Chris Piech
CS109

Section #4
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Section #4 Solutions

1. **Are we due for an earthquake?**:

   a. What is the probability of no 8+ earthquakes in four years? Let X be the time until an earthquake. $X \sim \text{Exp}(\lambda = 0.002)$.

$$\begin{aligned} P(X \geq 4) &= 1 - P(X < 4) \\ &= 1 - F_X(4) \\ &= 1 - [1 - e^{-0.002 \cdot 4}] \\ &= e^{-0.008} \approx 0.992 \end{aligned}$$

   b. What is the probability of no 8+ earthquakes in the 113 years?

$$\begin{aligned} P(X \geq 113) &= 1 - P(X < 113) \\ &= 1 - F_X(113) \\ &= 1 - [1 - e^{-0.002 \cdot 113}] \\ &= e^{-0.226} \approx 0.798 \end{aligned}$$

   c. What is $P(X > 113 | X > 109)$?

$$\begin{aligned} P(X > 113 | X > 109) &= \frac{P(X > 113, X > 109)}{P(X > 109)} \\ &= \frac{P(X > 113)}{P(X > 109)} = \frac{1 - F_X(113)}{1 - F_X(109)} \\ &= \frac{e^{-0.002 \cdot 113}}{e^{-0.002 \cdot 109}} = e^{-0.008} \approx 0.992 \end{aligned}$$

   d. It turns out that exponentials are what we call a "memoryless distribution." If X is an exponential random variable, it holds that $P(X > s + t | X > t) = P(X > s)$.

2. **ReCaptcha**

   a. What the the probability density function of a robot clicking $X = x$ mm from the left of the box and $Y = y$ mm from the top of the box?

$$f_{X,Y}(x, y) = \begin{cases} \frac{1}{100} & \text{if } 0 < x, y < 10 \\ 0 & \text{else} \end{cases}$$

b. What is the probability density function of a human clicking $X = x$ mm from the left of the box and $Y = y$ mm from the top of the box?

$$
\begin{aligned}
f_{X,Y}(x, y) &= f_X(x) f_Y(y) && \text{independence} \\
&= \frac{1}{(2\sqrt{2\pi})^2} e^{-\frac{(x-5)^2}{8}} e^{-\frac{(y-5)^2}{8}} && \text{normal PDF} \\
&= \frac{1}{8\pi} e^{-\frac{(x-5)^2}{8}} e^{-\frac{(y-5)^2}{8}} && \text{normal PDF}
\end{aligned}
$$

c. The visitor clicks in the box at ($x = 6$ mm, $y = 6$ mm). What is Google's new belief that the visitor is a robot?

We can start by expanding Bayes theorem for the new belief of a Robot

$$
P(\text{Robot}|X = 6, Y = 6) = \frac{f(X = 6, Y = 6|\text{Robot})P(\text{Robot})}{f(X = 6, Y = 6)}
$$

The two terms on the top are both ones that we can calculate from formulas that we have. The denominator is more problematic: it asks, what is the density of a click two pixels away if we don't know whether the user is a Robot or a Human. The answer is to use the law of total probability, just like in the past.

To make life a bit easier, lets define Click to be the event that the user clicked at location $X = 6, Y = 6$. Since all users are either humans or robots, $f(\text{Click}) = f(\text{Robot, Click}) + f(\text{Human, Click})$. As such:

$$
\begin{aligned}
P(\text{Robot}|\text{Click}) &= \frac{f(\text{Click}|\text{Robot})P(\text{Robot})}{f(\text{Click})} \\
&= \frac{f(\text{Click}|\text{Robot})P(\text{Robot})}{f(\text{Click}|\text{Robot})P(\text{Robot}) + f(\text{Click}|\text{Human})P(\text{Human})} \\
&= \frac{\frac{1}{100} \cdot 0.2}{\frac{1}{100} \cdot 0.2 + \frac{1}{8\pi} e^{-\frac{(1)^2}{8}} e^{-\frac{(1)^2}{8}} \cdot 0.8} \\
&= \frac{\frac{1}{100} \cdot 0.2}{\frac{1}{100} \cdot 0.2 + \frac{1}{8\pi} e^{-\frac{1}{4}} \cdot 0.8} \\
&= \frac{0.002}{0.002 + 0.0248} \approx 0.075
\end{aligned}
$$

3. It's Complicated

a. For each assignment to R, sum over all the values that S can take on that is consistent with that assignment.

Single = 0.44
In a Relationship = 0.47
It's Complicated = 0.09

b. Single = 0.125
In a Relationship = 0.875
It's Complicated = 0.00

c. Freshman = 0.33
Sophomore = 0.39
Junior = 0.48
Senior = 0.88
5+ = 0.59