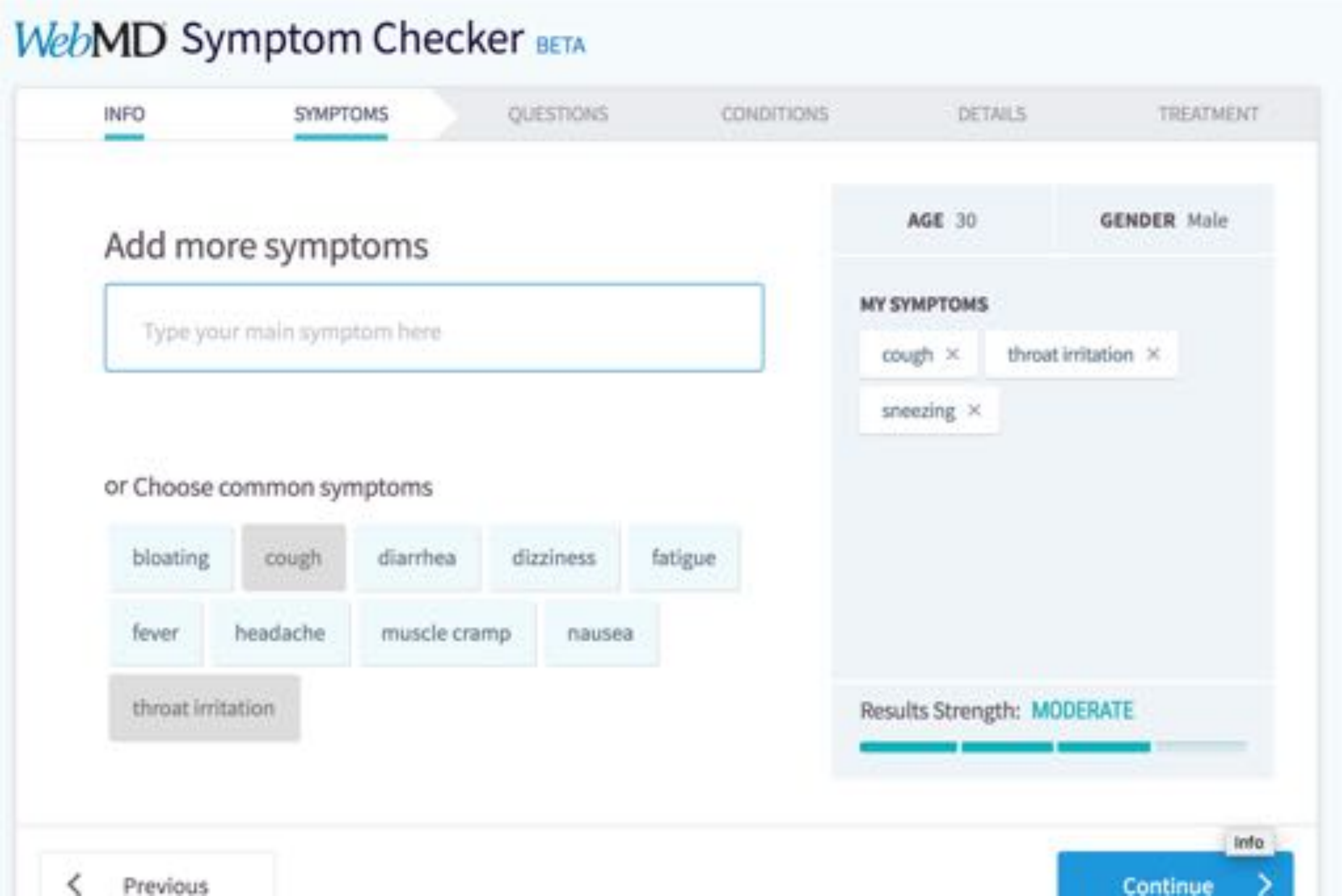
Parameter Estimation

Chris Piech
CS109, Stanford University

General “Inference”

WebMD
HEALTH
NOW

General “Inference”



Conditions that match your symptoms

UNDERSTANDING YOUR RESULTS 1

Influenza (flu) adults

Moderate match

Pneumococcal Infections

Moderate match

H1N1 Flu Virus (Swine Flu)

Moderate match

Bacterial Pneumonia

Moderate match

Sepsis (blood infection)

Moderate match

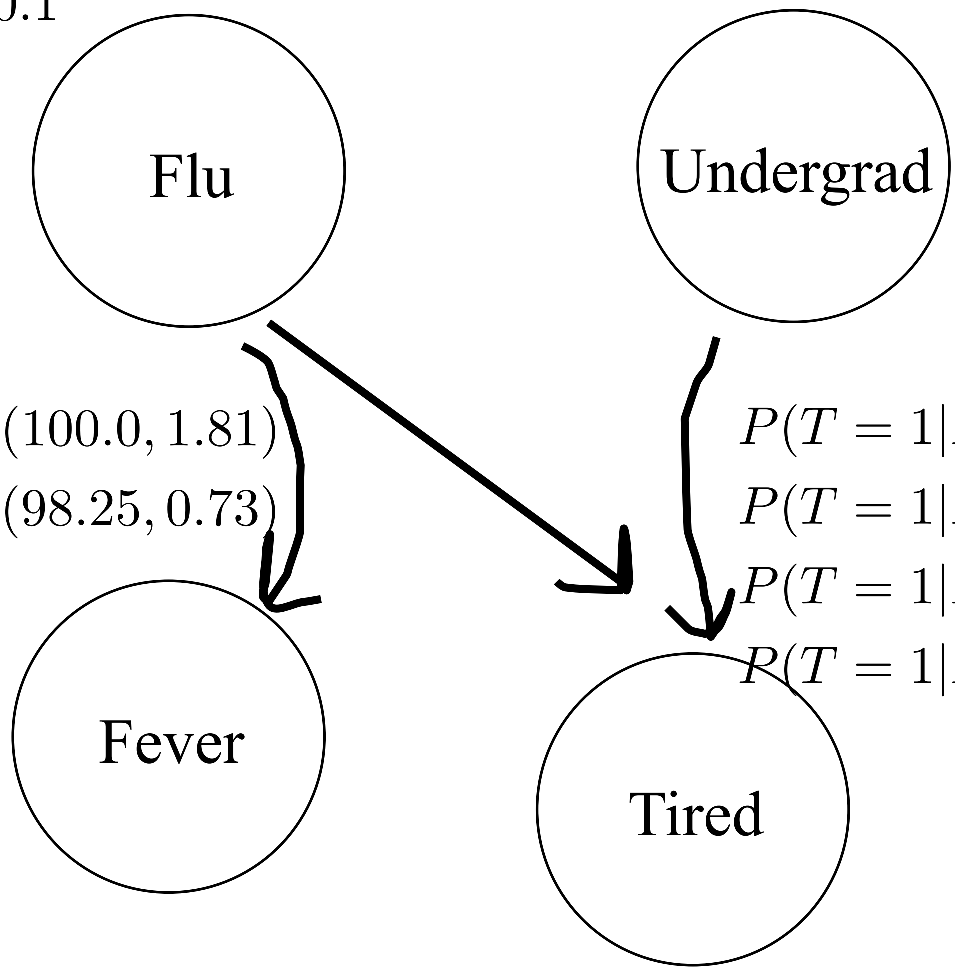
Gender **Male** Age **30** Edit

My Symptoms Edit

fever 103f to 104f, dizziness, throat irritation, migraine headache

Start Over

Probabilistic Model



$P(Fl = 1) = 0.1$

$P(U = 1) = 0.8$

$Fev|Flu = 0 \sim N(100.0, 1.81)$

$Fev|Flu = 1 \sim N(98.25, 0.73)$

$P(T = 1|Flu = 0, U = 0) = 0.1$

$P(T = 1|Flu = 0, U = 1) = 0.8$

$P(T = 1|Flu = 1, U = 0) = 0.9$

$P(T = 1|Flu = 1, U = 1) = 1.0$

Alg #1: Joint Sampling

```
N_SAMPLES = 100000

# Program: Joint Sa
# ---------------------
# we can answer any
# with multivariate
# where conditioned
def main():
    obs = getObserv
    print 'Observat

    samples = sampl
    prob = probFluG
    print 'Pr(Flu)
```

webMd — -bash — 38×22

```
[0, 0, 0, 0]
[0, 1, 0, 1]
[1, 0, 1, 0]
[1, 1, 1, 1]
[0, 1, 0, 1]
[0, 1, 0, 0]
[0, 0, 0, 0]
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[0, 0, 0, 0]
[1, 1, 1, 1]
[0, 0, 0, 0]
[0, 0, 0, 0]
[1, 1, 1, 1]
[0, 1, 0, 0]
Observation =  [None, None, None, 1]
Pr(Flu | Obs) =  0.140635888502
> 
```

Each one of these is one posterior sample:

[Flu, Ugrad, Fever, Tired]

Alg #2: MCMC

```
[1, 1, 101.0, 1]
[1, 1, 101.0, 1]
[0, 1, 101.0, 0]
[0, 0, 101.0, 0]
[1, 0, 101.0, 1]
[1, 0, 101.0, 0]
[1, 0, 101.0, 1]
[1, 0, 101.0, 1]
[1, 1, 101.0, 1]
[1, 1, 101.0, 1]
[1, 1, 101.0, 1]
[1, 1, 101.0, 1]
[1, 1, 101.0, 1]
[1, 1, 101.0, 1]
[1, 1, 101.0, 1]
[1, 0, 101.0, 1]
[1, 1, 101.0, 1]
[1, 1, 101.0, 1]
Pr(Flu) = 0.9773
>
```

MCMC is a way to sample with conditioned variables fixed

Each one of these is one posterior sample:

[Flu, Undergrad, Fever, Tired]

Alg #2: MCMC

```
All Samples = []
```

Flu Undergrad Fever Tired

```
Initial Sample = [0, 0, 101.0, 1]
```

Alg #2: MCMC

```
All Samples = []
```

Flu Undergrad Fever Tired

$S^{(0)}$ = [0, 0, 101.0, 1]

Alg #2: MCMC

All Samples = [$S^{(0)}$]

Flu Undergrad Fever Tired

$S^{(0)}$ = [0, 0, 101.0, 1]

From S_t make S_{t+1}

Alg #2: MCMC

```
All Samples = [S(0)]
```

Flu Undergrad Fever Tired

$S^{(1)} = [0,\ 0,\ 101.0,\ 1]$

$$P(Flu = 1 | \text{All others})$$

$$= P(Flu = 1 | Und = 0, Fev = 98.3, Tir = 1)$$

$$= 0.21$$

$$Flu_1 = \text{Sample}\Big[P(Flu = 1 | \text{All others})\Big]$$

Alg #2: MCMC

```
All Samples = [S(0)]
```

Flu Undergrad Fever Tired

```
S(1) = [1, 0, 101.0, 1]
```

$$P(Flu = 1 | \text{All others})$$
$$= P(Flu = 1 | Und = 0, Fev = 98.3, Tir = 1)$$
$$= 0.21$$

$$Flu_1 = \text{Sample}\Big[P(Flu = 1 | \text{All others})\Big]$$

Alg #2: MCMC

```
All Samples = [S(0)]
```

Flu Undergrad Fever Tired

$S^{(1)}$ = [1, 0, 101.0, 1]

$$P(Und = 1 | \text{All others})$$
$$= P(Und = 1 | Flu = 1, Fev = 98.3, Tir = 1)$$
$$= 0.91$$

$$Und_1 = \text{Sample}\Big[P(Und = 1 | \text{All others})\Big]$$

Alg #2: MCMC

```
All Samples = [S(0)]
```

Flu · Undergrad · Fever · Tired

```
S(1) = [1, 1, 101.0, 1]
```

$$P(Und = 1 | \text{All others})$$

$$= P(Und = 1 | Flu = 1, Fev = 98.3, Tir = 1)$$

$$= 0.91$$

$$Und_1 = \text{Sample}\Big[P(Und = 1 | \text{All others})\Big]$$

Alg #2: MCMC

```
All Samples = [S(0)]
```

Flu
Undergrad
Fever
Tired

$S^{(1)}$ = [1, 1, 101.0, 1]

Let's say you are conditioning on fever being 101.0...
then don't change that value

Alg #2: MCMC

All Samples = [$S^{(0)}$]

Flu Undergrad Fever Tired

$S^{(1)}$ = [1, 1, 101.0, 1]

Alg #2: MCMC

All Samples = [$S^{(0)}$]

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Alg #2: MCMC

All Samples = [$S^{(0)}$]

Flu Undergrad Fever Tired

$S^{(1)}$ = [1, 1, 101.0, 1]

