Noah Arthurs
CS109

Poisson Distribution

Poisson random variables will be the third main discrete distribution that we expect you to know well. After introducing Poisson, we will quickly introduce three more. I want you to be comfortable with being told the semantics of a distribution, given the key formulas (for expectation, variance and PMF) and then using it.

1 Binomial in the Limit

Algorithmic ride sharing started as an interesting computer science research project and now with companies like Lyft and Uber, has entered the daily lives of people around the world. One of the key questions that needed to be solved was: what is the probability of getting one request, two requests, etc from a particular location. Consider Bernal Heights bellow. By looking through historical records we can conclude that the average number of requests per minute is $\lambda = 5$. Can we calculate the probabilities we want from just this average? Let's start with a clever application of the Binomial.

Formally, let X be the number of requests in the next minute. What is $P(X = k)$?

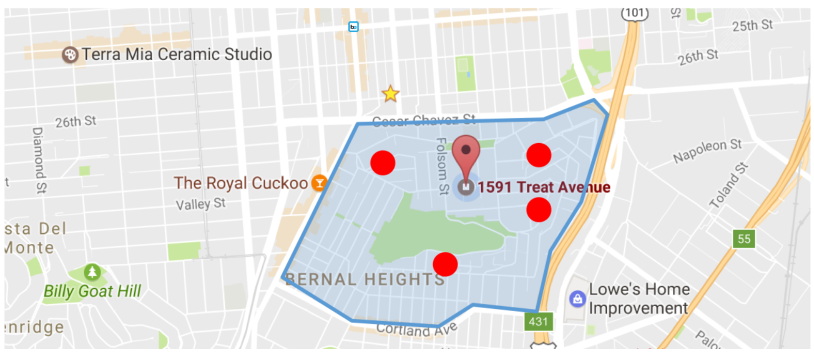
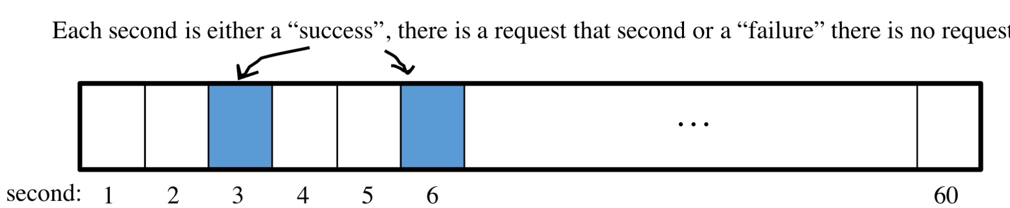


Figure 1: For this area, you have observed a historical average of $\lambda = 5$ requests per minute.

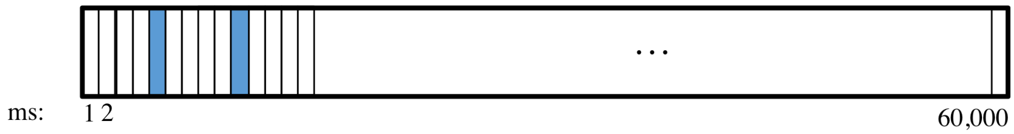
Let's do something very clever. We can take that next minute and split it up into seconds. For each of the 60 seconds we can have a Bernoulli event which is a success if there is a request in that second, and a failure if there is no request. Since we know that the average rate is 5 requests per minute, if we think that a request is equally likely for any second, the probability of success for each Bernoulli should be $5/60$:



This is great. We can now think of X, the number of requests in the next minute as the sum of 60 independent Bernoulli, or in other words, a Binomial: $X \sim \text{Bin}(n = 60, p = 5/60)$ and as such we can solve our original question:

$$P(X = k) = \binom{60}{k} (5/60)^k (1 - 5/60)^{60-k} \qquad \text{Since } X \text{ is a Binomial with } n = 60 \text{ and } p = 5/60$$

Fantastic! But at this point you are probably wondering: why break the minute into seconds, that feels quite arbitrary. And perhaps more concerning, its entirely possible to have two requests in the same second. We can do better than seconds. Let's create a Bernoulli event for every milli-second:



