accept the next one who is better than all of them.

What's the best value of *k*?

xkcd by Randall Munroe

Hiring and Engineer

n candidates, must say yes/no **immediately** after each interview.
Reject the first *k*, accept the next who is better than all of them.
What's the best value of *k*?

B: event that you hire the best engineer
X: position of the best engineer on the interview schedule

What is the $P(B|X = i)$?

Hiring and Engineer

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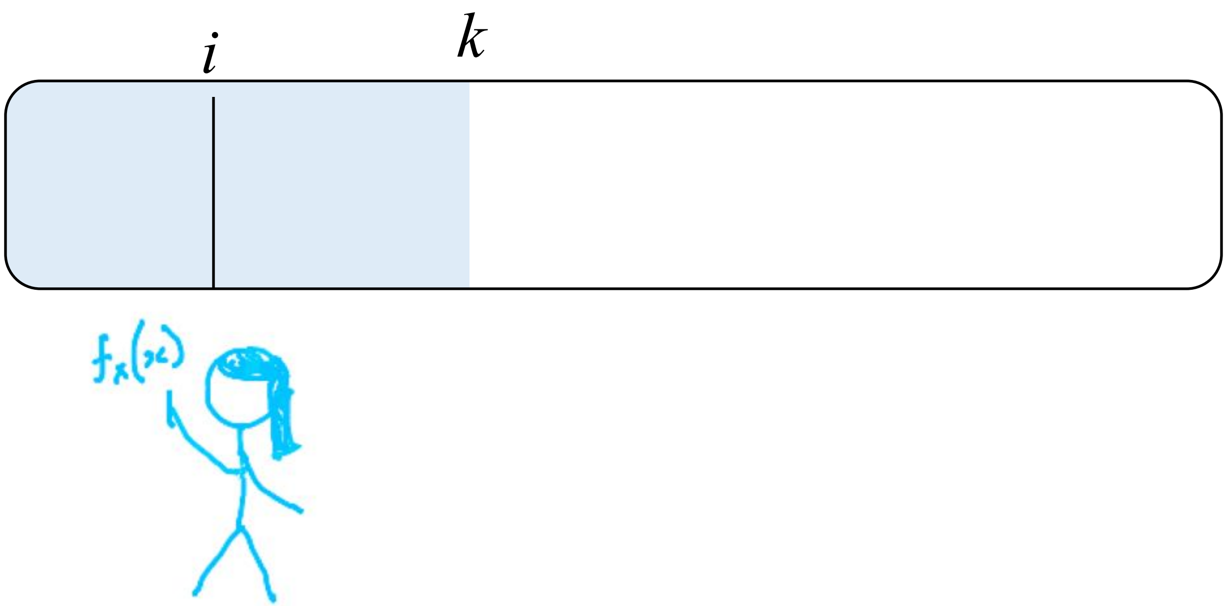

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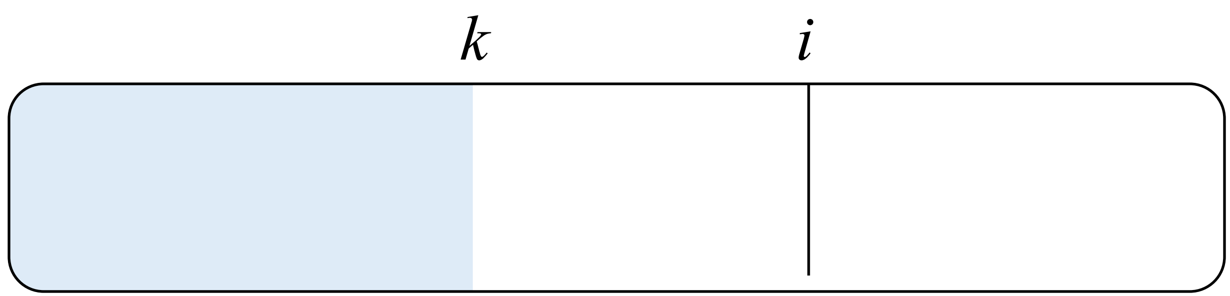

Hint: where is the best among the first $i-1$ candidates?