Alg #2: MCMC

All Samples = [$S^{(0)}$, $S^{(1)}$]

Flu Undergrad Fever Tired

$S^{(1)}$ = [1, 1, 101.0, 1]

Alg #2: MCMC

```
All Samples = [S(0), S(1)]
```

Flu Undergrad Fever Tired

```
S(2) = [1, 1, 101.0, 1]
```

$$P(Flu = 1 | \text{All others})$$

$$Flu_1 = \text{Sample}\Big[P(Flu = 1 | \text{All others})\Big]$$

Alg #2: MCMC

```
All Samples = [S(0), S(1)]
```

Flu Undergrad Fever Tired

```
S(2) = [1, 1, 101.0, 1]
```

$$P(Flu = 1 | \text{All others})$$

$$Flu_1 = \text{Sample}\Big[P(Flu = 1 | \text{All others})\Big]$$

Alg #2: MCMC

`All Samples = [S^(0), S^(1)]`

Flu Undergrad Fever Tired

$S^{(2)}$ = [1, 1, 101.0, 1]

$$P(Flu = 1 | \text{All others})$$

$$Flu_1 = \text{Sample}\Big[P(Flu = 1 | \text{All others})\Big]$$

Alg #2: MCMC

All Samples = [$S^{(0)}$, $S^{(1)}$]

Flu   Undergrad   Fever   Tired

$S^{(2)}$ = [1, 0, 101.0, 1]

Alg #2: MCMC

All Samples = [$S^{(0)}$, $S^{(1)}$]

Flu   Undergrad   Fever   Tired

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Alg #2: MCMC

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Flu Undergrad Fever Tired

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Alg #2: MCMC

All Samples = $[S^{(0)}, S^{(1)}]$

Flu Undergrad Fever Tired

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Alg #2: MCMC

All Samples = $[S^{(0)}, S^{(1)}, S^{(2)}]$

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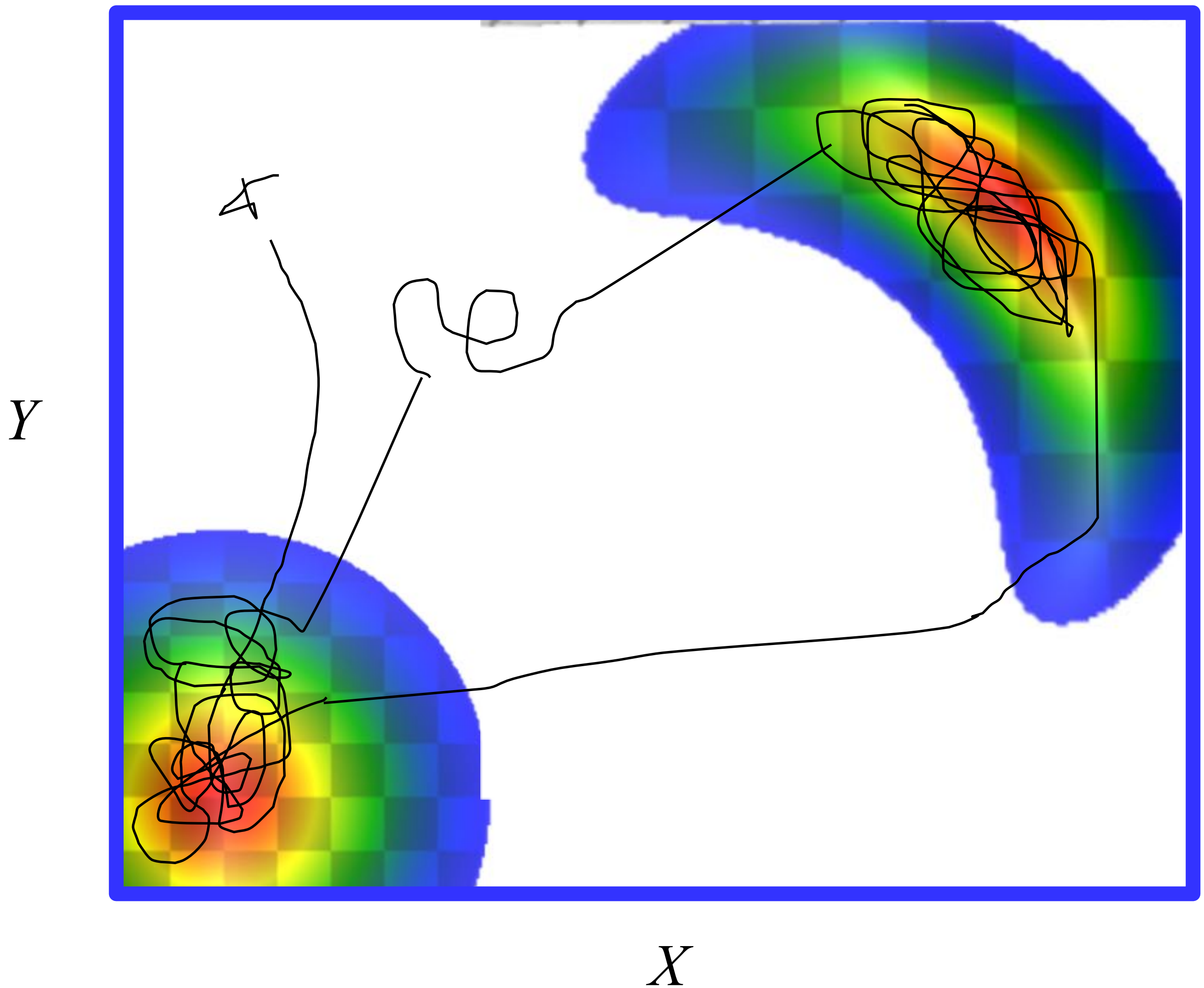
Alg #2: MCMC



BAE's Theorem?

$$P(A \mid B\,E) = \frac{P(B \mid A\,E)\,P(A \mid E)}{P(B \mid E)}$$

$P(F = 1 \mid$ all other rvs$)$

Know: P(symptom | flu, undergrad)   P(flu)   P(undergrad)

Flu is independent of undergrad

Tired and fever are conditionally independent given flu, undergrad

---

$$P(F = 1 | \text{all other rvs})$$

$$= P(F = 1 | \text{symptoms}, U = u)$$

$$= \frac{P(\text{symptoms} | F = 1, U = u) P(F = 1 | U = u)}{P(\text{symptoms} | U = u)}$$

