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CS109

Section #1
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Section #1 Solutions

1. a. This problem is much easier to solve if you try and calculate the probability that there are zero matches from the n people in section. We can solve this problem by setting up a sample space with equally likely outcomes. If the sample space is, how many ways can we assign birthdays to the n students, the outcomes are equally likely, and the number of outcomes can be calculated using the product rule. Note that in this setting I am thinking of the students as being distinct and ordered:

$$|S| = (365)^n$$

How many outcomes from that sample space (assignments of birthdays to students) have no birthday matches? Again we can use the product rule. There are 365 choices of birthdays for the first student, 364 for the second (since it can't be the same birthday as the first student) and so on.

$$|E| = (365) \cdot (364) \ldots (365 - n + 1)$$

$$\begin{aligned} P(\text{birthday match}) &= 1 - P(\text{no matches}) \\ &= 1 - \frac{|E|}{|S|} \\ &= 1 - \frac{(365) \cdot (364) \ldots (365 - n + 1)}{(365)^n} \end{aligned}$$

Interesting values. ($n = 13 : p \approx 0.19$), ($n = 23 : p \approx 0.5$), ($n = 70 : p \geq 0.99$)

b. Again, let's solve this problem by working out the probability of the complement of the event. We can use the exact same sample space as in the above problem, but now for each student there are (365 - 8) = 357 possible days:

$$|E| = (357)^n$$

$$\begin{aligned} P(\text{birthday during section day}) &= 1 - P(\text{no birthdays during section}) \\ &= 1 - \frac{|E|}{|S|} \\ &= 1 - \frac{(357)^n}{(365)^n} \end{aligned}$$

For a section of size 13, $p \approx 0.25$. For a group of size 200, $p \approx 0.99$.

2. This problem requires an application of Bayes' theorem.

$$P(X_1|H) = \frac{P(H|X_1)P(X_1)}{P(H)}$$

The hardest part is to calculate $P(H)$. You can do this using the expanded Bayes' formula, which allows us to calculate $P(H)$ as the weighted sum of the likelihoods of H under the two different song cases. Given that there are two possible songs in our mini Shazam:

$$\begin{aligned} P(X_1|H) &= \frac{P(H|X_1)P(X_1)}{P(H|X_1)P(X_1) + P(H|X_2)P(X_2)} \\ &= \frac{0.50 \cdot 0.80}{0.50 \cdot 0.80 + 0.90 \cdot 0.20} \\ &\approx 0.69 \end{aligned}$$

3. a. We can directly leverage that these are independent events:

$$\begin{aligned} P(M_1 \cap M_2 \cap M_3) &= P(M_1) \cdot P(M_2) \cdot P(M_3) && \text{Independence} \\ &= p_1 \cdot p_2 \cdot p_3 \end{aligned}$$

b. You have two options. You can either use the inclusion exclusions principle, or, you could use De Morgan's rule to turn your 'or' probability into an 'and' probability.

$$\begin{aligned} P(M_1 \cup M_2 \cup M_3) &= 1 - P((M_1 \cup M_2 \cup M_3)^C) && P(E) = 1 - P(E^C) \\ &= 1 - P(M_1^C \cap M_2^C \cap M_3^C) && \text{DeMorgan's} \\ &= 1 - P(M_1^C) \cdot P(M_2^C) \cdot P(M_3^C) && \text{Independence} \\ &= 1 - (1 - p_1) \cdot (1 - p_2) \cdot (1 - p_3) \end{aligned}$$

Alternatively:

$$\begin{aligned} P(M_1 \cup M_2 \cup M_3) &= P(M_1) + P(M_2) + P(M_3) && \text{Inclusion Exclusion} \\ &\quad - P(M_1 \cap M_2) - P(M_1 \cap M_3) - P(M_1 \cap M_2) \\ &\quad + P(M_1 \cap M_2 \cap M_3) \\ &= p_1 + p_2 + p_3 - p_1p_2 - p_1p_3 - p_2p_3 + p_1p_2p_3 && \text{Independence} \end{aligned}$$

4. a. There are 3 possibilities (X, O, empty) for each of the 9 spaces on the board.

$$3^9 = 19683$$

b. From the 9 total spots on the board, choose 3 places to place an X. From the 6 remaining open spots, choose 2 to place an O. This simplifies to

$$\binom{9}{3} \cdot \binom{6}{2} = \frac{9!}{3! \cdot 2! \cdot 4!}$$

c. Use the solution to the last part as your sample space. The outcomes in the event space can be counted using the product rule. Step 1 is to chose which line X has filled (there are 8 such lines). Step 2 is to chose the two locations where O has played out of the remaining 6 locations.

$$|S| = \frac{9!}{3! \cdot 2! \cdot 4!}$$
$$|E| = 8 \cdot \binom{6}{2}$$
$$p = |E|/|S|$$