Hiring and Engineer

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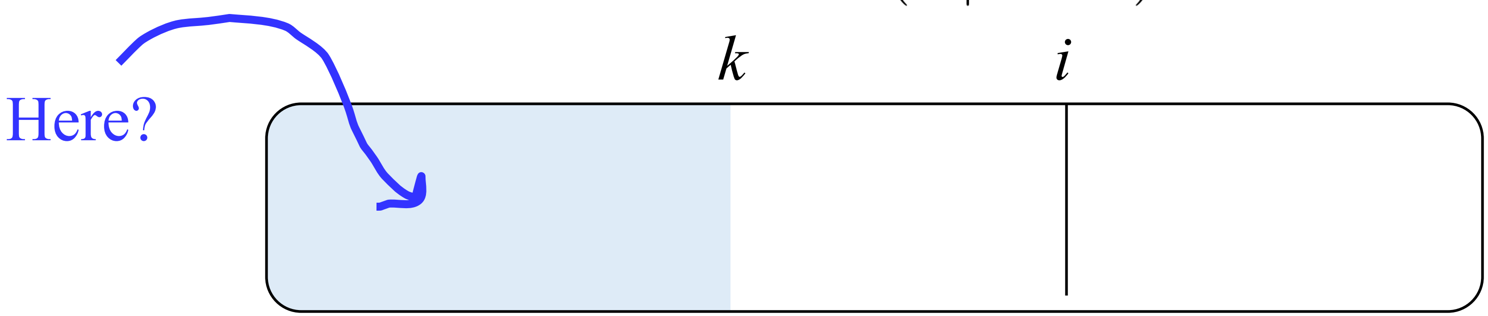

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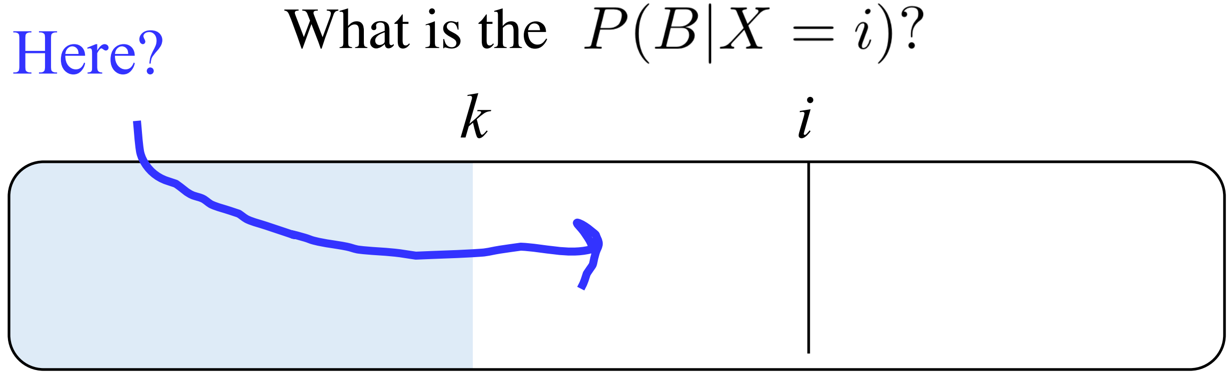

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Hiring and Engineer

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Reject the first *k*, accept the next who is better than all of them.
What's the best value of *k*?

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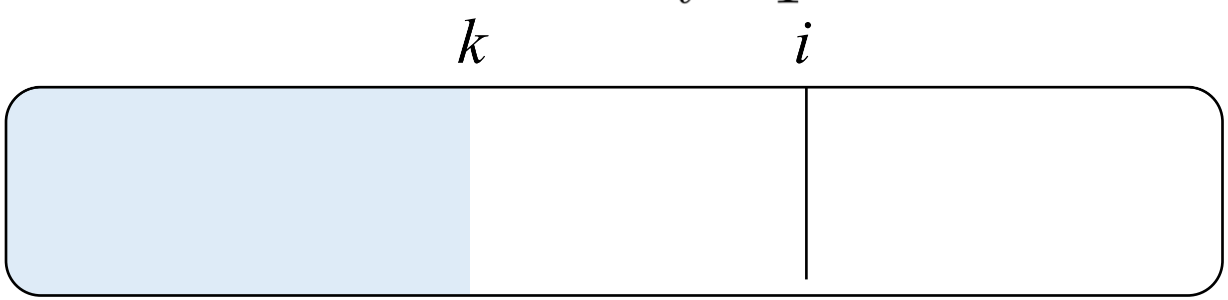
$$P(B|X = i) = \frac{k}{i-1} \text{ if } i > k$$