$$\propto P(\text{symptoms} | F = 1, U = u) P(F = 1 | U = u)$$

$$\propto P(F = 1) P(\text{symptoms} | F = 1, U = u)$$

$$\propto P(F = 1) \prod_i P(\text{symptom}_i | F = 1, U = u)$$

$$P(F=1|\text{all other rvs}) \propto P(F=1)\prod_i P(\text{symptom}_i|F=1,U=u)$$

```python
def sampleFlu(sample):
    f1 = getFluPr1(sample)
    f0 = getFluPr0(sample)
    p1 = f1 / (f1 + f0)
    return bern(p1)

def getFluPr0(sample):
    [_, und, fev, tir] = sample
    pFlu0 = 0.9
    pFev = getPrFeverX(fev, flu=0)
    pTir = getPrTiredX(tir, und=und, flu=0)
    return pFlu0 * pFev * pTir

def getFluPr1(sample):
    [_, und, fev, tir] = sample
    pFlu1 = 0.1
    pFev = getPrFeverX(fev, flu=1)
    pTir = getPrTiredX(tir, und=und, flu=1)
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```

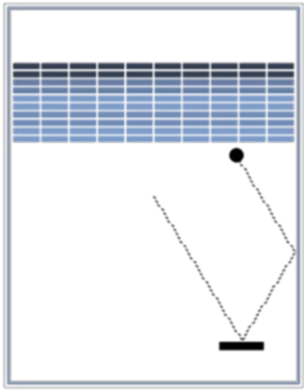
See you soon!

Summary

General Inference Summary

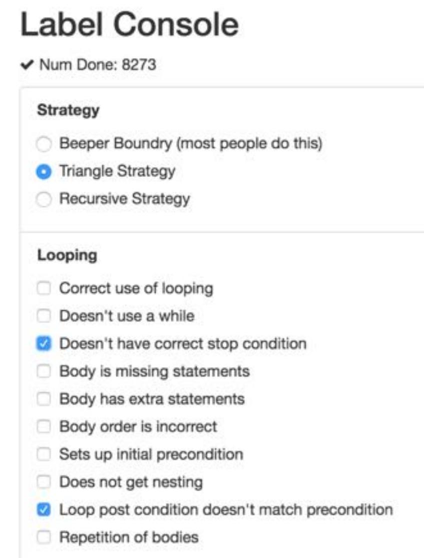
- **Straight Math** is fast, but can be prohibitively hard for complex models (see hw).
- **Joint Sampling** is really easy to program but fails for continuous variables (and when what you are conditioning on is rare)
- **MCMC** works well when conditioning on rare events, but is *much* harder to code / derive.
- All sampling is **slow.**

Insight



Let x be a students program

Let y be student mistakes



Feedback task:

$$P(y|x)$$

Hard for humans
Hard for computers

Joint sample:

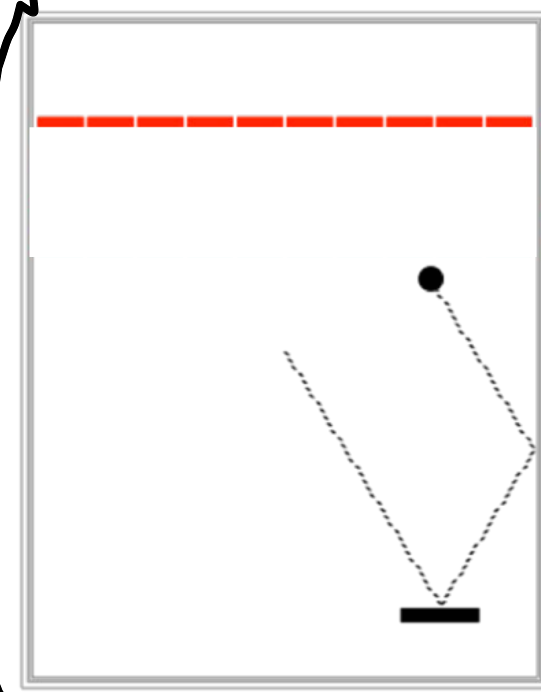
$$(\hat{x}, \hat{y}) \sim P(x, y)$$

Easy for humans

Imagine Students

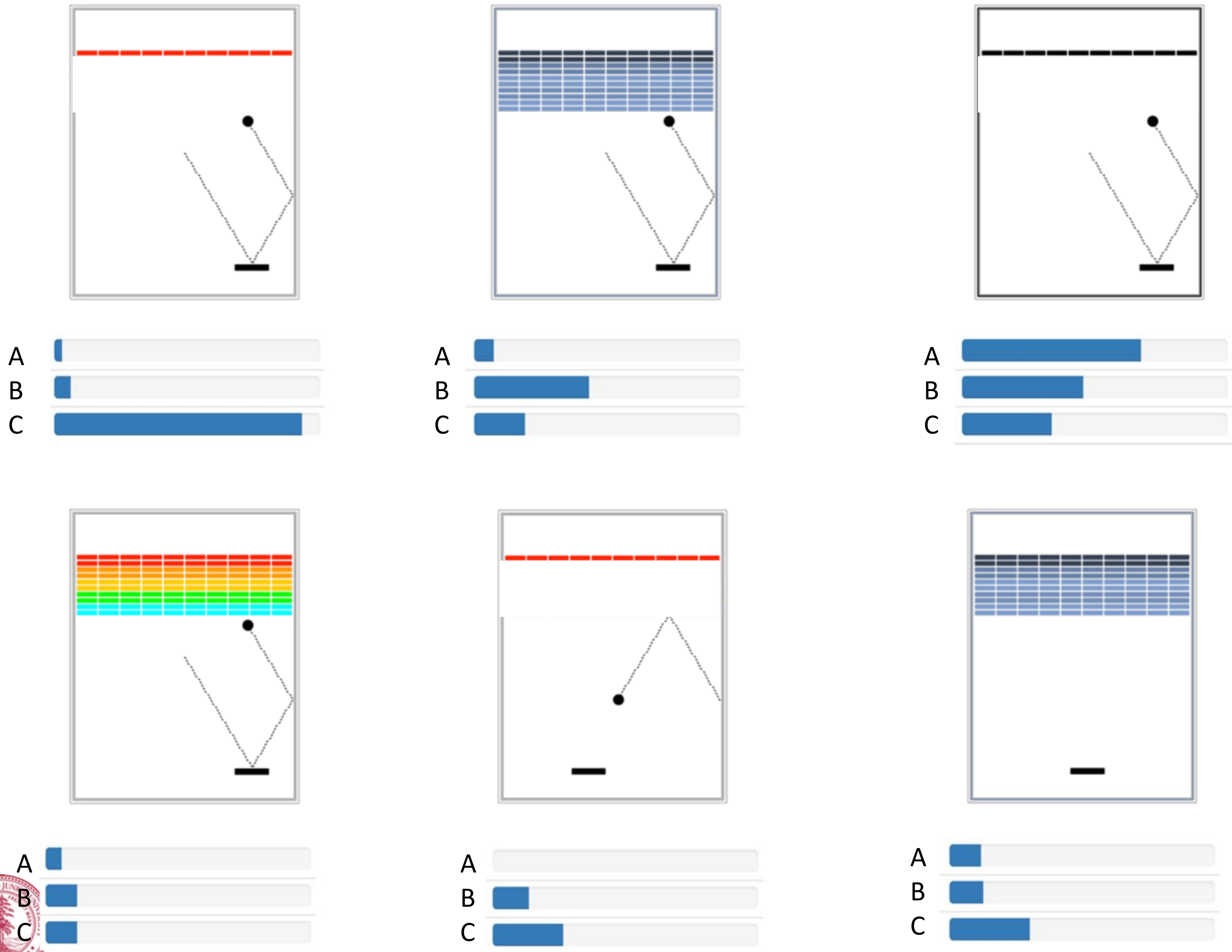
$$(\hat{x}, \hat{y}) \sim P(x, y)$$

- Struggle with double for loops
- Confuses logic for deleting bricks

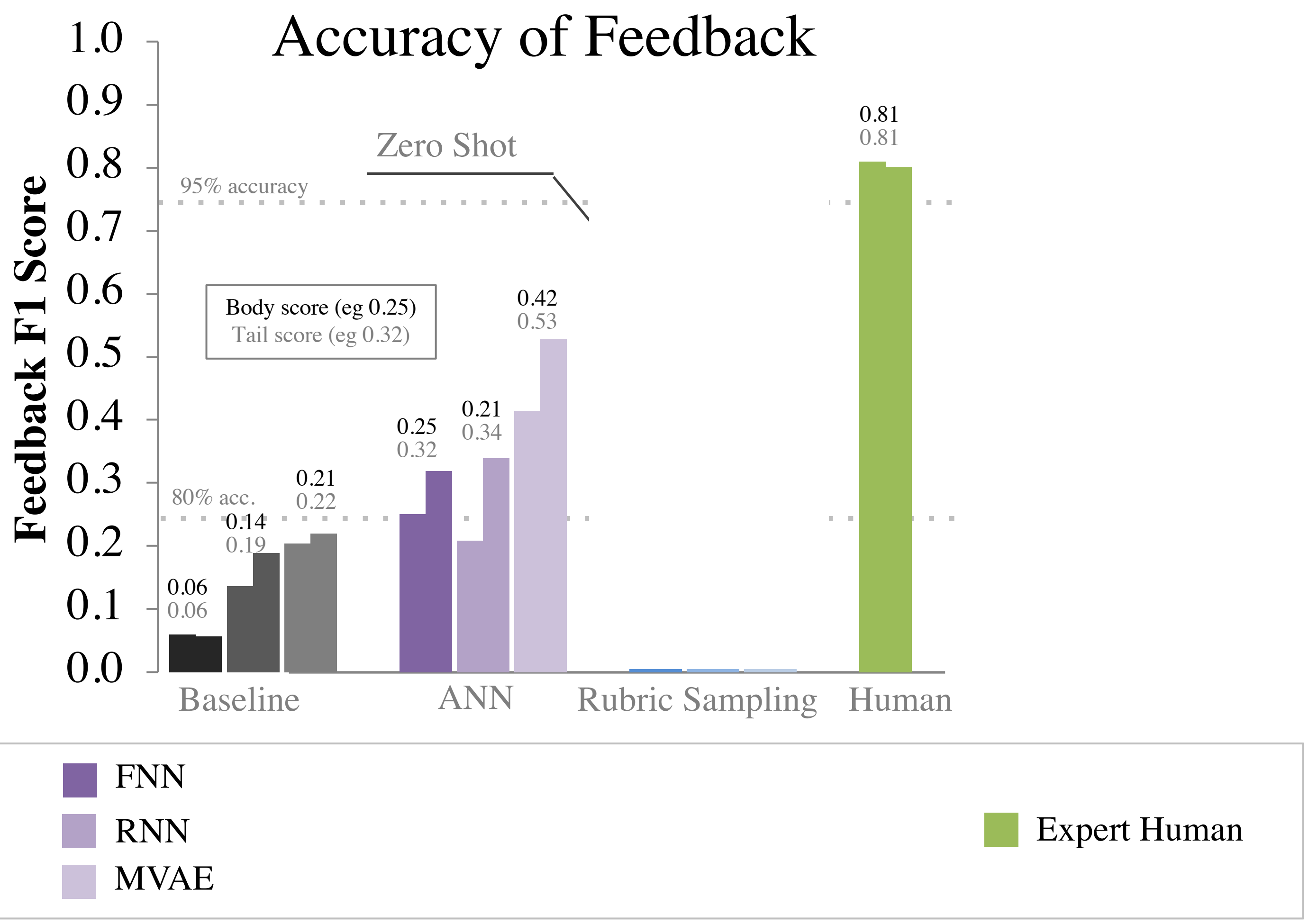


Mike Wu, Noah Goodman, Chris Piech. Zero Shot Learning for Code Education. AAAI 2019

Start Imagining Students



Zero Shot Learning



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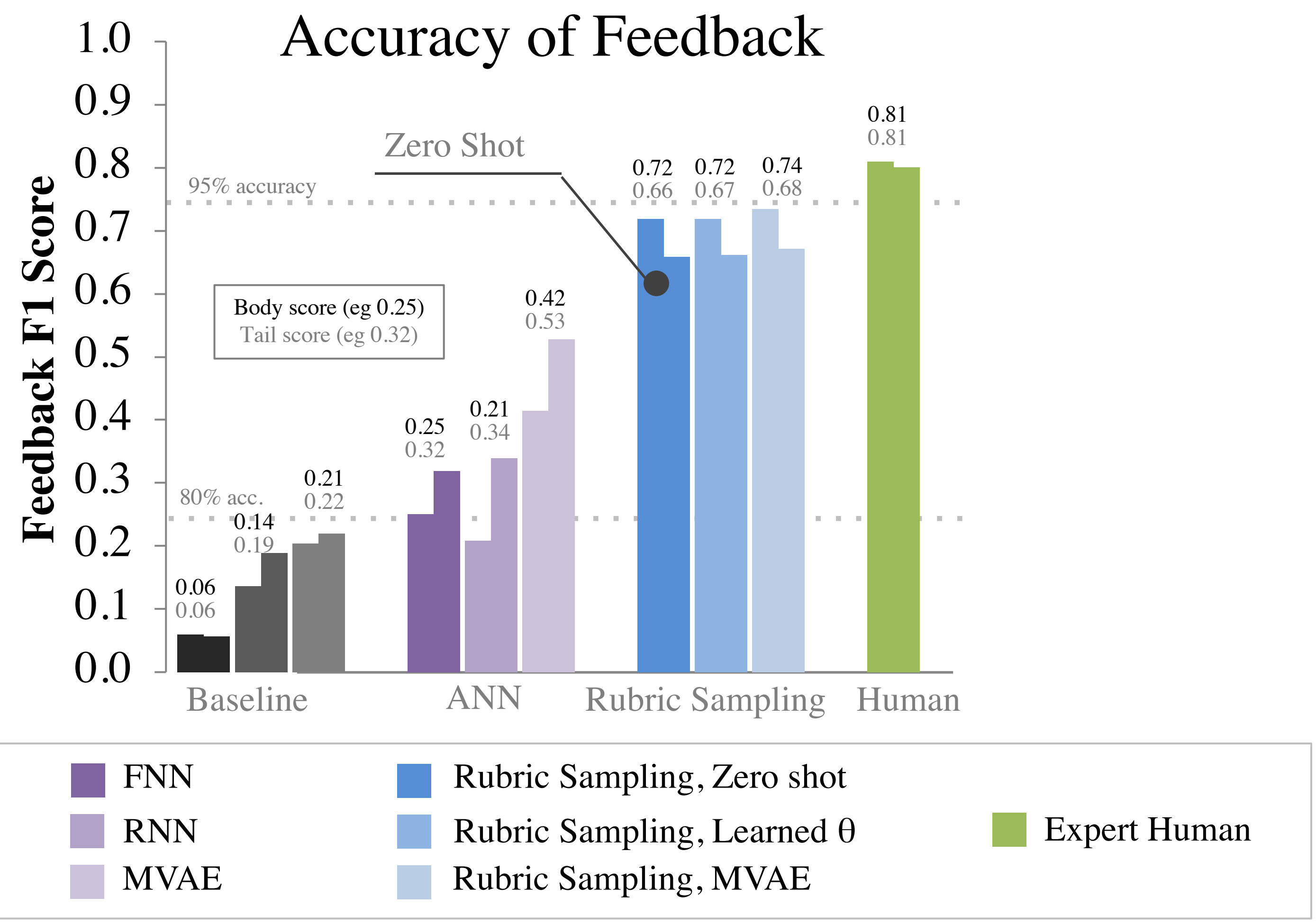
Zero Shot Learning

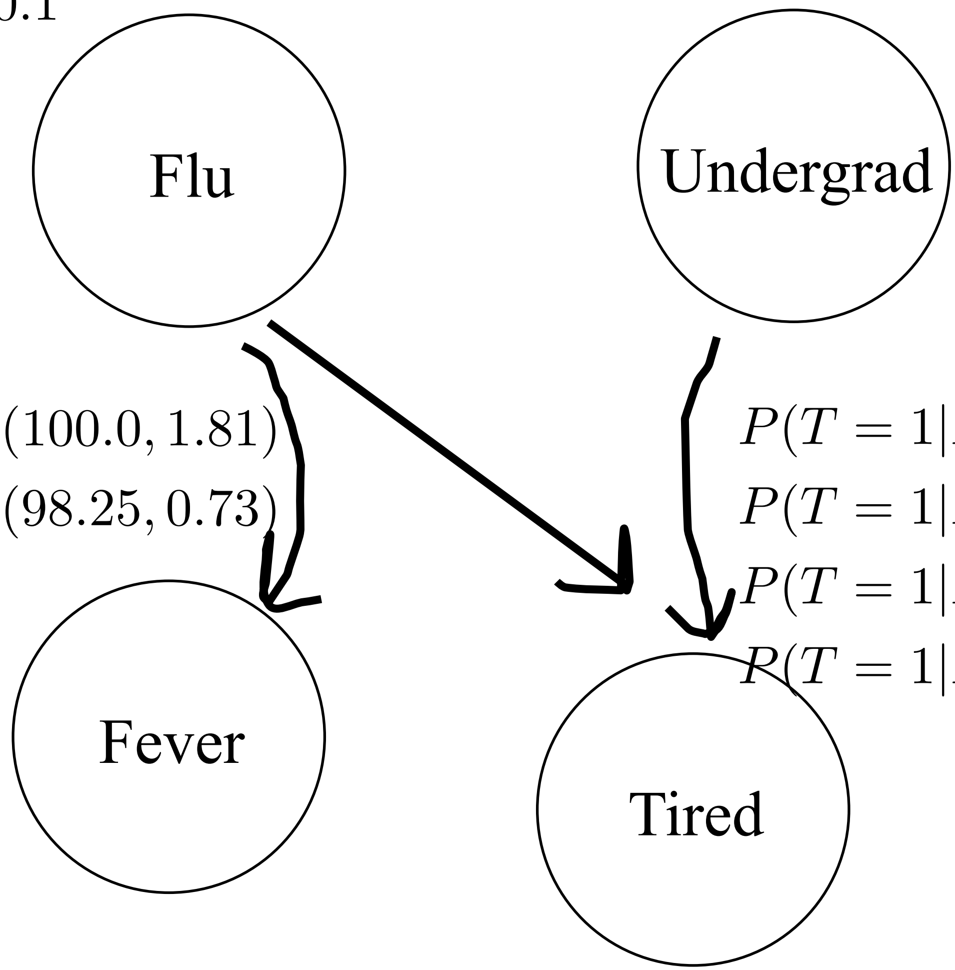


Model	Amount of Correct Feedback
Predicting from output	1,483,157 (86.0%)
Rubric sampling with MVAE	**1,610,020** (93.7%)
Expert human	1,658,162 (96.2%)

Where Do The Numbers Come From?

$$P(Fl = 1) = 0.1$$

$$P(U = 1) = 0.8$$



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$$P(T = 1|Flu = 0, U = 0) = 0.1$$

$$P(T = 1|Flu = 0, U = 1) = 0.8$$

$$P(T = 1|Flu = 1, U = 0) = 0.9$$

$$P(T = 1|Flu = 1, U = 1) = 1.0$$

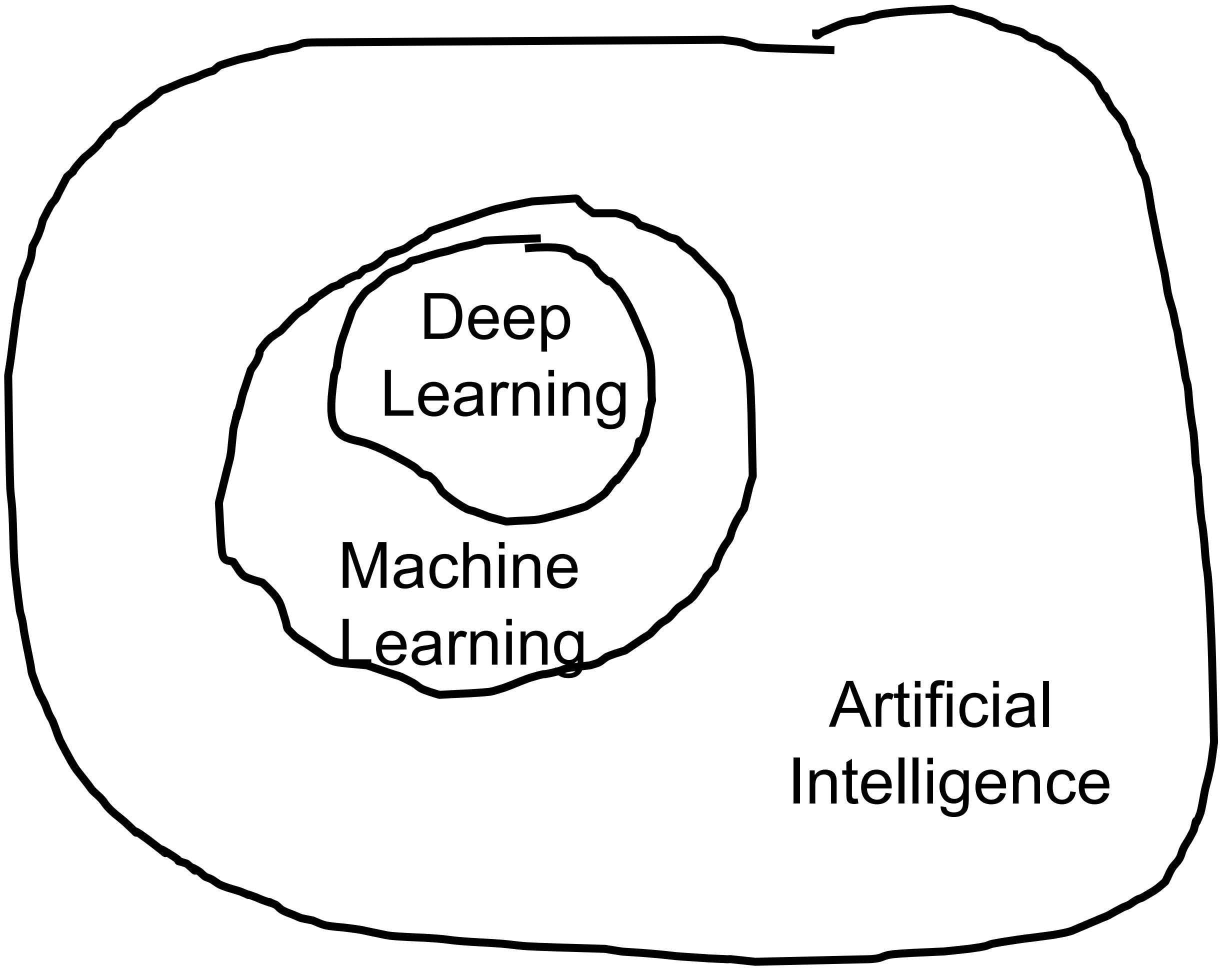
Pedagogical Pause

At this point, if you are given a *model*, with all the involved probabilities, you can make predictions