This seems better. Its much less likely that we have two events in the same milli second. Again we can model X as a Binomial. This time $X \sim \text{Bin}(n = 60000, p = 5/n)$. Of course, we would be even more accurate if we thought about having closer and closer to infinite divisions of that next minute. Consider having n divisions as $n \to \infty$:

$$X \sim \text{Bin}(n, p = \lambda/n)$$

$$\begin{aligned}
P(X = k) &= \lim_{n\to\infty} \binom{n}{k} (\lambda/n)^k (1-\lambda/n)^{n-k} \\
&= \lim_{n\to\infty} \frac{n!}{(n-k)!k!} \cdot \frac{\lambda^k}{n^k} \cdot \frac{(1-\lambda/n)^n}{(1-\lambda/n)^k} && \text{By expanding each term} \\
&= \lim_{n\to\infty} \frac{n!}{(n-k)!k!} \cdot \frac{\lambda^k}{n^k} \cdot \frac{e^{-\lambda}}{1} && \text{By definition of natural exp} \\
&= \lim_{n\to\infty} \frac{n!}{(n-k)!n^k} \cdot \frac{\lambda^k}{k!} \cdot \frac{e^{-\lambda}}{1} && \text{Rearranging terms} \\
&= \lim_{n\to\infty} \frac{n^k}{n^k} \cdot \frac{\lambda^k}{k!} \cdot \frac{e^{-\lambda}}{1} && \text{Limit analysis} \\
&= \frac{\lambda^k e^{-\lambda}}{k!} && \text{Simplifying}
\end{aligned}$$

This equation was made simpler by observing how some of the terms evaluate when n tends towards infinity:

$$\frac{n!}{(n-k)!} = n^k \qquad (1-\lambda/n)^n = e^{-\lambda} \qquad (1-\lambda/n)^k = 1$$

This final result is that:

$$P(X = k) = \frac{\lambda^k}{k!} e^{-\lambda}$$

2 Poisson Random Variable

A Poisson random variable is ideal for calculating the number of occurrences of an event in a time interval, when you only know the historical rate of that event.

Here are the key formulas you need to know for Poisson. If $Y \sim Poi(\lambda)$:

$$P(Y = i) = \frac{\lambda^i}{i!} e^{-\lambda}$$
$$E[Y] = \lambda$$
$$Var(Y) = \lambda$$

Example

Historically there is an average of 2.79 earthquakes around the world per year. What is the probability of exactly one earthquake next year? Let X be the number of earthquakes next year.

$$
\begin{aligned}
X &\sim \text{Poi}(\lambda = 2.79) \\
P(X = k) &= \frac{\lambda^k}{k!} e^{-\lambda} && \text{By the Poisson PMF} \\
P(X = 1) &= \frac{2.79^1}{1!} e^{-2.79} \approx 0.17 && \text{Since } \lambda = 2.79
\end{aligned}
$$

Example

Consider requests to a web server in 1 second. In the past, server load averages 2 hits/second. Let X = # hits server receives in a second. What is $P(X < 5)$?

$$
\begin{aligned}
X &\sim \text{Poi}(\lambda = 2) \\
P(X < 5) &= \sum_{i=0}^{4} P(X = i) \\
&= \sum_{i=0}^{4} e^{-\lambda} \frac{\lambda^i}{i!} && \text{Since } X \text{ is Poisson} \\
&= \sum_{i=0}^{4} e^{-2} \frac{2^i}{i!} \approx 0.95 && \text{Since } \lambda = 2
\end{aligned}
$$

3 Approximating Binomial

This same logic that we used to prove that a Binomial in the limit as $n \to \infty$ and $p = \lambda/n$ can be used to show that a Poisson is a great approximation for a Binomial when the Binomial has extreme values of n and p. A Poisson random variable approximates Binomial where n is large, p is small, and $\lambda = np$ is "moderate". Interestingly, to calculate the things we care about (PMF, expectation, variance) we no longer need to know n and p. We only need to provide λ which we call the rate.

There are different interpretations of "moderate". The accepted ranges are $n > 20$ and $p < 0.05$ or $n > 100$ and $p < 0.1$.

Example

Let's say you want to send a bit string of length $n = 10^4$ where each bit is independently corrupted with $p = 10^{-6}$. What is the probability that the message will arrive uncorrupted? You can solve this using a Poisson with $\lambda = np = 10^4 10^{-6} = 0.01$. Let $X \sim Poi(0.01)$ be the number of corrupted bits. Using the PMF for Poisson:

$$
\begin{aligned}
P(X = 0) &= \frac{\lambda^i}{i!} e^{-\lambda} \\
&= \frac{0.01^0}{0!} e^{-0.01} \\
&\sim 0.9900498
\end{aligned}
$$

We could have also modelled X as a binomial such that $X \sim Bin(10^4, 10^{-6})$. That would have been impossible to calculate on a computer but would have resulted in the same number (up to the millionth decimal).