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Hiring and Engineer

n candidates, must say yes/no **immediately** after each interview.
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B: event that you hire the best engineer
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$$P_k(B) = \sum_{i=1}^{n} P_k(B|X=i)P(X=i)$$ By the law of total expectation

$$= \frac{1}{n}\sum_{i=1}^{n} P_k(B|X=i)$$

$$= \frac{1}{n}\sum_{i=k+1}^{n} \frac{k}{i-1}$$ since we know $P_k(Best|X=i)$

$$\approx \frac{1}{n}\int_{i=k+1}^{n} \frac{k}{i-1} di$$ By Riemann Sum approximation

$$= \frac{k}{n}\ln(i=1)\Big|_{k+1}^{n} = \frac{k}{n}\ln\frac{n-1}{k} \approx \frac{k}{n}\ln\frac{n}{k}$$

Hiring and Engineer

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$$P_k(B) = \sum_{i=1}^{n} P_k(B|X = i)P(X = i)$$

By the law of total expectation

$$\approx \frac{k}{n} \ln \frac{n}{k}$$

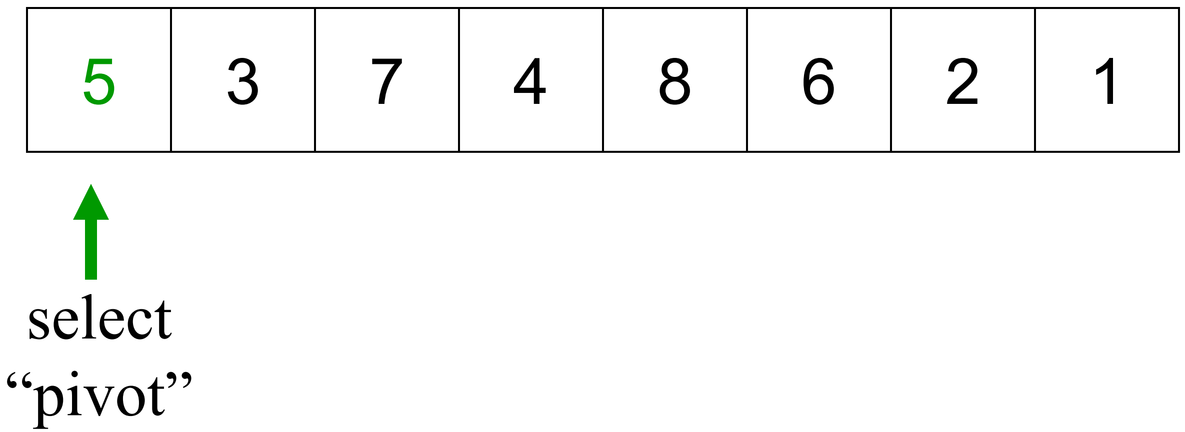
Fun fact. Optimized when: $k = \frac{n}{e}$

That’s all folks!

Let's Do Some Sorting!

5	3	7	4	8	6	2	1

QuickSort



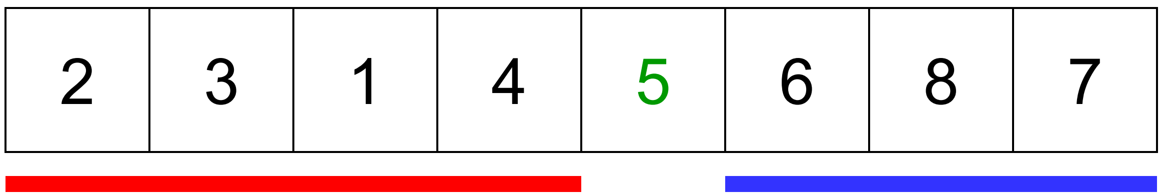
Recursive Insight

5	3	7	4	8	6	2	1

Partition array so:

- everything smaller than pivot is on left
- everything greater than or equal to pivot is on right
- pivot is in-between

Recursive Insight



Partition array so:

- everything smaller than pivot is on left
- everything greater than or equal to pivot is on right
- pivot is in-between