But what if you want to *learn* the probabilities in the model?

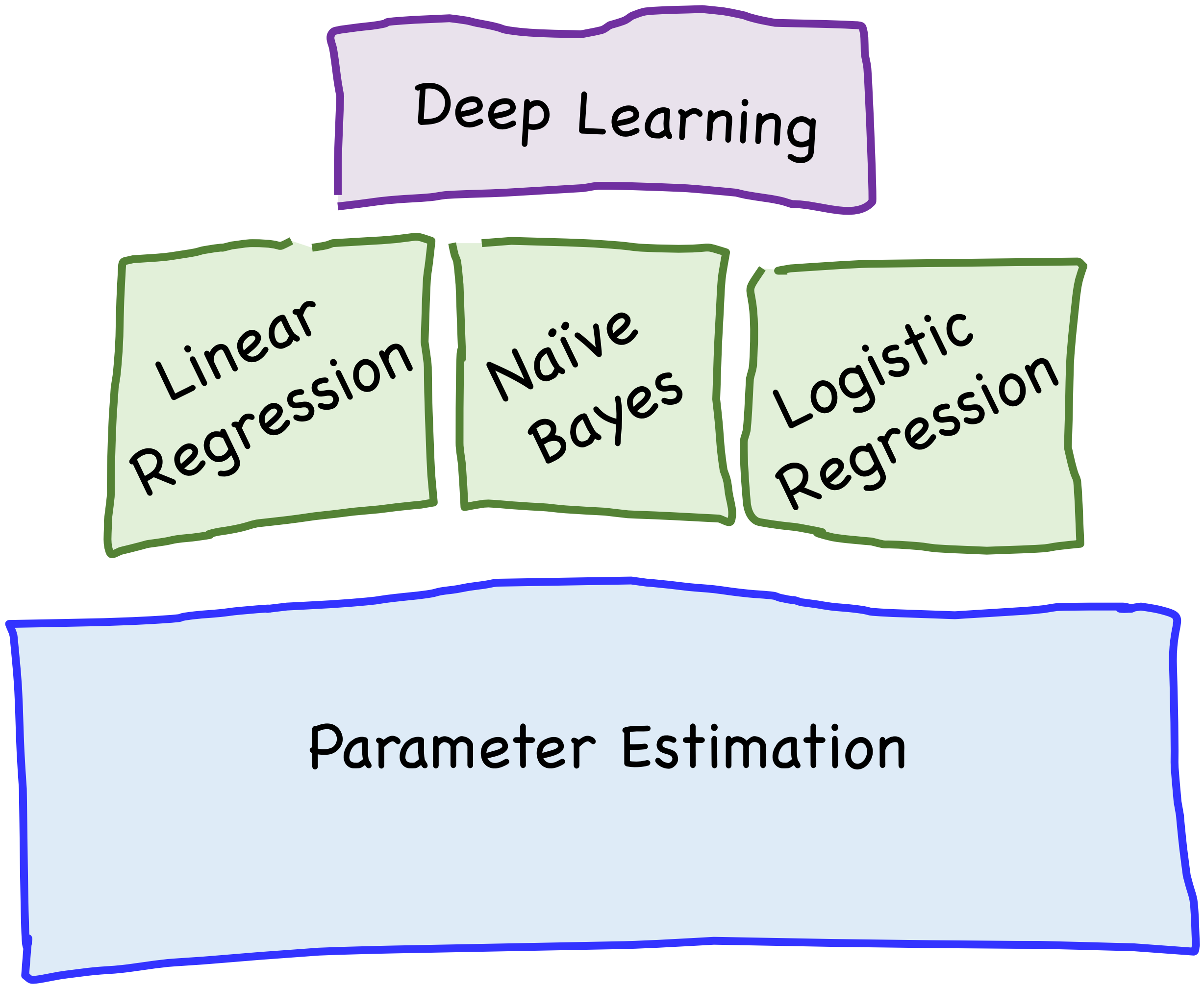
Machine Learning

AI and Machine Learning

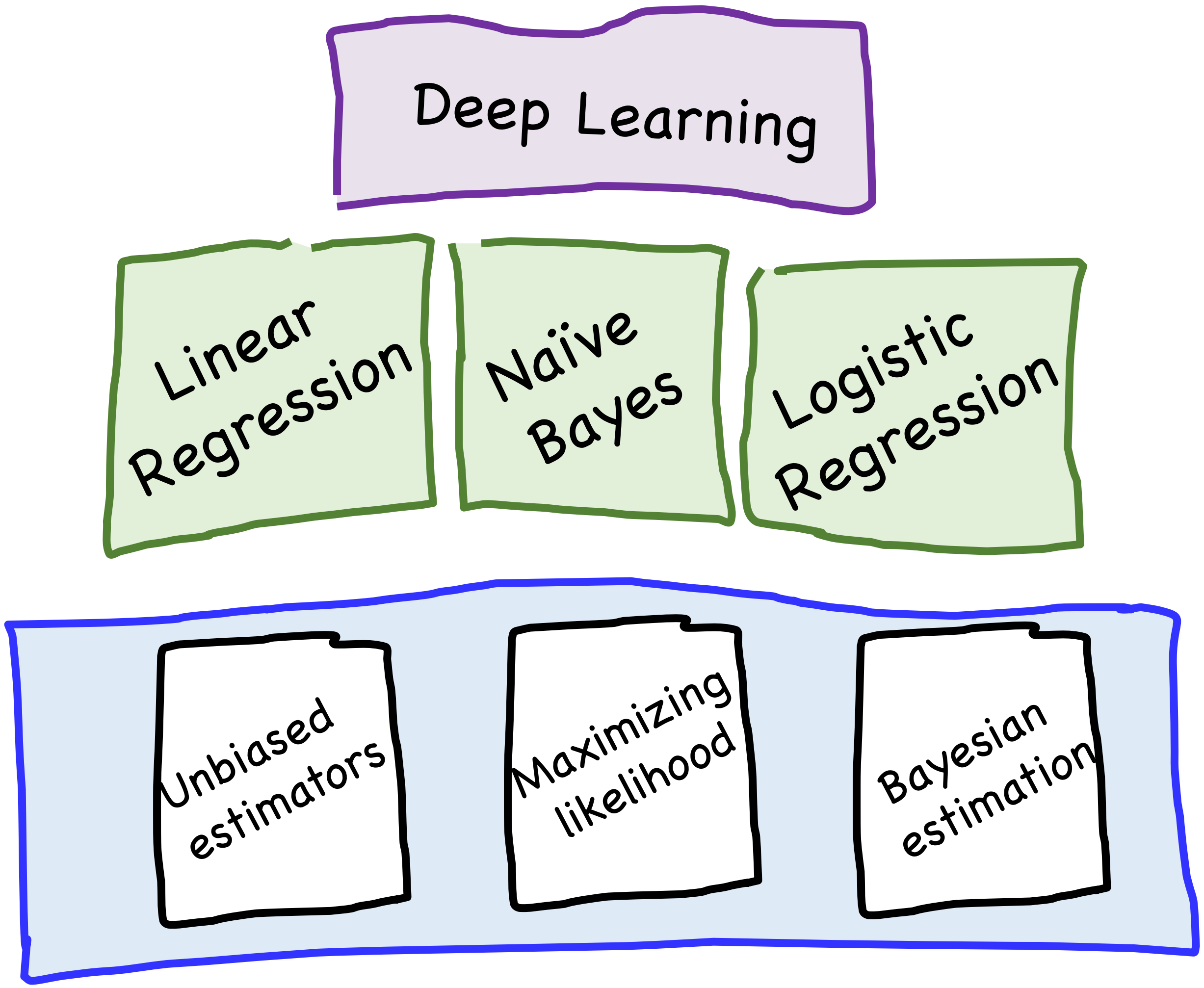


ML: Rooted in probability theory

Our Path


Deep Learning
Linear Regression
Naïve Bayes
Logistic Regression
Parameter Estimation

Our Path


Deep Learning
Linear Regression
Naïve Bayes
Logistic Regression
Unbiased estimators
Maximizing likelihood
Bayesian estimation

Jump Straight to Deep Learning?

Tensor Flow

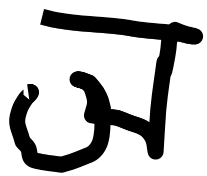
Jump Straight to Deep Learning?

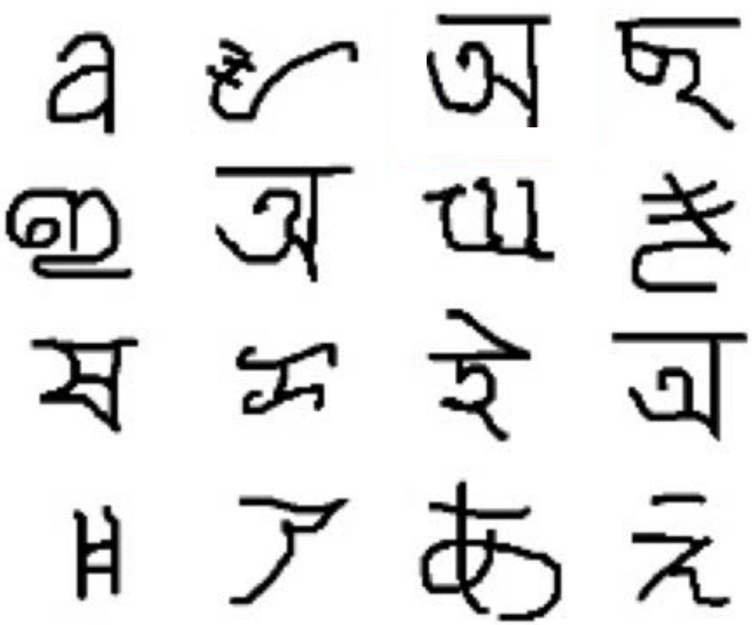
Understand the theory to help you debug

But another reason…

Machine Learning Uses a Lot of Data

One Shot Learning

Single training example: 

Test set: 

One Shot Learning

Single training example:

Computers struggle...

… especially for **human** problems.

Understand the theory
to push on the **grand challenges**


Walt Disney
PICTURES

Once upon a time…

…there was parameter estimation

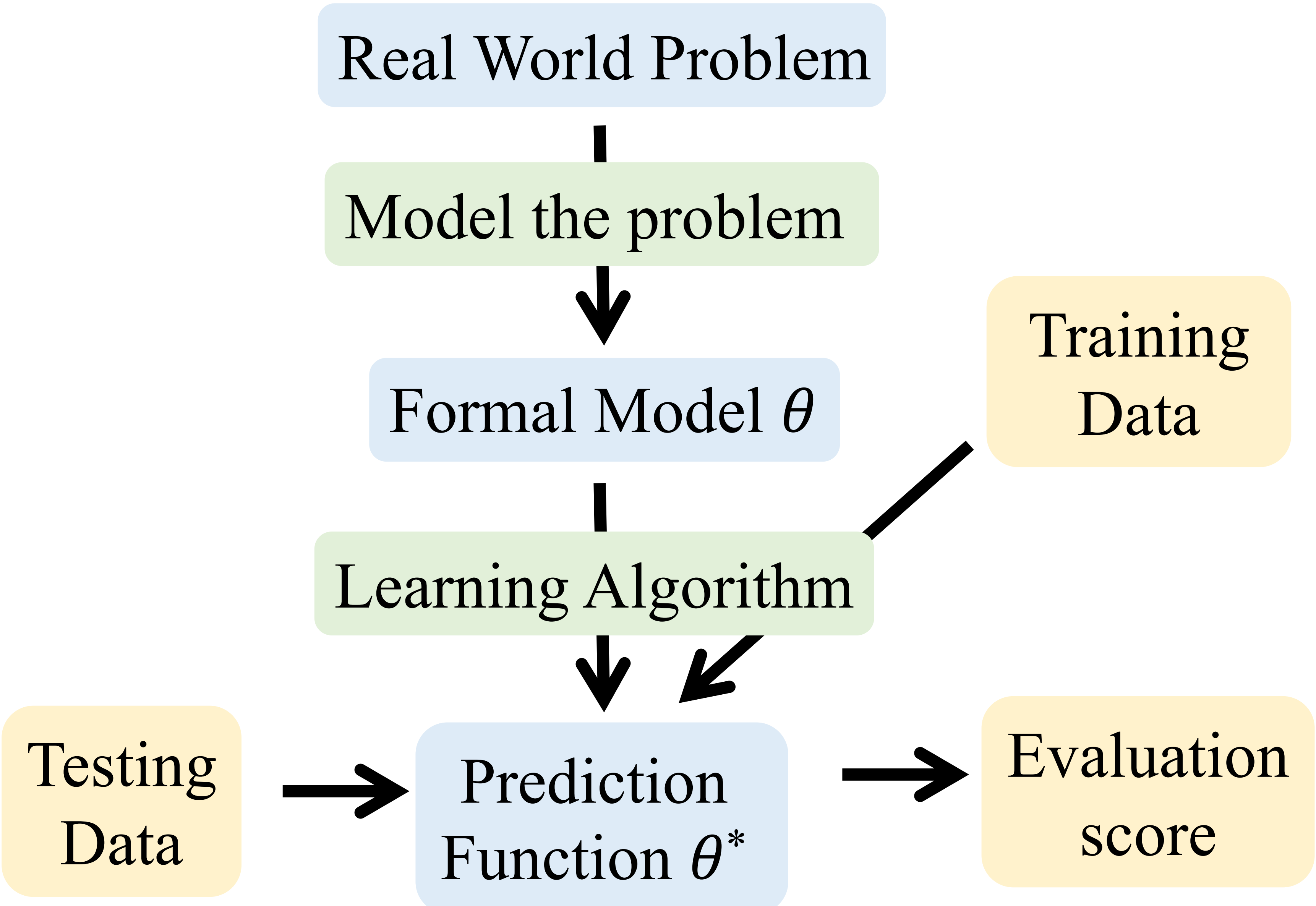
What are Parameters?

- Consider some probability distributions:
  - $\text{Ber}(p)$ — $\theta = p$
  - $\text{Poi}(\lambda)$ — $\theta = \lambda$
  - $\text{Uni}(\alpha, \beta)$ — $\theta = (\alpha, \beta)$
  - $\text{Normal}(\mu, \sigma^2)$ — $\theta = (\mu, \sigma^2)$
  - $Y = mX + b$ — $\theta = (m, b)$
  - etc…
- Call these "parametric models"
- Given model, **parameters** yield actual distribution
  - Usually refer to parameters of distribution as θ
  - Note that θ that can be a vector of parameters

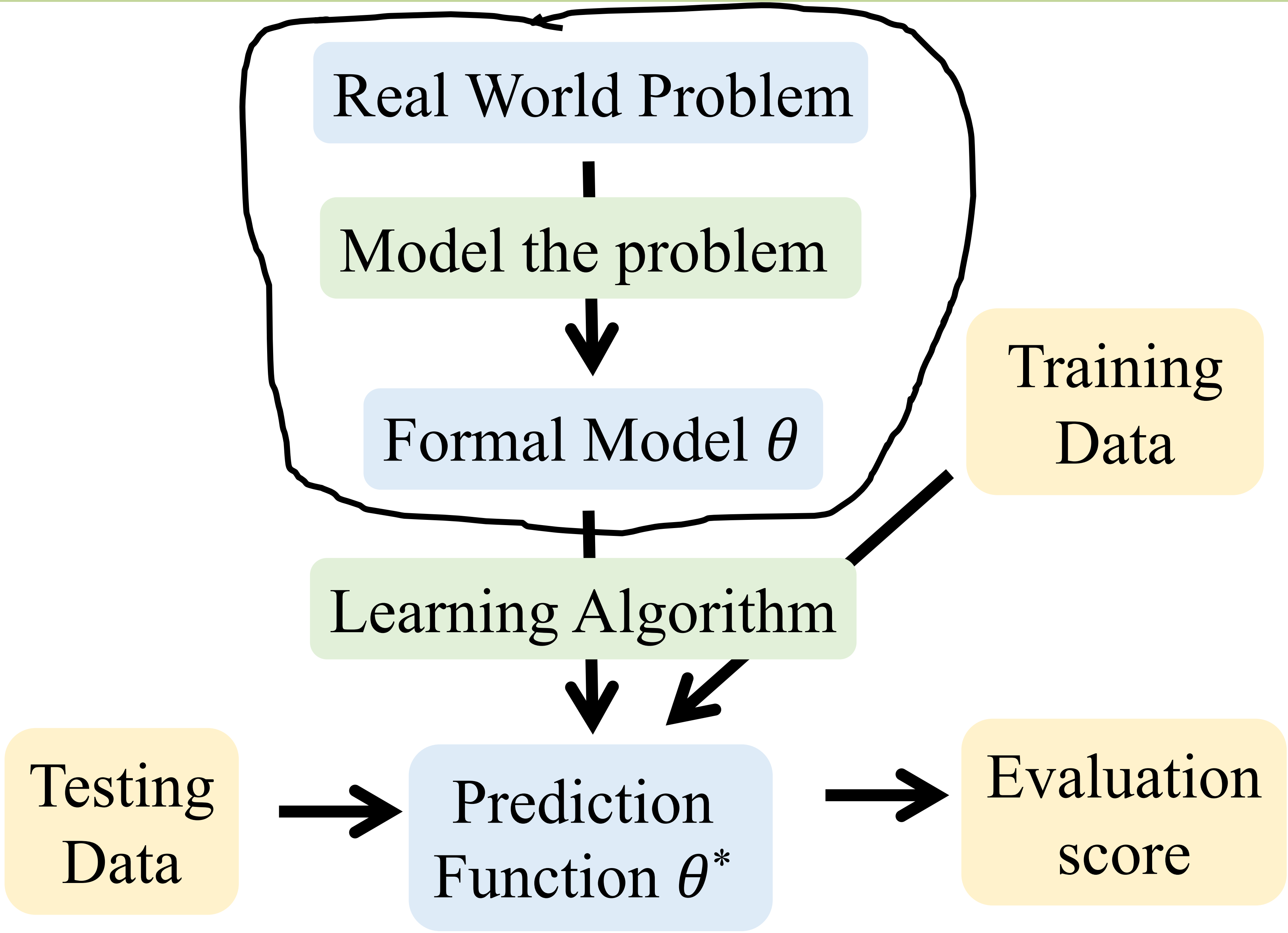
Why Do We Care?

- In real world, don't know "true" parameters
  - But, **we do get to observe data**
    - E.g., number of times coin comes up heads, lifetimes of disk drives produced, number of visitors to web site per day, etc.
  - Need to estimate model parameters from data
  - "Estimator" is random variable estimating parameter
- Estimate of parameters allows:
  - Better understanding of process producing data
  - Future **predictions** based on model
  - Simulation of processes

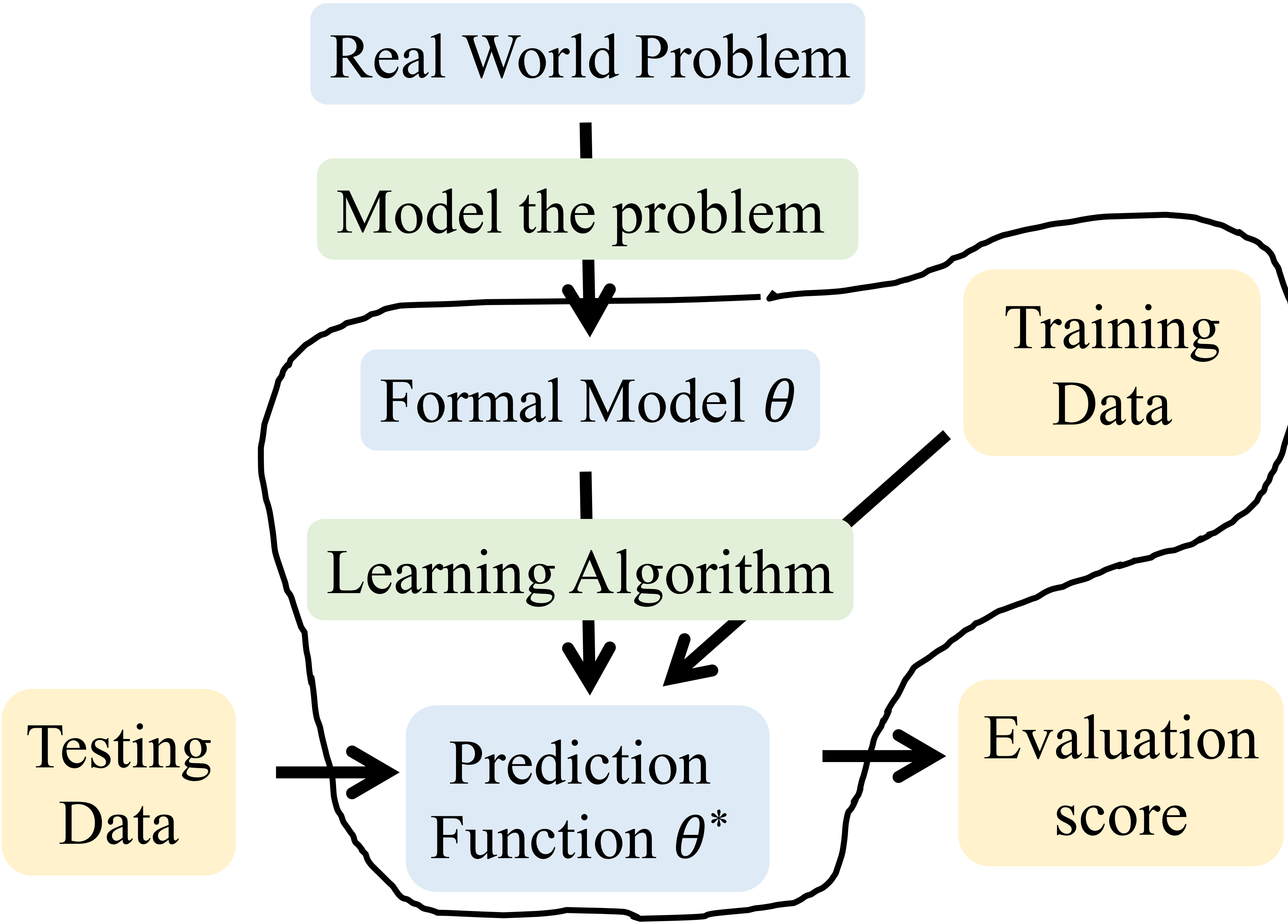
Supervised Learning



Modelling


Real World Problem
Model the problem
Formal Model θ
Training Data
Learning Algorithm
Testing Data
Prediction Function θ^*
Evaluation score

Training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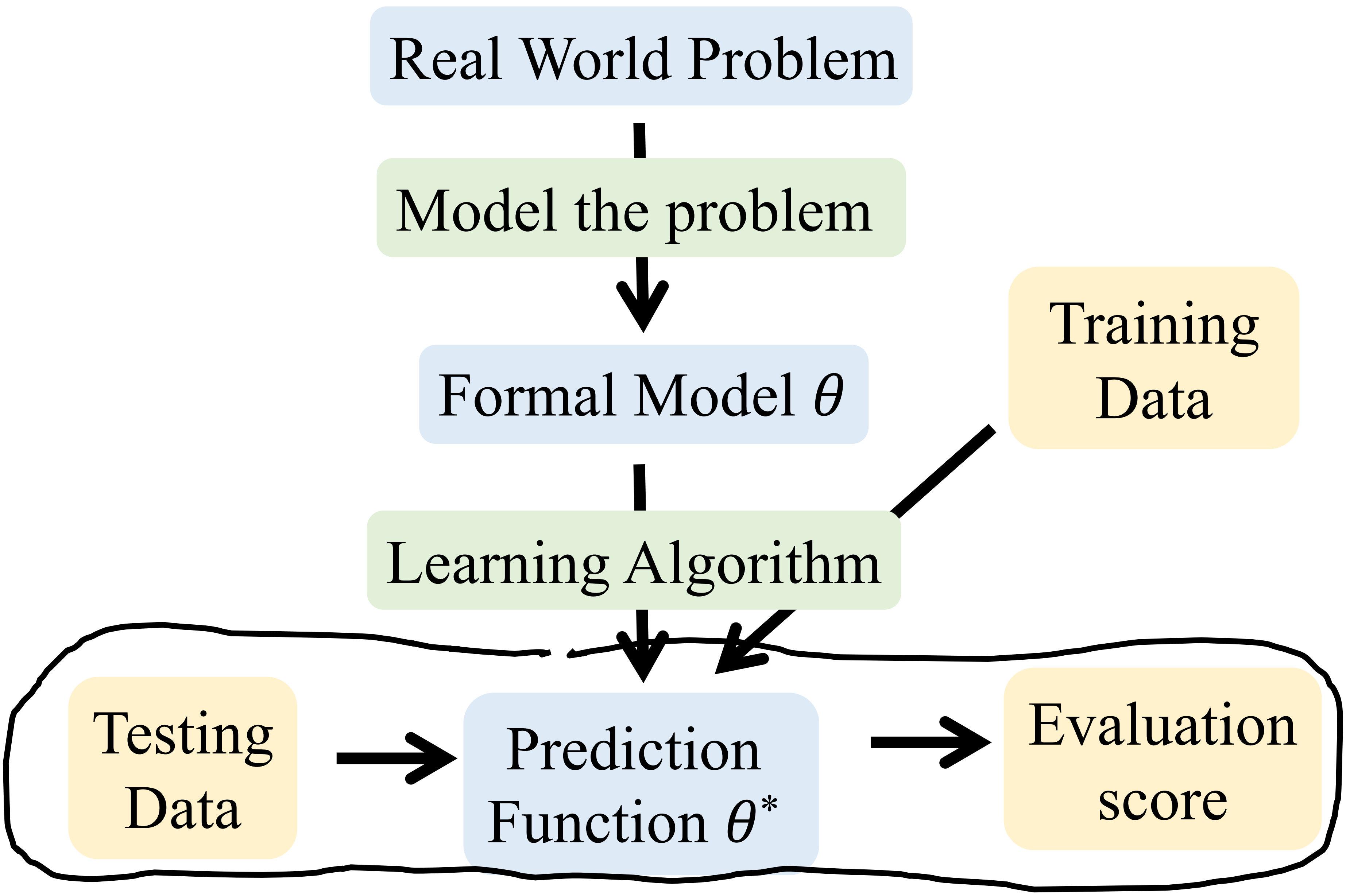

Real World Problem
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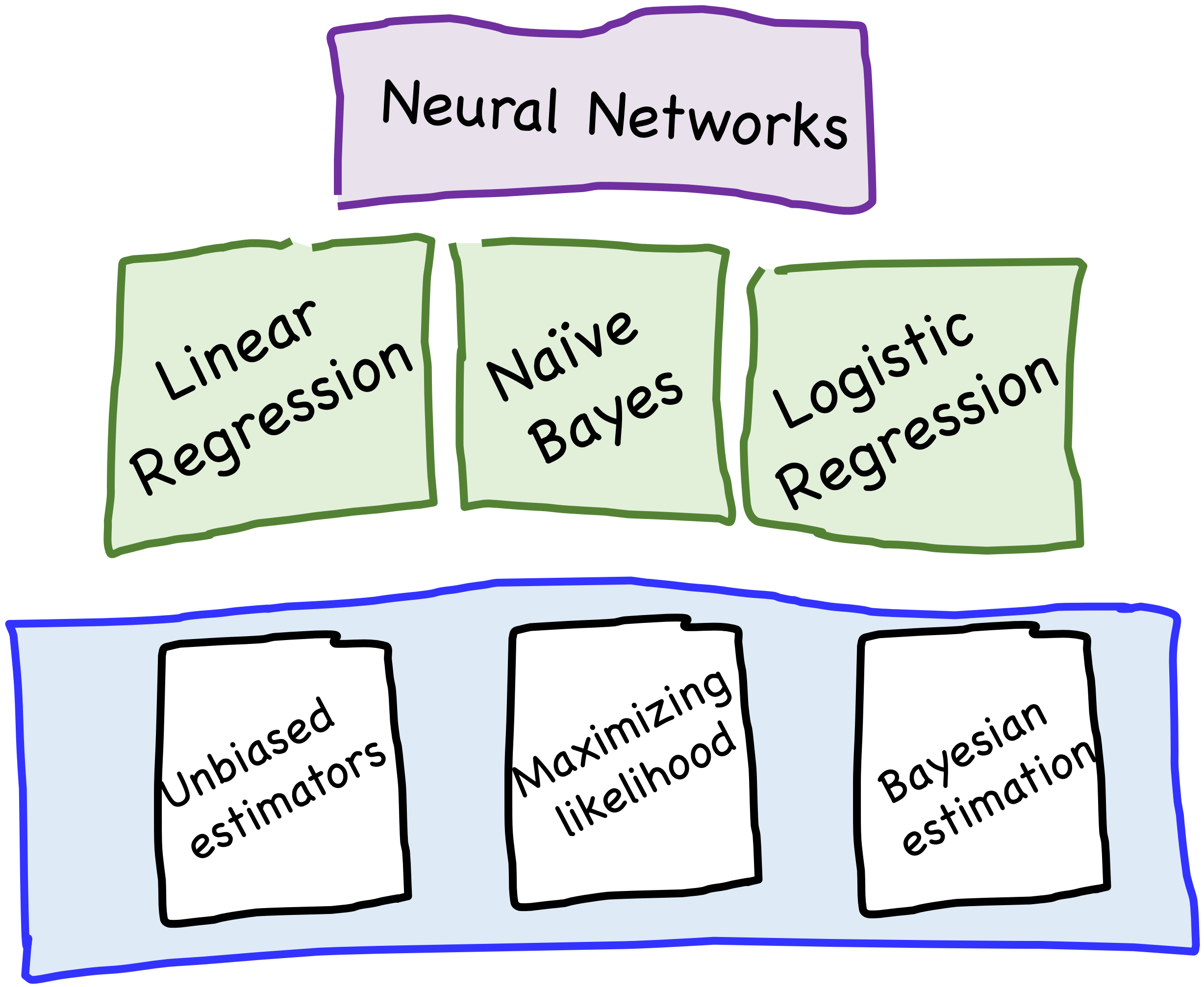
Basis for learning from data

Our Path



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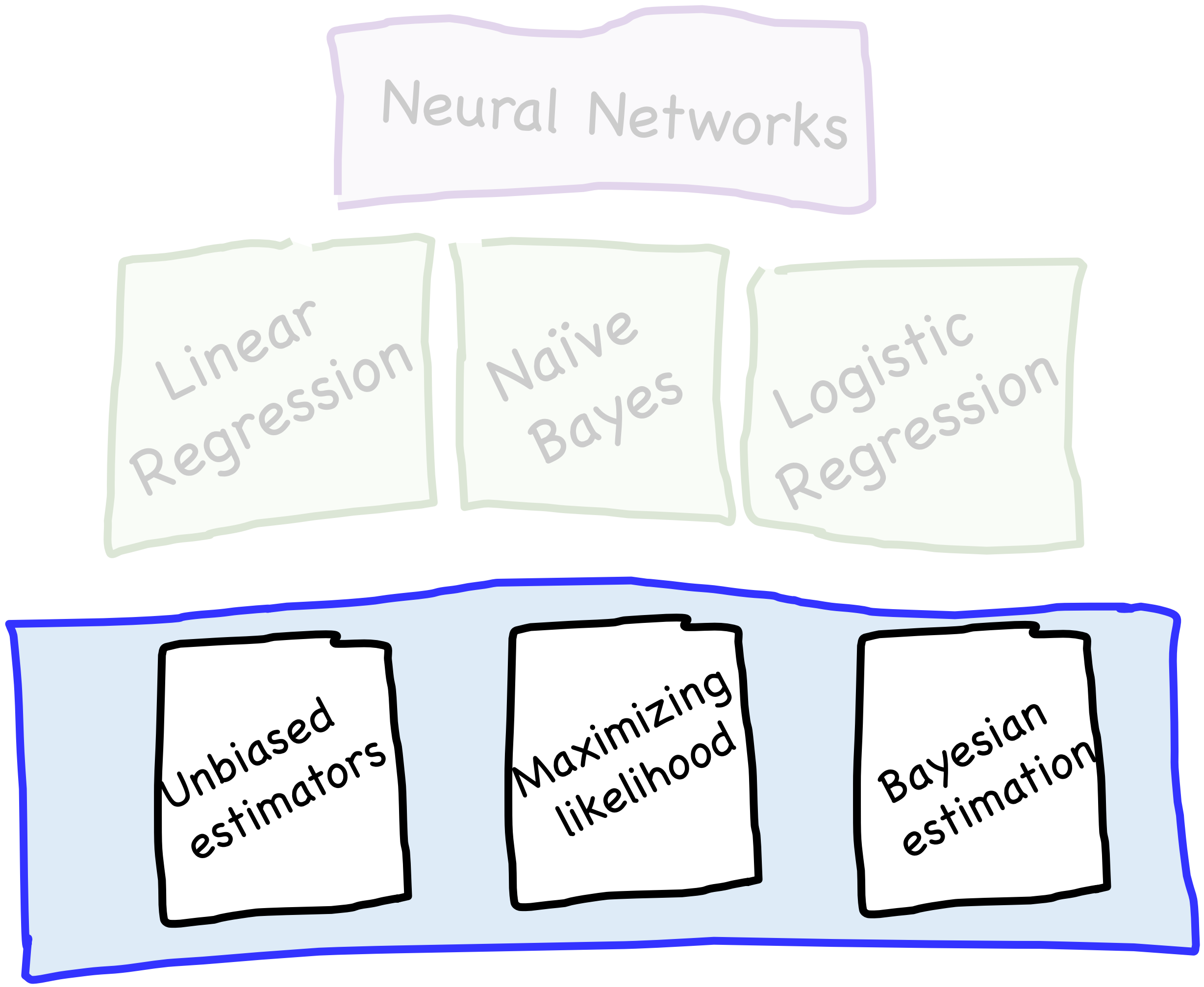
Parameter Estimation



Recall Sample Mean + Variance?

- Consider *n* I.I.D. random variables $X_1, X_2, \ldots X_n$
  - X_i have distribution *F* with $E[X_i] = \mu$ and $Var(X_i) = \sigma^2$
  - We call sequence of X_i a **sample** from distribution *F*
  - Recall sample mean: $\bar{X} = \sum_{i=1}^{n} \frac{X_i}{n}$ where $E[\bar{X}] = \mu$

    $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$ as $n \to \infty$
  - Recall sample variance:

$$S^2 = \sum_{i=1}^{n} \frac{(X_i - \bar{X})^2}{n-1} = \text{undefined}$$

Estimate parameters for Bernoulli and Normal

Limited tool: how could we use that for fitting a “Mixture of Gaussians”?

Great idea in Machine Learning

Likelihood of Data

- Consider *n* I.I.D. random variables $X_1, X_2, \ldots, X_n$
  - X_i is a sample from density function $f(X_i \mid \theta)$
    - Note: now explicitly specify parameter θ of distribution

Likelihood question:
How likely is the data given the samples?

$$\text{Likelihood}(\theta) = f(\text{Samples}|\theta)$$

Demo


L-R
30
40
50
60

Likelihood of Data

- Consider *n* I.I.D. random variables $X_1, X_2, ..., X_n$
  - X_i is a sample from density function $f(X_i \mid \theta)$
    - Note: now explicitly specify parameter θ of distribution
  - We want to determine how "likely" the observed data $(x_1, x_2, ..., x_n)$ is based on density $f(X_i \mid \theta)$
  - Define the **<u>Likelihood function</u>**, $L(\theta)$:

$$L(\theta) = \prod_{i=1}^{n} f(X_i \mid \theta)$$

    - This is just a product since X_i are I.I.D.
  - Intuitively: what is probability of observed data using density function $f(X_i \mid \theta)$, for some choice of θ

Maximum Likelihood Estimator

- The **Maximum Likelihood Estimator** (MLE) of θ, is the value of θ that maximizes $L(\theta)$
  - More formally: $\theta_{MLE} = \arg\max_{\theta} L(\theta)$

Likelihood (of data given parameters):

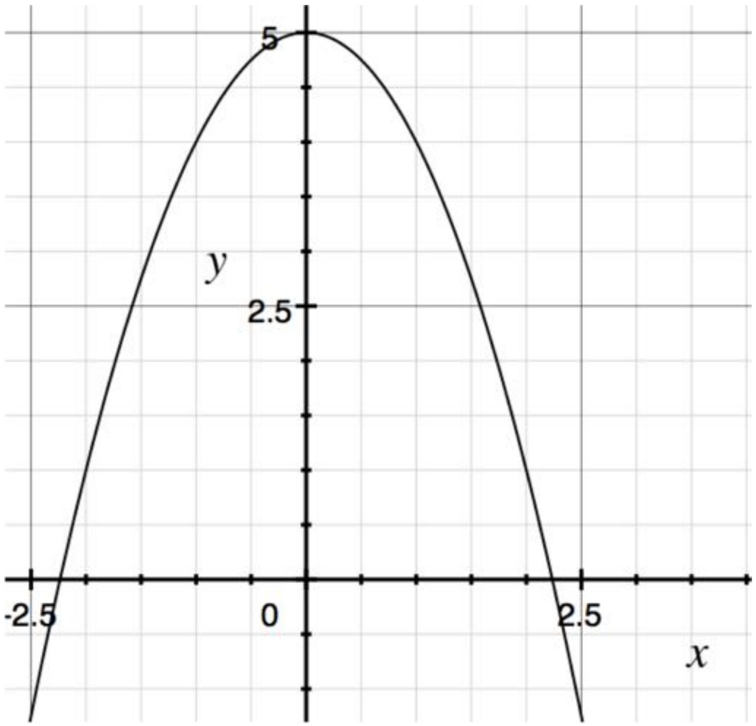
$$L(\theta) = \prod_{i=1}^{n} f(X_i \mid \theta)$$

Argmax

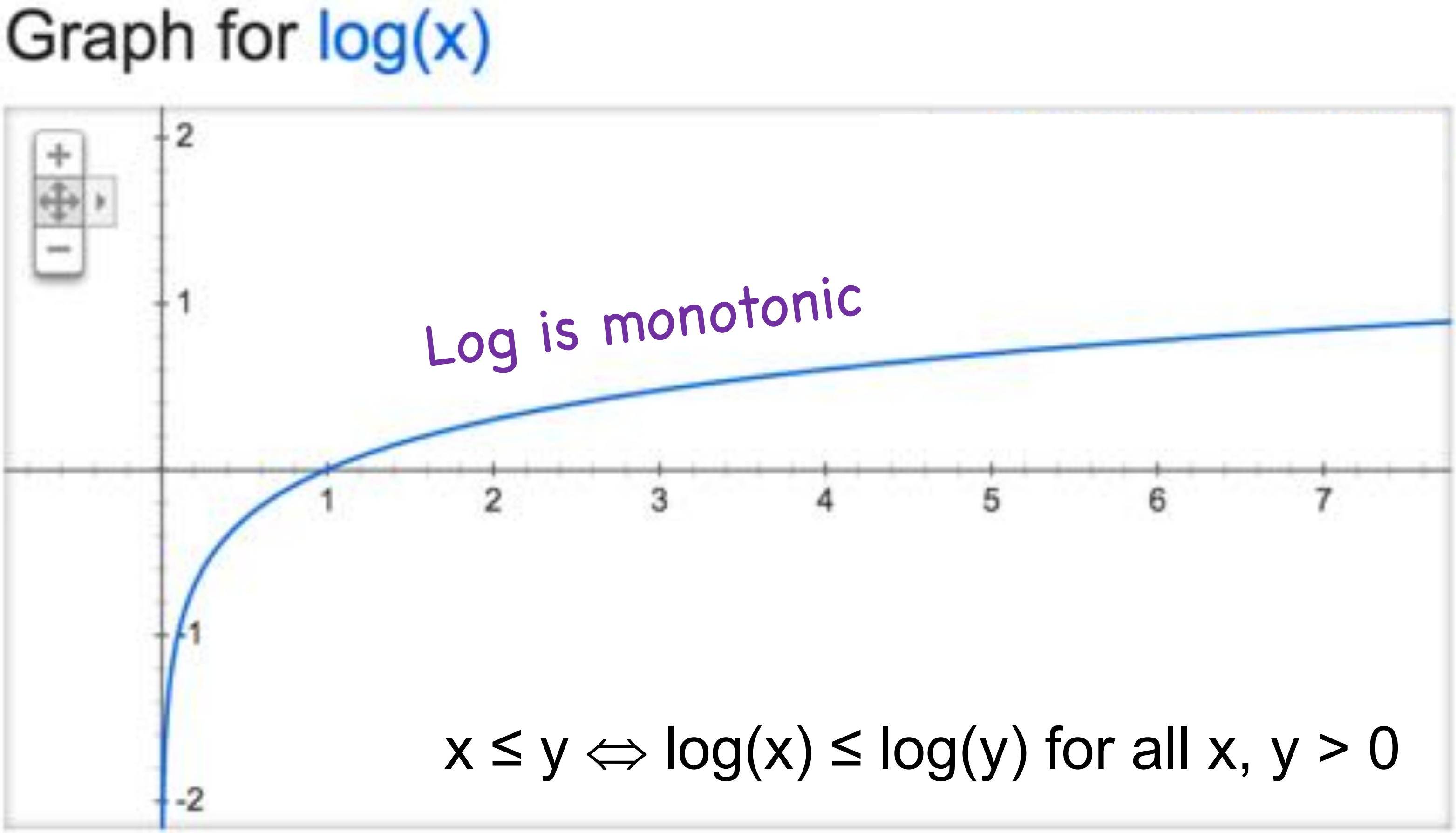
$$f(x) = -x^2 + 5$$

$$\max_x \; -x^2 + 5 = 5$$

$$\operatorname{argmax}_x \; -x^2 + 5 = 0$$



Argmax of Log



Claim: $\underset{x}{\operatorname{argmax}}\ f(x) = \underset{x}{\operatorname{argmax}}\ \log f(x)$

Argmax of Log

$$\underset{x}{\operatorname{argmax}} \; f(x) = \underset{x}{\operatorname{argmax}} \; \log f(x)$$

Log I Love You

$$\log(ab) = \log(a) + \log(b)$$

Natural Log

$$\log(x)$$

$$\log_e(x)$$

$$\ln(x)$$

Maximum Likelihood Estimator

- The **Maximum Likelihood Estimator** (MLE) of θ, is the value of θ that maximizes $L(\theta)$