Recursive Insight

2	3	1	4	5	6	8	7

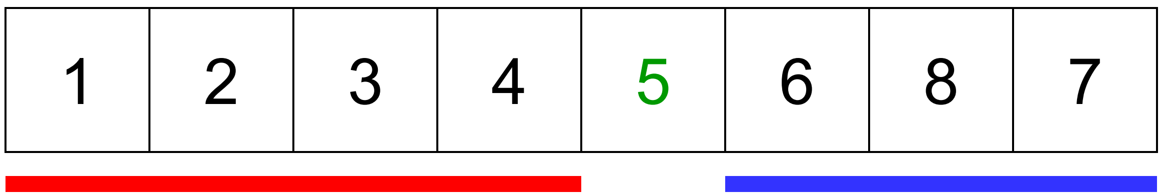
Now recursive sort “red” sub-array

Recursive Insight

1	2	3	4	5	6	8	7

Now recursive sort “red” sub-array

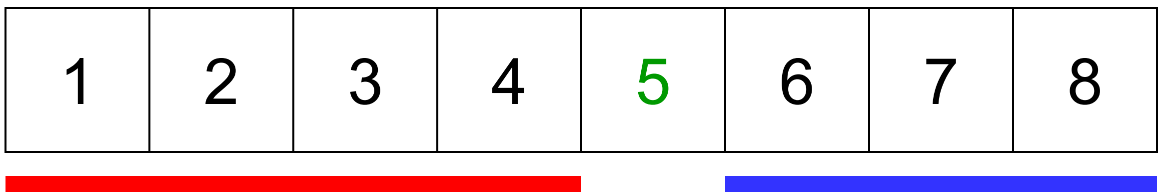
Recursive Insight



Now recursive sort "red" sub-array

Then, recursive sort "blue" sub-array

Recursive Insight



Now recursive sort "red" sub-array

Then, recursive sort "blue" sub-array

Recursive Insight

1	2	3	4	5	6	7	8

Everything is sorted!

```
void Quicksort(int arr[], int n)
{
   if (n < 2) return;

   int boundary = Partition(arr, n);

   // Sort subarray up to pivot
   Quicksort(arr, boundary);

   // Sort subarray after pivot to end
   Quicksort(arr + boundary + 1, n – boundary - 1);
}
```

"boundary" is the index of the pivot

Partition

Pivot

Before

5	3	7	4	8	6	2	1

After

2	3	1	4	5	6	8	7

Does one comparison for every element in the array and the pivot.

Complexity of quicksort is determined by number of comparisons made to pivot

Complexity QuickSort

- QuickSort is O(n log n), where n = # elems to sort
  - But in “worst case” it can be $O(n^2)$
  - Worst case occurs when every time pivot is selected, it is maximal or minimal remaining element

Expected Running Time of QuickSort

- Let X = # comparisons made when sorting n elems
  - E[X] gives us expected running time of algorithm
  - Given V_1, V_2, ..., V_n in random order to sort
  - Let Y_1, Y_2, ..., Y_n be V_1, V_2, ..., V_n in sorted order

When are Y_a and Y_b are compared?

Lets Imagine Our Array in Sorted Order

		Y_a		Y_b	
1	3	5	7	9	11
Y_1	Y_2	Y_3	Y_4	Y_5	Y_6

Whether or not they are compared depends on pivot choice

Lets Imagine Our Array in Sorted Order

Y_a Y_b

1 3 5 7 9 11

Whether or not they are compared
depends on pivot choice

P(Y_a and Y_b ever compared)

Y_a Y_b

1 3 5 7 9 11

Consider pivot choice: Y_a

They are compared

P(Y_a and Y_b ever compared)

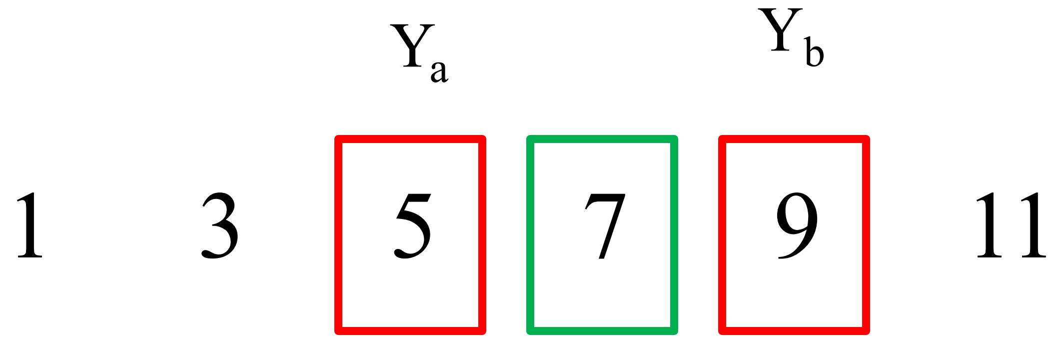
Y_a Y_b

1 3 [5] 7 [9] 11

Consider pivot choice: Y_b

They are compared

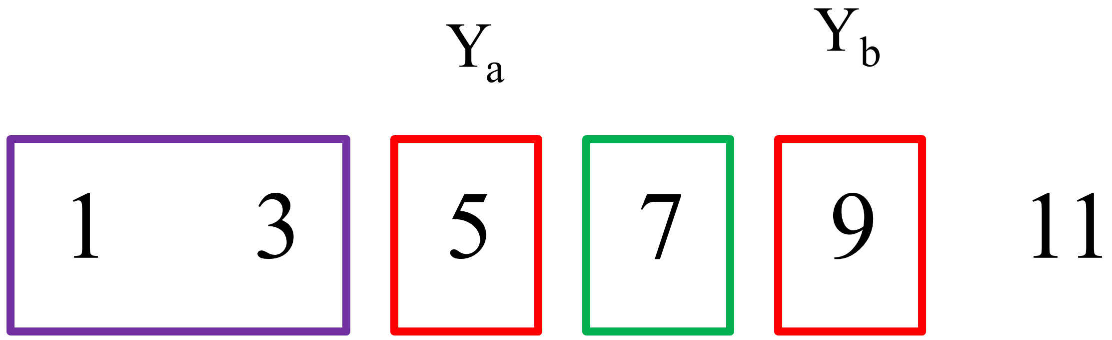
P(Y_a and Y_b ever compared)



Consider pivot choice: 7

They are **not** compared

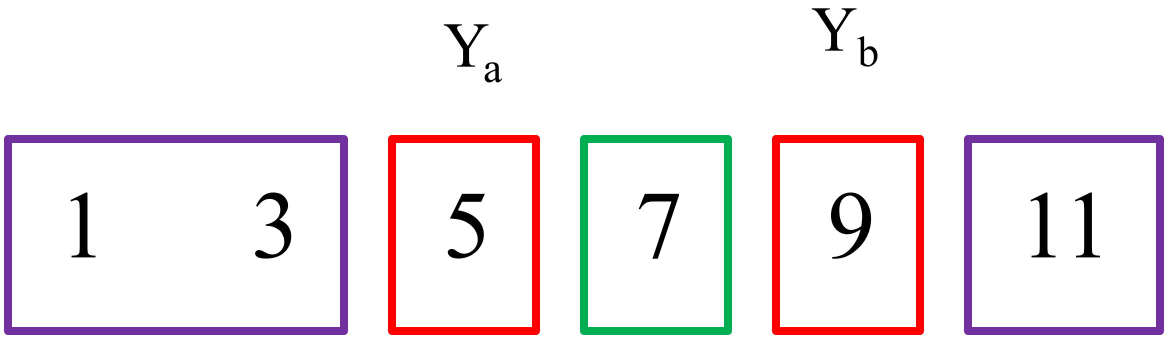
P(Y_a and Y_b ever compared)



Consider pivot choice: < Y_a

Whether or not they are compared depends on future pivots

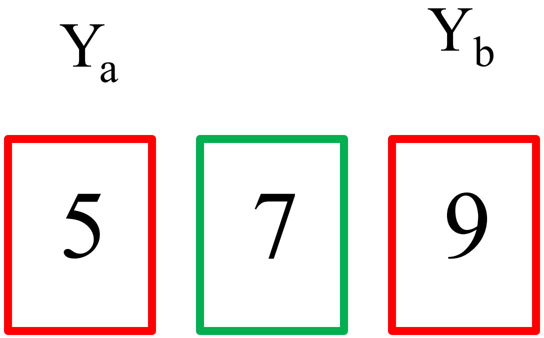
P(Y_a and Y_b ever compared)



Consider pivot choice: > Y_b

Whether or not they are compared depends on future pivots

P(Y_a and Y_b ever compared)



Are Y_a and Y_b compared?