  - More formally: $\theta_{MLE} = \arg\max_{\theta} L(\theta)$
  - More convenient to use **log-likelihood function**, $LL(\theta)$:

$$LL(\theta) = \log L(\theta) = \log \prod_{i=1}^{n} f(X_i \mid \theta) = \sum_{i=1}^{n} \log f(X_i \mid \theta)$$

  - θ that maximizes $LL(\theta)$ also maximizes $L(\theta)$
    - Formally: $\arg\max_{\theta} LL(\theta) = \arg\max_{\theta} L(\theta)$
    - Similarly, for any positive constant c (not dependent on θ):

$$\arg\max_{\theta}(c \cdot LL(\theta)) = \arg\max_{\theta} LL(\theta) = \arg\max_{\theta} L(\theta)$$

Story so far: We can chose parameters by finding the argmax of the log likelihood of our data

Maximum Likelihood

$$L(\theta) = \prod_{i=1}^{n} f(X_i \mid \theta)$$

$$LL(\theta) = \sum_{i=1}^{n} \log f(X_i|\theta)$$

$$\hat{\theta} = \underset{\theta}{\text{argmax}}\ LL(\theta)$$

But how do we compute argmax?

Option #1: Straight optimization

Computing the MLE

- General approach for finding MLE of θ
  - Determine formula for $LL(\theta)$
  - Differentiate $LL(\theta)$ w.r.t. (each) θ : $\frac{\partial LL(\theta)}{\partial \theta}$
  - To maximize, set $\frac{\partial LL(\theta)}{\partial \theta} = 0$
  - Solve resulting (simultaneous) equations to get θ_{MLE}
    - Make sure that derived $\hat{\theta}_{MLE}$ is actually a maximum (and not a minimum or saddle point). E.g., check $LL(\theta_{MLE} \pm \varepsilon) < LL(\theta_{MLE})$
      - This step often ignored in expository derivations
      - So, we'll ignore it here too (and won't require it in this class)

Maximizing Likelihood with Bernoulli

- Consider I.I.D. random variables $X_1, X_2, ..., X_n$
  - $X_i \sim \text{Ber}(p)$
  - Probability mass function, $f(X_i \mid p)$:

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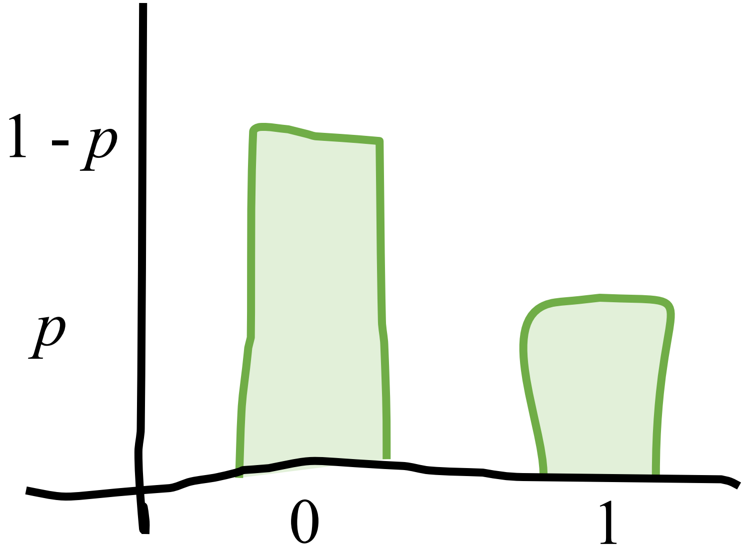
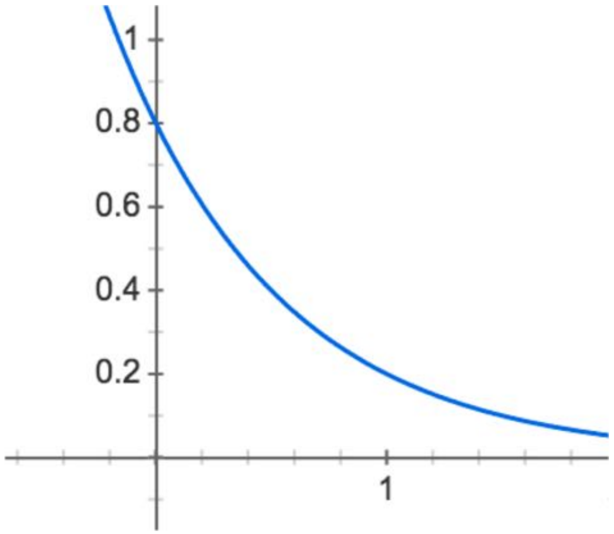
PMF of Bernoulli



$$f(X_i \mid p) = p^{x_i}(1-p)^{1-x_i}$$

PMF of Bernoulli (p = 0.2)



$$f(x) = 0.2^x(1-0.2)^{1-x}$$

Bernoulli PMF

$$X \sim \mathrm{Ber}(p)$$

$$f(X = x|p) = p^x (1-p)^{1-x}$$

Maximizing Likelihood with Bernoulli

- Consider I.I.D. random variables $X_1, X_2, ..., X_n$
  - $X_i \sim \text{Ber}(p)$
  - Probability mass function, $f(X_i \mid p)$, can be written as:

$$f(X_i \mid p) = p^{x_i}(1-p)^{1-x_i} \quad \text{where } x_i = 0 \text{ or } 1$$

  - Likelihood: $L(\theta) = \prod_{i=1}^{n} p^{X_i}(1-p)^{1-X_i}$
  - Log-likelihood:

$$LL(\theta) = \sum_{i=1}^{n} \log(p^{X_i}(1-p)^{1-X_i}) = \sum_{i=1}^{n} \left[X_i(\log p) + (1-X_i)\log(1-p)\right]$$

$$= Y(\log p) + (n-Y)\log(1-p) \quad \text{where } Y = \sum_{i=1}^{n} X_i$$

  - Differentiate w.r.t. p, and set to 0:

$$\frac{\partial LL(p)}{\partial p} = Y\frac{1}{p} + (n-Y)\frac{-1}{1-p} = 0 \quad \Rightarrow \quad p_{MLE} = \frac{Y}{n} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

Isn't that the same as unbiased estimator?

Yes. For Bernoulli.

Maximum Likelihood Algorithm

1. Decide on a model for the distribution of your samples. Define the PMF / PDF for your sample.

2. Write out the log likelihood function.

3. State that the optimal parameters are the argmax of the log likelihood function.

4. Use an optimization algorithm to calculate argmax

Maximizing Likelihood with Poisson

- Consider I.I.D. random variables X_1, X_2, ..., X_n
  - $X_i \sim \text{Poi}(\lambda)$
  - PMF: $f(X_i \mid \lambda) = \frac{e^{-\lambda} \lambda^{x_i}}{x_i!}$ Likelihood: $L(\theta) = \prod_{i=1}^{n} \frac{e^{-\lambda} \lambda^{X_i}}{X_i!}$
  - Log-likelihood:

$$LL(\theta) = \sum_{i=1}^{n} \log\left(\frac{e^{-\lambda} \lambda^{X_i}}{X_i!}\right) = \sum_{i=1}^{n} \left[-\lambda \log(e) + X_i \log(\lambda) - \log(X_i!) \right]$$

$$= -n\lambda + \log(\lambda) \sum_{i=1}^{n} X_i - \sum_{i=1}^{n} \log(X_i!)$$

  - Differentiate w.r.t. λ, and set to 0:

$$\frac{\partial LL(\lambda)}{\partial \lambda} = -n + \frac{1}{\lambda} \sum_{i=1}^{n} X_i = 0 \quad \Rightarrow \quad \lambda_{MLE} = \frac{1}{n} \sum_{i=1}^{n} X_i$$

Its so general!

Maximizing Likelihood with Normal

- Consider I.I.D. random variables $X_1, X_2, ..., X_n$
  - $X_i \sim N(\mu, \sigma^2)$
  - PDF: $f(X_i \mid \mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(X_i-\mu)^2/(2\sigma^2)}$
  - Log-likelihood:

$$LL(\theta) = \sum_{i=1}^{n} \log\left(\frac{1}{\sqrt{2\pi}\sigma} e^{-(X_i-\mu)^2/(2\sigma^2)}\right) = \sum_{i=1}^{n} \left[-\log(\sqrt{2\pi}\sigma) - (X_i - \mu)^2/(2\sigma^2)\right]$$

  - First, differentiate w.r.t. μ, and set to 0:

$$\frac{\partial LL(\mu,\sigma^2)}{\partial \mu} = \sum_{i=1}^{n} 2(X_i - \mu)/(2\sigma^2) = \frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i - \mu) = 0$$

  - Then, differentiate w.r.t. σ, and set to 0:

$$\frac{\partial LL(\mu,\sigma^2)}{\partial \sigma} = \sum_{i=1}^{n} -\frac{1}{\sigma} + 2(X_i - \mu)^2/(2\sigma^3) = -\frac{n}{\sigma} + \sum_{i=1}^{n}(X_i - \mu)^2/(\sigma^3) = 0$$

Being Normal, Simultaneously

- Now have two equations, two unknowns:

$$\frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i-\mu)=0 \qquad -\frac{n}{\sigma}+\sum_{i=1}^{n}(X_i-\mu)^2/(\sigma^3)=0$$

- First, solve for μ_{MLE}:

$$\frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i-\mu)=0 \;\Rightarrow\; \sum_{i=1}^{n}X_i=n\mu \;\Rightarrow\; \mu_{MLE}=\frac{1}{n}\sum_{i=1}^{n}X_i$$