Keep repeating pivot choice until you get a pivot
In the range [Y_a, Y_b] inclusive

Expected Running Time of QuickSort

- Let X = # comparisons made when sorting n elems
  - E[X] gives us expected running time of algorithm
  - Given $V_1, V_2, ..., V_n$ in random order to sort
  - Let $Y_1, Y_2, ..., Y_n$ be $V_1, V_2, ..., V_n$ in sorted order
  - Let $I_{a,b}$ = 1 if Y_a and Y_b are compared, 0 otherwise
  - Order where $Y_b > Y_a$, so we have: $X = \sum_{a=1}^{n-1} \sum_{b=a+1}^{n} I_{a,b}$

Expected Running Time of QuickSort

Aside: $$X = \sum_{a=1}^{n-1} \sum_{b=a+1}^{n} I_{a,b}$$

When a = 1 $\quad I_{1,2} + I_{1,3} + \ldots + I_{1,n}$

When a = 2 $\quad + I_{2,3} + \ldots + I_{2,n}$

When a = n-1 $\quad + I_{n-1,n}$

Contains a comparison between each i and j
(where i does not equal j)
exactly once

Expected Running Time of QuickSort

- Let X = # comparisons made when sorting n elems
  - E[X] gives us expected running time of algorithm
  - Given $V_1, V_2, ..., V_n$ in random order to sort
  - Let $Y_1, Y_2, ..., Y_n$ be $V_1, V_2, ..., V_n$ in sorted order
  - Let $I_{a,b}$ = 1 if Y_a and Y_b are compared, 0 otherwise
  - Order where $Y_b > Y_a$, so we have: $X = \sum_{a=1}^{n-1} \sum_{b=a+1}^{n} I_{a,b}$

$$E[X] = E\left[\sum_{a=1}^{n-1} \sum_{b=a+1}^{n} I_{a,b}\right] = \sum_{a=1}^{n-1} \sum_{b=a+1}^{n} E[I_{a,b}] = \sum_{a=1}^{n-1} \sum_{b=a+1}^{n} P(Y_a \text{ and } Y_b \text{ ever compared})$$

P(Ya and Yb ever compared)

- Consider when Y_a and Y_b are directly compared
  - We only care about case where pivot chosen from set: $\{Y_a, Y_{a+1}, Y_{a+2}, \ldots, Y_b\}$
  - From that set either Y_a and Y_b must be selected as pivot (with equal probability) in order to be compared
  - So,

$$P(Y_a \text{ and } Y_b \text{ ever compared}) = \frac{2}{b-a+1}$$

$$E[X] = \sum_{a=1}^{n-1} \sum_{b=a+1}^{n} P(Y_a \text{ and } Y_b \text{ ever compared}) = \sum_{a=1}^{n-1} \sum_{b=a+1}^{n} \frac{2}{b-a+1}$$

Bring it on Home (i.e. Solve the Sum)

$$E[X] = \sum_{a=1}^{n-1} \sum_{b=a+1}^{n} \frac{2}{b-a+1}$$

$$\sum_{b=a+1}^{n} \frac{2}{b-a+1} \approx \int_{a+1}^{n} \frac{2}{b-a+1}\, db$$

Recall: $\int \frac{1}{x}\, dx = \ln(x)$

Thanks Riemann

$$= 2\ln(b-a+1)\Big|_{a+1}^{n} = 2\ln(n-a+1) - 2\ln(2)$$

$$\approx 2\ln(n-a+1) \quad \text{for large } n$$

$$E[X] \approx \sum_{a=1}^{n-1} 2\ln(n-a+1) \approx 2\int_{a=1}^{n-1} \ln(n-a+1)\, da$$

Let $y = n - a + 1$

$$= -2\int_{y=n}^{2} \ln(y)\, dy$$

$$= -2(y\ln(y) - y)\Big|_{n}^{2}$$

Recall:

$$\int \ln(x)\, dx = x\ln(x) - x$$

$$= -2[(2\ln(2) - 2) - (n\ln(n) - n)] \approx 2n\ln(n) - 2n = O(n\log n)$$

Ahhh ☺