- Then, solve for σ^2_{MLE}:

$$-\frac{n}{\sigma}+\sum_{i=1}^{n}(X_i-\mu)^2/(\sigma^3)=0 \;\Rightarrow\; n\sigma^2=\sum_{i=1}^{n}(X_i-\mu)^2$$

$$\sigma^2_{MLE}=\frac{1}{n}\sum_{i=1}^{n}(X_i-\mu_{MLE})^2$$

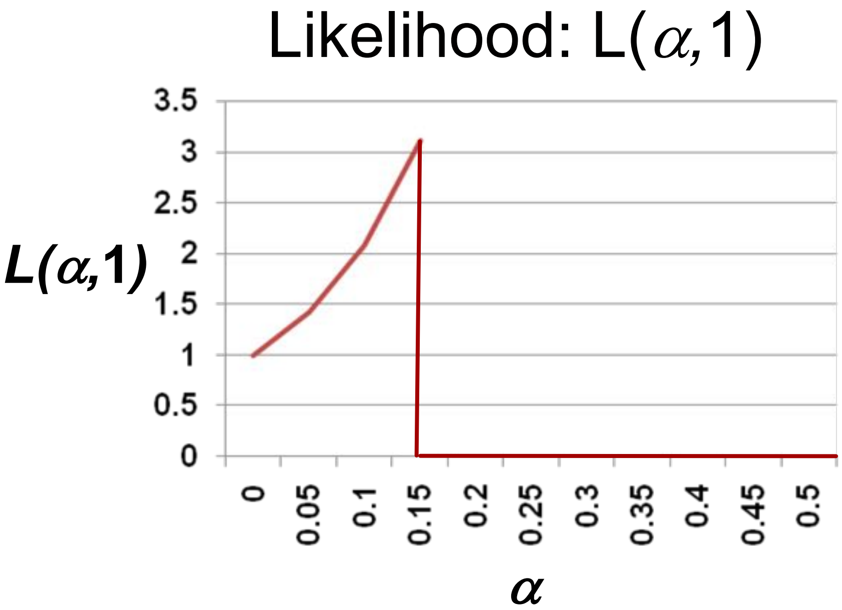
- Note: μ_{MLE} unbiased, but σ^2_{MLE} biased

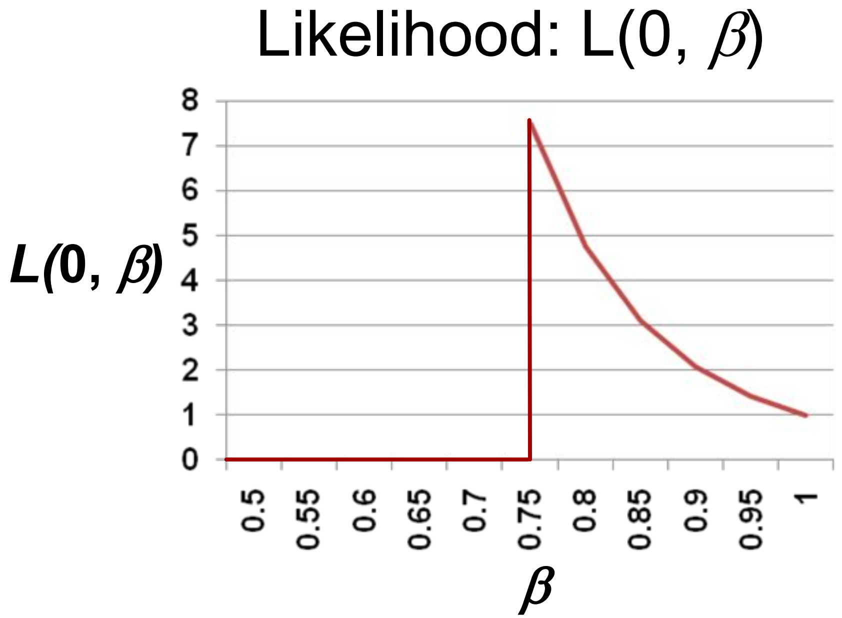
Maximizing Likelihood with Uniform

- Consider I.I.D. random variables $X_1, X_2, ..., X_n$
  - $X_i \sim \text{Uni}(\alpha, \beta)$
  - PDF: $f(X_i \mid \alpha, \beta) = \begin{cases} \frac{1}{\beta - \alpha} & \alpha \le x_i \le \beta \\ 0 & \text{otherwise} \end{cases}$
  - Likelihood: $L(\theta) = \begin{cases} \left(\frac{1}{\beta - \alpha}\right)^n & \alpha \le x_1, x_2, ..., x_n \le \beta \\ 0 & \text{otherwise} \end{cases}$
    - Constraint $\alpha \le x_1, x_2, \ldots, x_n \le \beta$ makes differentiation tricky
    - Intuition: want interval size $(\beta - \alpha)$ to be as small as possible to maximize likelihood function for each data point
    - But need to make sure all observed data contained in interval
      - If all observed data not in interval, then $L(\theta) = 0$
  - Solution: $\alpha_{MLE} = \min(x_1, \ldots, x_n)$ $\beta_{MLE} = \max(x_1, \ldots, x_n)$

Understanding MLE with Uniform

- Consider I.I.D. random variables $X_1, X_2, ..., X_n$
  - $X_i \sim \text{Uni}(0, 1)$
  - Observe data:
    - 0.15, 0.20, 0.30, 0.40, 0.65, 0.70, 0.75





Small Samples = Problems

- How do small samples affect MLE?
  - In many cases, $\mu_{MLE} = \frac{1}{n}\sum_{i=1}^{n} X_i$ = sample mean
    - Unbiased. Not too shabby…
  - As seen with Normal, $\sigma^2_{MLE} = \frac{1}{n}\sum_{i=1}^{n}(X_i - \mu_{MLE})^2$
    - Biased. Underestimates for small *n* (e.g., 0 for n = 1)
  - As seen with Uniform, $\alpha_{MLE} \geq \alpha$ and $\beta_{MLE} \leq \beta$
    - Biased. Problematic for small *n* (e.g., $\alpha = \beta$ when n = 1)
  - Small sample phenomena intuitively make sense:
    - Maximum likelihood $\Rightarrow$ best explain data we've seen
    - Does not attempt to generalize to unseen data

Properties of MLE

- Maximum Likelihood Estimators are generally:
  - **Consistent**: $\lim_{n\to\infty} P(|\hat{\theta}-\theta|<\varepsilon)=1$ for $\varepsilon > 0$
  - Potentially biased (though asymptotically less so)
  - **Asymptotically optimal**
    - Has smallest variance of "good" estimators for large samples
  - **Often used in practice** where sample size is large relative to parameter space
    - But be careful, there are some very large parameter spaces